Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 7 Issue 2(2015), Pages 37-45.

ON THE HYERS-ULAM-GĂVRUTA STABILITY OF A PEXIDER HOSSZÜS FUNCTIONAL EQUATION

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Abstract. In this paper, we will investigate the Ulam-Rassias-Găvruta stability of the pexiderized Hosszú's functional equation

$$f(x+y+\alpha xy)=g(x)+h(y)+k(xy)$$

for all $x, y \in \mathbb{R}$ and $\alpha \in \mathbb{R}^*$.

1. Introduction

The stability problem of functional equations has been originally raised by S. M. Ulam. In 1940, he posed the following problem: Give conditions in order for a linear mapping near an approximately additive mapping to exist (see [18]).

In 1941, this problem was solved by D. H. Hyers [6] for the first time. This problem has been further generalized and solved by Th. M. Rassias [13]. Thereafter, the stability problem of functional equations has been extended in various directions and studied by several mathematicians (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12]). In [2] we have the study of the Hyers-Ulam stability of the Hosszüs functional equation

$$f(x+y+xy)=f(xy)+f(x)+f(y), \quad x, y \in \mathbb{R}.$$

The main purpose of this work is to study the Ulam-Rassias-Găvruta stability of the pexider Hosszüs functional equation

$$f(x+y+\alpha xy)=g(x)+h(y)+k(xy), \tag{1.1}$$

for all $x, y \in \mathbb{R}$ and $\alpha \in \mathbb{R}^*$, with the condition that

$$|f(x)+f(-x)| \leq \delta+\eta(x).$$

Many investigations were used to establish Hyers-Ulam stability of this equation in the case $\alpha = 1$ (see [1], [2], [11], [13]). In this paper, we will investigate the Ulam-Rassias-Găvruta stability of the pexider Hosszú's functional equation (1.1) and we give some applications.

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2000 *Mathematics Subject Classification.* 39B22, 39B82.
Key words and phrases. Hyers-Ulam-Rassias stability and Hosszú's equation.

Submitted August 18, 2014. Published January 2, 2015.

This paper is a continuation, of the previous work by B. Bouikhalene, J. M. Rassias, A. Charifi and S. Kabbaj of under title "on the approximate solution of Hosszüs functional equation" see [2].

The paper is organized as follows: in the second section after this introduction we give some notions, preliminary results. In the fourth section, we give general solution of eq (1.1), in the fifth section, we study the stability of eq (1.1) and the last section contains some applications.

2. Notations

Throughout this paper we use following notations:

- δ, δ' are two positive real numbers.
- $\eta : \mathbb{R} \longrightarrow \mathbb{R}^+$.
- $\psi : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}^+$ is one application and $\psi_0 = \psi,\ \psi_n(x, y) = \psi_{n-1}(2^\epsilon x, 2^\epsilon y)$ with $n \in \mathbb{N}^*$ and $\epsilon \in \{-1, 1\}$.
- For some application $f : \mathbb{R} \longrightarrow \mathbb{R}$, we define $\theta : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ by $\theta(x, y) = 5\delta' + 2|f(1)| + \psi(x, 2y + 1) + \psi(x + y + xy, 1) + 2\psi(x, y) + \psi(y, 1)$ if one of numbers x or y is non null and $\theta(0, 0) = \delta' + \psi(0, 0)$.
- $\widetilde{\psi}(x, y) = \sum_{i=\frac{1-\epsilon}{2}}^{+\infty} \frac{\psi_{i-1}(2^\epsilon x, 2^\epsilon y)}{2^{i\epsilon+\frac{1-\epsilon}{2}}}$ and consequently $\widetilde{\theta}(x, y) = 5\delta' + 2|f(0)| + \widetilde{\psi}(x, 2y + 1) + \widetilde{\psi}(x + y + xy, 1) + 2\widetilde{\psi}(x, y) + \widetilde{\psi}(y, 1)$.

3. Preliminary results

Lemma 3.1. *Let δ be a positive real number and let $f, g, h, k : \mathbb{R} \longrightarrow \mathbb{R}$, $\varphi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ and $\eta : \mathbb{R} \longrightarrow \mathbb{R}^+$ be functions satisfying the following inequality*

$$|f(x + y + xy) - g(x) - h(y) - k(xy)| \leq \varphi(x, y), \quad x, y \in \mathbb{R}, \tag{3.1}$$

and

$$|f(x) + f(-x)| \leq \delta + \eta(x).$$

Then f, g, h, k satisfies the following inequalities

i) $|f(0) - g(0) - h(0) - k(0)| \leq \varphi(0, 0)$,
ii) $|g(x) - f(x)| \leq \varphi(x, 0) + |g(0)| + |f(0)| + \varphi(0, 0)$,
iii) $|f(x) - h(x)| \leq \varphi(0, 0) + |f(0)| + |h(0)| + \varphi(0, x)$,
iv) $|f(x) - k(x)| \leq \delta + \eta(x) + \varphi(-x, -1) + \varphi(-x, 0) + \varphi(0, 0) + |f(0)| + |g(0)| + |h(-1)| + |f(-1)|$.

Proof. i) Letting $x = y = 0$ in (3.1) we obtain

$$|f(0) - g(0) - h(0) - k(0)| \leq \varphi(0, 0).$$

ii) Replacing y by 0 in (3.1) we get

$$|g(x) - g(0) + f(0) - f(x)| \leq \varphi(x, 0) + \varphi(0, 0),\ x \in \mathbb{R}$$

then we obtain the result.
iii) Replacing (x, y) by $(0, x)$ in (3.1) we obtain

$$|f(x) - f(0) + h(0) - h(x)| \leq \varphi(0, 0) + \varphi(0, x),\ x \in \mathbb{R},$$

and we get the result.
iv) Putting $y = -1$ in (3.1) we find

$$|f(-1) - g(x) - h(-1) - k(-x)| \leq \varphi(x, -1),$$

and using the previous inequalities, we get the result. □

Lemma 3.2. *Let $f, g, h, k : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the functional inequality (3.1). Then f satisfies the inequality*

$$|f(x + y + xy) - f(x) - f(y) - f(xy)| \leq \delta^{'} + \psi(x, y) \tag{3.2}$$

for all $x, y \in \mathbb{R}$, where

$$\delta^{'} = \delta + 3\varphi(0, 0) + |f(-1)| + |h(-1)| + 3|f(0)| + 2|g(0)| + |h(0)|$$

and

$$\psi(x, y) = \varphi(x, y) + \varphi(x, 0) + \varphi(0, y) + \varphi(-xy, -1) + \varphi(-xy, 0) + \eta(xy).$$

Proof. By Lemma 3.1 and f, g, h, k satisfies (3.1) we get

$$\begin{aligned} |f(x + y + xy) - f(x) - f(y) - f(xy)| &\leq |f(x + y + xy) - g(x) - h(y) - k(xy)| \\ &\quad + |f(x) - g(x)| + |f(y) - h(y)| + |f(xy) - k(xy)| \\ &\leq \delta^{'} + \psi(x, y), \end{aligned}$$

where $\delta^{'} = \delta + 3\varphi(0, 0) + |f(-1)| + |h(-1)| + 3|f(0)| + 2|g(0)| + |h(0)|$ and $\psi(x, y) = \varphi(x, y) + \varphi(x, 0) + \varphi(0, y) + \varphi(-xy, -1) + \varphi(-xy, 0) + \eta(xy)$.
Which complete the proof of lemma. □

4. GENERAL SOLUTION OF EQ (1.1)

In this section we give general solution of the functional equation (1.1).

Theorem 4.1. *The functions f, g, h, k satisfies the functional equation (1.1) and f is odd function if only if there exists an additive function $T : \mathbb{R} \to \mathbb{R}$ such that*

$$\begin{aligned} f(x) &= T(\alpha x), \\ g(x) &= T(\alpha x) - h(0) - k(0), \\ h(x) &= T(\alpha x) - g(0) - k(0) \end{aligned}$$

and

$$k(x) = T(\alpha^2 x) + 2k(0) + g(0) + h(0),$$

for all $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}^*$.

Proof. Assume that the functions f, g, h, k satisfying (1.1) and f is odd function. Replacing in the first y by 0 in (1.1) we get

$$g(x) = f(x) - h(0) - k(0),$$

in the second x by 0 and y by x we obtain

$$h(x) = f(x) - g(0) - k(0).$$

Putting $y = \frac{-1}{\alpha}$ in (1.1) and using the previous equalities we find

$$k(x) = f(\alpha x) + 2k(0) + g(0) + h(0),$$

for all $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}^*$. This implies that

$$f(x + y + \alpha xy) = f(x) + f(y) + f(\alpha xy), \tag{4.1}$$

for all $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}^*$. Replacing (x, y) by $(\frac{x}{\alpha}, \frac{y}{\alpha})$ in (4.1)and we put $F(x) = f(\frac{x}{\alpha})$ we obtain

$$F(x + y + xy) = F(x) + F(y) + F(xy), \tag{4.2}$$

for all $x, y \in \mathbb{R}$. By Theorem 4.1 in [2], there there exists an additive function $T : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) = T(x),$$

which implies that

$$f(x) = T(\alpha x),$$

for all $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}^*$.
The converse is obvious. The proof of Theorem is complete. □

5. Stability of eq (1.1)

In this section we establish the Ulam-Rassias-Găvruta stability of equation (1.1).

Theorem 5.1. *Let δ be a positive real number and α a real non-zero and let $f, g, h, k : \mathbb{R} \longrightarrow \mathbb{R}$, $\varphi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ and $\eta : \mathbb{R} \longrightarrow \mathbb{R}^+$ be functions satisfying the following inequality*

$$|f(x + y + \alpha xy) - g(x) - h(y) - k(xy)| \leq \varphi(x, y), \quad x, y \in \mathbb{R} \tag{5.1}$$

and

$$|f(x) + f(-x)| \leq \delta + \eta(x). \tag{5.2}$$

Assume that $\widetilde{\psi}(x, y) < +\infty$. Then there exists a unique quadruplet (T, U, V, W) of functions, $T, U, V, W : \mathbb{R} \longrightarrow \mathbb{R}$ solution of equation (1.1) such that

$$|f(x) - T(x)| \leq \widetilde{\theta}(x, \frac{1}{2\alpha}) + \widetilde{\theta}(x, \frac{-1}{2\alpha}) + |f(0)| + 2(\delta' + \widetilde{\psi}(\frac{x}{2}, \frac{-1}{\alpha})),$$

$$|g(x) - U(x)| \leq \widetilde{\theta}(x, \frac{1}{2\alpha}) + \widetilde{\theta}(x, \frac{-1}{2\alpha}) + 2(\delta' + \widetilde{\psi}(\frac{x}{2}, \frac{-1}{\alpha})) + \varphi(x, 0) + \varphi(0, 0) + |f(0)| + 2|g(0)|,$$

$$|h(x) - V(x)| \leq \widetilde{\theta}(x, \frac{1}{2\alpha}) + \widetilde{\theta}(x, \frac{-1}{2\alpha}) + 2(\delta' + \widetilde{\psi}(\frac{x}{2}, \frac{-1}{\alpha})) + \varphi(0, x) + \varphi(0, 0) + |f(0)| + 2|h(0)|,$$

$$\begin{aligned}|k(x) - W(x)| \leq\ & \widetilde{\theta}(\alpha x, \frac{1}{2\alpha}) + \widetilde{\theta}(\alpha x, \frac{-1}{2\alpha}) + 2|f(0)| + 2(\delta' + \widetilde{\psi}(\frac{\alpha x}{2}, \frac{-1}{\alpha})) \\ & + \delta + \varphi(-\alpha x, \frac{-1}{\alpha}) + \varphi(-\alpha x, 0) + \varphi(0, 0) + \eta(\alpha x) + \\ & 2|g(0)| + |h(0)| + |h(\frac{-1}{\alpha})| + |f(\frac{-1}{\alpha})|,\end{aligned}$$

where

$$\delta' = \delta + 3\varphi(0, 0) + |f(\frac{-1}{\alpha})| + |h(\frac{-1}{\alpha})| + 3|f(0)| + 2|g(0)| + |h(0)|,$$

$$\psi(x, y) = \varphi(x, y) + \varphi(x, 0) + \varphi(0, y) + \varphi(-xy, -1) + \varphi(-xy, 0) + \eta(xy)$$

for all $x, y \in \mathbb{R}$. Moreover
T is additive, $U(x) = T(x) - a_1$, $V(x) = T(x) - a_2$, $W(x) = T(\alpha x) + a_1 + a_2$ with $a_1 = f(0) + g(0)$ and $a_2 = f(0) + h(0)$ two fixed numbers.

Proof. Putting $F(x) = f(\frac{x}{\alpha})$, $G(x) = g(\frac{x}{\alpha})$, $H(x) = h(\frac{x}{\alpha})$ and $K(x) = k(\frac{x}{\alpha^2})$ for all $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}^*$. Then the inequalities (5.1) become

$$\begin{aligned}|F(x+y+xy) - G(x) - H(y) - K(xy)| &\leq \varphi(\frac{x}{\alpha}, \frac{y}{\alpha}) \\ &=: \phi(x,y) \end{aligned} \tag{5.3}$$

and

$$|F(x) + F(-x)| \leq \delta + \eta(\frac{x}{\alpha}) := \delta + \eta^{'}(x), \tag{5.4}$$

for all $x, y \in \mathbb{R}$ and $\alpha \in \mathbb{R}^*$.
By Lemma 3.1 and 3.2 we get

$$|F(x+y+xy) - F(x) - F(y) - F(xy)| \leq \delta^{'} + \psi^{'}(x,y),$$

with

$$\delta^{'} = \delta + 3\phi(0,0) + |F(-1)| + |H(-1)| + 3|F(0)| + 2|G(0)| + |H(0)|$$

and

$$\begin{aligned}\psi^{'}(x,y) &= \phi(x,y) + \phi(x,0) + \phi(0,y) + \phi(-xy,-1) + \phi(-xy,0) + \eta^{'}(xy) \\ &= \psi(\frac{x}{\alpha}, \frac{y}{\alpha}).\end{aligned}$$

By Theorem 4.2 in [2], there exists a unique additive function $T^{'} : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$|F(x) - T'(x)| \leq \widetilde{\theta}'(x, \frac{1}{2}) + \widetilde{\theta}'(x, \frac{-1}{2}) + |F(0)| + 2(\delta^{'} + \widetilde{\psi'}(\frac{x}{2}, -1)),$$

where

$$\begin{aligned}\widetilde{\theta}'(x,y) &= 5\delta^{'} + 2|F(0)| + \widetilde{\psi'}(x, 2y+1) + \widetilde{\psi'}(x+y+xy, 1) + 2\widetilde{\psi'}(x,y) + \widetilde{\psi'}(y,1) \\ &= \widetilde{\theta}(\frac{x}{\alpha}, \frac{y}{\alpha}),\end{aligned}$$

for all $x, y \in \mathbb{R}$. Then

$$|F(x) - T'(x)| \leq \widetilde{\theta}(\frac{x}{\alpha}, \frac{1}{2\alpha}) + \widetilde{\theta}(\frac{x}{\alpha}, \frac{-1}{2\alpha}) + |F(0)| + 2(\delta^{'} + \widetilde{\psi}(\frac{x}{2\alpha}, \frac{-1}{\alpha})),$$

which implies that

$$|f(x) - T(x)| \leq \widetilde{\theta}(x, \frac{1}{2\alpha}) + \widetilde{\theta}(x, \frac{-1}{2\alpha}) + |f(0)| + 2(\delta^{'} + \widetilde{\psi}(\frac{x}{2}, \frac{-1}{\alpha})),$$

for all $x \in \mathbb{R}$, where $T(x) = T^{'}(\alpha x)$. Letting $y = 0$ in (5.3) and using lemma 3.1 (i) we get

$$|G(x) - F(x) + a_1| \leq \varphi(\frac{x}{\alpha}, 0) + \varphi(0,0) + 2|G(0)|, \ \forall x \in \mathbb{R} \tag{5.5}$$

where $a_1 = F(0) + G(0) = g(0) + f(0)$. Then

$$\begin{aligned}|G(x) - T'(x) + a_1| &\leq |F(x) - T'(x)| + |G(x) - F(x) + a_1| \\ &\leq \widetilde{\theta}(\frac{x}{\alpha}, \frac{1}{2\alpha}) + \widetilde{\theta}(\frac{x}{\alpha}, \frac{-1}{2\alpha}) + |F(0)| + 2(\delta^{'} + \widetilde{\psi}(\frac{x}{2\alpha}, \frac{-1}{\alpha})), \\ &+ \varphi(\frac{x}{\alpha}, 0) + \varphi(0,0) + 2|G(0)|,\end{aligned}$$

which implies that

$$|g(x)-U(x)| \leq \widetilde{\theta}(x,\frac{1}{2\alpha})+\widetilde{\theta}(x,\frac{-1}{2\alpha})+2(\delta^{'}+\widetilde{\psi}(\frac{x}{2},\frac{-1}{\alpha}))+\varphi(x,0) + \varphi(0,0)+|f(0)|+2|g(0)|,$$

for all $x\in\mathbb{R}$, where $U(x)=T^{'}(\alpha x)-a_1$.
Replacing (x,y) by $(0,x)$ in (5.3) and using Lemma 3.1 (i) we obtain

$$|H(x)-F(x)+a_2| \leq \varphi(0,\frac{x}{\alpha})+\varphi(0,0)+2|H(0)|,\ \forall x\in\mathbb{R} \tag{5.6}$$

where $a_2=F(0)+H(0)=h(0)+f(0)$. Then

$$\begin{aligned}|H(x)-T'(x)+a_2| &\leq |F(x)-T'(x)|+|H(x)-F(x)+a_2|\\ &\leq \widetilde{\theta}(\frac{x}{\alpha},\frac{1}{2\alpha})+\widetilde{\theta}(\frac{x}{\alpha},\frac{-1}{2\alpha})+|F(0)|+2(\delta^{'}+\widetilde{\psi}(\frac{x}{2\alpha},\frac{-1}{\alpha})),\\ &+\varphi(0,\frac{x}{\alpha})+\varphi(0,0)+2|H(0)|,\end{aligned}$$

which implies that

$$|h(x)-V(x)| \leq \widetilde{\theta}(x,\frac{1}{2\alpha})+\widetilde{\theta}(x,\frac{-1}{2\alpha})+2(\delta^{'}+\widetilde{\psi}(\frac{x}{2},\frac{-1}{\alpha}))+\varphi(0,x) +\varphi(0,0)+|f(0)|+2|h(0)|,$$

for all $x\in\mathbb{R}$, where $V(x)=T^{'}(\alpha x)-a_2$. Next, we replace (x,y) by $(-x,-1)$ in (5.3) we get

$$|K(x)+G(-x)+H(-1)-F(-1)| \leq \varphi(\frac{-x}{\alpha},\frac{-1}{\alpha}) \tag{5.7}$$

for all $x\in\mathbb{R}$.
It follow from (5.4), (5.5) and (5.7) that

$$\begin{aligned}|K(x)-F(x)-a_1-a_2| &\leq |K(x)+G(-x)+H(-1)-F(-1)|+|F(-1)|+|H(-1)|\\ &+|-G(-x)+F(-x)-a_1|+|F(-x)+F(x)|+|a_2|\\ &\leq \varphi(\frac{-x}{\alpha},\frac{-1}{\alpha})+|F(-1)|+|H(-1)|+\varphi(\frac{-x}{\alpha},0)+\varphi(0,0)\\ &+2|G(0)|+\delta+\eta(\frac{x}{\alpha})+|F(0)|+|H(0)|.\end{aligned}$$

Thus

$$\begin{aligned}|K(x)-T'(x)-a_1-a_2| &\leq |K(x)-F(x)-a_1-a_2|+|F(x)-T'(x)|\\ &\leq \widetilde{\theta}(\frac{x}{\alpha},\frac{1}{2\alpha})+\widetilde{\theta}(\frac{x}{\alpha},\frac{-1}{2\alpha})+2(\delta^{'}+\widetilde{\psi}(\frac{x}{2\alpha},\frac{-1}{\alpha}))+\varphi(\frac{-x}{\alpha},\frac{-1}{\alpha})\\ &+\varphi(\frac{-x}{\alpha},0)+\eta(\frac{x}{\alpha})+\delta+\varphi(0,0)+2|F(0)|+2|G(0)|\\ &+|H(0)|+|F(-1)|+|H(-1)|\end{aligned}$$

for all $x\in\mathbb{R}$. Hence

$$\begin{aligned}|k(x)-W(x)| \leq\ &\widetilde{\theta}(\alpha x,\frac{1}{2\alpha})+\widetilde{\theta}(\alpha x,\frac{-1}{2\alpha})+2|f(0)|+2(\delta^{'}+\widetilde{\psi}(\frac{\alpha x}{2},\frac{-1}{\alpha}))\\ &+\delta+\varphi(-\alpha x,\frac{-1}{\alpha})+\varphi(-\alpha x,0)+\varphi(0,0)+\eta(\alpha x)+\\ &2|g(0)|+|h(0)|+|h(\frac{-1}{\alpha})|+|f(\frac{-1}{\alpha})|,\end{aligned}$$

for all $x \in \mathbb{R}$, where $W(x) = T^{'}(\alpha^2 x) + a_2 + a_2$. Moreover T is a unique additive function. Since $a_1 = f(0) + g(0)$ and $a_2 = f(0) + h(0)$ two fixed numbers, thus the triple (U, V, W) is unique and the quadruplet (T, U, V, W) is the solution of equation (1.1). The proof of Theorem is complete. □

Remark. *If $\alpha = 1$ we find a particular case of the Theorem 5.1.*

6. Applications

In this section we give some applications of the Theorem 5.1.

Corollary 6.1. *Let δ, γ be positive real numbers and α a real non-zero, and let $f, g, h, k : \mathbb{R} \longrightarrow \mathbb{R}$ be functions satisfying the following inequality*

$$|f(x + y + \alpha xy) - g(x) - h(y) - k(xy)| \leq \gamma, \quad x, y \in \mathbb{R}$$

and

$$|f(x) + f(-x)| \leq \delta.$$

Then there exists a unique quadruplet (T, U, V, W) of functions, $T, U, V, W : \mathbb{R} \longrightarrow \mathbb{R}$ solution of equation (1.1) such that

$$|f(x) - T(x)| \leq \frac{25}{2}\delta^{'} + 4|f(\frac{1}{\alpha})|,$$

$$|g(x) - U(x)| \leq \frac{25}{2}\delta^{'} + 4|f(\frac{1}{\alpha})| + 2\gamma + 2|g(0)|,$$

$$|h(x) - V(x)| \leq \frac{25}{2}\delta^{'} + 4|f(\frac{1}{\alpha})| + 2\gamma + 2|h(0)|,$$

$$|k(x) - W(x)| \leq \frac{25}{2}\delta^{'} + 4|f(\frac{1}{\alpha})| + \delta + 3\gamma + |f(0)| + 2|g(0)| + |h(0)| + |f(\frac{-1}{\alpha})| + |h(\frac{-1}{\alpha})|,$$

for all $x \in \mathbb{R}$, where

$$\delta' = 8\gamma + |f(\frac{-1}{\alpha})| + |h(\frac{-1}{\alpha})| + \delta + 3|f(0)| + 2|g(0)| + |h(0)|.$$

Moreover T is additive, $U(x) = T(x) - a_1$, $V(x) = T(x) - a_2$, $W(x) = T(\alpha x) + a_1 + a_2$ with $a_1 = f(0) + g(0)$ and $a_2 = f(0) + h(0)$ two fixed numbers.

Remark. *If we take $\varphi(x, y) = \gamma(|x|^p + |y|^p)$ where $p \neq 1, \frac{1}{2}$ and $\gamma \in \mathbb{R}^+$ in the Theorem 5.1, with the condition that $|f(x) + f(-x)| \leq \delta$, we find Ulam-Rassias stability of equation (1.1).*

Corollary 6.2. *Let α a real non-zero and let $f, h, k : \mathbb{R} \longrightarrow \mathbb{R}$ and $\varphi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ be functions satisfying the following inequality*

$$|f(x + y + \alpha xy) + f(-x) - h(y) - k(xy)| \leq \varphi(x, y), \quad x, y \in \mathbb{R}. \tag{6.1}$$

Assume that $\widetilde{\psi}(x, y) < +\infty$. Then there exists a unique triplet (T, V, W) of functions, $T, V, W : \mathbb{R} \longrightarrow \mathbb{R}$ solution of equation $f(x+y+\alpha xy)+f(-x)-h(y)-k(xy) = 0$ such that

$$|f(x) - T(x)| \leq \widetilde{\theta}(x, \frac{1}{2\alpha}) + \widetilde{\theta}(x, \frac{-1}{2\alpha}) + |f(0)| + 2(\delta^{'} + \widetilde{\psi}(\frac{x}{2}, \frac{-1}{\alpha})),$$

$$|h(x) - V(x)| \leq \widetilde{\theta}(x, \frac{1}{2\alpha}) + \widetilde{\theta}(x, \frac{-1}{2\alpha}) + |f(0)| + 2(\delta^{'} + \widetilde{\psi}(\frac{x}{2}, \frac{-1}{\alpha})) + \varphi(0, x) + \varphi(0, 0) + 2|h(0)|,$$

and

$$|k(x) - W(x)| \leq \widetilde{\theta}(\alpha x, \frac{1}{2\alpha}) + \widetilde{\theta}(\alpha x, \frac{-1}{2\alpha}) + 4|f(0)| + 2(\delta^{'} + \widetilde{\psi}(\frac{\alpha x}{2}, \frac{-1}{\alpha}))+ \varphi(-\alpha x, \frac{-1}{\alpha}) + \varphi(-\alpha x, 0) + \varphi(0,0) + \varphi(\alpha x, 0) + 2|h(0)| + |k(0)| + |f(\frac{-1}{\alpha})| + |h(\frac{-1}{\alpha})|,$$

where

$$\delta^{'} = \delta + 3\varphi(0,0) + |f(\frac{-1}{\alpha})| + |h(\frac{-1}{\alpha})| + 5|f(0)| + |h(0)|,$$
$$\psi(x,y) = \varphi(x,y) + \varphi(x,0) + \varphi(0,y) + \varphi(-xy,-1) + \varphi(-xy,0) + \eta(xy),$$
$$\delta + \eta(x) = \varphi(x,0) + |h(0)| + |k(0)|,$$

for all $x, y \in \mathbb{R}$. Moreover
T is additive, $V(x) = T(x) - a_2$, and $W(x) = T(\alpha x) + a_2$ with $a_2 = f(0) + h(0)$ a fixed number.

Proof. Replacing y by 0 in (6.1) we get

$$|f(x) + f(-x)| \leq \varphi(x,0) + |h(0)| + |k(0)|,$$

and we consider $g(x) = -f(-x)$ and $\delta + \eta(x) = \varphi(x,0) + |h(0)| + |k(0)|$, then by Theorem 5.1 we get the result. □

Corollary 6.3. *Let α a real non-zero and let $f, g, k : \mathbb{R} \longrightarrow \mathbb{R}$ and $\varphi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ be functions satisfying the following inequality*

$$|f(x + y + \alpha xy) + f(-y) - g(x) - k(xy)| \leq \varphi(x,y), \quad x, y \in \mathbb{R}. \tag{6.2}$$

Assume that $\widetilde{\psi}(x,y) < +\infty$. Then there exists a unique triplet (T, U, W) of functions, $T, U, W : \mathbb{R} \longrightarrow \mathbb{R}$ solution of equation $f(x+y+\alpha xy)-g(x)+f(-y)-k(xy) = 0$ such that

$$|f(x) - T(x)| \leq \widetilde{\theta}(x, \frac{1}{2\alpha}) + \widetilde{\theta}(x, \frac{-1}{2\alpha}) + |f(0)| + 2(\delta^{'} + \widetilde{\psi}(\frac{x}{2}, \frac{-1}{\alpha})),$$

$$|g(x) - U(x)| \leq \widetilde{\theta}(x, \frac{1}{2\alpha}) + \widetilde{\theta}(x, \frac{-1}{2\alpha}) + |f(0)| + 2(\delta^{'} + \widetilde{\psi}(\frac{x}{2}, \frac{-1}{\alpha})) + \varphi(x,0) + \varphi(0,0) + 2|g(0)|,$$

and

$$|k(x) - W(x)| \leq \widetilde{\theta}(\alpha x, \frac{1}{2\alpha}) + \widetilde{\theta}(\alpha x, \frac{-1}{2\alpha}) + 3|f(0)| + 2(\delta^{'} + \widetilde{\psi}(\frac{\alpha x}{2}, \frac{-1}{\alpha})) + \varphi(-\alpha x, \frac{-1}{\alpha}) + \varphi(-\alpha x, 0) + \varphi(0,0) + \varphi(0, \alpha x) + 3|g(0)| + |k(0)| + |f(\frac{-1}{\alpha})| + |h(\frac{-1}{\alpha})|,$$

where

$$\delta^{'} = \delta + 3\varphi(0,0) + |f(\frac{-1}{\alpha})| + |f(\frac{1}{\alpha})| + 4|f(0)| + 2|g(0)|,$$
$$\psi(x,y) = \varphi(x,y) + \varphi(x,0) + \varphi(0,y) + \varphi(-xy,-1) + \varphi(-xy,0) + \eta(xy),$$
$$\delta + \eta(x) = \varphi(0,x) + |g(0)| + |k(0)|,$$

for all $x, y \in \mathbb{R}$. Moreover
T is additive, $U(x) = T(x) - a_1$ and $W(x) = T(\alpha x) + a_1$ with $a_1 = f(0) + h(0)$ a fixed number.

Proof. Replacing x by 0 and y by x in (6.2) we get

$$|f(x) + f(-x)| \leq \varphi(0, x) + |g(0)| + |k(0)|,$$

and we consider $h(y) = -f(-y)$ and $\delta + \eta(x) = \varphi(0, x) + |g(0)| + |k(0)|$, then by Theorem 5.1 we get the result. □

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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