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On the Unification of Two Families of Multiple Twisted Type Polynomials by Using p-Adic q-Integral at $q=-1$

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Abstract. The present paper deals with unification of the multiple twisted Euler and Genocchi numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q=-1$. Some earlier results of Ozden's papers in terms of unification of the multiple twisted Euler and Genocchi numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q=-1$ can be deduced. We apply the method of generating function and p-adic q-integral representation on $\mathbb{Z}_p$, which are exploited to derive further classes of Euler polynomials and Genocchi polynomials. To be more precise we summarize our results as follows, we obtain some relations between Ozden's generating function and fermionic p-adic q-integral on $\mathbb{Z}_p$ at $q=-1$. Furthermore we derive Witt's type formula for the unification of twisted Euler and Genocchi polynomials. Also we derive distribution formula (Multiplication Theorem) for multiple twisted Euler and Genocchi numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q=-1$ which yields a deeper insight into the effectiveness of this type of generalizations. Furthermore we define unification of multiple twisted zeta function and we obtain an interpolation formula between unification of multiple twisted zeta function and unification of the multiple twisted Euler and Genocchi numbers at negative integers. Our new generating function possess a number of interesting properties which we state in this paper.

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1. Introduction, definitions and notations

Bernoulli numbers were introduced by Jacques Bernoulli (1654–1705), in the second part of his treatise published in 1713, "*Ars conjectandi*", at the time, Bernoulli numbers were used for writing the infinite series expansions of hyperbolic and trigonometric functions. Van den berg was the first to discuss finding recurrence formulae for the Bernoulli numbers with arbitrary sized gaps (1881). Ramanujan showed how gaps of size 7 could be

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found, and explicitly wrote out the recursion for gaps, of size 6. Lehmer in 1934 extended these methods to Euler numbers, Genocchi numbers and Lucas numbers (1934) and calculated the 196-th Bernoulli numbers. The study of generalized Bernoulli, Euler and Genocchi numbers and polynomials and their combinatorial relations has received much attention [1, 2, 4, 5, 7, 8, 25–29, 40]. Generalized Bernoulli polynomials, generalized Euler polynomials and generalized Genocchi numbers and polynomials are the signs of very strong bond between elementary number theory, complex analytic number theory, Homotopy theory (stable Homotopy groups of spheres), differential topology (differential structures on spheres), theory of modular forms (Eisenstein series), p-adic analytic numbers theory (p-adic L-functions), quantum physics(quantum Groups). p-adic numbers were invented by Kurt Hensel around the end of the nineteenth century. In spite of their being already one hundred years old, these numbers are still today enveloped in an aura of mystery within the scientific community. The p-adic integral was used in mathematical physics, for instance, the functional equation of the q-zeta function, q-stirling numbers and q-Mahler theory of integration with respect to the ring $\mathbb{Z}_p$ together with Iwasawa's p-adic q-L functions. Also the p-adic interpolation functions of the Bernoulli and Euler polynomials have been treated by Tsumura [39] and Young [41]. Kim [10–24] also studied on p-adic interpolation functions of these numbers and polynomials. In [3], Carlitz originally constructed q-Bernoulli numbers and polynomials. These numbers and polynomials are studied by many authors (see cf. [6, 10–24, 30, 31, 33, 35]). In the last decade, a surprising number of papers appeared proposing new generalizations of the Bernoulli, Euler and Genocchi polynomials to real and complex variables. In [6, 10–24], Kim studied some families of multiple Bernoulli, Euler and Genocchi numbers and polynomials. By using the fermionic p-adic invariant integral on $\mathbb{Z}_p$, he constructed p-adic Bernoulli, Euler and Genocchi numbers and polynomials of higher order. A unification (and generalization) of Bernoulli polynomials and Euler polynomials with a, b and c parameters first was introduced and investigated by Luo [27–29]. After Luo and Srivastava defined unification (and generalization) of Apostol type Bernoulli polynomials with a, b and c parameters of higher order [29]. After Ozden *et al.* [31] unified and extended the generating functions of the generalized Bernoulli polynomials, the generalized Euler polynomials and the generalized Genocchi polynomials associated with the positive real parameters a and b and the complex parameter. Also they, by applying the Mellin transformation to the generating function of the unification of Bernoulli, Euler and Genocchi polynomials, constructed a unification of the Zeta functions. Actually, their definition provides a generalization and unification of the Bernoulli, Euler and Genocchi polynomials and also of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, which were considered in many earlier investigations by (among others) Srivastava *et al.* [36–38], Karande [9]. Also, they, by using a Dirichlet character, defined unification of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials and numbers. Kim in [20], constructed Apostol-Euler numbers and polynomials by using fermionic expression of p-adic q-integral at $q = -1$. In this paper by his method we derive several properties for unification of the multiple twisted Euler and Genocchi numbers and polynomials.

Let p be a fixed odd prime number. Throughout this paper we use the following notations, by $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. The

p-adic absolute value is defined by $|p|_p = 1/p$. In this paper, we assume $|q-1|_p < 1$ as an indeterminate. $[x]_q$ is a q-extension of x which is defined by $[x]_q = (1-q^x)/(1-q)$, we note that $\lim_{q\to 1}[x]_q = x$.

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient

$$F_f(x,y) = \frac{f(x)-f(y)}{x-y}$$

has a limit $f'(a)$ as $(x,y) \to (a,a)$ and denote this by $f \in UD(\mathbb{Z}_p)$.

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, let us begin with the following expression

$$\frac{1}{[p^N]} \sum_{0\le x<p^N} f(x)q^x = \sum_{0\le x<p^N} f(x)\mu_q\left(x+p^N\mathbb{Z}_p\right),$$

represents p-adic q-analogue of Riemann sums for f. The integral of f on $\mathbb{Z}_p$ will be defined as the limit $(N\to\infty)$ of these sums, when it exists. The p-adic q-integral of function $f \in UD(\mathbb{Z}_p)$ is defined by Kim

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x. \tag{1.1}$$

The bosonic integral is considered by Kim as the bosonic limit $q \to 1$, $I_1(f) = \lim_{q\to 1} I_q(f)$. Similarly, the fermionic p-adic integral on $\mathbb{Z}_p$ is considered by Kim as follows:

$$I_{-q}(f) = \lim_{q\to -q} I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x).$$

Assume that $q \to 1$, then we have fermionic p-adic fermionic integral on $\mathbb{Z}_p$ as follows

$$I_{-1}(f) = \lim_{q\to -1} I_q(f) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x. \tag{1.2}$$

If we take $f_1(x) = f(x+1)$ in (1.2), then we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0). \tag{1.3}$$

Let p be a fixed prime. For a fixed positive integer d with $(p,d) = 1$, we set

$$X = X_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p, \quad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p$$

and

$$a + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a\left(\bmod dp^N\right)\right\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$.

Definition 1.1. [32] *A unification $y_{n,\beta}(x:k,a,b)$ of the Bernoulli, Euler and Genochhi polynomials is given by the following generating function:*

$$F_{a,b}(x;t;k,\beta) = \frac{2\left(\frac{t}{2}\right)^k}{\beta^b e^t - a^b} e^{xt} = \sum_{n=0}^{\infty} y_{n,\beta}(x:k,a,b)\frac{t^n}{n!} \quad \left(\left|t+\log\left(\frac{\beta}{a}\right)\right| < 2\pi;\ x\in\mathbb{R}\right)$$

$$\left(k\in\mathbb{N}^*; a,b\in\mathbb{R}^+; \beta\in\mathbb{C}\right), \tag{1.4}$$

where as usual $\mathbb{R}^{+}$, and $\mathbb{C}$ denote the sets of positive real numbers and complex numbers, respectively, $\mathbb{R}$ being the set of real numbers.

Observe that, if we put $x=0$ in the generating function (1.4), then we obtain the corresponding unification of the generating functions of Bernoulli, Euler and Genocchi numbers. So, we have

$$y_{n,\beta}(0:k,a,b)=y_{n,\beta}(k,a,b).$$

We are now ready to give a relationship between the Ozden's generating function and the fermionic p-adic q-integral on $\mathbb{Z}_p$ at $q=-1$ by the following theorem:

Theorem 1.1. *The following relationship holds:*

$$a^{-b}\left(\frac{t}{2}\right)^{k}\int_{\mathbb{Z}_p}(-1)^{x+1}\left(\frac{\beta}{a}\right)^{bx}e^{tx}d\mu_{-1}(x)=\sum_{n=0}^{\infty}y_{n,\beta}(k,a,b)\frac{t^n}{n!}. \tag{1.5}$$

Proof. We set $f(x)=a^{-b}\left(\frac{t}{2}\right)^{k}(-1)^{x+1}\left(\frac{\beta}{a}\right)^{bx}e^{tx}$ in (1.3), it is easy to show the following assertion

$$\begin{aligned} a^{-b}\left(\frac{t}{2}\right)^{k}\left(-\left(\frac{\beta}{a}\right)^{b}e^{t}+1\right)\int_{\mathbb{Z}_p}(-1)^{x+1}\left(\frac{\beta}{a}\right)^{bx}e^{tx}d\mu_{-1}(x) &= -\frac{2\left(\frac{t}{2}\right)^{k}}{a^{b}} \\ a^{-b}\left(\frac{t}{2}\right)^{k}\int_{\mathbb{Z}_p}(-1)^{x+1}\left(\frac{\beta}{a}\right)^{bx}e^{tx}d\mu_{-1}(x) &= \frac{2\left(\frac{t}{2}\right)^{k}}{\beta^{b}e^{t}-a^{b}}. \end{aligned}$$

So, we complete the proof of Theorem. ∎

Theorem 1.2. *Then the following identity holds:*

$$\int_{\mathbb{Z}_p}(-1)^{x+1}\left(\frac{\beta}{a}\right)^{bx}x^{n-k}d\mu_{-1}(x)=2^{k}a^{b}\frac{(n-k)!}{n!}y_{n,\beta}(k,a,b).$$

Proof. From (1.5) and by using the Taylor expansion of e^{tx}, we readily see that,

$$\sum_{n=0}^{\infty}\left(2^{-k}a^{-b}\int_{\mathbb{Z}_p}(-1)^{x+1}\left(\frac{\beta}{a}\right)^{bx}x^{n}d\mu_{-1}(x)\right)\frac{t^{n+k}}{n!}=\sum_{n=0}^{\infty}y_{n,\beta}(k,a,b)\frac{t^n}{n!}.$$

By comparing coefficients of t^n in the both sides of the above equation, we arrive at the desired result. ∎

Similarly, we obtain the following theorem for unification of the Euler and Genocchi polynomials as follows:

Theorem 1.3. *The following identity holds:*

$$\int_{\mathbb{Z}_p}(-1)^{y+1}\left(\frac{\beta}{a}\right)^{by}(x+y)^{n}d\mu_{-1}(y)=2^{k}a^{b}\frac{n!}{(n+k)!}y_{n+k,\beta}(x:k,a,b). \tag{1.6}$$

From the binomial theorem in (1.6), we possess the following theorem:

Theorem 1.4. *The following relation holds:*

$$\frac{y_{n+k,\beta}(x:k,a,b)}{\binom{n+k}{k}}=\sum_{m=0}^{n}\frac{\binom{n}{m}}{\binom{m+k}{k}}y_{m+k,\beta}(k,a,b)x^{n-m}.$$

Proof. By using (1.6) and binomial theorem, we express the following relation

$$\sum_{m=0}^{n}\binom{n}{m}\left(\int_{\mathbb{Z}_p}(-1)^{y+1}\left(\frac{\beta}{a}\right)^{by}y^m d\mu_{-1}(y)\right)x^{n-m}=2^k a^b\frac{n!}{(n+k)!}y_{n+k,\beta}(x:k,a,b).$$

By using p-adic q-integral on $\mathbb{Z}_p$ at $q=-1$, we arrive at the desired proof of the theorem. ∎

Now, we consider symmetric properties of this type of polynomials as follows:

Theorem 1.5. *The following relation holds:*

$$y_{n,\beta^{-1}}\left(1-x:k,a^{-1},b\right)=(-1)^{k+n+1}\beta^b a^b y_{n,\beta}(x:k,a,b).$$

Proof. We set $x\to 1-x$, $\beta\to\beta^{-1}$ and $a\to a^{-1}$ into (1.6). That is

$$\int_{\mathbb{Z}_p}(-1)^{y+1}\left(\frac{\beta^{-1}}{a^{-1}}\right)^{by}(1-x+y)^n d\mu_{-1}(y)$$
$$=(-1)^n\int_{\mathbb{Z}_p}(-1)^{y+1}\left(\frac{\beta}{a}\right)^{-by}(x-1+y)^n d\mu_{-1}(y)=(-1)^{k+n+1}\beta^b a^b y_{n,\beta}(x:k,a,b).$$

Thus, we complete proof of the theorem. ∎

Ozden has obtained distribution formula for $y_{n,\beta}(x:k,a,b)$. We will also obtain distribution formula by using p-adic q-integral on $\mathbb{Z}_p$ at $q=-1$.

Theorem 1.6. *The following identity holds:*

$$y_{n,\beta}(x:k,a,b)=a^{b(d-1)}d^{n-k}\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{bj}y_{n,\beta^d}\left(\frac{x+j}{d}:k,a^d,b\right).$$

Proof. By using definition of the p-adic integral on $\mathbb{Z}_p$, we compute

$$2^k a^b\frac{n!}{(n+k)!}y_{n+k,\beta}(x:k,a,b)$$
$$=\int_{\mathbb{Z}_p}(-1)^{y+1}\left(\frac{\beta}{a}\right)^{by}(x+y)^n d\mu_{-1}(y)=\lim_{N\to\infty}\sum_{y=0}^{dp^N-1}(-1)^{y+1}\left(\frac{\beta}{a}\right)^{by}(x+y)^n(-1)^y$$
$$=d^n\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{bj}\lim_{N\to\infty}\sum_{y=0}^{p^N-1}(-1)^{y+1}\left(\frac{\beta}{a}\right)^{bdy}\left(\frac{x+j}{d}+y\right)^n(-1)^y$$
$$=d^n\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{bj}\int_{\mathbb{Z}_p}(-1)^{y+1}\left(\frac{\beta^d}{a^d}\right)^{by}\left(\frac{x+j}{d}+y\right)^n d\mu_{-1}(y)$$
$$=d^n\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{bj}2^k a^{db}\frac{n!}{(n+k)!}y_{n+k,\beta^d}\left(\frac{x+j}{d}:k,a^d,b\right).$$

Substituting n by $n-k$, we obtain the desired result and so proof is complete. ∎

Remark 1.1. This distribution for $y_{n,\beta}(x:k,a,b)$ is also introduced by Ozden cf. [32].

Definition 1.2. [31] *Let χ be a Dirichlet character with conductor $d\in\mathbb{N}$. The generating functions of the generalized Bernoulli, Euler and Genocchi polynomials with parameters a, b, β and k have been defined by Ozden, Simsek and Srivastava as follows:*

$$\mathscr{F}_{\chi,\beta}(t,k,a,b)$$

$$= 2\left(\frac{t}{2}\right)^k \sum_{j=1}^{d} \frac{\chi(j)\left(\frac{\beta}{a}\right)^j e^{jt}}{\beta^{bd}e^{dt} - a^{bd}}$$
$$= \sum_{n=0}^{\infty} y_{n,\chi,\beta}(x:k,a,b)\frac{t^n}{n!}, \qquad \left(\left|t + b\log\left(\frac{\beta}{a}\right)\right| < 2\pi;\ d,k \in \mathbb{N};\ a,b \in \mathbb{R}^+;\ \beta \in \mathbb{C}\right).$$

By using p-adic integral on $\mathbb{Z}_p$, we can obtain Definition 1.2 in terms of p-adic q-integral on $\mathbb{Z}_p$ at $q = -1$, as follows:

Theorem 1.7. *Let χ be a Dirichlet's character with conductor $d \in \mathbb{N}$. Then the following relation holds*

$$a^{b(1-d)}\left(\frac{t}{2}\right)^k \int_{\mathbb{Z}_p} \chi(x)(-1)^{x+1}\left(\frac{\beta}{a}\right)^{bx} e^{tx}d\mu_{-1}(x) = 2^{1-k}t^k \sum_{j=1}^{d} \frac{\chi(j)\left(\frac{\beta}{a}\right)^{bj} e^{tj}}{\beta^{db}e^{dt} - a^{db}}. \tag{1.7}$$

Proof. From the definition of p-adic q-integral on $\mathbb{Z}_p$ at $q = -1$, we compute

$$a^{b(1-d)}\left(\frac{t}{2}\right)^k \int_{\mathbb{Z}_p} \chi(x)(-1)^{x+1}\left(\frac{\beta}{a}\right)^{bx} e^{tx}d\mu_{-1}(x)$$
$$= a^{b(1-d)}\left(\frac{t}{2}\right)^k \lim_{N\to\infty} \sum_{x=0}^{dp^N-1} \chi(x)(-1)^{x+1}\left(\frac{\beta}{a}\right)^{bx} e^{tx}(-1)^x$$
$$= \frac{1}{d^k}\sum_{j=1}^{d} \chi(j)\left(\frac{\beta}{a}\right)^{bj} e^{tj}\left(\frac{1}{a^{db}}\left(\frac{td}{2}\right)^k \lim_{N\to\infty}\sum_{x=0}^{p^N-1}(-1)^{x+1}\left(\frac{\beta^d}{a^d}\right)^{bx} e^{tdx}(-1)^x\right)$$
$$= \frac{1}{d^k}\sum_{j=1}^{d} \chi(j)\left(\frac{\beta}{a}\right)^{bj} e^{tj}\left(\frac{2\left(\frac{td}{2}\right)^k}{\beta^{db}e^{dt} - a^{db}}\right) = 2^{1-k}t^k \sum_{j=1}^{d} \frac{\chi(j)\left(\frac{\beta}{a}\right)^{bj} e^{tj}}{\beta^{db}e^{dt} - a^{db}}.$$

Thus, we arrive at the desired result. ∎

By expression of (1.7), we get the following equation

$$a^{b(1-d)}\left(\frac{t}{2}\right)^k \int_{\mathbb{Z}_p} \chi(x)(-1)^{x+1}\left(\frac{\beta}{a}\right)^{bx} e^{tx}d\mu_{-1}(x) = \sum_{n=0}^{\infty} y_{n,\chi,\beta}(x:k,a,b)\frac{t^n}{n!}. \tag{1.8}$$

We are now ready to give distribution formula for generalized Euler and Genocchi polynomials by using p-adic q-integral on $\mathbb{Z}_p$ at $q = -1$ by means of theorem.

Theorem 1.8. *For any $n,k,d \in \mathbb{N}$ $a,b \in \mathbb{R}^+$; $\beta \in \mathbb{C}$, we have*

$$y_{n,\chi,\beta}(x:k,a,b) = d^{n-k}\sum_{j=0}^{d-1} \chi(j)\left(\frac{\beta}{a}\right)^{bj} y_{n,\beta^d}\left(\frac{x+j}{d}:k,a^d,b\right).$$

Proof. By expression of (1.8), we compute as follows assertion

$$\sum_{n=0}^{\infty} y_{n,\chi,\beta}(x:k,a,b)\frac{t^n}{n!}$$
$$= a^{b(1-d)}\left(\frac{t}{2}\right)^k \int_{\mathbb{Z}_p} \chi(y)(-1)^{y+1}\left(\frac{\beta}{a}\right)^{by} e^{t(x+y)}d\mu_{-1}(y)$$

$$= a^{b(1-d)}\left(\frac{t}{2}\right)^k \lim_{N\to\infty}\sum_{y=0}^{dp^N-1}\chi(y)(-1)^{y+1}\left(\frac{\beta}{a}\right)^{by} e^{t(x+y)}(-1)^y$$
$$= \frac{1}{d^k}\sum_{j=0}^{d-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\left(\frac{1}{a^{db}}\left(\frac{dt}{2}\right)^k \lim_{N\to\infty}\sum_{y=0}^{p^N-1}(-1)^{y+1}\left(\frac{\beta^d}{a^d}\right)^{by} e^{dt\left(\frac{x+j}{d}+y\right)}(-1)^y\right)$$
$$= \frac{1}{d^k}\sum_{j=0}^{d-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\left(\frac{1}{a^{db}}\left(\frac{dt}{2}\right)^k \int_{\mathbb{Z}_p}(-1)^{y+1}\left(\frac{\beta^d}{a^d}\right)^{by} e^{dt\left(\frac{x+j}{d}+y\right)}d\mu_{-1}(y)\right)$$
$$= \frac{1}{d^k}\sum_{j=0}^{d-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\left(\sum_{n=0}^{\infty}d^n y_{n,\beta^d}\left(\frac{x+j}{d}:k,a^d,b\right)\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(d^{n-k}\sum_{j=0}^{d-1}\chi(j)\left(\frac{\beta}{a}\right)^{bj} y_{n,\beta^d}\left(\frac{x+j}{d}:k,a^d,b\right)\right)\frac{t^n}{n!}.$$

So, we complete the proof of theorem. ∎

2. New properties on the unification of multiple twisted Euler and Genocchi polynomials

In this section, we introduce a unification of the twisted Euler and Genocchi polynomials. We assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. For $n \in \mathbb{N}$, by the definition of the p-adic integral on $\mathbb{Z}_p$, we have

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{x=0}^{n-1} f(x)(-1)^{n-1-x} \tag{2.1}$$

where $f_n(x) = f(x+n)$.

Let $T_p = \cup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n} = C_{p^\infty}$ be the locally constant space, where $C_{p^n} = \{w \mid w^{p^n} = 1\}$ is the cylic group of order p^n. For $w \in T_p$, we denote the locally constant function by

$$\phi_w : \mathbb{Z}_p \to \mathbb{C}_p, \quad x \to w^x, \tag{2.2}$$

If we set $f(x) = \phi_w(x)\, a^{-b}(t/2)^k(-1)^{x+1}(\beta/a)^{bx}e^{tx}$, then we have

$$a^{-b}\left(\frac{t}{2}\right)^k \int_{\mathbb{Z}_p}\phi_w(x)(-1)^{x+1}\left(\frac{\beta}{a}\right)^{bx} e^{tx}d\mu_{-1}(x) = \frac{2\left(\frac{t}{2}\right)^k}{w\beta^b e^t - a^b}. \tag{2.3}$$

We now define unification of twisted Euler and Genocchi polynomials as follows:

$$\frac{2\left(\frac{t}{2}\right)^k}{w\beta^b e^t - a^b} = \sum_{n=0}^{\infty} y_{n,w,\beta}(k,a,b)\frac{t^n}{n!}.$$

We note that by substituting $w = 1$, we obtain Ozden's generating function (1.4). From (2.2) and (2.3), we obtain Witt's type formula for a unification of twisted Euler and Genocchi polynomials as follows:

$$a^{-b}2^{-k}\int_{\mathbb{Z}_p}\phi_w(x)(-1)^{x+1}\left(\frac{\beta}{a}\right)^{bx} x^n d\mu_{-1}(x) = \frac{y_{n+k,w,\beta}(k,a,b)}{k!\binom{n+k}{k}} \tag{2.4}$$

for each $w \in T_p$ and $n \in \mathbb{N}$.

We now establish Witt's type formula for the unification of multiple twisted Euler and Genocchi polynomials by the following theorem.

Definition 2.1. *Let be $w \in T_p$, $n,h,k \in \mathbb{N}$ $a,b \in \mathbb{R}^+$; $\beta \in \mathbb{C}$, we define*

$$\begin{aligned} &a^{-hb}2^{-hk}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{h-times}\phi_w\left(x_1+\ldots+x_h\right)(-1)^{x_1+\ldots+x_h+h} \\ &\times\left(\frac{\beta}{a}\right)^{b(x_1+\ldots+x_h)}\left(x_1+\ldots+x_h\right)^n d\mu_{-1}\left(x_1\right)\ldots d\mu_{-1}\left(x_h\right)=\frac{y^{(h)}_{n+kh,w,\beta}\left(k,a,b\right)}{(kh)!\binom{n+kh}{kh}}. \end{aligned} \tag{2.5}$$

Remark 2.1. Taking $h=1$ into (2.5), we get the unification of the twisted Euler and Genocchi polynomials $y_{n,w,\beta}\left(k,a,b\right)$.

Remark 2.2. By substituting $h=1$ and $w=1$, we obtain a special case of the unification of Euler and Genocchi polynomials $y_{n,\beta}\left(k,a,b\right)$.

Theorem 2.1. *For any $w \in T_p$, $n,h,k \in \mathbb{N}$ $a,b \in \mathbb{R}^+$; $\beta \in \mathbb{C}$,*

$$\frac{y^{(h)}_{n+kh,w,\beta}\left(k,a,b\right)}{(kh)!\binom{n+kh}{kh}}=\sum_{\substack{l_1+\ldots+l_h=n\\ l_1,\ldots,l_h\geq 0}}\frac{n!}{l_1!\ldots l_h!}\prod_{i=1}^{h}\frac{y^{(h)}_{l_i+kh,w,\beta}\left(k,a,b\right)}{(kh)!\binom{l_i+kh}{kh}}.$$

Proof. By using definition of the multiple twisted a unification of Euler and Genocchi numbers and polynomials, and, definition of

$$\left(x_1+x_2+\ldots+x_h\right)^n=\sum_{\substack{l_1+\ldots+l_h=n\\ l_1,\ldots,l_h\geq 0}}\frac{n!}{l_1!\ldots l_h!}x_1^{l_1}x_2^{l_2}\ldots x_h^{l_h},$$

we see that

$$\begin{aligned} &a^{-hb}2^{-hk}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{h-times}\left\{\begin{array}{c}\phi_w\left(x_1+\ldots+x_h\right)(-1)^{x_1+\ldots+x_h+h}\left(\frac{\beta}{a}\right)^{b(x_1+\ldots+x_h)}\\ \times\left(x+x_1+\ldots+x_h\right)^n\end{array}\right\}d\mu_{-1}\left(x_1\right)\ldots d\mu_{-1}\left(x_h\right) \\ &=\sum_{\substack{l_1+\ldots+l_h=n\\ l_1,\ldots,l_h\geq 0}}\frac{n!}{l_1!\ldots l_h!}\left(a^{-b}2^{-k}\int_{\mathbb{Z}_p}w^{x_1}\left(\frac{\beta}{a}\right)^{bx_1}x_1^{l_1}d\mu_{-1}\left(x_1\right)\right)\times \\ &\quad\ldots\times\left(a^{-b}2^{-k}\int_{\mathbb{Z}_p}w^{x_h}\left(\frac{\beta}{a}\right)^{bx_h}x_h^{l_h}d\mu_{-1}\left(x_h\right)\right) \\ &=\sum_{\substack{l_1+\ldots+l_h=n\\ l_1,\ldots,l_h\geq 0}}\frac{n!}{l_1!\ldots l_h!}\prod_{j=1}^{h}\frac{y^{(h)}_{l_i+kh,w,\beta}\left(k,a,b\right)}{(kh)!\binom{l_i+kh}{kh}}. \end{aligned}$$

Thus, we arrive at the desired result. ∎

From these formulas, we can define the unification of the twisted Euler and Genocchi polynomials as follows:

$$\left(\frac{2\left(\frac{t}{2}\right)^k}{w\beta^b e^t-a^b}\right)^h e^{xt}=\sum_{n=0}^{\infty}y^{(h)}_{n,w,\beta}\left(x:k,a,b\right)\frac{t^n}{n!}, \tag{2.6}$$

So from the above, we get the Witt's type formula for $y_{n,w,\beta}^{(h)}(x:k,a,b)$ as follows.

Theorem 2.2. *For any $w \in T_p$, $n,h,k \in \mathbb{N}$ $a,b \in \mathbb{R}^+$; $\beta \in \mathbb{C}$, we get*

$$a^{-hb}2^{-hk}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{h-times}\left\{\begin{array}{c}\phi_w(x_1+\ldots+x_h)(-1)^{x_1+\ldots+x_h+h}\left(\frac{\beta}{a}\right)^{b(x_1+\ldots+x_h)}\\ \times(x+x_1+\ldots+x_h)^n\end{array}\right\}d\mu_{-1}(x_1)\ldots d\mu_{-1}(x_h)$$
$$=\frac{y_{n+kh,w,\beta}^{(h)}(x:k,a,b)}{(kh)!\binom{n+kh}{kh}}.$$

Note that

$$(x+x_1+x_2+\ldots+x_h)^n=\sum_{\substack{l_1+\ldots+l_h=n\\ l_1,\ldots,l_h\geq 0}}\frac{n!}{l_1!\ldots l_h!}x_1^{l_1}x_2^{l_2}\ldots(x+x_h)^{l_h}. \tag{2.7}$$

We obtain the sum of powers of consecutive a unification of multiple twisted Euler and Genocchi polynomials as follows:

Theorem 2.3. *For any $w \in T_p$, $n,h,k \in \mathbb{N}$ $a,b \in \mathbb{R}^+$; $\beta \in \mathbb{C}$, we get*

$$\frac{y_{n+kh,w,\beta}^{(h)}(x:k,a,b)}{(kh)!\binom{n+kh}{kh}}=\sum_{\substack{l_1+\ldots+l_h=n\\ l_1,\ldots,l_h\geq 0}}\frac{n!}{l_1!\ldots l_h!}\frac{y_{l_h+kh,w,\beta}^{(h)}(x:k,a,b)}{(kh)!\binom{l_h+kh}{kh}}\prod_{j=1}^{h-1}\frac{y_{l_i+kh,w,\beta}^{(h)}(k,a,b)}{(kh)!\binom{l_i+kh}{kh}}.$$

Proof. By Theorem 2.2 and (2.7), we see that,

$$a^{-hb}2^{-hk}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{h-times}\left\{\begin{array}{c}\phi_w(x_1+\ldots+x_h)(-1)^{x_1+\ldots+x_h+h}\left(\frac{\beta}{a}\right)^{b(x_1+\ldots+x_h)}\\ \times(x+x_1+\ldots+x_h)^n\end{array}\right\}d\mu_{-1}(x_1)\ldots d\mu_{-1}(x_h)$$
$$=\sum_{\substack{l_1+\ldots+l_h=n\\ l_1,\ldots,l_h\geq 0}}\frac{n!}{l_1!\ldots l_h!}\left(a^{-b}2^{-k}\int_{\mathbb{Z}_p}w^{x_1}\left(\frac{\beta}{a}\right)^{bx_1}x_1^{l_1}d\mu_{-1}(x_1)\right)\times$$
$$\ldots\times\left(a^{-b}2^{-k}\int_{\mathbb{Z}_p}w^{x_h}\left(\frac{\beta}{a}\right)^{bx_h}(x+x_h)^{l_h}d\mu_{-1}(x_h)\right)$$
$$=\sum_{\substack{l_1+\ldots+l_h=n\\ l_1,\ldots,l_h\geq 0}}\frac{n!}{l_1!\ldots l_h!}\frac{y_{l_h+kh,w,\beta}^{(h)}(x:k,a,b)}{(kh)!\binom{l_h+kh}{kh}}\prod_{j=1}^{h-1}\frac{y_{l_i+kh,w,\beta}^{(h)}(k,a,b)}{(kh)!\binom{l_i+kh}{kh}}.$$

So, we complete the proof of the theorem. ∎

3. Unification of multiple twisted Zeta functions

Our goal in this section is to establish a unification of multiple twisted zeta functions which interpolates of unification of multiple twisted Euler and Genocchi polynomials at negative integers. For $q \in \mathbb{C}$, $|q| < 1$ and $w \in T_p$, the unification of multiple twisted Euler and Genocchi polynomials are considered as follows:

$$\left(\frac{2\left(\frac{t}{2}\right)^k}{w\beta^b e^t-a^b}\right)^h=\sum_{n=0}^{\infty}y_{n,w,\beta}^{(h)}(k,a,b)\frac{t^n}{n!},\quad \left|t+\log\left(w\left(\frac{\beta}{a}\right)^b\right)\right|<2\pi. \tag{3.1}$$

By (3.1), we easily see that,

$$\sum_{n=0}^{\infty} y_{n,w,\beta}^{(h)}(k,a,b)\frac{t^n}{n!} = 2^h\left(\frac{t}{2}\right)^{kh}\left(\frac{1}{w\beta^b e^t - a^b}\right)\cdots\left(\frac{1}{w\beta^b e^t - a^b}\right)$$
$$= 2^h\left(\frac{t}{2}\right)^{kh}(-1)^h\sum_{n_1=0}^{\infty} w^{n_1}\left(\frac{\beta}{a}\right)^{bn_1} e^{n_1 t}\cdots\sum_{n_h=0}^{\infty} w^{n_h}\left(\frac{\beta}{a}\right)^{bn_h} e^{n_h t}$$
$$= 2^h\left(\frac{t}{2}\right)^{kh}(-1)^h\sum_{n_1,\dots,n_h=0}^{\infty}\phi_w(n_1+\dots+n_h)\left(\frac{\beta}{a}\right)^{b(n_1+\dots+n_h)} e^{(n_1+\dots+n_h)t}.$$

By using the Taylor expansion of $e^{(n_1+\dots+n_h)t}$ and by comparing the coefficients of t^n in the both sides of the above equation, we obtain that

(3.2)
$$\frac{y_{n+kh,w,\beta}^{(h)}(k,a,b)}{(kh)!\binom{n+kh}{kh}} = 2^{h(1-k)}(-1)^h\sum_{\substack{n_1,\dots,n_h\geq 0\\ n_1+\dots+n_h\neq 0}}^{\infty}\phi_w(n_1+\dots+n_h)\left(\frac{\beta}{a}\right)^{b(n_1+\dots+n_h)}(n_1+\dots+n_h)^n.$$

From (3.2), we can define unification of multiple twisted zeta functions as follows:

$$\zeta_{\beta,w}^{(h)}(s:k,a,b) = 2^{h(1-k)}(-1)^h\sum_{\substack{n_1,\dots,n_h=0\\ n_1+\dots+n_h\neq 0}}^{\infty}\frac{\phi_w(n_1+\dots+n_h)\left(\frac{\beta}{a}\right)^{b(n_1+\dots+n_h)}}{(n_1+\dots+n_h)^s}$$

for all $s\in\mathbb{C}$. We also obtain the following theorem in which the unification of multiple twisted zeta functions interpolate the unification of multiple twisted Euler and Genocchi polynomials at negative integers.

Theorem 3.1. *For any $w\in T_p$, $n,h,k\in\mathbb{N}$ $a,b\in\mathbb{R}^+$; $\beta\in\mathbb{C}$, we obtain*

$$\zeta_{\beta,w}^{(h)}(-n:k,a,b) = \frac{y_{n+kh,w,\beta}^{(h)}(k,a,b)}{(kh)!\binom{n+kh}{kh}}.$$

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