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EXISTENCE OF SOLUTION TO A COUPLED SYSTEM OF HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This article is concerned with the study of coupled systems of fractional order hybrid differential equations. We use hybrid fixed point theorem due to Dhage and develop sufficient conditions for existence of solutions to the system. We provide an example to demonstrate our main results.

1. Introduction

Fractional order differential equations are available in various disciplines of science and technology, we refer to [4, 6, 15, 19, 22, 25] for some of the applications. These equations have solutions under certain conditions. The area devoted to investigate sufficient conditions for existence of positive solutions to these equations is well studied and plenty of research papers are available in literature, we refer to some of them [5, 16, 18, 20, 23, 7, 21] and the references therein. Recently, existence of solutions to boundary value problems for coupled systems of fractional order differential equations have also attracted some attentions, we refer to [1, 3, 24, 26, 27].

The area of differential equations devoted to quadratic perturbations of nonlinear problems, also known as hybrid differential equations, is one of the most important area and have attracted considerable attention from researchers. It is because, the class of hybrid differential equations that includes the perturbations of original differential equations in different ways, have fundamental importance as they include several dynamical systems as special cases. The study of hybrid differential equations is implicit in the works of Krasnoseĺskii, Dhage and Lakshmikantham and extensively studied by many researchers, we refer [9, 13, 14, 17, 28]. Dhage and Lakshmikanathm [9] studied existence and uniqueness results for the following first order hybrid differential equation

$$\begin{cases} \dfrac{d}{dt}\Big(\dfrac{u(t)}{f(t,u(t))}\Big) = g(t,u(t)), \quad \text{a.e.}\, t \in I = [0,T], \\ u(t_0) = u_0 \in \mathbb{R}, \end{cases}$$

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where $f \in C(I\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $g \in C\,(I \times \mathbb{R},\mathbb{R})$. Further they also developed some fundamentals inequalities for hybrid differential equations. Y.Zhao et al [28], studied existence and uniqueness results for the following hybrid differential equations involving Riemann-Liouville differential operators

$$\begin{cases} D^{\alpha}_{0+}\big(\dfrac{u(t)}{f(t,u(t)}\big) = g(t,u(t),\ \text{a.e.}\, t \in [0,T], \\ u(0) = 0, \end{cases}$$

where $0 < \alpha < 1,\ f \in C\,(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C\,(I \times \mathbb{R},\mathbb{R})$. K. Hilal et al [13] extended the results to the following boundary value problem for fractional hybrid differential equations involving Caputo's derivative

$$\begin{cases} D^{\alpha}_{0+}\big(\dfrac{u(t)}{f(t,u(t))}\big) = g(t,u(t)), \text{a.e.}\, t \in [0,T], \\ a\dfrac{u(0)}{f(0,u(0))} + b\dfrac{u(T)}{f(T,u(T))} = c, \end{cases}$$

where $0 < \alpha < 1$, a,b,c are real constants with $a+b \neq 0$ and $f \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in C(I \times \mathbb{R},\mathbb{R})$. To the best of our knowledge, the area devoted to the study of coupled systems of hybrid fractional order differential equations is not well studied and very few articles are available in literature. For example, in [2], the authors studied existence and uniqueness results for the following coupled systems of boundary value problems for hybrid fractional differential equations

$$\begin{cases} {}^{c}D^{\alpha}\big(\dfrac{u(t)}{f_1(t,u(t),v(t)}\big) = h_1(t,u(t),v(t)),\ 0 < t < 1, \\ {}^{c}D^{\beta}\big(\dfrac{v(t)}{f_2(t,u(t),v(t)}\big) = h_2(t,u(t),v(t)),\ 0 < t < 1, \\ u(0) = u(1) = 0,\ v(0) = v(1) = 0, \end{cases}$$

where $\alpha,\beta \in (1,2],^{c}D$ is the Caputo's fractional derivative, $f_i \in C\,(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and and $h_i \in C\,(I \times \mathbb{R} \times \mathbb{R},\mathbb{R})\,, i = 1,2$. The system was extended by Baleanu et al [12] to multi-point hybrid system and studied sufficient conditions for existence and uniqueness of solutions to the following coupled system

$$\begin{aligned} D^{\omega}\big(\frac{x(t)}{\mathcal{H}(t,x(t),z(t))}\big) &= -\mathcal{K}_1(t,x(t),z(t)), \quad \omega \in (2,3], \\ D^{\epsilon}\big(\frac{z(t)}{\mathcal{G}(t,x(t),z(t))}\big) &= -\mathcal{K}_2(t,x(t),z(t)), \quad \epsilon \in (2,3], \\ \frac{x(t)}{\mathcal{H}(t,x(t),z(t))}|_{t=1} &= 0,\ D^{\mu}\big(\frac{x(t)}{\mathcal{H}(t,x(t),z(t))}\big)|_{t=\delta_1} = 0,\ x^{(2)}(0) = 0, \\ \frac{z(t)}{\mathcal{G}(t,x(t),z(t))}|_{t=1} &= 0,\ D^{\nu}\big(\frac{z(t)}{\mathcal{G}(t,x(t),z(t))}\big)|_{t=\delta_2} = 0,\ z^{(2)}(0) = 0, \end{aligned}$$

where $t \in [0,1]$, δ_1, δ_2, μ, $\nu \in (0,1)$, and D^{ω}, D^{ϵ}, D^{μ} and D^{ν} are Caputo's fractional derivatives of order ω, ϵ, μ and ν respectively, $\mathcal{K}_1,\mathcal{K}_2 \in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $\mathcal{G},\mathcal{H} \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} - \{0\})$.

Motivated by the aforementioned work, in this article, we study existence of solutions to the following more general coupled systems of nonlinear hybrid differential equations subject to nonhomogeneous boundary conditions

$$\begin{cases} {}^cD^\alpha\big(\dfrac{u(t)-f_1(t,u(t),v(t))}{f_2(t,u(t),v(t))}\big)=\phi(t,u(t),v(t)),\ a.e\ t\in I=[0,1],\\ {}^cD^\beta\big(\dfrac{v(t)-g_1(t,u(t),v(t))}{g_2(t,u(t),v(t))}\big)=\psi(t,u(t),v(t)),\ a.e\ t\in I=[0,1],\\ u(0)=a,\ u(1)=b,\ v(0)=c, v(1)=d, \end{cases} \tag{1.1}$$

where $\alpha,\beta\in(1,2],{}^cD$ is the Caputo's fractional derivative, a,b,c,d are real constants and the nonlinear functions $f_2,g_2: I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+\setminus\{0\},\ f_1,g_1,\phi,\psi: I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions. We use standard local fixed point theory developed in [8, 10, 11] involving three operators in Banach algebras to establish necessary and sufficient conditions for existence of positive solutions to coupled system. We also provide an example to demonstrate our main results.

2. Preliminaries

The space $U=C([0,1],\mathbb{R})$ of all continuous functions from $[0,1]\to\mathbb{R}$ is a Banach space under the norm $\|u\|=sup\{|u(t)|:t\in[0,1]\}$. The product space $X=U\times V$ is a Banach space under the norm $\|(u,v)\|=\|u\|+\|v\|$. The following results play important rule in our main result.

Theorem 2.1. *(Dhage [8]). Let W be a closed convex and bounded subset of the Banach space X and let $T,Q:X\to X$ and $S:W\to X$ be three operators such that*

(i) T and Q are Lipschitz with the Lipschitz constants γ and δ respectively;
(ii) S is compact and continuous;
(iii) $x=TxSy+Qx$ for all $y\in W\Rightarrow x\in W$, and
(iv) $\gamma M+\delta<1$, where $M=\|S(W)\|=\sup\{\|Sx\|:x\in W\}$,

then the operator equations $Tx.Sx+Qx=x$ has a solution in W .

Now, we list the following conditions:

(C_1) There exist constants $K_i\in\mathbb{R}^+$ such that

$$|f_i(t,u,v)-f_i(t,\bar u,\bar v)|\le K_i(|u-\bar u|+|v-\bar v|),\ \text{for all}\ \ t\in I,\ u,\bar u,v,\bar v\in\mathbb{R},\ i=1,2;$$

(C_2) There exist constants $L_i\in\mathbb{R}^+$ such that

$$|g_i(t,u,v)-g_i(t,\bar u,\bar v)|\le L_i(|u-\bar u|+|v-\bar v|),\ \text{for all}\ \ t\in I,\ u,\bar u,v,\bar v\in\mathbb{R},\ i=1,2;$$

(C_3) $f_2,\ g_2$ satisfy $f_2(0,a,c)\ne0, f_2(1,b,d)\ne0,\ g_2(0,a,c)\ne0,\ g_2(1,b,d)\ne0.$

Definition. An operator $A:X\to X$ is μ-Lipschitz if there exists continuous nondecreasing functions $\mu_A:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\|A(u)-A(v)\|\le\mu_A(\|u-v\|),$$

for all $u,\ v\in X$ with $\mu_A(0)=0$. In particularly, if $\mu_A=\mu>0$, then A is called $\mu-$ Lipschitz with constant μ. Further if $\mu<1$, then A is a strict contraction.

Lemma 2.2. *The following result holds for fractional differential equations*

$$I^\alpha[{}^cD^\alpha h(t)]=h(t)+c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1},$$

for arbitrary $c_i\in\mathbb{R}$, $\quad i=0,1,2,...,n-1$.

3. Main Results

In this section, we discuss existence of solutions to the coupled system (1.1).

Theorem 3.1. *If $h : I \to \mathbb{R}$ is α times integrable function, then the hybrid boundary value problem (HBVP)*

$$\begin{aligned} &{}^cD^\alpha\Big(\frac{u(t) - f_1(t, u(t), v(t))}{f_2(t, u(t), v(t))}\Big) = \phi(t, u(t), v(t)),\ a.e.\, t \in I = [0, 1],\\ &u(0) = a,\ u(1) = b, \end{aligned} \tag{3.1}$$

is equivalent to the following integral equation,

$$\begin{aligned} u(t) &= f_1(t, u(t), v(t)) + f_2(t, u(t), v(t))\Big[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s, u(s), v(s))ds + \frac{a - f_1(0, a, v(0))}{f_2(0, a, v(0))}\Big] \\ &+ t\Big[\frac{b - f_1(1, b, v(1))}{f_2(1, b, v(1))} - \frac{a - f_1(0, a, v(0))}{f_2(0, a, v(0))} - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s, u(s), v(s))ds\Big] \\ &= f_1(t, u(t), v(t)) + f_2(t, u(t), v(t))\times \\ &\Big[\int_0^1 G_1(t, s)\phi(s, u(s), v(s))ds + (1-t)\frac{a - f_1(0, a, v(0))}{f_2(0, a, v(0))} + t\frac{b - f_1(1, b, v(1))}{f_2(1, b, v(1))}\Big], \end{aligned} \tag{3.2}$$

Proof. Applying I^α on ${}^cD^\alpha\big(\frac{u(t)-f_1(t,u(t),v(t))}{f_2(t,u(t),v(t))}\big) = \phi(t, u(t), v(t))$ and using Lemma 2.2, we obtain

$$\frac{u(t) - f_1(t, u(t), v(t))}{f_2(t, u(t), v(t))} = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s, u(s), v(s))ds + c_1 + c_2 t \tag{3.3}$$

which implies that

$$u(t) = f_1(t, u(t), v(t)) + f_2(t, u(t), v(t))\Big[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s, u(s), v(s))ds + c_1 + c_2 t\Big]. \tag{3.4}$$

Using the boundary conditions $u(0) = a,\ u(1) = b$, we obtain

$$\begin{aligned} c_1 &= \frac{a - f_1(0, a, v(0))}{f_2(0, a, v(0))}, \\ c_2 &= \frac{b - f_1(1, b, v(1))}{f_2(1, b, v(1))} - \frac{a - f_1(0, a, v(0))}{f_2(0, a, v(0))} + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s, u(s), v(s))ds. \end{aligned}$$

Hence,(3.4) takes the form

$$\begin{aligned} u(t) &= f_1(t, u(t), v(t)) + f_2(t, u(t), v(t))\Big[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s, u(s), v(s))ds + \frac{a - f_1(0, a, v(0))}{f_2(0, a, v(0))} \\ &+ t\Big(\frac{b - f_1(1, b, v(1))}{f_2(1, b, v(1))} - \frac{a - f_1(0, a, v(0))}{f_2(0, a, v(0))} - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s, u(s), v(s))ds\Big)\Big] \\ &= f_1(t, u(t), v(t)) + f_2(t, u(t), v(t))\times \\ &\Big[\int_0^1 G_1(t, s)\phi(s, u(s), v(s))ds + (1-t)\frac{a - f_1(0, a, v(0))}{f_2(0, a, v(0))} + t\frac{b - f_1(1, b, v(1))}{f_2(1, b, v(1))}\Big]. \end{aligned} \tag{3.5}$$

□

Similarly form the second part of (1.1), we obtain

$$v(t) = g_1(t,u(t),v(t)) + g_2(t,u(t),v(t))\Big[\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\psi(s,u(s),v(s))ds + \frac{c-g_1(0,u(0),c)}{g_2(0,u(0),c)}\Big]$$
$$+\, t\Big[\frac{c-g_1(1,u(1),d)}{g_2(1,u(1),d)} - \frac{c-g_1(0,u(0),c)}{g_2(0,u(0),c)} - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\psi(s,u(s),v(s))ds\Big]$$
$$= g_1(t,u(t),v(t)) + g_2(t,u(t),v(t))\times$$
$$\Big[\int_0^1 G_2(t,s)\psi(s,u(s),v(s))ds + (1-t)\frac{c-g_1(0,u(0),c)}{g_2(0,u(0),c)} + t\frac{c-g_1(1,u(1),d)}{g_2(1,u(1),d)}\Big], \tag{3.6}$$

We use the notations

$A = |\frac{a-f_1(0,a,c)}{f_2(0,a,c)}| + |\frac{b-f_1(1,b,d)}{f_2(1,b,d)}|, B = |\frac{c-g_1(0,a,c)}{g_2(0,a,c)}| + |\frac{d-g_1(1,b,d)}{g_2(1,b,d)}|$ and for $i=1,2$,
$F_i = \sup_{t\in[0,1]} |f_i(t,0,0)|,\ G_i = \sup_{t\in[0,1]} |g_i(t,0,0)|,\ G_i^* = \max\{|G_i(t,s)| : (t,s)\in[0,1]\times[0,1]\}$.

Assume that ϕ, ψ satisfy the the following condition on X

(C_4) $\Delta = K_1\big(A+G_1^*\int_0^1|\phi(s,u(s),v(s))|ds\big)+L_1\big(B+G_2^*\int_0^1|\psi(s,u(s),v(s))|ds\big)+K_2+L_2<1$.

Theorem 3.2. *Assume that* $(C_1)-(C_4)$ *hold, then the coupled system* (1.1) *has a solution on* $I\times I$.

Proof. Choose $\rho \geq \frac{F_2+G_2+\Delta-K_2-L_2}{1-\Delta}$ and define a closed ball $W=\{(u,v)\in U\times V : \|(u,v)\|\leq\rho\}\subset X$. Define the operators $T=(T_1,T_2), Q=(Q_1,Q_2) : X\to X$ and $S=(S_1,S_2): W\to X$ by

$$T_1(u,v) = f_1(t,u,v),\ T_2(u,v) = g_1(t,u,v),\ Q_1(u,v) = f_2(t,u,v),\ Q_2(u,v) = g_2(t,u,v),$$
$$S_1(u,v) = \int_0^1 G_1(t,s)\phi(s,u,v)ds + (1-t)\frac{a-f_1(0,a,c)}{f_2(0,a,c)} + t\frac{b-f_1(1,b,d)}{f_2(1,b,d)},$$
$$S_2(u,v) = \int_0^1 G_2(t,s)\psi(s,u,v)ds + (1-t)\frac{c-g_1(0,a,c)}{g_2(0,a,c)} + t\frac{d-g_1(1,b,d)}{g_2(1,b,d)}. \tag{3.7}$$

The coupled systems of hybrid integral equations (3.5) and (3.6) can be written as the system of operator equations

$$\big(T_1(u,v)(t)S_1(u,v)(t)+Q_1(u,v)(t), T_2(u,v)(t)S_2(u,v)(t)+Q_2(u,v)(t)\big) = (u,v)(t),\ t\in[0,1] \tag{3.8}$$

which implies that

$$T_1(u,v)(t)S_1(u,v)(t)+Q_1(u,v)(t) = u(t),\ t\in[0,1]$$
$$T_2(u,v)(t)S_2(u,v)(t)+Q_2(u,v)(t) = v(t),\ t\in[0,1].$$

Now we prove that the operators T,S,Q satisfy the conditions of Theorem (2.1). To do this, we show that $T=(T_1,T_2)$ and $Q=(Q_1,Q_2)$ are Lipschitz operators on X with Lipschitz constants K_1+L_1 and K_2+L_2 respectively. For $(u,v)\in X$, using (C_1), we have

$$|T_1(u,v)(t)-T_1(\bar u,\bar v)(t)| = |f_1(t,u(t),v(t)) - f_1(t,\bar u(t),\bar v(t))|$$
$$\leq K_1[|u(t)-\bar u(t)|+|v(t)-\bar v(t)|] \leq K_1[\|u-\bar u\|+\|v-\bar v\|], \forall t\in[0,1].$$

Taking supremum over t, we get

$$\|T_1(u,v)-T_1(\bar{u},\bar{v})\| \leq K_1[\|u-\bar{u}\|+\|v-\bar{v}\|], \forall (u,v),(\bar{u},\bar{v}) \in X.$$

Similarly, we can show that T_2 is Lipschitz with Lipschitz constant L_1, that is,

$$\|T_2(u,v)-T_2(\bar{u},\bar{v})\| \leq L_1[\|u-\bar{u}\|+\|v-\bar{v}\|], \forall (u,v),(\bar{u},\bar{v}) \in X.$$

Hence, it follows that

$$\|T(u,v)-T(\bar{u},\bar{v})\| \leq (K_1+L_1)[\|u-\bar{u}\|+\|v-\bar{v}\|], \forall (u,v),(\bar{u},\bar{v}) \in X,$$

that is, T is Lipschitz with Lipschitz constant K_1+L_1. Similarly, it is easy to show that Q is Lipschitz on X with Lipschitz constant K_2+L_2.
Now we show that $S=(S_1,S_2)$ is compact and continuous operator from W to X. For continuity of S, let (u_n,v_n) be a sequence in W converging to a point $(u,v)\in W$. By Lebesgue Dominated convergence theorem, we have

$$\lim_{n\to\infty} S_1(u_n,v_n)(t) = \lim_{n\to\infty}\int_0^1 G_1(t,s)\phi(s,u_n(s),v_n(s))ds + (1-t)\frac{a-f_1(0,a,c)}{f_2(0,a,c)} + t\frac{b-f_1(1,b,d)}{f_2(1,b,d)}$$
$$= \int_0^1 G_1(t,s)\phi(s,u(s),v(s))ds + (1-t)\frac{a-f_1(0,a,c)}{f_2(0,a,c)} + t\frac{b-f_1(1,b,d)}{f_2(1,b,d)} = S_1(u,v)(t), \forall t\in[0,1],$$

Similarly, we obtain

$$\lim_{n\to\infty} S_2(u_n,v_n)(t) = S_2(u,v)(t), \forall t\in[0,1].$$

Hence $S(u_n,v_n)=(S_1(u_n,v_n),S_2(u_n,v_n))$ is converges to $S(u,v)$ point wise on [0,1]. Next we show that $\{S(u_n,v_n)\}$ is equi-continuous sequence of functions in X. Choose $t_1,t_2\in[0,1]$ such that $t<\tau$, then

$$|S_1(u_n,v_n)(t)-S_1(u_n,v_n)(\tau)| \leq \Big|\int_0^1 \big(G_1(t,s)-G_1(\tau,s)\big)\phi(s,u_n(s),v_n(s))ds\Big|$$
$$+ (t-\tau)\Big(\frac{b-f_1(1,b,d)}{f_2(1,b,d)} - \frac{a-f_1(0,a,c)}{f_2(0,a,c)}\Big)$$
$$\leq \int_0^1 \big(G_1(t,s)-G_1(\tau,s)\big)|\phi(s,u_n(s),v_n(s))|ds + (t-\tau)\Big(\frac{b-f_1(1,b,d)}{f_2(1,b,d)} - \frac{a-f_1(0,a,c)}{f_2(0,a,c)}\Big)$$
$$\to 0 \text{ uniformly for all } n\in N \text{ as } t\to\tau,$$

which implies that $S_1(u_n,v_n)(t)\to S_1(u,v)(t)$ uniformly and hence is uniformly continuous on X. Similarly we can prove that S_2 is uniformly continuous. Thus S is uniformly continuous on X. Further we show that S is compact operator on W. For $(u,v)\in W$, using (C_4), we have

$$|S_1(u,v)(t)| = \Big|t\Big(\frac{b-f_1(1,b,d)}{f_2(1,b,d)} - \frac{a-f_1(0,a,c)}{f_2(0,a,c)}\Big) + \int_0^1 G_1(t,s)\phi(s,u(s),v(s))ds\Big|$$
$$\leq \Big|\Big(\frac{b-f_1(1,b,d)}{f_2(1,b,d)} - \frac{a-f_1(0,a,c)}{f_2(0,a,c)}\Big)\Big| + G_1^*\int_0^1 |\phi(s,u(s),v(s))|ds,$$

which implies that

$$\|S_1(u,v)\| \leq \Big|\Big(\frac{b-f_1(1,b,d)}{f_2(1,b,d)} - \frac{a-f_1(0,a,c)}{f_2(0,a,c)}\Big)\Big| + G_1^*\int_0^1 |\phi(s,u(s),v(s))|ds, \forall (u,v)\in W.$$

Hence, S_1 is uniformly bounded on W. Similarly we can show that S_2 is uniformly bounded on W. Hence S is uniformly bounded on W. For $t < \tau \in I$ and $(u, v) \in W$, we have

$$
\begin{aligned}
&|S_1(u,v)(t) - S_1(u,v)(\tau)| \leq \Big| \int_0^1 [G_1(t,s) - G_1(\tau,s)]\phi(s,u(s),v(s))ds\Big| \\
&+ \big|(t-\tau)\big(\frac{b - f_1(1,b,d)}{f_2(1,b,d)} - \frac{a - f_1(0,a,c)}{f_2(0,a,c)}\big)\big| \\
&\leq G_1^* \int_0^1 |\phi(s,u(s),v(s))|ds + (t-\tau)\Big|\frac{b - f_1(1,b,d)}{f_2(1,b,d)} - \frac{a - f_1(0,a,c)}{f_2(0,a,c)}\Big| \\
&\to 0 \text{ uniformly for all } (u,v) \in W, \text{ as } t \to \tau.
\end{aligned}
$$

Similarly,

$$
|S_2(u,v)(t) - S_2(u,v)(\tau)|\big| \to 0 \text{ uniformly for all } (u,v) \in W, \text{ as } t \to \tau.
$$

Hence, it follows that

$$
|S(u,v)(t) - S(u,v)(\tau)|\big| \to 0 \text{ uniformly for all } (u,v) \in W, \text{ as } t \to \tau.
$$

By Arzelá-Ascoli theorem, S is compact and continuous operator on W which implies that $S(W)$ is compact subset of X.
Now we show that the last condition of Theorem2.1 also hold. For $(u, v) \in X$ and $(x, y) \in W$ such that $(u, v) = (T_1(u, v)S_1(x, y) + Q_1(u, v), T_2(u, v)S_2(x, y) + Q_2(u, v))$. Then, we have

$$
\begin{aligned}
&|u(t)| = |T_1(u,v)(t)S_1(x,y)(t) + Q_1(u,v)(t)| \leq |T_1(u,v)S_1(x,y)| + |Q_1(u,v)| \\
&\leq \big[|f_2(t,u,v) - f_2(t,0,0)| + |f_2(t,0,0)|\big] + \big[|f_1(t,u,v) - f_1(t,0,0)| + |f_1(t,0,0)|\big] \\
&\Big(\int_0^1 |G_1(t,s)\phi(s,x(s),y(s))|ds + \Big|\frac{a - f_1(0,a,c)}{f_2(0,a,c)}\Big| + \Big|\frac{b - f_1(1,b,d)}{f_2(1,b,d)}\Big|\Big),
\end{aligned}
$$

which implies that

$$
\|u\| \leq K_2[\|u\| + \|v\|] + F_2 + \big[K_1[\|u\| + \|v\|] + F_1\big]\big(G_1^* \int_0^1 |\phi(s,u(s),v(s))|ds + A\big). \tag{3.9}
$$

Similarly, we obtain

$$
\|v\| \leq L_2[\|u\| + \|v\|] + G_2 + \big[L_1[\|u\| + \|v\|] + G_1\big]\big(G_2^* \int_0^1 |\psi(s,u(s),v(s))|ds + B\big). \tag{3.10}
$$

From (3.9) and (3.10), we obtain

$$
\|(u,v)\| \leq \frac{F_2 + G_2 + \Delta - K_2 - L_2}{1 - \Delta} \leq \rho, \tag{3.11}
$$

which implies that $(u,v)\in W$. From(3.11), it is obvious that last condition is also satisfied. Finally, we have

$$R=\|S(W)\|=\sup\{\|(S(u,v)\|:(u,v)\in W\}=\sup\{\|S_1(u,v)\|+|S_2(u,v)\|:(u,v)\in W\}$$
$$\leq A+G_1^*\int_0^1|\phi(s,u(s),v(s))\|ds+B+G_2^*\int_0^1|\psi(s,u(s),v(s))\|ds.$$

Moreover

$$K_1\big(A+G_1^*\int_0^1|\phi(s,u(s),v(s))|ds\big)+L_1\big(B+G_2^*\int_0^1|\psi(s,u(s),v(s))|ds\big)+K_2+L_2<1.$$

Thus all the conditions of Theorem (2.1) are satisfied, so the operator equation $T(u,v)S(u,v)+Q(u,v)=(u,v)$ has a solution in W, which implies that the coupled system (1.1) has a solution. □

4. Example

Example 5.1. Consider the following hybrid coupled system of FDEs

$$\begin{cases} {}^cD^{\frac{3}{2}}\left(\dfrac{u(t)-\frac{e^{-t}+|\sin u(t)|+|\cos v(t)|}{40}}{\frac{t+|\cos(u(t))|+|\sin(v(t))|}{20}}\right)=\dfrac{t+\sqrt{u(t)+v(t)}}{40+t^2},\ t\in[0,1],\\ {}^cD^{\frac{3}{2}}\left(\dfrac{v(t)-\frac{e^{-2t}+|\sin(2u(t))|+|\cos^2(v(t))|}{20}}{\frac{t+|\sin(u(t))|+|\cos(v(t))|}{10}}\right)=\dfrac{t^2+\sqrt{u(t)-v(t)}}{32+t},\ t\in[0,1],\\ u(0)=0,\ u(1)=1,\ v(0)=0,\ v(1)=1. \end{cases}\tag{4.1}$$

From (4.1) we have

$$f_1(t,u(t),v(t))=\frac{e^{-t}+|\sin(u(t))|+|\cos(v(t))|}{40},\ f_2(t,u(t),v(t))=\frac{t+|\cos(u(t))|+|\sin(v(t))|}{20}$$
$$g_1(t,u(t),v(t))=\frac{e^{-2t}+|\sin(2u(t))|+|\cos^2(v(t))|}{20},\ g_2(t,u(t),v(t))=\frac{t+|\sin(u(t))|+|\cos(v(t))|}{10},$$
$$\phi(t,u(t),v(t))=\frac{t+\sqrt{u(t)+v(t)}}{40+t^2},\ \psi(t,u(t),v(t))=\frac{t^2+\sqrt{u(t)-v(t)}}{32+t}.$$

Now it is easy to calculate $K_1=\frac{1}{40},\ K_2=\frac{1}{20},\ L_1=\frac{1}{10},\ L_2=\frac{1}{20}$, and $G_i^*=\frac{8}{15\sqrt{\pi}}$ for $i=1,2$ and $A=B=\frac{11}{20},\ \int_0^1|\phi(s,u(s),v(s))|ds=\frac{1}{40},\ \int_0^1|\psi(s,u(s),v(s))|ds=\frac{1}{32}$.

$$K_1\left(A+G_1^*\int_0^1|\phi(s,u(s),v(s))|ds\right)+L_1\left(B+G_2^*\int_0^1|\psi(s,u(s),v(s))|ds\right)+K_2+L_2$$
$$=\frac{1}{40}\left(\frac{11}{20}+\frac{8}{15\sqrt{\pi}}\frac{1}{40}\right)+\frac{1}{10}\left(\frac{11}{20}+\frac{8}{15\sqrt{\pi}}\frac{1}{32}\right)+\frac{1}{20}+\frac{1}{20}<1.$$

Moreover all other conditions of Theorem 3.2 are easy to verify. Hence by Theorem 3.2, the coupled system (4.1) has a solution.

Competing interests

We declare that none of the authors has the competing interests regarding this Manuscript.

Authors contribution

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