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A NOTE ON NEW OSTROWSKI TYPE INEQUALITIES USING A GENERALIZED KERNEL

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Abstract. In this paper, Based on general form of 3-step kernel, new versions of Ostrowski's type integral inequality are developed. We investigate the new Ostrowski's integral inequalities for differentiable mapping f with first derivative belongs to two different Lebesgue spaces. Moreover, the case when $f'' \in L_2$ is considered. Some applications to cumulative distribution function, and to composite quadrature rules are also given.

1. Introduction

With reference to their applications, integral inequalities play an important role in several branches of mathematics and statistics. In 1938, Ostrowski [1] introduced an interesting integral inequality. His inequality measures the deviation of a function from its itegral mean. Consequently, associated with a differentiable mapping there has been an extensive research history of related results. The classical integral inequality of Ostrowski was presented as follows:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be continuous mapping on $[a, b]$ and differentiable on (a, b), whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e.*

$$\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)| < \infty$$

then for all $x \in [a, b]$

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty . \tag{1.1}$$

In 2003, UJević [2] proved the following Ostrowski type inequality:

Theorem 1.2. *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a differentiable mapping in Int I, and let $a, b \in$ Int I, $a < b$. If $\exists\ \gamma, \Gamma \in \mathbb{R}$ such that $\gamma \le$*

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$f'(x) \leq \Gamma$, *for all* $x \in [a,b]$ *and* $f' \in L_1(a,b)$, *then we have*

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{2}(S-\gamma) \tag{1.2}$$

and

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{2}(\Gamma - S), \tag{1.3}$$

where

$$S = \frac{f(b) - f(a)}{b-a}.$$

Recently, motivated by [2] and by utilizing 3-step linear kernel, Liu [3] investigated (1.2) and (1.3) as follows:

Theorem 1.3. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a differentiable mapping in* (a,b). *If* $f' \in L_1[a,b]$ *and* $\gamma \leq f'(x) \leq \Gamma$, *for all* $x \in [a,b]$, *then for all* $x \in \left[a, \frac{a+b}{2}\right]$, *we have*

$$\left| \frac{f(x) - f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] (S - \gamma) \tag{1.4}$$

and

$$\left| \frac{f(x) - f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] (\Gamma - S), \tag{1.5}$$

where

$$S = \frac{f(b) - f(a)}{b-a}.$$

For other related results, the reader may be refer to [4]-[13].

In this paper, we point out some integral inequalities of Ostrowski type by introducing a general form of 3-step linear kernel. We establish our new estimations of the left hand side of both (1.4) and (1.5) by employing a differentiable mapping f with derivative belongs to two different Lebesgue spaces, namely, $f' \in L_1[a,b]$ and $f' \in L_2[a,b]$. Moreover, the case when $f'' \in L_2[a,b]$ is also carried out. Most of the new version of Ostrowski type inequality in our paper will be obtained by using both Hölder's integral inequality and Diaz-Metcalf inequality. Finally, we apply our results for both cumulative distribution function and composite quadrature rules.

2. Main Results

Before we introduce our main results for a general form of 3-step linear kernel, we commence with the following lemma:

Lemma 2.1. *Consider the kernel*

$$K(x,t)=\begin{cases} t-\left(a+h\frac{b-a}{2}\right), & t\in[a,x] \\ t-\left(\frac{a+b}{2}-h\frac{b-a}{2}\right), & t\in(x,a+b-x] \\ t-\left(b-h\frac{b-a}{2}\right), & t\in(a+b-x,b] \end{cases} \tag{2.1}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$ *and* $h\in[0,1]$, *then the following identity holds:*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b K(x,t)f'(t)\,dt \\ =\ &\frac{1}{2}\left[(1-2h)f(x)+f(a+b-x)+h(f(a)+f(b))\right] \\ &-\frac{1}{b-a}\int_a^b f(t)\,dt. \end{aligned} \tag{2.2}$$

Proof: From (2.1), we have

$$\begin{aligned} &\int_a^b K(x,t)f'(t)\,dt \\ =\ &\int_a^x \left(t-\left(a+h\frac{b-a}{2}\right)\right)f'(t)\,dt \\ &+\int_x^{a+b-x}\left(t-\left(\frac{a+b}{2}-h\frac{b-a}{2}\right)\right)f'(t)\,dt \\ &+\int_{a+b-x}^b \left(t-\left(b-h\frac{b-a}{2}\right)\right)f'(t)\,dt \\ =\ &\left(\frac{b-a}{2}\right)(1-2h)f(x)+h\left(\frac{b-a}{2}\right)[f(a)+f(b)] \\ &+\left(\frac{b-a}{2}\right)f(a+b-x)-\int_a^b f(t)\,dt. \end{aligned}$$

Hence, we obtain (2.2). Now, with the use of (2.2), we state and prove the following cases:

2.1 The case when $f'\in L_1[a,b]$ and f' is bounded

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping in (a,b). If $f' \in L_1[a,b]$ and $\gamma \le f'(x) \le \Gamma$, $\forall x \in [a,b]$, then $\forall x \in \left[a, \frac{a+b}{2}\right]$ and $h \in [0,1]$ we have*

$$\begin{aligned} & \left| \frac{1}{2}\left[(1-2h) f(x) + f(a+b-x) + h(f(a)+f(b))\right] \right. \\ & \left. -h\left(\frac{a+b}{2} - x\right) S - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \le\ & m(x,h)(S-\gamma) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} & \left| \frac{1}{2}\left[(1-2h) f(x) + f(a+b-x) + h(f(a)+f(b))\right] \right. \\ & \left. -h\left(\frac{a+b}{2} - x\right) S - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \le\ & m(x,h)(\Gamma - S), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} & m(x,h) \\ =\ & \frac{1}{2}\left[\left(\frac{a+b}{2} - x\right) + h(x-a) + \left|(x-a) + h(x-b)\right| \right. \\ & \left. + \left|\left(x - \frac{a+b}{2}\right) - h(x-a) + \left|(x-a)+h(x-b)\right|\right|\right], \end{aligned} \tag{2.5}$$

$S = (f(b)-f(a))/(b-a)$, $\gamma = \inf\limits_{t\in[a,b]} f'(t)$, *and* $\Gamma = \sup\limits_{t\in[a,b]} f'(t)$.

Proof: From (2.2) and the facts

$$\frac{1}{b-a}\int_a^b f'(t)\,dt = \frac{f(b)-f(a)}{b-a} \tag{2.6}$$

and

$$\int_a^b K(x,t)dt = h\frac{b-a}{2}(a+b-2x), \tag{2.7}$$

it follows that

$$\begin{aligned} & \frac{1}{b-a}\int_a^b K(x,t) f'(t)\,dt - \frac{1}{(b-a)^2}\int_a^b K(x,t)dt \int_a^b f'(t)\,dt \\ =\ & \frac{1}{2}\left[(1-2h) f(x) + f(a+b-x) + h(f(a)+f(b))\right] \\ & -h\left(\frac{a+b}{2} - x\right)\frac{f(b)-f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt. \end{aligned} \tag{2.8}$$

We denote

$$R_n(x) = \frac{1}{b-a}\int_a^b K(x,t)f'(t)\,dt - \frac{1}{(b-a)^2}\int_a^b K(x,t)dt\int_a^b f'(t)\,dt. \tag{2.9}$$

If $C \in \mathbb{R}$ is an arbitrary constant, then we have

$$R_n(x) = \frac{1}{b-a}\int_a^b (f'(t) - C)\left[K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right] dt. \tag{2.10}$$

Furthermore, we have

$$|R_n(x)| \leq \frac{1}{b-a}\max_{t\in[a,b]}\left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right| \int_a^b |f'(t) - C|\,dt. \tag{2.11}$$

Now, to compute

$$\begin{aligned}
&\max_{t\in[a,b]}\left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right| \\
=\;& \max_{t\in[a,b]}\left\{|(x-a)+h(x-b)|, \left|\left(\frac{a+b}{2}-x\right)+h(x-a)\right|, |h(x-a)|\right\} \\
=\;& \max_{t\in[a,b]}\left\{|(x-a)+h(x-b)|, \left(\frac{a+b}{2}-x\right)+h(x-a)\right\} \\
=\;& \frac{1}{2}\left[\left(\frac{a+b}{2}-x\right)+h(x-a)+|(x-a)+h(x-b)|\right. \\
&\left. + \left|\left(x-\frac{a+b}{2}\right)-h(x-a)+|(x-a)+h(x-b)|\right|\right].
\end{aligned} \tag{2.12}$$

We also have

$$\int_a^b |f'(t) - \gamma|\,dt = (S-\gamma)(b-a) \tag{2.13}$$

and

$$\int_a^b |f'(t) - \Gamma|\,dt = (\Gamma - S)(b-a). \tag{2.14}$$

Therefore, we obtain (2.3) and (2.4) by using (2.8)-(2.14) and choosing $C = \gamma$ and $C = \Gamma$ in (2.11), respectively.

Remark. *Choosing $h = 0$ in both (2.3) and (2.4) respectively, yields*

$$\begin{aligned}
&\left|\frac{[f(x)+f(a+b-x)]}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq\;& \left[\frac{b-a}{4} + \left|x - \frac{3a+b}{4}\right|\right](S-\gamma)
\end{aligned} \tag{2.15}$$

and

$$\left|\frac{[f(x)+f(a+b-x)]}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right](\Gamma-S). \tag{2.16}$$

Noting that (2.15) and (2.16) are similar to those obtained by [3].

Corollary 2.3. *Under the assumptions of Theorem (4),*

choosing $x=\frac{3a+b}{4}$, *yields*

$$\left|\frac{1}{2}\left[(1-2h)f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+h(f(a)+f(b))\right] - h\left(\frac{b-a}{4}\right)S-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{b-a}{8}\left[(1+h)+|1-3h|+|(1+h)-|1-3h||\right](S-\gamma) \tag{2.17}$$

and

$$\left|\frac{1}{2}\left[(1-2h)f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+h(f(a)+f(b))\right] - h\left(\frac{b-a}{4}\right)S-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{b-a}{8}\left[(1+h)+|1-3h|+|(1+h)-|1-3h||\right](\Gamma-S), \tag{2.18}$$

choosing $x=a$, *yields*

$$\left|\frac{1}{2}\left[(1-2h)f(a)+f(b)+h(f(a)+f(b))\right]-h\left(\frac{b-a}{2}\right)S -\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{b-a}{2}\left[\left(\frac{1}{2}+h\right)+\left|\frac{1}{2}-h\right|\right](S-\gamma) \tag{2.19}$$

and

$$\left|\frac{1}{2}\left[(1-2h)f(a)+f(b)+h(f(a)+f(b))\right]-h\left(\frac{b-a}{2}\right)S -\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{b-a}{2}\left[\left(\frac{1}{2}+h\right)+\left|\frac{1}{2}-h\right|\right](\Gamma-S), \tag{2.20}$$

choosing $x = \frac{a+b}{2}$, *yields*

$$\begin{aligned} & \left| \frac{1}{2}\left[(1-2h) f\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right) + h\left(f(a)+f(b)\right)\right] \right. \\ & \left. -\frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq \; & \frac{b-a}{4}\left[1+\left|1-2h\right|\right](S-\gamma) \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} & \left| \frac{1}{2}\left[(1-2h) f\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right) + h\left(f(a)+f(b)\right)\right] \right. \\ & \left. -\frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq \; & \frac{b-a}{4}\left[1+\left|1-2h\right|\right](\Gamma-S). \end{aligned} \tag{2.22}$$

A new inequality of Ostrowski's type may be stated as follows:

Corollary 2.4. *Let* f *be as in Theorem (4). Additionally, if* f *is symmetric about* $x = \frac{a+b}{2}$, *then* $\forall x \in \left[a, \frac{a+b}{2}\right]$, *we have*

$$\begin{aligned} & \left| (1-h) f(x) + \frac{h}{2}\left(f(a)+f(b)\right) - h\left(\frac{a+b}{2}-x\right) S \right. \\ & \left. -\frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq \; & m(x,h)(S-\gamma) \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} & \left| (1-h) f(x) + \frac{h}{2}\left(f(a)+f(b)\right) - h\left(\frac{a+b}{2}-x\right) S \right. \\ & \left. -\frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq \; & m(x,h)(\Gamma-S). \end{aligned} \tag{2.24}$$

Remark. *Choosing* $h = 1$ *in both (2.23) and (2.24) with* $x = \frac{a+b}{2}$ *yields*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{b-a}{2}(S-\gamma) \tag{2.25}$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{b-a}{2}(\Gamma-S). \tag{2.26}$$

2.2 The case when $f'' \in L_2[a,b]$

Theorem 2.5. *Let $f : [a,b] \to \mathbb{R}$ be a twice continuously differentiable mapping in (a,b) with $f'' \in L_2[a,b]$. Then $\forall x \in \left[a, \frac{a+b}{2}\right]$ and $h \in [0,1]$ we have*

$$\begin{aligned} & \left| \frac{1}{2}\left[(1-2h) f(x) + f(a+b-x) + h(f(a)+f(b))\right] \right. \\ & \left. -h\left(\frac{a+b}{2} - x\right) S - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq & \frac{(b-a)^{\frac{1}{2}}}{\sqrt{2}\pi}\left[\frac{(b-a)^2}{48}\left(8h^3+1\right) + \left(x - \left(\frac{3a+b}{4} + h\frac{b-a}{2}\right)\right)^2 \right. \\ & + (1-2h)\left(\frac{(1-2h)^2(b-a)^2}{48} + \left(x - \frac{3a+b}{4}\right)^2\right) \\ & \left. - \frac{h^2}{2}(a+b-2x)^2\right]^{\frac{1}{2}} \|f''\|_2 . \end{aligned} \tag{2.27}$$

Proof: Let $R_n(x)$ be defined by (2.9). From (2.8), we get

$$\begin{aligned} R_n(x) = & \frac{1}{2}\left[(1-2h) f(x) + f(a+b-x) + h(f(a)+f(b))\right] \\ & -h\left(\frac{a+b}{2} - x\right)\frac{f(b)-f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt. \end{aligned} \tag{2.28}$$

If we choose $C = f'((a+b)/2)$ in (2.10) and use the Cauchy inequality, then we get

$$\begin{aligned} |R_n(x)| = & \frac{1}{b-a}\int_a^b \left| f'(t) - f'\left(\frac{(a+b)}{2}\right)\right| \left| K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right| dt \\ \leq & \frac{1}{b-a}\left(\int_a^b \left(f'(t) - f'\left(\frac{a+b}{2}\right)\right)^2 dt\right)^{\frac{1}{2}} \\ & \times \left(\int_a^b \left(K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right)^2 dt\right)^{\frac{1}{2}}. \end{aligned} \tag{2.29}$$

Now, by using Diaz-Metcalf inequality [10], we get

$$\int_a^b \left(f'(t) - f'\left(\frac{a+b}{2}\right)\right)^2 dt \leq \frac{(b-a)^2}{\pi^2}\|f''\|_2^2 . \tag{2.30}$$

Moreover, we also have

$$\begin{aligned}
&\int_a^b \left(K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right)^2 dt \\
= \quad & \frac{b-a}{2}\left[\frac{(b-a)^2}{48}\left(8h^3+1\right) + \left(x - \left(\frac{3a+b}{4} + h\frac{b-a}{2}\right)\right)^2\right. \\
&+(1-2h)\left(\frac{(1-2h)^2(b-a)^2}{48} + \left(x - \frac{3a+b}{4}\right)^2\right) \\
&\left.-\frac{h^2}{2}(a+b-2x)^2\right]. \qquad (2.31)
\end{aligned}$$

Therefore, by using (2.28), (2.29), (2.30), and (2.31), we obtain (2.27).

Remark. *Choosing $h = 0$ in (2.27), yields*

$$\begin{aligned}
&\left|\frac{1}{2}\left[f(x) + f(a+b-x)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq \quad & \frac{(b-a)^{\frac{1}{2}}}{\pi}\left[\frac{(b-a)^2}{48} + \left(x - \frac{3a+b}{4}\right)^2\right]^{\frac{1}{2}} \left\|f''\right\|_2. \qquad (2.32)
\end{aligned}$$

Corollary 2.6. *Under the assumption of Theorem (5),*

choosing $x = \frac{3a+b}{4}$, yields

$$\begin{aligned}
&\left|\frac{1}{2}\left[(1-2h)f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + h\left(f(a)+f(b)\right)\right]\right. \\
&\left. - h\left(\frac{b-a}{4}\right)S - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq \quad & \frac{(b-a)^{\frac{3}{2}}}{4\sqrt{3}\pi}\left[9h^2 - 3h + 1\right]^{\frac{1}{2}} \left\|f''\right\|_2, \qquad (2.33)
\end{aligned}$$

choosing $x = a$, yields

$$\begin{aligned}
&\left|\frac{1}{2}\left[(1-2h)f(a) + f(b) + h\left(f(a)+f(b)\right)\right]\right. \\
&\left. - h\left(\frac{b-a}{2}\right)S - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq \quad & \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}\pi}\left\|f''\right\|_2, \qquad (2.34)
\end{aligned}$$

choosing $x=\frac{a+b}{2}$, *yields*

$$\begin{aligned}&\left|\frac{1}{2}\left[(1-2h)\,f\left(\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)+h\left(f\left(a\right)+f\left(b\right)\right)\right]\right.\\&\left.-\frac{1}{b-a}\int_a^b f\left(t\right)dt\right|\\ \leq\;&\frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}\pi}\left[3h^2-3h+1\right]^{\frac{1}{2}}\left\|f\,''\right\|_2. \qquad (2.35)\end{aligned}$$

Additional new inequality of Ostrowski's type may be stated as follows:

Corollary 2.7. *Let* f *be as in Theorem (5). Additionally, if* f *is symmetric about* $x=\frac{a+b}{2}$, *then* $\forall x\in\left[a,\frac{a+b}{2}\right]$ *we have*

$$\begin{aligned}&\left|(1-h)\,f\left(x\right)+\frac{h}{2}\left(f\left(a\right)+f\left(b\right)\right)-h\left(\frac{a+b}{2}-x\right)S\right.\\&\left.-\frac{1}{b-a}\int_a^b f\left(t\right)dt\right|\\ \leq\;&\frac{(b-a)^{\frac{1}{2}}}{\sqrt{2}\pi}\left[\frac{(b-a)^2}{48}\left(8h^3+1\right)+\left(x-\left(\frac{3a+b}{4}+h\frac{b-a}{2}\right)\right)^2\right.\\&+(1-2h)\left(\frac{(1-2h)^2\,(b-a)^2}{48}+\left(x-\frac{3a+b}{4}\right)^2\right)\\&\left.-\frac{h^2}{2}\left(a+b-2x\right)^2\right]^{\frac{1}{2}}\left\|f\,''\right\|_2. \qquad (2.36)\end{aligned}$$

Remark. *Choosing* $h=1$ *in (2.36) with* $x=\frac{a+b}{2}$ *yields*

$$\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_a^b f\left(t\right)dt\right|\leq\frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}\pi}\left\|f\,''\right\|_2. \qquad (2.37)$$

2.3 The case when $f\,'\in L_2\left[a,b\right]$

Theorem 2.8. *Let* $f : [a,b] \to \mathbb{R}$ *be an absolutely continuous mapping in* (a,b) *with* $f' \in L_2[a,b]$. *Then* $\forall x \in \left[a, \frac{a+b}{2}\right]$ *and* $h \in [0,1]$ *we have*

$$\begin{aligned}
&\left|\frac{1}{2}\left[(1-2h) f(x) + f(a+b-x) + h(f(a)+f(b))\right]\right. \\
&\left. - h\left(\frac{a+b}{2} - x\right) S - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq\ & (b-a)^{-\frac{1}{2}}\left[\frac{(b-a)^2}{48}\left(8h^3+1\right) + \left(x - \left(\frac{3a+b}{4} + h\frac{b-a}{2}\right)\right)^2\right. \\
&+ (1-2h)\left(\frac{(1-2h)^2(b-a)^2}{48} + \left(x - \frac{3a+b}{4}\right)^2\right) \\
&\left. - \frac{h^2}{2}(a+b-2x)^2\right]^{\frac{1}{2}}\sqrt{\frac{\sigma(f')}{2}},
\end{aligned} \tag{2.38}$$

where $\sigma(f')$ *is defined by*

$$\sigma(f') = \|f\|_2^2 - \frac{(f(b)-f(a))^2}{b-a} = \|f\|_2^2 - S^2(b-a).$$

Proof: Let $R_n(x)$ be defined by (2.9). If we choose $C = \frac{1}{b-a}\int_a^b f'(t)\,dt$ in (2.10) and use the Cauchy inequality and (2.31), then we have

$$\begin{aligned}
& |R_n(x)| \\
=\ & \frac{1}{b-a}\int_a^b \left|f'(t) - \frac{1}{b-a}\int_a^b f'(t)\,dt\right|\left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right| dt \\
\leq\ & \frac{1}{b-a}\left(\int_a^b\left(f'(t) - \frac{1}{b-a}\int_a^b f'(t)\,dt\right)^2 dt\right)^{\frac{1}{2}} \\
& \times\left(\int_a^b\left(K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right)^2 dt\right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
\leq\ & \sqrt{\frac{\sigma(f')}{2}}(b-a)^{-\frac{1}{2}}\left[\frac{(b-a)^2}{48}\left(8h^3+1\right) + \left(x - \left(\frac{3a+b}{4} + h\frac{b-a}{2}\right)\right)^2\right. \\
&\left. +(1-2h)\left(\frac{(1-2h)^2(b-a)^2}{48} + \left(x - \frac{3a+b}{4}\right)^2\right) - \frac{h^2}{2}(a+b-2x)^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Remark. *Choosing $h = 0$ in (2.38) yields*

$$\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq (b-a)^{-\frac{1}{2}}\left[\frac{(b-a)^2}{48}+\left(x-\frac{3a+b}{4}\right)^2\right]^{\frac{1}{2}}\sqrt{\sigma(f')}. \tag{2.39}$$

Corollary 2.9. *Under the assumption of Theorem (6),*

choosing $x = \frac{3a+b}{4}$, yields

$$\left|\frac{1}{2}\left[(1-2h)f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+h\left(f(a)+f(b)\right)\right] - h\left(\frac{b-a}{4}\right)S-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^{\frac{1}{2}}}{4\sqrt{3}}\left[9h^2-3h+1\right]^{\frac{1}{2}}\sqrt{\sigma(f')}, \tag{2.40}$$

choosing $x = a$, yields

$$\left|\frac{1}{2}\left[(1-2h)f(a)+f(b)+h\left(f(a)+f(b)\right)\right]-h\left(\frac{b-a}{2}\right)S - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^{\frac{1}{2}}}{2\sqrt{3}}\sqrt{\sigma(f')}, \tag{2.41}$$

choosing $x = \frac{a+b}{2}$, yields

$$\left|\frac{1}{2}\left[(1-2h)f\left(\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)+h\left(f(a)+f(b)\right)\right] -\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^{\frac{1}{2}}}{2\sqrt{3}}\left[3h^2-3h+1\right]^{\frac{1}{2}}\sqrt{\sigma(f')}. \tag{2.42}$$

Further inequality of Ostrowski's type may be stated as follows:

Corollary 2.10. *Let f be as in Theorem (6). Additionally, if f is symmetric about $x=\frac{a+b}{2}$, then $\forall x\in\left[a,\frac{a+b}{2}\right]$ we have*

$$\begin{aligned}
&\left|(1-h)\,f(x)+\frac{h}{2}\,(f(a)+f(b))-h\left(\frac{a+b}{2}-x\right)S\right.\\
&\left.-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\
\leq\;& (b-a)^{-\frac{1}{2}}\left[\frac{(b-a)^2}{48}\left(8h^3+1\right)+\left(x-\left(\frac{3a+b}{4}+h\frac{b-a}{2}\right)\right)^2\right.\\
&+(1-2h)\left(\frac{(1-2h)^2(b-a)^2}{48}+\left(x-\frac{3a+b}{4}\right)^2\right)\\
&\left.-\frac{h^2}{2}\,(a+b-2x)^2\right]^{\frac{1}{2}}\sqrt{\frac{\sigma(f\,')}{2}}. \qquad (2.43)
\end{aligned}$$

Remark. *Choosing $h=1$ in (2.43) with $x=\frac{a+b}{2}$ yields*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{(b-a)^{\frac{1}{2}}}{2\sqrt{3}}\sqrt{\sigma(f\,')}. \qquad (2.44)$$

3. Application to Cumulative Distribution Function

Let X be a random variable taking values in the finite interval $[a,b]$ with the probability density function $f:[a,b]\to[0,1]$ and cumulative distribution function such that

$$F(x)=\Pr(X\leq x)=\int_a^x f(t)\,dt,$$

$$F(b)=\Pr(X\leq b)=\int_a^b f(t)\,dt=1.$$

Then the following theorem holds:

Theorem 3.1. *Let X and F be as above. Then with the assumption of Theorem (4), we have*

$$\begin{aligned}
&\left|\frac{1}{2}\left[(1-2h)\,F(x)+F(a+b-x)\right]\right.\\
&\left.+\frac{1}{b-a}\left(h(x-a)-(b-E(X))\right)\right|\\
\leq\;& m(x,h)\left(\frac{1}{b-a}-\gamma\right) \qquad (3.1)
\end{aligned}$$

and

$$\begin{aligned} & \left| \frac{1}{2} \left[(1-2h) F(x) + F(a+b-x) \right] \right. \\ & \left. + \frac{1}{b-a} \left(h(x-a) - (b - E(X)) \right) \right| \\ \leq \; & m(x,h) \left(\Gamma - \frac{1}{b-a} \right), \end{aligned} \tag{3.2}$$

$\forall\, x \in \left[a, \frac{a+b}{2}\right]$ *and* $h \in [0,1]$, *where* $E(X)$ *is the expectation of* X*.*

Proof: By (2.3) and (2.4) on choosing $f = F$ and taking into account

$$E(X) = \int_a^b t f(t)\, dt = b - \int_a^b F(t)\, dt$$

we can obtain inequality (3.1) and (3.2).

Corollary 3.2. *Under the assumption of Theorem (7) with* $x = \frac{a+b}{2}$ *and* $h = 1$, *we have*

$$\left| E(X) - \frac{a+b}{2} \right| \leq \frac{(b-a)^2}{2} \left(\frac{1}{b-a} - \gamma \right)$$

and

$$\left| E(X) - \frac{a+b}{2} \right| \leq \frac{(b-a)^2}{2} \left(\Gamma - \frac{1}{b-a} \right).$$

Theorem 3.3. *Let* X *and* F *be as above. Then with the assumption of Theorem (5), we have*

$$\begin{aligned} & \left| \frac{1}{2} \left[(1-2h) F(x) + F(a+b-x) \right] + \frac{1}{b-a} \left(h(x-a) - (b - E(X)) \right) \right| \\ \leq \; & \frac{(b-a)^{\frac{1}{2}}}{\sqrt{2}\pi} \left[\frac{(b-a)^2}{48} \left(8h^3 + 1 \right) + \left(x - \left(\frac{3a+b}{4} + h \frac{b-a}{2} \right) \right)^2 \right. \\ & + (1-2h) \left(\frac{(1-2h)^2 (b-a)^2}{48} + \left(x - \frac{3a+b}{4} \right)^2 \right) \\ & \left. - \frac{h^2}{2} (a+b-2x)^2 \right]^{\frac{1}{2}} \|f\|_2, \end{aligned} \tag{3.3}$$

$\forall\, x \in \left[a, \frac{a+b}{2}\right]$ *and* $h \in [0,1]$.

Proof: By (2.27) on choosing $f = F$ and taking into account

$$E(X) = \int_a^b t f(t)\, dt = b - \int_a^b F(t)\, dt,$$

we obtain (3.3).

Corollary 3.4. *Under the assmuption of Theorem (8) with* $x = \frac{a+b}{2}$ *and* $h = 1$, *we have*

$$\left| E(X) - \frac{a+b}{2} \right| \leq \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}\pi} \|f'\|_2.$$

Theorem 3.5. *Let X and F be as above. Then with the assumption of Theorem (6), we have*

$$\begin{aligned} & \left|\frac{1}{2}\left[(1-2h)F(x)+F(a+b-x)\right]+\frac{1}{b-a}\left(h(x-a)-(b-E(X))\right)\right| \\ \leq & \quad (b-a)^{-\frac{1}{2}}\left[\frac{(b-a)^2}{48}\left(8h^3+1\right)+\left(x-\left(\frac{3a+b}{4}+h\frac{b-a}{2}\right)\right)^2\right. \\ & +(1-2h)\left(\frac{(1-2h)^2(b-a)^2}{48}+\left(x-\frac{3a+b}{4}\right)^2\right) \\ & \left.-\frac{h^2}{2}(a+b-2x)^2\right]^{\frac{1}{2}}\sqrt{\frac{\sigma(f)}{2}}, \end{aligned} \tag{3.4}$$

where $\sigma(f)$ is defined by

$$\sigma(f)=\|f\|_2^2-\frac{1}{b-a}.$$

Proof: By (2.38) on choosing $f=F$ and taking into account

$$E(X)=\int_a^b tf(t)\,dt=b-\int_a^b F(t)\,dt,$$

we obtain (3.4).

Corollary 3.6. *Under the assmuption of Theorem (9) with $x=\frac{a+b}{2}$ and $h=1$, we have*

$$\left|E(X)-\frac{a+b}{2}\right|\leq\frac{(b-a)^{\frac{1}{2}}}{2\sqrt{3}}\sqrt{\sigma(f)}.$$

4. Application to Composite Quadrature Rules

Let $I_n: a=x_0<x_1<x_2<....<x_{n-1}<x_n=b$ be a partition of the interval $[a,b]$ and $\delta_i=x_{i+1}-x_i$ $(i=0,1,.....,n-1)$. Consider the following general quadrature rule:

$$\begin{aligned} & S(f,I_n) \\ = & \quad \frac{1}{2}\sum_{i=0}^{n-1}\left[(1-2h)f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right. \\ & \left.+hf(x_i)+f(x_{i+1})\right]\delta_i-\frac{h}{4}\sum_{i=0}^{n-1}\left[f(x_{i+1})-f(x_i)\right]\delta_i. \end{aligned} \tag{4.1}$$

Theorem 4.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping in (a,b). If $f'\in L_1[a,b]$ and $\gamma\leq f'(x)\leq\Gamma$, $\forall x\in[a,b]$, then $\forall x\in\left[a,\frac{a+b}{2}\right]$ and $h\in[0,1]$ we have*

$$\int_a^b f(x)\,dx=S(f,I_n)+R(f,I_n),$$

where $S(f, I_n)$ is defined by by formula (4.1), and the remainder $R(f, I_n)$ satisfies the estimates

$$\begin{aligned} & R(f, I_n) \\ \leq & \left[\frac{(1+h)+|1-3h|+|(1+h)-|1-3h||}{8}\right] \sum_{i=0}^{n-1} (S_i - \gamma)\, \delta_i^2 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} & R(f, I_n) \\ \leq & \left[\frac{(1+h)+|1-3h|+|(1+h)-|1-3h||}{8}\right] \sum_{i=0}^{n-1} (\Gamma - S_i)\, \delta_i^2, \end{aligned} \tag{4.3}$$

where $S_i = (f(x_{i+1}) - f(x_i))/\delta_i$, $i = 0, 1, ..., n-1$.

Proof: Applying (2.17) and (2.18) to the interval $[x_i, x_{i+1}]$, then, respectively, we get

$$\begin{aligned} & \left| \frac{1}{2}\left[(1-2h)\, f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right) \right.\right. \\ & \left. + h\left[f(x_i) + f(x_{i+1})\right]\right]\delta_i - \frac{h}{4}\left[f(x_{i+1}) - f(x_i)\right]\delta_i \\ & \left. - \int_{x_i}^{x_{i+1}} f(t)\, dt \right| \\ \leq & \left[\frac{(1+h)+|1-3h|+|(1+h)-|1-3h||}{8}\right] (S_i - \gamma)\, \delta_i^2 \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{1}{2}\left[(1-2h)\, f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right) \right.\right. \\ & \left. + h\left[f(x_i) + f(x_{i+1})\right]\right]\delta_i - \frac{h}{4}\left[f(x_{i+1}) - f(x_i)\right]\delta_i \\ & \left. - \int_{x_i}^{x_{i+1}} f(t)\, dt \right| \\ \leq & \left[\frac{(1+h)+|1-3h|+|(1+h)-|1-3h||}{8}\right] (\Gamma - S_i)\, \delta_i^2, \end{aligned}$$

for all $i = 0, 1, ..., n-1$. Now summing over i from 0 to $n-1$ and using the triangle inequality, we get both (4.2) and (4.3).

Theorem 4.2. *Let $f : [a, b] \to \mathbb{R}$ be a twice continuously differentiable mapping in (a, b) with $f'' \in L_2[a, b]$. Then $\forall x \in \left[a, \frac{a+b}{2}\right]$ and $h \in [0, 1]$ we have*

$$\int_a^b f(x)\, dx = S(f, I_n) + R(f, I_n),$$

where $S(f, I_n)$ is defined by by formula (4.1), and the remainder $R(f, I_n)$ satisfies the estimate

$$R(f, I_n) \le \frac{\left[9h^2 - 3h + 1\right]^{\frac{1}{2}}}{4\sqrt{3}\pi} \|f''\|_2 \sum_{i=0}^{n-1} \delta_i^{\frac{5}{2}}. \tag{4.4}$$

Proof: Applying (2.33) to the interval $[x_i, x_{i+1}]$, we get

$$\begin{aligned} & \left| \frac{1}{2}\left[(1-2h) f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right.\right. \\ & +h\left[f(x_i) + f(x_{i+1})\right]\delta_i - \frac{h}{4}\left[f(x_{i+1}) - f(x_i)\right]\delta_i \\ & \left. - \int_{x_i}^{x_{i+1}} f(t)\, dt \right| \\ \le \; & \frac{\left[9h^2 - 3h + 1\right]^{\frac{1}{2}}}{4\sqrt{3}\pi} \|f''\|_2 \, \delta_i^{\frac{5}{2}}, \end{aligned}$$

for all $i = 0, 1, ..., n-1$. Now summing over i from 0 to $n-1$ and using the triangle inequality, we get (4.4).

Theorem 4.3. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping in (a, b) with $f' \in L_2[a, b]$. Then $\forall x \in \left[a, \frac{a+b}{2}\right]$ and $h \in [0, 1]$ we have*

$$\int_a^b f(x)\, dx = S(f, I_n) + R(f, I_n),$$

where $S(f, I_n)$ is defined by by formula (4.1), and the remainder $R(f, I_n)$ satisfies the estimate

$$R(f, I_n) \le \frac{\left[9h^2 - 3h + 1\right]^{\frac{1}{2}}}{4\sqrt{3}} \sum_{i=0}^{n-1} \left[\|f\|_2^2 - S_i^2 \delta_i\right]^{\frac{1}{2}} \delta_i^{\frac{3}{2}}, \tag{4.5}$$

where $S_i = (f(x_{i+1}) - f(x_i))/\delta_i$, $i = 0, 1, ..., n-1$.

Proof: Applying (2.40) to the interval $[x_i, x_{i+1}]$, we get

$$\begin{aligned} & \left| \frac{1}{2}\left[(1-2h) f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right.\right. \\ & +h\left[f(x_i) + f(x_{i+1})\right]\delta_i - \frac{h}{4}\left[f(x_{i+1}) - f(x_i)\right]\delta_i \\ & \left. - \int_{x_i}^{x_{i+1}} f(t)\, dt \right| \\ \le \; & \frac{\left[9h^2 - 3h + 1\right]^{\frac{1}{2}}}{4\sqrt{3}} \left[\|f\|_2^2 - S_i^2 \delta_i\right]^{\frac{1}{2}} \delta_i^{\frac{3}{2}}, \end{aligned}$$

for all $i = 0, 1, ..., n-1$. Now summing over i from 0 to $n-1$ and using the triangle inequality, we get (4.5).

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