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RETRO BANACH FRAMES, ALMOST EXACT RETRO BANACH FRAMES IN BANACH SPACES

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Abstract. In this paper, we give characterizations of retro Banach frames in Banach spaces. The notion of almost exact retro Banach frame is defined and a characterization of retro Banach frame has been given. Also results exhibiting relationship between frames, almost exact retro Banach frames and Riesz bases has been proved. Finally, we give some perturbation results of retro Banach frames and an almost exact retro Banach frames for Banach spaces.

1. Introduction and Preliminaries

Duffin and Schaeffer [7] introduced the notion of frames. Let $\mathcal{H}$ be a real (or complex) separable Hilbert space with inner product $\langle .,\rangle$. A countable sequence $\{f_k\} \subset \mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$, if there exist numbers $A,\ B > 0$ such that

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

The scalars A and B are called the *lower* and *upper frame bounds* of the frame, respectively. They are not unique. The inequality in (1.1) is called the *frame inequality* of the frame. Feichtinger and Gröcheing [9] extended the notion of frames to Banach space and defined the notion of atomic decomposition. Gröcheing [10] introduced a more general concept for Banach spaces called Banach frame. Casazza, Christensen and Stoeva [3] studied E_d-frame and E_d-Bessel sequence. For more on the theory of frames, one may refer to [5]. Recall that a BK-space is, by definition, a Banach (scalar) sequence space in which the coordinate functionals are continuous.

Definition 1.1. [3] Let E be a Banach space and E_d be a BK-space. A sequence $\{f_n\}_{n=1}^{\infty} \subseteq E^*$ is called an *E_d-frame* for E if

(1) $\{f_n(x)\} \in E_d$, for all $x \in E$,
(2) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A \parallel x \parallel_E \leq \parallel \{f_n(x)\} \parallel_{E_d} \leq B \parallel x \parallel_E, \text{ for all } x \in E. \tag{1.2}$$

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A and B are called E_d-*frame* bounds. If at least (a) and the upper bound condition in (1.2) are satisfied, then $\{f_n\}$ is called an E_d-*Bessel sequence* for E. If $\{f_n\}$ is an E_d-*Bessel sequence* for E, then a $U : E \to E_d$ given by

$$U(x) = \{f_n(x)\}, \text{ for } x \in E$$

is a bounded linear operator and U is called the *analysis operator* associated to E_d-*Bessel sequence* $\{f_n\}$. If $\{f_n\}$ is an E_d-*frame* and there exists a sequence $\{x_n\} \subseteq E$ such that $x = \sum\limits_{n=1}^{\infty} f_n(x)x_n$, for all $x \in E$, then a pair (x_n, f_n) is called an *atomic decomposition* for E with respect E_d. Further, if $\{f_n\}$ is an E_d-*frame* for E and there exists a bounded linear operator $S : E_d \longrightarrow E$ such that $S(\{f_n(x)\}) = x$ for all $x \in$E, then a pair $(\{f_n\}, S)$ is called a *Banach frame* for E with respect to E_d.

In [14] Stoeva defined and studied E_d-Riesz bases.

Definition 1.2. [14] Let E be a Banach space and E_d be a BK-space. The sequence $\{x_n\}_{n=1}^{\infty} \subseteq E$ is called an E_d-Riesz basis for E, if it is complete in E and there exist constants $0 < A \leq B < \infty$ such that for every $\{c_n\}_{n=1}^{\infty} \in E_d$ one has

$$A\|\{c_n\}_{n=1}^{\infty}\|_{E_d} \leq \|\sum_{n=1}^{\infty} c_n x_n\|_E \leq B\|\{c_n\}_{n=1}^{\infty}\|_{E_d} \tag{1.3}$$

The number A (resp. B) in (1.3) is called a lower (resp. upper) E_d-Riesz basis bound.

Next, we give few results in the form of lemmas which will be used in the subsequent work.

Lemma 1.3. [3] *Let E_d be a BK-space for which the canonical unit vectors $\{e_n\}$ form a Schauder basis. Then the space $Y_d = \{\{h(e_n)\} : h \in E_d^*\}$ with norm $\|\{h(e_n)\}\|_{Y_d} = \|h\|_{E_d^*}$ is a BK-space isometrically isomorphic to E_d^*. Also, every continuous linear functional Φ on E_d has the form $\Phi\{c_n\} = \sum\limits_{n=1}^{\infty} c_n d_n$, where $\{d_n\} \in Y_d$ is uniquely determined by $d_n = \Phi(e_n)$, and $\|\Phi\| = \|\{\Phi(e_n)\}\|_{Y_d}$.*

Lemma 1.4. [15] *Let X, Y be Banach spaces and $S \in B(X, Y)$. Then the following are equivalent.*

(1) *S has a pseudoinverse operator $S^\dagger$. i.e. $S^\dagger$: $SS^\dagger S = S$.*
(2) *There exist closed subspaces W, Z of X, Y such that*

$$X = kerS \oplus W, \ Y = S(X) \oplus Z.$$

Lemma 1.5. [14] *Let E_d be BK-space which has a sequence of canonical unit vectors as basis and $\{x_n\}_{n=1}^{\infty} \subseteq E$ be a sequence. Then, $\{x_n\}$ is a Riesz basis if and only if the operator T, given by $T\{\alpha_n\}_{n=1}^{\infty} = \sum\limits_{n=1}^{\infty} \alpha_n x_n$ is an isomorphism of E_d onto E.*

Lemma 1.6. [2] *If E is a Banach space, $\lambda_1, \lambda_2 \in [0, 1)$ and $S : E \to E$ is a linear operator satisfying*

$$\|x - S(x)\| \leq \lambda_1\|x\| + \lambda_2\|S(x)\|, \text{ for all } \in E.$$

Then, S is a bounded invertible operator.

Throughout this paper, E will denote a Banach space over the scaler field $\mathbb{K}$ (which is $\mathbb{R}$ or $\mathbb{C}$), E^* the conjugate space of E, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E. Further, E_d denotes a BK-space which has a sequence of canonical unit vectors $\{e_n\}_{n=1}^{\infty}$ as basis, E_d^* the conjugate space of E_d and $Y_d = \{\{h(e_n)\} : h \in E_d^*\}$ denotes a BK-space which is defined in Lemma 1.3.

2. Main Results

In this section, we begin with defining few definitions. In [11, 12] Jain, Kaushik, Vashisht had introduced and studied retro Banach frames for Banach spaces. Further, P.A.Terekhin [16] introduced and studied the notion of frames for Banach spaces.

Definition 2.1. [16] Let E be a Banach space and E_d be a BK-space. A sequence $\{x_n\}_{n=1}^{\infty} \setminus \{0\} \subseteq E$ is called a *frame* for E with respect to E_d if

(1) $\{f(x_n)\} \in Y_d$ for all $f \in E^*$,

(2) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{Y_d} \leq B\|f\|_{E^*}, \text{ for all } f \in E^*. \tag{2.1}$$

We refer (2.1) as the *frame inequalities*. If at least (a) and the upper bound condition in (2.1) are satisfied, then $\{x_n\}$ is called *Bessel sequence* for E with respect to E_d. If $\{x_n\}$ is a frame for E with respect to E_d and there exists a bounded linear operator $\mathcal{J} : Y_d \to E^*$ such that $\mathcal{J}(\{f(x_n)\}) = f$ for all $f \in E^*$, then a pair $(\{x_n\}, \mathcal{J})$ is called a *retro Banach frame* for E^* with respect to Y_d. The operator $\mathcal{J} : Y_d \to E^*$ is called the *reconstruction operator*. If removal one element x_k renders the collection $\{x_n\}_{n\neq k}$ no longer a retro Banach frame, then $(\{x_n\}, \mathcal{J})$ is called an exact retro Banach frame.

In the following result, we give a necessary and sufficient conditions for the existence of frame in E.

Theorem 2.2. *$\{x_n\}$ is a frame for E with respect to E_d if and only if there exists a bounded linear operator $\mathcal{T} : E_d \to E$ from E_d onto E for which $\mathcal{T}(e_n) = x_n$ for all $n \in \mathbb{N}$.*

Proof. Let $f \in E^*$, B be upper bound of frame $\{x_n\}$. By Lemma 1.3, $\{f(x_n)\}=\{\Phi_f(e_n)\}$ for some $\Phi_f \in E_d^*$ and $\|\{f(x_n)\}\| = \|\Phi_f\|$. Let $n, m \in \mathbb{N}$ with $n \leq m$ and $\{c_n\} \in E_d$, then

$$\begin{aligned}
\|\sum_{k=n}^{m} c_k x_k\| &= \sup_{f\in E^*,\|f\|=1} |\sum_{k=n}^{m} c_k f(x_k)| = \sup_{f\in E^*,\|f\|=1} |\sum_{k=n}^{m} c_k \Phi_f(e_k)| \\
&= \sup_{f\in E^*,\|f\|=1} |\Phi_f(\sum_{k=n}^{m} c_k e_k)| \leq \sup_{f\in E^*,\|f\|=1} \|\Phi_f\|\|\sum_{k=n}^{m} c_k e_k\| \\
&= \sup_{f\in E^*,\|f\|=1} \|\{f(x_n)\}\|\|\sum_{k=n}^{m} c_k e_k\| \leq B\|\sum_{k=n}^{m} c_k e_k\|.
\end{aligned}$$

Hence, $\mathcal{T} : E_d \to E$ given by $\mathcal{T}\{c_n\} = \sum\limits_{n=1}^{\infty} c_n x_n$, $\{c_n\} \in E_d$ is well defined bounded linear operator from E_d into E. Moreover, $\mathcal{T}(e_n) = x_n$ for all $n \in \mathbb{N}$. By Lemma 1.3 and for $f \in E^*$ we have

$$\|\{f(x_n)\}\| = \|\{f(\mathcal{T}(e_n))\}\| = \|\{\mathcal{T}^*(f)(e_n)\}\| = \|\mathcal{T}^* f\|$$

and from the frame inequalities we have $\mathcal{T}^*$ is one-one and $\mathcal{T}^*(E^*)$ is closed. Thus by Theorem in [13], p 103], $\mathcal{T}$ is onto.

Conversely, let $\mathcal{T} : E_d \to E$ be well defined bounded linear operator from E_d onto E with $\mathcal{T}(e_n) = x_n$ for all $n \in \mathbb{N}$. So $\mathcal{T}^*$ is one-one and $\mathcal{T}^*(E^*)$ is closed by Theorem in [13], p 103]. Again, by Lemma in [8], p 487], there exists constant $C > 0$ such that $\|f\| \leq C\|\mathcal{T}^*(f)\|$ for all $f \in E^*$. Let $f \in E^*$. Then

$$\{f(x_n)\} = \{f(\mathcal{T}(e_n))\} = \{\mathcal{T}^* f(e_n)\} \in Y_d.$$

Also, by using Lemma 1.3 and for $f \in E^*$ we have

$$\|f\| \leq C\|\mathcal{T}^*(f)\| = \|\{\mathcal{T}^* f(e_n)\}\| = \|\{f(\mathcal{T}(e_n))\}\| = \|\{f(x_n)\}\|.$$

To show the upper inequality,

$$\|\{f(x_n)\}\| = \|\{\mathcal{T}^* f(e_n)\}\| = \|\mathcal{T}^*(f)\| \leq \|\mathcal{T}\|\|f\|, \text{ for all } f \in E^*.$$

Hence, $\{x_n\}$ is a frame for E with respect to E_d. □

Remark 2.3. Note that $\{x_n\} \subseteq E$ is a *Bessel sequence* for E if and only if there exists a bounded linear operator $\mathcal{T} : E_d \to E$ from E_d into E for which $\mathcal{T}(e_n) = x_n$ for all $n \in \mathbb{N}$. The operator $\mathcal{T}$ is called the *synthesis operator* associated with *Bessel sequence* $\{x_n\}$ and $\mathcal{R} : E^* \to Y_d$ given by

$$\mathcal{R}(f) = \{f(x_n)\}, \text{ for } f \in E^*.$$

is called the *analysis operator* associated with *Bessel sequence* $\{x_n\}$. From Lemma 1.3, we know that Y_d is isometrically isomorphic E_d^*. Let $j_d : Y_d \to E_d^*$ be isometrically isomorphism from Y_d onto E_d^*. Then, $\mathcal{T}^* = j_d \circ \mathcal{R}$.

Next, we give the following characterization of retro Banach frame in Banach spaces.

Theorem 2.4. *Let $\{x_n\}$ be frame for E with respect to E_d with bounds A and B. Let $\mathcal{T} : E_d \to E$ and $\mathcal{R} : E^* \to E_d^*$ be synthesis and analysis operators associated to frame $\{x_n\}$. Then, the following conditions are equivalent.*

1. *$\mathcal{T}^*(E^*)$ is complemented subspace of E_d^*.*
2. *There exists a bounded linear operator $\mathcal{J} : Y_d \to E^*$ such that $(\{x_n\}, \mathcal{J})$ is a retro Banach frame for E^* with respect to Y_d.*
3. *$\mathcal{T}^*$ has pseudoinverse $(\mathcal{T}^*)^\dagger$.*
4. *$ker\mathcal{T}$ is complemented subspace of E_d.*
5. *$\mathcal{T}$ has pseudoinverse $\mathcal{T}^\dagger$.*
6. *There exists an E_d-Bessel $\{f_n\}_{n=1}^\infty \subseteq E^*$ such that*
$$x = \sum_{n=1}^{\infty} f_n(x)x_n, \text{ for all } x \in E.$$
7. *$\mathcal{R}$ has pseudoinverse $\mathcal{R}^\dagger$.*

Proof. (5) ⇒ (2) By given hypotheses, $\mathcal{T}$ has a pseudoinverse $\mathcal{T}^\dagger : E \to E_d$ and $\mathcal{T}\mathcal{T}^\dagger$ is a projection from E onto $\mathcal{T}(E_d)$. But, $\mathcal{T}(E_d) = E$. So, $\mathcal{T}\mathcal{T}^\dagger = I_E$ and $I_{E^*} = (\mathcal{T}^*)^\dagger\mathcal{T}^*$. Take $\mathcal{J} = (\mathcal{T}^*)^\dagger j_d : Y_d \to E^*$ and we have

$$f = (\mathcal{T}^*)^\dagger\mathcal{T}^*(f) = (\mathcal{T}^*)^\dagger j_d\mathcal{R}(f) = \mathcal{J}\{f(x_n)\}, \text{ for all } f \in E^*.$$

Hence, $(\{x_n\}, \mathcal{J})$ is a retro Banach frame for E^* with respect to Y_d.
(2) ⇒ (3) Given that $f = \mathcal{J}(\mathcal{R}(f)) = \mathcal{J}j_d^{-1}\mathcal{T}^*(f)$ for all $f \in E^*$. Thus $\mathcal{T}^*\mathcal{J}j_d^{-1}\mathcal{T}^* =$

$\mathcal{T}^*$. Thus, $\mathcal{J}j_d^{-1}$ is a pseudoinverse of $\mathcal{T}^*$.
(1) $\Leftrightarrow$ (3) Since $ker\mathcal{T}^* = \{0\}$ and by Lemma 1.4, it follows.
(3) $\Leftrightarrow$ (5) obvious.
(4) $\Leftrightarrow$ (5) Since, $\mathcal{T}(E_d) = E$ and by Lemma 1.4, it follows.
(7) $\Leftrightarrow$ (3) Obvious.
(6) $\Rightarrow$ (5) By hypothesis, there exists an E_d-Bessel sequence $\{f_n\} \subseteq E^*$ such that

$$x = \sum_{n=1}^{\infty} f_n(x)x_n, \text{ for all } x \in E.$$

Let $U : E \to E_d$ be the associated analysis operator of E_d-Bessel sequence $\{f_n\}$ given by $U(x) = \{f_n(x)\}$, $x \in E$. Then, $\mathcal{T}U = I_E$ and $\mathcal{T}U\mathcal{T} = \mathcal{T}$. Hence, $\mathcal{T}$ has pseudoinverse.

(5) $\Rightarrow$ (6) $\mathcal{T}\mathcal{T}^\dagger$ is a projection from E onto $\mathcal{T}(E_d) = E$. So $\mathcal{T}\mathcal{T}^\dagger = I_E$. Take $f_n = (\mathcal{T}^\dagger)^*(l_n)$, $n \in \mathbb{N}$, where $\{l_n\} \subseteq E_d^*$ is a sequence of coordinate functionals on E_d. So, for $x \in E$, we have

$$f_n(x) = (\mathcal{T}^\dagger)^*(l_n(x)) = l_n(\mathcal{T}^\dagger(x)).$$

This gives $\{f_n(x)\} = \mathcal{T}^\dagger(x) \in E_d$, for all $x \in E$. Further

$$\|\{f_n(x)\}\| \leq \|\mathcal{T}^\dagger\|\|x\|, \text{ for all } x \in E.$$

Thus, $\{f_n\}$ is an E_d-Bessel sequence for E. Also, for $x \in E$, we have

$$x = \mathcal{T}\mathcal{T}^\dagger(x) = \mathcal{T}(\{f_n(x)\}) = \sum_{n=1}^{\infty} f_n(x)x_n.$$

□

Remark 2.5. Let $(\{x_n\}, \mathcal{J})$ be a retro Banach frame for E^* with respect Y_d. By Theorem 2.4, there exists an E_d-Bessel sequence $\{f_n\}$ such that $x = \sum\limits_{n=1}^{\infty} f_n(x)x_n$, for all $x \in \mathbb{N}$. We will call $\{f_n\}$ as the *associated E_d-Bessel sequence* to retro Banach frame $(\{x_n\}, \mathcal{J})$. Let U be the analysis operator of $\{f_n\}$. So, $I_E = \mathcal{T}U$. Also, $U\mathcal{T}U = U$ and $\mathcal{T}U\mathcal{T} = \mathcal{T}$. Moreover, $U\mathcal{T}$ is a projection from E_d onto $U(E)$. So, $E_d = U(E) \oplus kerU\mathcal{T}$. It is obvious that $ker\mathcal{T} \subseteq kerU\mathcal{T}$. Let $\alpha \in kerU\mathcal{T}$, then $U\mathcal{T}(\alpha) = 0$ and $\mathcal{T}U\mathcal{T}(\alpha) = 0$. Thus, $ker\mathcal{T} = kerU\mathcal{T}$. Hence, $E_d = U(E) \oplus ker\mathcal{T}$.

Theorem 2.6. *Let $(\{x_n\}, \mathcal{J})$ be a retro Banach frame for E^* with respect to Y_d. Then, removal of one element x_k from $\{x_n\}_{n=1}^{\infty}$ leaves $\{x_n\}_{n\neq k}$ either incomplete or a frame.*

Proof. Since $(\{x_n\}, \mathcal{J})$ is a retro Banach frame, so by Theorem 2.4 there exists an E_d-Bessel sequence $\{f_n\} \subseteq E^*$ and for $k \in \mathbb{N}$ we have

$$x_k = \sum_{n=1}^{\infty} f_n(x_k)x_n = \mathcal{T}(\{f_n(x_k)\}_{n=1}^{\infty})$$

In the first case let $f_n(x_n) = 1$, for all $n \in \mathbb{N}$. Also, we have $x_k = \mathcal{T}(e_k)$. Then

$$0 = \mathcal{T}(\{f_n(x_k)\}_{n=1}^{\infty} - e_k) = \mathcal{T}(\{f_n(x_k)\}_{n\neq k}).$$

Therefore, $\{f_n(x_k)\}_{n\neq k} \in ker\mathcal{T}$. Let U be the analysis operator of $\{f_n\}$ and $\{f_n(x_k)\}_{n=1}^{\infty} \in U(E)$. Therefore, $\{f_n(x_k)\}_{n\neq k} \in U(E)$. Moreover, from the Remark 2.5 $ker\mathcal{T} \cap U(E) = \{0\}$. Thus, $f_n(x_k) = 0$, for all $n \neq k$. In this case $\{f_n\}$ is a biorthogonal system for $\{x_n\}$ and hence $\{x_n\}_{n\neq k}$ is incomplete.

In the second case, suppose that there is a $k \in \mathbb{N}$ such that $f_k(x_k) \neq 1$. Then we have

$$x = \sum_{n=1}^{\infty} f_n(x)x_n = f_k(x)x_k + \sum_{n\neq k} f_n(x)x_n, \text{ for all } x \in E.$$

and also

$$x_k = \sum_{n=1}^{\infty} f_n(x_k)x_n = f_k(x_k)x_k + \sum_{n\neq k} f_n(x_k)x_n.$$

So, $x_k = A \sum\limits_{n\neq k} f_n(x_k)x_n$, where $A = \dfrac{1}{1 - f_k(x_k)}$. From the above equations we have

$$\begin{aligned} x &= \sum_{n\neq k} (f_k(x)Af_n(x_k) + f_n(x))x_n \\ &= \sum_{n\neq k} (f_k(x)Af_n(x_k) + f_n(x))\mathcal{T}(e_n) \\ &= \mathcal{T}(\sum_{n\neq k} (f_k(x)Af_n(x_k) + f_n(x))e_n) \\ &= \mathcal{T}(\{f_k(x)Af_n(x_k) + f_n(x)\}_{n\neq k}) \end{aligned}$$

But $\{f_k(x)Af_n(x_k) + f_n(x)\}_{n\neq k} \in E_d$, so $\mathcal{T}$ is bounded linear operator from E_d onto E for which $\mathcal{T}(e_n) = x_n$, for all $n \neq k$. So, by Theorem 2.2 $\{x_n\}_{n\neq k}$ is a frame for E with respect to E_d. □

Definition 2.7. A retro Banach frame $(\{x_n\}, \mathcal{J})$ is said to be almost exact if on removal of one element x_k from $\{x_n\}_{n=1}^{\infty}$ leaves $\{x_n\}_{n\neq k}$ not longer a frame.

Next, we give relation between almost retro Banach frame and Schauder basis in Banach spaces.

Theorem 2.8. *Let $(\{x_n\}, J)$ be almost exact retro Banach frame for E^*. Then, $\{x_n\}$ is a Schauder basis for E.*

Proof. Let $\{f_n\}$ be associated E_d-Bessel sequence of retro Banach frame $(\{x_n\}, J)$. In view of Theorem 2.6, $\{f_n\}$ is biorthogonal to $\{x_n\}$. Also, from Theorem 2.4, $x = \sum\limits_{n=1}^{\infty} f_n(x)x_n$, for all $x \in E$.

Hence, $\{x_n\}$ is a Schauder basis for E. □

Remark 2.9. Let $(\{x_n\}, \mathcal{J})$ be a retro Banach frame which is not almost exact. Then, $\{x_n\}$ is not a basis. Indeed, by Definition 2.7, there is a $k \in \mathbb{N}$ such that $\{x_n\}_{n\neq k}$ is a frame. So, $\overline{span}\{x_n\}_{n\neq k} = E$. But, no proper subset of a basis can be complete. Hence, $\{x_n\}$ cannot be a basis.

Next, we give the characterization of almost exact retro Banach frame for E^*.

Theorem 2.10. *Let $\{x_n\}$ be a frame for E with respect to E_d and $\mathcal{T}$ as its synthesis operator. Then the following are equivalent.*

(1) *$\{x_n\}$ is an E_d-Riesz basis.*
(2) *$\mathcal{T}$ is one-one.*

(3) *There exists a bounded linear operator $\mathcal{J} : Y_d \to E^*$ from Y_d into E^* such that $(\{x_n\}, \mathcal{J})$ is almost exact retro Banach frame for E^*.*

Proof. (1) $\Leftrightarrow$ (2) Obvious.
(3) $\Rightarrow$ (2) We know that $\mathcal{T}$ is bounded linear operator from E_d onto E for which $\mathcal{T}(e_n) = x_n$, for all $n \in \mathbb{N}$. From Theorem 2.8, $\{x_n\}$ is a Schauder basis for E. Let $\{a_n\} \in E_d$ such that $\mathcal{T}(\{a_n\}) = 0$. Then, $\sum\limits_{n=1}^{\infty} a_n x_n = 0$. Thus, $a_n = 0$ for all $n \in \mathbb{N}$. Hence, $\mathcal{T}$ is one-one.

(2) $\Rightarrow$ (3) By given hypothesis, $\mathcal{T}$ is invertible, so also $\mathcal{T}^*$ is also invertible. Take $\mathcal{J} = (\mathcal{T}^*)^{-1} j_d : Y_d \to E^*$, then

$$\mathcal{J}(\{f(x_n)\}) = (\mathcal{T}^*)^{-1} j_d \mathcal{R}(f) = (\mathcal{T}^*)^{-1}\mathcal{T}^*(f) = f, \text{ for all } f \in E^*.$$

Also, it is clear that $\{x_n\}$ is a Schauder basis for E. So for $k \in \mathbb{N}$, $\{x_n\}_{n \neq k}$ is incomplete in E. Hence, $\{x_n\}_{n \neq k}$ is not a frame. □

Remark 2.11. If $(\{x_n\}, J)$ is an almost exact retro Banach frame for E^*, then $\{x_n\}$ is minimal and there exists a sequence $\{f_n\} \subseteq E^*$ which is biorthogonal to $\{x_n\}$.

3. Perturbation of frames and retro Banach frames

Perturbation theory is a very important tool in various area of applied mathematics. In frame theory, it begin with the fundamental perturbation result of Paley and Wiener. P.G.Casazza and O.Christensen [2] studied the perturbation of operator and its application to frame theory. Also, O.Christensen and C.Heil [4], Y.C.Zhu and S.Y.Wang [17] and T. Stoeva [6] gave various results related to the perturbation of atomic decompositions and Banach frames in Banach spaces. In this section, we give some perturbation results related to almost exact retro Banach frames and retro Banach frames in Banach spaces.

Theorem 3.1. *Let $(\{x_n\}, \mathcal{J})$ be an almost exact retro Banach frame for E^* and $\{f_n\} \subset E^*$ be its associated E_d-Bessel sequence. If for every non zero element $x_0 \in E$ there exists a reconstruction operator $J_1 : Y_d \to E^*$ such that $(\{x_n + x_0\}, J_1)$ is a retro Banach frame for E^* with respect to Y_d, then there exists an x_0 such that the retro Banach frame $(\{x_n + x_0\}, J_1)$ is not almost exact.*

Proof. Since, $\{f_n\}$ is orthogonal to $\{x_n\}$ so $f_j(x_i) = \delta_{ji}$, for all $j, i \in \mathbb{N}$. Suppose, $(\{x_n + x_0\}, J_1)$ is almost exact. By Remark 2.11, there a sequence $\{g_n\} \subseteq E^*$ which is orthogonal to $\{x_n + x_0\}$ such that $g_j(x_i + x_0) = \delta_{ji}$ for all $j, i \in \mathbb{N}$. As $x_0 \neq 0$, so there exists $p \in \mathbb{N}$ such that $f_p(x_0) \neq 0$. Let $m \in \mathbb{N}$ such that $m \geq p$ and $a_1,\ a_2,\ a_3, ...,\ a_m$ be any m scalars. Then

$$\begin{aligned} |\sum_{k=1}^{m} a_k f_p(x_k + x_0)| &= |a_p + \sum_{k=1}^{m} a_k f_p(x_0)| \\ &\geq |\sum_{k=1}^{m} a_k||f_p(x_0)| - |a_p| \\ &= |\sum_{k=1}^{m} a_k||f_p(x_0)| - |\sum_{k=1}^{m} a_k g_p(x_k + x_0)|. \end{aligned}$$

This gives us

$$|\sum_{k=1}^{m} a_k| \leq (\frac{\|f_p\| + \|g_p\|}{|f_p(x_0)|})\|\sum_{k=1}^{m} a_k(x_k + x_0)\|.$$

Therefore, by Theorem 5 in [1], p35] there exist linear functional $f \in E^*$ on E such that $f(x_n + x_0) = 1$, for all $n \in \mathbb{N}$. If $f(x_0) = 1$, then $f(x_n) = 0$ for all $n \in \mathbb{N}$. But $\overline{span}\{x_n\}_{n=1}^{\infty} = E$, so $f = 0$ which a contradiction. Thus $f(x_0) \neq 1$. Now take $g = \dfrac{1}{1 - f(x_0)} f$. Then $g \in E^*$ such that $g(x_n) = \dfrac{1}{1 - f(x_0)} f(x_n) = 1$, for all $n \in \mathbb{N}$. Let $y \in E$ such that $g(y) \neq 0$. Take $x_0 = \dfrac{-1}{g(y)} y$ which is a non zero element in E. Then, $g(x_n + x_0) = 1 + \dfrac{-1}{g(y)} g(y) = 0$, for all $n \in \mathbb{N}$. Since, $\overline{span}\{x_n + x_0\}_{n=1}^{\infty} = E$, so $g = 0$ which is a contradiction. Hence, retro Banach frame $(\{x_n + x_0\}, J_1)$ is not almost exact. $\square$

Theorem 3.2. *Let $(\{x_n\}, \mathcal{J})$ be a retro Banach frame for E^* with respect to Y_d with bounds A and B. Let there exist constants $\lambda, \mu > 0$ and $\beta \in [0, 1)$ satisfying $\lambda + \mu\|\mathcal{T}^\dagger\| < 1$, where $\mathcal{T}$ is the associated synthesis operator of retro Banach frame $(\{x_n\}, \mathcal{J})$ and $\mathcal{T}^\dagger$ is the pseudoinverse of $\mathcal{T}$(see Theorem 2.4). Let a sequence $\{y_n\} \subset E$ satisfy the conditions $\{f(y_n)\} \in Y_d$ for all $f \in E^*$ and*

$$\|\sum \alpha_n(x_n - y_n)\| \leq \lambda\|\sum \alpha_n x_n\| + \beta\|\sum \alpha_n y_n\| + \mu\|\alpha\|$$

for all finite sequences $\alpha = \{\alpha_n\} \in E_d$. Then there exists a bounded linear operator $\mathcal{J}_1 : Y_d \to E^*$ such that $(\{y_n\}, J_1)$ is a retro Banach frame for E^* with respect to Y_d with bounds $\dfrac{1 - (\lambda + \mu\|\mathcal{T}^\dagger\|)}{(1 + \beta)\|\mathcal{T}^\dagger\|}$ and $\dfrac{(1 + \lambda)B + \mu}{1 - \beta}$.

Proof. By Theorem 2.4, $\mathcal{T}$ has a pseudoinverse $\mathcal{T}^\dagger$ such that $\mathcal{T}\mathcal{T}^\dagger(x) = x$, for all $x \in E$. By given conditions, for all finite sequences $\alpha = \{\alpha_n\} \in E_d$ we have

$$\|\sum \alpha_n y_n\| \leq \frac{(1 + \lambda)\|\sum \alpha_n x_n\| + \mu\|\alpha\|}{1 - \beta} \leq \frac{(1 + \lambda)B + \mu}{1 - \beta}\|\sum \alpha_n e_n\|.$$

By Cauchy, it follows that the series $\|\sum\limits_{n=1}^{\infty} \alpha_n y_n\|$ is convergent for all $\alpha \in E_d$ and also that the operator $\mathcal{T}_1 : E_d \to E$, given by $\mathcal{T}_1(\alpha) = \sum\limits_{n=1}^{\infty} \alpha_n y_n$, is well defined; moreover, for every $\alpha \in E_d$

$$\|\mathcal{T}_1(\alpha)\| \leq \frac{(1 + \lambda)B + \mu}{1 - \beta}\|\sum_{n=1}^{\infty} \alpha_n e_n\|.$$

Thus, $\{y_n\}$ is a Bessel sequence for E with respect to E_d and $\mathcal{T}_1$ is the associated operator of $\{y_n\}$. Let $\mathcal{R}_1(f) = \{f(y_n)\}$ be the analysis operator of $\{y_n\}$. Then for all $f \in E^*$ we have

$$\|\{f(y_n)\}\| = \|\mathcal{R}_1(f)\| = \|j_d^{-1}\mathcal{T}_1^*(f)\| = \|\mathcal{T}_1^* f\| \leq \frac{(1 + \lambda)B + \mu}{1 - \beta}\|f\|$$

By the given condition and for $x \in E$, we have

$$\begin{aligned}\|x - \mathcal{T}_1\mathcal{T}^\dagger(x)\| &= \|(\mathcal{T} - \mathcal{T}_1)\mathcal{T}^+(x)\| \\ &\leq \lambda\|\mathcal{T}\mathcal{T}^+(x)\| + \beta\|\mathcal{T}_1\mathcal{T}^\dagger(x)\| + \mu\|\mathcal{T}^\dagger(x)\| \\ &\leq (\lambda + \mu\|\mathcal{T}^\dagger\|)\|x\| + \beta\|\mathcal{T}_1\mathcal{T}^\dagger(x)\|\end{aligned}$$

Take $L = \mathcal{T}_1\mathcal{T}^\dagger$, by Lemma 1.6 L is invertible and $L^* = (\mathcal{T}^\dagger)^*\mathcal{T}_1^*$. Now for $f \in E^*$ we have

$$\|f\| = \|(L^*)^{-1}L^*(f)\| \leq \|(L^*)^{-1}\|\|(\mathcal{T}^\dagger)^*\|\|\{f(y_n)\}\|.$$

Also, for $x \in E$ we obtain

$$\begin{aligned}\|L(x)\| &\geq \|x\| - \|(I_E - L)(x)\| \\ &\geq \|x\| - ((\lambda + \mu\|\mathcal{T}^\dagger\|)\|x\| + \beta\|L(x)\|)\end{aligned}$$

From here we get

$$\frac{1 - (\lambda + \mu\|\mathcal{T}^\dagger\|)}{1 + \beta}\|x\| \leq \|L(x)\|, \text{ for } x \in E$$

and

$$\|L^{-1}\| \leq \frac{1 + \beta}{1 - (\lambda + \mu\|\mathcal{T}^\dagger\|)}.$$

Thus, for $f \in E^*$ we have $\dfrac{1 - (\lambda + \mu\|\mathcal{T}^\dagger\|)}{(1 + \beta)\|\mathcal{T}^\dagger\|}\|f\| \leq \|\{f(y_n)\}\|.$

Hence, $\{y_n\}$ is a frame for E with bounds $\dfrac{1 - (\lambda + \mu\|\mathcal{T}^\dagger\|)}{(1 + \beta)\|\mathcal{T}^\dagger\|}$, $\dfrac{(1 + \lambda)B + \mu}{1 - \beta}$. Since, L is invertible so $I_E = LL^{-1} = \mathcal{T}_1\mathcal{T}^+L^{-1}$. That gives us $\mathcal{T}_1 = \mathcal{T}_1\mathcal{T}^\dagger L^{-1}\mathcal{T}_1$. Thus $\mathcal{T}_1$ has pseudoinverse $\mathcal{T}_1^\dagger = \mathcal{T}^\dagger L^{-1}$. Hence, by Theorem 2.4 there exists a bounded linear operator $\mathcal{J}_1 : Y_d \to E^*$ such that $(\{y_n\}, \mathcal{J}_1)$ is retro Banach frame for E^* with respect to Y_d. □

Theorem 3.3. *Let $(\{x_n\}, \mathcal{J})$ be a retro Banach frame for E^* with respect to Y_d with bounds A and B and $\mathcal{T}$, $\mathcal{R}$ be its associated synthesis operator, analysis operator. Let $\lambda, \beta \in [0, 1)$ and $\mu \geq 0$ with $\lambda\|Q\| + \beta\|I - Q\| + \mu\|(\mathcal{T}^*)^\dagger\| < 1$, where $Q = \mathcal{R}\mathcal{R}^\dagger$ is a projection from Y_d onto $\mathcal{R}(E^*)$ and $(\mathcal{T}^*)^\dagger$ is the pseudoinverse of $\mathcal{T}^*$. If a sequence $\{y_n\} \subset E$ satisfies the conditions $\{f(y_n)\} \in Y_d$ for all $f \in E^*$ and $\|\{f(x_n) - f(y_n)\}\| \leq \lambda\|\{f(x_n)\}\| + \beta\|\{f(y_n)\}\| + \mu\|f\|$, for all $f \in E^*$. Then, there exists a bounded linear operator $\mathcal{J}_1 : Y_d \to E^*$ such that $(\{y_n\}, \mathcal{J}_1)$ is a retro Banach frame for E^* with respect to Y_d with bounds $\dfrac{(1 - \lambda)\|(\mathcal{T}^*)^\dagger\|^{-1} - \mu}{1 + \beta}$ and $\dfrac{(1 + \lambda)B + \mu}{1 - \beta}$.*

Proof. By Theorem 2.4, $\mathcal{T}\mathcal{T}^\dagger = I_E$ and therefore $(\mathcal{T}^*)^\dagger\mathcal{T}^* = I_{E^*}$. So, for all $f \in E^*$ we have

$$\|f\| \leq \|(\mathcal{T}^*)^\dagger\|\|\mathcal{T}^*(f)\| = \|(\mathcal{T}^*)^\dagger\|\|j_d\mathcal{R}(f)\| = \|(\mathcal{T}^*)^\dagger\|\|\mathcal{R}(f)\|$$

Define $\mathcal{R}_1 : E^* \to Y_d$ as $\mathcal{R}_1(f) = \{f(y_n)\}$, for $f \in E^*$. By given condition we have

$$|||\mathcal{R}(f) - \mathcal{R}_1(f)||| \leq \lambda\|\mathcal{R}(f)\| + \beta\|\mathcal{R}_1(f)\| + \mu\|f\|, \ f \in E^*.$$

It follows that

$$\begin{aligned}\|\mathcal{R}_1(f)\| &\leq \frac{1+\lambda}{1-\beta}\|\mathcal{R}(f)\| + \frac{\mu}{1-\beta}\|f\| \\ &\leq \frac{(1+\lambda)B+\mu}{1-\beta}\|f\|, \text{ for all } f \in E^*\end{aligned}$$

Therefore, $\{y_n\}$ is a Bessel sequence for E with respect to E_d with $\mathcal{R}_1$ as its analysis operator. Moreover,

$$\begin{aligned}\|\mathcal{R}_1(f)\| &\geq \frac{1-\lambda}{1+\beta}\|\mathcal{R}(f)\| - \frac{\mu}{1+\beta}\|f\| \\ &\geq \frac{(1-\lambda)\|(\mathcal{T}^*)^\dagger\|^{-1}-\mu}{1+\beta}\|f\|, \text{ for all } f \in E^*.\end{aligned}$$

But $1 > \lambda\|Q\|+\beta\|I-Q\|+\mu\|(\mathcal{T}^*)^\dagger\| \geq \lambda+\mu\|(\mathcal{T}^*)^\dagger\| \geq 0$. So $\frac{(1-\lambda)\|(\mathcal{T}^*)^\dagger\|^{-1}-\mu}{1+\beta} > 0$. Therefore $\{y_n\}$ is a frame for E with respect to E_d with bounds $\frac{(1-\lambda)\|(\mathcal{T}^*)^\dagger\|^{-1}-\mu}{1+\beta}$ and $\frac{(1+\lambda)B+\mu}{1-\beta}$. Let $\mathcal{T}_1$ be synthesis operator of frame $\{y_n\}$. As we know, $\mathcal{R}$ has pseudoinverse $\mathcal{R}^\dagger = (\mathcal{T}^*)^\dagger j_d$. Let $\alpha \in Y_d$ and we have

$$\begin{aligned}\|(\mathcal{R}_1-\mathcal{R})\mathcal{R}^\dagger(\alpha)\| &\leq \lambda\|\mathcal{R}\mathcal{R}^\dagger(\alpha)\| + \beta\|\mathcal{R}_1\mathcal{R}(\alpha)\| + \mu\|\mathcal{R}^\dagger(\alpha)\| \\ &\leq (\lambda\|Q\| + \mu\|\mathcal{R}^\dagger\|)\|\alpha\| \\ &+ \beta\|\alpha + (\mathcal{R}_1-\mathcal{R})\mathcal{R}^\dagger(\alpha) - (I^*_{Y_d} - \mathcal{R}\mathcal{R}^\dagger(\alpha)\| \\ &\leq (\lambda\|Q\| + \beta\|I_{Y_d} - Q\| + \mu\|R^\dagger\|)\|\alpha\| \\ &+ \beta\|I_{Y_d} + (\mathcal{R}_1-\mathcal{R})\mathcal{R}^\dagger(\alpha)\|.\end{aligned}$$

Let $L = I_{Y_d} + (\mathcal{R}_1-\mathcal{R})\mathcal{R}^\dagger$, so L is a bounded linear operator from Y_d into Y_d. Also, for $\alpha \in Y_d$ we have

$$\|L(\alpha)-\alpha\| \leq (\alpha\|Q\| + \beta\|I_{Y_d} - Q\| + \mu\|\mathcal{R}^\dagger\|)\|\alpha\| + \beta\|L(\alpha)\|.$$

Thus, by Lemma 1.6 L is invertible. Also $(\mathcal{T}^*)^\dagger T^* = I_{E^*}$, so we have $\mathcal{R}^\dagger\mathcal{R} = (\mathcal{T}^*)^{-1} j_d j_d^{-1}\mathcal{T}^* = (\mathcal{T}*)^\dagger\mathcal{T}^* = I_{E^*}$

$$L\mathcal{R} = (I_{E^*} + (\mathcal{R}_1-\mathcal{R})(\mathcal{R})^\dagger)\mathcal{R} = \mathcal{R}_1$$

But $E_d^* = \mathcal{T}^*(E^*) \oplus Z$, for Z is a closed subspace of E_d^*. So also $Y_d = j^{-1}(E_d^*) = j^{-1}(\mathcal{T}^*(E^*) \oplus j^{-1}(Z) = \mathcal{R}(E^*) \oplus j^{-1}(Z)$. From here we have

$$\begin{aligned}Y_d &= L(Y_d) = L(\mathcal{R}(E^*) \oplus j_d^{-1}(Z)) = L\mathcal{R}(E^*) \oplus L(j_d^{-1}(Z)) \\ &= R_1(E^*) \oplus L(j_d^{-1}(Z)).\end{aligned}$$

Thus,

$$E_d^* = j_d(Y_d) = j_d\mathcal{R}_1(E^*) \oplus j_d L(j_d^{-1}(Z) = \mathcal{T}_1^*(E^*) \oplus j_d L(j_d^{-1}(Z).$$

Hence, by Theorem 2.4, there exists a bounded linear operator $\mathcal{J}_1 : Y_d \to E^*$ such that $(\{y_n\}, \mathcal{J}_1)$ is a retro Banach frame for E^* with respect to Y_d. □

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