The Borel radius and the S radius of the K-quasimeromorphic mapping in the unit disc

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Abstract: By using Ahlfors' theory of covering surface, a fundamental inequality for the K-quasimeromorphic mapping in the unit disc is established. As an application, some results on the Borel radius and the S radius dealing with multiple values of the K-quasimeromorphic mapping in the unit disc are obtained.

Keywords: K-quasimeromorphic mapping; in the unit disc; Borel radius; S radius

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1 Introduction

In 1997, the value distribution theory of meromorphic functions due to Nevanlinna (see [1],[2] and [3] for standard references) has been extended to the corresponding theory of the K-quasimeromorphic mapping by Sun and Yang [4]. The K-quasimeromorphic mapping is a more widespread function than the meromorphic function, but it has no derivative, even the partial derivative does not exist everywhere. They established a fundamental inequality on the complex plane and used it to prove the existence theorem of the Borel direction and the filling disc theorem of the K-quasimeromorphic mapping. In 1999, Gao [5] established a fundamental inequality dealing with multiple values on the complex plane and improved some results of [4].

Recently, the singular direction is one of the interesting topics studied in the theory of value distribution of the K-quasimeromorphic mapping on the complex plane such as Julia direction, Borel direction, Nevanlinna direction and S direction, see [6-11]. Their existence theorems and some

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connections between them have also been established, which extends the relative properties of meromorphic function on the complex plane. In 2004, Yang and Liu [10] used a fundamental inequality of an angular domain on the complex plane to confirm the existence of a Borel direction of the K-quasimeromorphic mappings of the zero order. Later, Wu and Sun [8] proved the existence of a S direction for the K-quasimeromorphic mapping on the complex plane, which was inspired by the idea of the T-direction [12] for the meromorphic function.

Theorem A. Let $f(z)$ be K-quasimeromorphic mapping on the complex plane and satisfy $\overline{\lim\limits_{r\to\infty}}\frac{S(r,f)}{(\ln r)^2} = +\infty$, then there exists a ray $\arg z = \theta$ (namely S direction) such that for any $\varepsilon > 0$,

$$\overline{\lim_{r\to\infty}}\frac{\overline{n}(\Omega(\theta-\varepsilon,\theta+\varepsilon),r,a)}{S(r,f)} > 0$$

holds for all $a \in \mathbf{C}_\infty := \mathbf{C} \cup \infty$, except for two possible exceptional values.

It is well known that if f is a transcendental meromorphic function defined in $|z| < 1$, it will share some properties with the one on the complex plane, see [2], [13] and [14]. Thus a natural question is: Is there a Borel radius or a S radius for the K-quasimeromorphic mapping in $|z| < 1$? However, until now only a few results on the singular radius of the K-quasimeromorphic mapping have been discussed, see [15] and [16]. So in this paper, we establish a more precise fundamental inequality for the K-quasimeromorphic mapping in the unit disc and confirm the existence of the Borel radius and the S radius (dealing with multiple values) for the K-quasimeromorphic mapping in the unit disc, which develop some results of [4], [7] and [11]. To do so, we recall some definitions and notations, which can be found in [4] and [6].

Definition 1.[4] Let $f(s)$ be a homeomorphism from D to D'. If for any rectangle $\{z = x + iy; a < x < b, c < y < d\}$ in D,

(i) $f(x+iy)$ is absolutely continuous of y for almost every fixed $x \in (a,b)$ and $f(x+iy)$ is absolutely continuous of x for almost every fixed $y \in (c,d)$;

(ii) there exists a constant $K \geq 1$ such that

$$|f_z(z)| + |f_{\overline{z}}(z)| \leq K(|f_z(z)| - |f_{\overline{z}}(z)|)$$

holds almost everywhere in D; then f is named an univalent K-quasimeromorphic mapping in D.

Definition 2.[4] Let f be a complex and continuous function in the region D. For a point z_0 in D, if there is a neighborhood $U(\subset D)$ and a positive integer n depending on z_0, such that

$$F(z) = \begin{cases} (f(z))^{1/n}, & f(z_0) = \infty \\ (f(z) - f(z_0))^{1/n} + f(z_0), & f(z_0) \neq \infty \end{cases}$$

is an univalent K-quasimeromorphic mapping, then f is named n-valent K-quasimeromorphic mappings at point z_0. If f is n-valent K-quasimeromorphic at every point of D, then f is called a K-quasimeromorphic mapping in D.

It is obvious that a meromorphic function is a 1-quasimeromorphic mapping. The composition function $g \circ f$ of a meromorphic function g and a K-quasimeromorphic mapping f is still a K-quasimeromorphic mapping. Let $n(r, a)$ be the number of zero points of $f(z) - a$ in disc $|z| \leq r$. If the multiple zeros are counted only once, then we use $\overline{n}(r, a)$. Let $n(\Omega(\varphi - \varepsilon, \varphi + \varepsilon), r, a)$ be the number of zero points $f(z) - a$ in $\{\varphi - \varepsilon < \arg z < \varphi + \varepsilon\} \cap \{|z| \leq r\}$. If the multiple zeros are counted only once, we use $\overline{n}(\Omega(\varphi - \varepsilon, \varphi + \varepsilon), r, a)$. Let $\overline{n}^{l)}(\Omega(\varphi - \varepsilon, \varphi + \varepsilon), r, a)$ be the number of distinct roots with the multiplicity $\leq l$ of $f(z) = a$ in the same region.

Let $\mathbf{V}$ be a Riemann sphere whose diameter is 1. $f(z) = u(x, y) + iv(x, y)$ is a K-quasimeromorphic mapping in the angular domain $E = \Omega(\varphi_1, \varphi_2) \cap \{|z| \leq r\}$. Set

$$S(E, f) = S(\Omega(\varphi_1, \varphi_2), r) = \frac{|F_r|}{|V|} = \frac{1}{\pi} \int\int_{z \in E} \frac{u_x v_y - v_x u_y}{(1 + |f(z)|^2)^2} r dr d\theta.$$

where $|F_r|$ is the area of the image of E on $\mathbf{V}$ and $|\mathbf{V}|$ is the area of $\mathbf{V}$. If $E = \{|z| \leq r\}$, then $S(E, f)$ can be replaced by $S(r, f)$.

Definition 3. Let $f(z)$ be a K-quasimeromorphic mapping defined in the unit disc. If $S(r, f) \to +\infty$ as $r \to 1^-$, then we call $f(z)$ transcendental. The order of the transcendental K-quasimeromorphic mapping in the unit disc is defined by

$$\rho = \overline{\lim_{r \to 1^-}} \frac{\ln S(r, f)}{-\ln(1 - r)}.$$

If $\rho = \lim\limits_{r \to 1^-} \frac{\ln S(r,f)}{-\ln(1-r)}$, then $f(z)$ is of regular growth. Especially, when $K = 1$, if $S(r, f)$ is replaced by $T(r, f) = \int_0^r S(t, f)/t dt$, then ρ is called the order of the meromorphic function $f(z)$.

Definition 4. Let $f(z)$ be a transcendental K-quasimeromorphic mapping defined in the unit disc. A radius $\Delta(\varphi) = \{z : \arg z = \varphi, |z| < 1\}$ is called a Borel radius of the order $\rho \in (0, +\infty)$ for the K-quasimeromorphic mapping $f(z)$ in the unit disc, provided that for any $\varepsilon \in (0, \pi)$,

$$\lim_{r \to 1^-} \frac{\ln \overline{n}(\Omega(\varphi - \varepsilon, \varphi + \varepsilon), r, a)}{\ln \frac{1}{1-r}} \geq \rho$$

holds for all $a \in \mathbf{C}_\infty$, except for two possible exceptional values.

Note that this definition of Borel radius meaningfully characterizes the growth of $f(z)$ only when $0 < \rho < \infty$. Inspired by the idea of [8], we give a definition of the S radius in the unit disc.

Definition 5. Let $f(z)$ be a transcendental K-quasimeromorphic mapping defined in the unit disc. A radius $\Delta(\varphi)$ is called a S radius of $f(z)$ dealing with multiple values $l(\geq 3)$, provided that for any $\varepsilon > 0$,

$$\overline{\lim_{r \to 1^-}} \frac{\overline{n}^{l)}(\Omega(\varphi - \varepsilon, \varphi + \varepsilon), r, a)}{S(r, f)} > 0$$

holds for all $a \in \mathbf{C}_\infty$, except for two possible exceptional values. If $l \longrightarrow +\infty$, $\Delta(\varphi)$ is called the S radius of $f(z)$ in the unit disc (S direction in the case of the complex plane).

2 Preliminary lemmas

Lemma 1.[5] Let $f(z)$ be a K-quasimeromorphic mapping in $|z| < R$ and $\{a_1, a_2, \cdots, a_q\}$ be $q(q \geq 3)$ distinct points with the mutual spherical distance no less than $\delta \in (0, 1/2)$, then for any $r \in (0, R)$, we have

$$(q - 2 - \frac{2}{l})S(r, f) \leq \sum_{v=1}^{q} \overline{n}^{l)}(R, a_v) + \frac{(960 + 2q\pi)^2}{\delta^6} \cdot \frac{2^5\pi^2 K}{q - 2 - \frac{2}{l}} \cdot \frac{R}{R - r}.$$

Lemma 2.[13] Let $f(z)$ be a meromorphic function defined in $|z| < 1$, then

$$\overline{\lim_{r \to 1^-}} \frac{\ln T(r, f)}{-\ln(1 - r)} = \rho \Longleftrightarrow \overline{\lim_{r \to 1^-}} \frac{\ln S(r, f)}{-\ln(1 - r)} = \rho + 1.$$

3 Main Results and their proofs

Theorem1. Let $f(z)$ be a K-quasimeromorphic mapping in $|z| < 1$. Δ and Δ_0 are two angular domains with the common vertex on the center of the unit disc, where

$$\Delta := \Omega(\varphi - \eta, \varphi + \eta) \subset \Delta_0 := \Omega(\varphi - \eta_0, \varphi + \eta_0),\ 0 < \eta < \eta_0,\ 0 \leq \varphi < 2\pi.$$

Then,

$$(q-2-\frac{2}{l})S(\Delta,r)\leq(1+\frac{2\ln 2}{\ln\frac{1}{1-r}})\sum_{v=1}^{q}\overline{n}^{l)}(\Delta_0,\frac{3+r}{4},a_v)+O(\ln\frac{1}{1-r}),\quad(1)$$

$$(q-2)S(\Delta,r)\leq(1+\frac{2\ln 2}{\ln\frac{1}{1-r}})\sum_{v=1}^{q}\overline{n}(\Delta_0,\frac{3+r}{4},a_v)+O(\ln\frac{1}{1-r}),\qquad(2)$$

where $a_1,a_2\cdots a_q$ are q distinct points in $\mathbf{V}$ with the mutual spherical distance not less than $\delta\in(0,1/2)$, $r_0\in(1/2,1)$ and $r\in(r_0,1)$.

Proof. Let $r_i=1-\frac{1-r_0}{2^i}$ $(i=0,1,\cdots)$, then $r_1=\frac{1+r_0}{2}$. For any $r\in(r_1,1)$, there exists $n\in N_+$ such that $r_n\leq r\leq r_{n+1}$. So we set

$$r_{i,j}=r_i+\frac{j(r_{i+1}-r_i)}{n}\quad(j=0,1,\cdots n-1),$$

then $r_{i,0}=r_i$ and $r_{i,n}=r_{i+1}$. For any positive integer $i\geq 2$, we set

$$\Delta_0(r_{i,j},r_{i+1,j+1}):=\Delta_0\cap\{r_{i,j}\leq|z|\leq r_{i+1,j+1}\}.$$

So we can easily see that there exists an integer $j_0\in[0,n-1]$ such that

$$\sum_{v=1}^{q}\overline{n}^{l)}(\bigcup_{i=0}^{n+1}\Delta_0(r_{i,j_0},r_{i,j_0+1}),a_v)\leq\frac{1}{n}\sum_{v=1}^{q}\overline{n}^{l)}(\Delta_0,r_{n+2},a_v),$$

where $\Delta_0(r_{i,j_0},r_{i,j_0+1})=\Delta_0\cap\{r_{i,j_0}\leq|z|\leq r_{i,j_0+1}\}$. Then, we set

$$\Delta(r_i',r_{i+1}'):=\Delta\cap\{r_i'\leq|z|\leq r_{i+1}'\}\subset\Delta_0(r_{i,j_0},r_{i+1,j_0+1}),$$

where

$$r_i'=\frac{r_{i,j_0}+r_{i,j_0+1}}{2},\quad r_{i+1}'=\frac{r_{i+1,j_0}+r_{i+1,j_0+1}}{2}.$$

Without loss of generality, we suppose that $\varphi=0$. Since

$$r_{i+1,j_0+1}-r_{i,j_0}=\frac{1-r_0}{2^{i+2}}(2+\frac{1-j_0}{n}),\qquad r_{i+1}'-r_i'=\frac{1-r_0}{2^{i+2}}(2-\frac{2j_0+1}{2n}),$$

then $\Delta_0(r_{i,j_0},r_{i+1,j_0+1})$ $(i=2,3,\cdots)$ can map mutually by some transforms such that their sub domains $\Delta(r_i',r_{i+1}')$ and whose centers $(\frac{r_i'+r_{i+1}'}{2},0)$ map each other, respectively. Through the Riemann mapping theorem, for any fixed i, we can map the $\Delta_0(r_{i,j_0},r_{i+1,j_0+1})$ on $|\xi|<1$ by a conformal mapping g such that the point $(\frac{r_i'+r_{i+1}'}{2},0)$ of $\Delta_0(r_{i,j_0},r_{i+1,j_0+1})$ becomes $\xi=0$, then the image of $\Delta(r'_i, r'_{i+1})$ is contained in $|\xi| \leq c < 1$, where $c(> 0)$ is a constant defined by η, η_0 and r_0, independent of i. Hence by Lemma 1, we have

$$(q-2-\frac{2}{l})S(c, f\circ g^{-1}) \leq \sum_{v=1}^{q} \overline{n}^{l)}(1, f\circ g^{-1} = a_v) + \frac{H}{1-c},$$

where H is a constant. Then,

$$(q-2-\frac{2}{l})S(\Delta(r'_i, r'_{i+1}), f) \leq \sum_{v=1}^{q} \overline{n}^{l)}(\Delta_0(r_{i,j_0}, r_{i+1,j_0+1}), a_v) + \frac{H}{1-c}, \quad i = 0, 1, \cdots, n.$$

Adding two sides of the above expression from $i = 0$ to n, we obtain

$$(q-2-\frac{2}{l})\sum_{i=0}^{n} S(\Delta(r'_i, r'_{i+1}), f) \leq \sum_{i=0}^{n}\sum_{v=1}^{q} \overline{n}^{l)}(\Delta_0(r_{i,j_0}, r_{i+1,j_0+1}), a_v) + \frac{H}{1-c}(n+1).$$

Since $r_n \leq r \leq r_{n+1}$, then

$$r_{n+2} = \frac{3+r_n}{4} \leq \frac{3+r}{4}, \quad \frac{1-r}{4} \leq 1-r_{n+2} \leq \frac{1}{2^{n+2}}, \quad \frac{1}{1-r} \leq \frac{1}{1-r_{n+1}} = \frac{2^{n+1}}{1-r_0}.$$

Hence

$$2^n \leq \frac{1}{1-r}, \quad n+1 \leq 1 + \frac{1}{\ln 2}\ln\frac{1}{1-r}, \quad \frac{1}{n} \leq \frac{2\ln 2}{\ln\frac{1}{1-r}} \quad (r \to 1^-).$$

When r is sufficiently close to 1^-, we have

$$\begin{aligned}
&(q-2-\tfrac{2}{l})S(\Delta, r) \leq (q-2-\tfrac{2}{l})\textstyle\sum_{i=0}^{n} S(\Delta(r'_i, r'_{i+1}), f) + (q-2-\tfrac{2}{l})S(\Delta, r_1) \\
&\leq \textstyle\sum_{i=0}^{n}\sum_{v=1}^{q} \overline{n}^{l)}(\Delta_0(r_{i,j_0}, r_{i+1,j_0+1}), a_v) + \frac{H(n+1)}{1-c} + (q-2-\tfrac{2}{l})S(\Delta, r_1) \\
&\leq \textstyle\sum_{i=0}^{n}\sum_{v=1}^{q} \overline{n}^{l)}(\Delta_0(r_{i,j_0}, r_{i,j_0+1}), a_v) + \sum_{i=0}^{n}\sum_{v=1}^{q} \overline{n}^{l)}(\Delta_0(r_{i,j_0+1}, r_{i+1,j_0+1}), a_v) + \frac{2H(n+1)}{1-c} \\
&\leq (1+\tfrac{1}{n})\textstyle\sum_{v=1}^{q} \overline{n}^{l)}(\Delta_0, r_{n+2}, a_v) + \frac{2H(n+1)}{1-c} \\
&\leq (1+\tfrac{2\ln 2}{\ln\frac{1}{1-r}})\textstyle\sum_{v=1}^{q} \overline{n}^{l)}(\Delta_0, \frac{3+r}{4}, a_v) + O(\ln\frac{1}{1-r}).
\end{aligned}$$

So (1) follows, the second inequality (2) can be obtained by the similar proof.

Remark 1. Theorem 1 gives a fundamental inequality for the K-quasimeromorphic mapping in an angular domain of the unit disc, which is more precise than that of [15]. If $K = 1$, it is also better than Theorem VII.14 of [2, P.291] and Lemma 3 of [13].

Theorem 2. Let $f(z)$ be a K-quasimeromorphic mapping in the unit disc with the order $\rho \in (0, +\infty)$, then $f(z)$ has a Borel radius of the order ρ.

Proof. Otherwise, for any $\varphi \in [0, 2\pi)$, there exists $\varepsilon_\varphi > 0$ and three distinct complex numbers $a_1, a_2, a_3 \in \mathbf{C}_\infty$ such that

$$\overline{\lim_{r\to 1^-}} \frac{\ln \overline{n}(\Omega(\varphi - \varepsilon_\varphi, \varphi + \varepsilon_\varphi), r, a_i)}{\ln \frac{1}{1-r}} = \rho_0 < \rho, \quad i = 1, 2, 3. \tag{3}$$

It is obvious that the open sets $\{(\varphi - \frac{\varepsilon_\varphi}{4}, \varphi + \frac{\varepsilon_\varphi}{4}) | \varphi \in [0, 2\pi)\}$ cover the unit disc. From the finite covering theorem, there exists a subsequence

$$(\varphi_1 - \frac{\varepsilon_{\varphi_1}}{4}, \varphi_1 + \frac{\varepsilon_{\varphi_1}}{4}), \cdots, (\varphi_n - \frac{\varepsilon_{\varphi_n}}{4}, \varphi_n + \frac{\varepsilon_{\varphi_n}}{4})$$

lying in $(\varphi_k - \varepsilon_{\varphi_k}, \varphi_k + \varepsilon_{\varphi_k}) \quad (k = 1, \cdots, n)$, such that for any $\varepsilon > 0$ and each k

$$\sum_{i=1}^{3} \overline{n}(\Omega(\varphi - \varepsilon_\varphi, \varphi + \varepsilon_\varphi), r, a_i) < 3(\frac{1}{1-r})^{\rho_0 + \varepsilon}.$$

By (2) of Theorem 1, it follows that

$$S(r, f) \le \sum_{k=1}^{n} S(\Omega(\varphi_k - \frac{\varepsilon_{\varphi_k}}{4}, \varphi_k + \frac{\varepsilon_{\varphi_k}}{4})$$

$$\le (1 + \frac{2\ln 2}{\ln \frac{1}{1-r}}) \sum_{k=1}^{n} \sum_{i=1}^{3} \overline{n}(\Omega(\varphi_k - \frac{\varepsilon_{\varphi_k}}{2}, \varphi_k + \frac{\varepsilon_{\varphi_k}}{2}), \frac{r+3}{4}, a_i) + O(\ln \frac{1}{1-r}).$$

Then, there is a positive constant C such that

$$S(r, f) \le C(1 + \frac{2\ln 2}{\ln \frac{1}{1-r}})(\frac{1}{1-r})^{\rho_0 + \varepsilon} + O(\ln \frac{1}{1-r}),$$

This is in contradiction to that $f(z)$ is of the order ρ. Hence we complete the proof.

Corollary 1. Let $f(z)$ be a meromorphic function in the unit disc with the order $\rho \in (0, +\infty)$, then $f(z)$ has a Borel radius of the order $\rho + 1$.

Remark 2. Why the Borel radius is of the order $\rho + 1$? In fact, from Lemma 2

$$\overline{n}(r, a) = O(S(r, f)) = O\Big(\frac{1}{1-r} T(r, f)\Big) = O\Big((\frac{1}{1-r})^{1+\rho+\varepsilon}\Big)$$

in general comes into existence when $r \to 1^-$. Hence for some ray to be a Borel radius for a function f, it means that the function f has a maximal number(relative to its growth) of a-points in an $\varepsilon-$neighborhood of that ray.

Theorem 3. Let $f(z)$ be a K-quasimeromorphic mapping defined in $|z| < 1$ and satisfy

$$\overline{\lim_{r\to 1^-}} \frac{S(r,f)}{\ln \frac{1}{1-r}} = +\infty, \tag{4}$$

then $f(z)$ can take any complex number infinite times, except for two possible exceptional values.

Proof. Otherwise, for any $\varphi \in [0, 2\pi)$ and $r \in (0,1)$, there exists $\varepsilon_0 > 0$ and three distinct complex numbers $a_1, a_2, a_3 \in \mathbf{C}_\infty$ such that

$$\sum_{j=1}^{3} \overline{n}(\Omega(\varphi - 2\varepsilon_0, \varphi + 2\varepsilon_0), r, a_j) \leq \sum_{j=1}^{3} n(r, a_j) = O(1).$$

By (2) of Theorem 1, we have

$$S(\Omega(\varphi - \varepsilon_0, \varphi + \varepsilon_0), r) \leq O\left(1 + \frac{2\ln 2}{\ln \frac{1}{1-r}}\right) + O(\ln \frac{1}{1-r}).$$

Since φ is arbitrary, from the similar proof of Theorem 2, we have

$$S(r,f) \leq C\left(1 + \frac{2\ln 2}{\ln \frac{1}{1-r}}\right) + O(\ln \frac{1}{1-r}),$$

where C is a positive constant. This is in contradiction to the hypothesis (4). Hence we complete the proof.

Corollary 2. Let $f(z)$ be a meromorphic function defined in $|z| < 1$ and satisfy (4), then $f(z)$ can take any complex number infinite times with at most two exceptional values.

Theorem 4. Let $f(z)$ be a K-quasimeromorphic mapping defined in $|z| < 1$. If $f(z)$ satisfies (4) and

$$\overline{\lim_{r\to 1^-}} \frac{S(\frac{3+r}{4}, f)}{S(r,f)} < +\infty, \tag{5}$$

then $f(z)$ has a S radius (dealing with multiple values $l(\geq 3)$).

Proof. From the condition of (4), there exists an increasing sequence $\{r_n\} \uparrow 1$ $(n \to \infty)$ such that $\lim_{n\to\infty} \frac{S(r_n,f)}{\ln \frac{1}{1-r_n}} = +\infty$.

Using the finite covering theorem on $[0, 2\pi)$, there must be some $\varphi_0 \in [0, 2\pi)$ such that for any $\varepsilon \in (0, \pi/4)$,

$$\overline{\lim_{n\to\infty}} \frac{S((\varphi_0 - \varepsilon, \varphi_0 + \varepsilon), r_n)}{S(r_n, f)} > 0.$$

Now we can predicatively say that the radius $\Delta(\varphi_0) = \{z : \arg z = \varphi_0, |z| < 1\}$ is a S radius of $f(z)$ dealing with multiple values l.

Otherwise, there are three distinct complex numbers $a_1, a_2, a_3 \in \mathbf{C}_\infty$ and a positive δ such that

$$\overline{\lim_{n\to\infty}} \frac{\sum_{j=1}^{3} \overline{n}^{l)}(\Omega(\varphi_0 - \delta, \varphi_0 + \delta), r_n, a_j)}{S(r_n, f)} = 0.$$

By (1) of Theorem 1, when $q = 3$, for any $0 < \varepsilon < \delta$, we have

$$\begin{aligned}
&(1 - \tfrac{2}{l}) S(\Omega(\varphi_0 - \varepsilon, \varphi_0 + \varepsilon), r_n) \\
&\leq (1 + \tfrac{2 \ln 2}{\ln \frac{1}{1-r_n}}) \sum_{j=1}^{3} \overline{n}^{l)}(\Omega(\varphi_0 - \delta, \varphi_0 + \delta), \tfrac{3+r_n}{4}, a_j) + O(\ln \tfrac{1}{1-r_n}).
\end{aligned}$$

Hence

$$\begin{aligned}
&(1 - \tfrac{2}{l}) \overline{\lim_{n\to\infty}} \tfrac{S(\Omega(\varphi_0 - \varepsilon, \varphi_0 + \varepsilon), r_n)}{S(r_n, f)} \\
&\leq \overline{\lim_{n\to\infty}} (1 + \tfrac{2 \ln 2}{\ln \frac{1}{1-r_n}}) \tfrac{\sum\limits_{j=1}^{3} \overline{n}^{l)}(\Omega(\varphi_0 - \delta, \varphi_0 + \delta), \frac{3+r_n}{4}, a_j)}{S(\frac{3+r_n}{4}, f)} \tfrac{S(\frac{3+r_n}{4}, f)}{S(r_n, f)} + \overline{\lim_{n\to\infty}} \tfrac{O(\ln \frac{1}{1-r_n})}{S(r_n, f)}.
\end{aligned}$$

It follows from (4) and (5) that $1 - \frac{2}{l} \leq 0$, we get a contradiction. Hence the radius $\Delta(\varphi_0)$ is a S radius of $f(z)$ dealing with multiple values l. By the similar proof, the radius $\Delta(\varphi_0)$ is also a S radius.

Corollary 3. Let $f(z)$ be a meromorphic function defined in $|z| < 1$ and satisfy the conditions of (4) and (5), then $f(z)$ has a S radius (dealing with multiple values $l(\geq 3)$).

Theorem 5. Let $f(z)$ be a K-quasimeromorphic mapping in the unit disc with order $\rho \in [0, +\infty)$ and of regular growth, then every S radius (dealing with multiple values) is a Borel radius of the order ρ.

Proof. Let $\Delta(\varphi_0) = \{z : \arg z = \varphi_0, |z| < 1\}$ be a S radius dealing with multiple values for $f(z)$ in the unit disc, then for any $\varepsilon \in (0, \pi/2)$ and each a (except for two possible exceptional values), we have

$$\overline{\lim_{r\to 1^-}} \frac{\overline{n}^{l)}(\Omega(\varphi_0 - \varepsilon, \varphi_0 + \varepsilon), r, a)}{S(r, f)} > \delta > 0.$$

Then, there exist $\{r_n\}$ for a sufficiently large n we have

$$\overline{n}^{l)}(\Omega(\varphi_0-\varepsilon,\varphi_0+\varepsilon),r_n,a)>\frac{\delta}{2}S(r_n,f).$$

Since $f(z)$ is of regular growth, it follows that

$$\overline{\lim_{r\to 1^-}}\frac{\ln\overline{n}^{l)}(\Omega(\varphi_0-\varepsilon,\varphi_0+\varepsilon),r,a)}{\ln\frac{1}{1-r}}\geq\overline{\lim_{n\to\infty}}\frac{\ln\overline{n}^{l)}(\Omega(\varphi_0-\varepsilon,\varphi_0+\varepsilon),r_n,a)}{\ln\frac{1}{1-r_n}}\geq\rho$$

holds for all $a\in\mathbf{C}_\infty$, except for two possible exceptional values.

Hence every S radius (dealing with multiple values) is a Borel radius of the order ρ.

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References

[1] L. Yang, Value distribution theory and its new reasearch, Springer-Verlag, 1993.

[2] M. Tsuji, Potential theory in modern function theory, Maruzen Co. LTD, Tokyo, 1959.

[3] J.F. Xu and H.X. Yi, On uniquess of meromorphic functions with shared four values in some angular domains. Bull. Malays. Math. Sci. Soc. (2) 32(2008), no. 1, 57-65.

[4] D.C. Sun and L. Yang. Value distribution of quasiconformal mappings. Science in China(Ser.A), 1997, 27(2), 132-139.

[5] Z.S. Gao, Multiple values of quasiconformal mappings, J. Math. (Wuhan), 19 (1999), No.2, 121-126.

[6] Y.Y. Kong and Y. Hong, On the growth of Laplace-Stieltjes transforms and the singular direction of complex analysis, Jinan University Press, Guangzhou, 2010.

[7] H.Y. Xu and T.S. Zhang, On the existence of T-direction and Nevanlinna direction of K-Quasi-meromorphic mapping dealing with multiple values. Bull. Malays. Math. Sci. Soc., (2) 33(2) (2010), 281-294.

[8] Z.J. Wu and D.C. Sun, On a new singular direction of quasiconformal mappings, Acta Mathematica Scientia, 2009(29B), 1453-1460.

[9] H.H. Chen, Some new results on planar harmonic mappings. Sci China Math, 2010, 53(3), 597-604.

[10] Y. Yang and M.S. Liu, On the Borel direction of K-quasimeromorphic mapping. Acta Mathematica Scientia, 2004(24B), 75-82.

[11] F.W. Deng, Some remarks on quasimeromorphic mapping. Acta Mathematica Scientia, 2003(23B): 419-425.

[12] H. Guo, J. Zheng and T.W. Ng, On a new singular direction of meromorphic functions, Bull. Austral. Math. Soc. 2004,69,277-287.

[13] Y.Y. Kong, On filling discs in Borel radius of meromorphic mapping with finite order in the unit circle, J. Math. Anal. Appl. 344 (2008) 1158-1164.

[14] T.B. Cao, The growth, oscillation and fixed points of solutions of complex linear differential equations in the unit disc, J. Math. Anal. Appl. 352 (2009) 739-748.

[15] F.W. Deng, On the Borel points of quasi-meromorphic mappings in the unit circle, Acta Mathematica Scientia, 2000(20), 562-567.

[16] Y.Y. Kong, A new singular radius of quasimeromorphic mappings on some Type-functions in the unit disc, Acta Mathematica Scientia, 2010(30A), 1640-1647.