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ON HYPOELLIPTICITY IN $\mathcal{G}$

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A b s t r a c t. We give a condition of sufficiency for the hypoellipticity of a family of equations with constant coefficients satisfied prescribed power growth rate with respect to $\varepsilon \in (0,1)$. *The framework is Colombeau algebra of generalized functions.*

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1. *Introduction*

We have considered in [5] the hypoellipticity of a differential equation with generalized constant coefficients and have given several necessary conditions. In order to explain our approach, consider a family of equations with constant coefficients

$$P_\varepsilon(D)G = \sum_{|\alpha|\le m} a_{\alpha,\varepsilon} D^\alpha G = F_\varepsilon,\ F_\varepsilon \in C^\infty(\Omega),\ \varepsilon \in (0,1)$$

If $P_\varepsilon(D)$ is hypoelliptic for fixed $\varepsilon \in (0,1)$, then the corresponding solution to the above equation, G_ε, is in $C^\infty(\Omega)$. If we suppose that $\sup\limits_{x\in K\subset\subset\Omega} |D^\alpha F_\varepsilon(x)|$

satisfies the power growth condition $\mathcal{O}(\varepsilon^{-N_K})$ for every α (N_K depends on a compact set K), then one can ask whether the derivatives of G_ε satisfy similar estimates on compact sets.

In this paper we will repeat necessary conditions for the hypoellipticity ([5]) and give appropriate sufficient conditions for it.

Another type of hypoellipticity was considered by Hörmann and Oberguggenberger (personal communication) who pointed to us that our sufficient conditions in [5] need some additional assumptions. In this paper we reconsider the following condition:

"There exist $N > 0$ and $q \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_q$ $A > 0$ there exist $\eta > 0$ and $B \in \mathbb{R}$ such that

$$|\tau| \geq A(\log|\sigma| + N\log\varepsilon) - B, \sigma + i\tau \in V(P_{\phi,\varepsilon}),\ \varepsilon \in (0,\eta),\text{"}$$

and show that above we need

$$|\tau| \geq A\log|\sigma| + N\log\varepsilon - B, \sigma + i\tau \in V(P_{\phi,\varepsilon}),\ \varepsilon \in (0,\eta).$$

The proof of this fact is the main goal of this paper.

2. *Colombeau algebras*

Let

$$\begin{aligned} \mathcal{A}_0(\mathbb{R}) &= \{\phi \in C_0^\infty | \int \phi(x)dx = 1, \operatorname{diam}(\operatorname{supp}\phi) = 1\}, \\ \mathcal{A}_q(\mathbb{R}) &= \{\phi \in \mathcal{A}_0 | \int x^\alpha \phi(x)dx = 0,\ 1 \leq \alpha \leq q,\ \alpha \in \mathbb{N}\},\ q \in \mathbb{N} \end{aligned}$$

and $\mathcal{A}_q(\mathbb{R}^n) = \{\phi(x_1,\dots,x_n) = \phi_1(x_1)\cdot\ldots\cdot\phi_1(x_n)|\ \phi_1 \in \mathcal{A}_q(\mathbb{R})\}$. Put $\phi_\varepsilon = (1/\varepsilon)\phi(\cdot/\varepsilon)$, where $\phi \in \mathcal{A}_0$.

Let Ω be an open subset of $\mathbb{R}^n$ and $\mathcal{E}(\Omega)$ be the space of functions $G : \mathcal{A}_0 \times (0,1) \times \Omega \to \mathbb{C}$ which are C^∞ for every $\phi \in \mathcal{A}_0$ and $\varepsilon \in (0,1)$. We will use the notation $G_{\phi,\varepsilon}$ for $(\phi,\varepsilon,x) \mapsto G_{\phi,\varepsilon}(x)$, $x \in \Omega$.

A family of smooth complex valued functions on Ω, $G_{\phi,\varepsilon}$, $\phi \in \mathcal{A}_0$, $\varepsilon \in (0,1)$, belongs to $\mathcal{E}_M(\Omega)$ if for every compact set $K \subset\subset \Omega$ and $\alpha \in \mathbb{N}_0^n$ there exist $N \in \mathbb{N}$ and $r = r(K,\alpha) \in \mathbb{R}$ such that

$$\sup_{x\in K} |\partial^\alpha G_{\phi,\varepsilon}(x)| = \mathcal{O}(\varepsilon^r),\ \varepsilon \to 0,\ \text{for every } \phi \in \mathcal{A}_N. \tag{1}$$

If $G_{\phi,\varepsilon}$ does not depend on x and (1) holds for $\alpha = 0$, then the space of corresponding families of complex numbers is denoted by $\mathbb{C}_M$. If $g \in \mathcal{D}'$, the corresponding element in $\mathcal{E}_M$ is given by $G_{\phi,\varepsilon} = g * \delta_{\phi,\varepsilon}$, where we use the notation $\delta_{\phi,\varepsilon} = \phi_\varepsilon$, $\phi \in \mathcal{A}_0$ since it is a delta net.

The space of all elements $G_{\phi,\varepsilon}$ in $\mathcal{E}_M(\Omega)$ which satisfy (1) independently of $\alpha \in \mathbb{N}_0$ is denoted by $\mathcal{E}_M^\infty$.

The space $\mathcal{E}_0(\Omega)$ (resp. $\mathbb{C}_0$) is the subspace of $\mathcal{E}_M(\Omega)$ (resp. $\mathbb{C}_M$) consisting of elements $G_{\phi,\varepsilon}$ with the property that for every $K \subset\subset \Omega$, $\alpha \in \mathbb{N}_0^n$ and $r \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that (1) holds (resp. (1) holds for $\alpha = 0$ and $G_{\phi,\varepsilon}$ does not depend on x) for every $\phi \in \mathcal{A}_N$.

The space of Colombeau's generalized functions on an open set $\Omega \subset \mathbb{R}^n$ is defined by $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{E}_0(\Omega)$ and $\overline{\mathbb{C}} = \mathbb{C}_M/\mathbb{C}_0$ is the ring of Colombeau's generalized complex numbers. Note that $\Omega \to \mathcal{G}(\Omega)$, $\Omega \subset \mathbb{R}^n$, is a sheaf.

$[G_{\phi,\varepsilon}]$ denotes the class in $\mathcal{G}$ (or $\overline{\mathbb{C}}$) determinated by the representative $G_{\phi,\varepsilon}$.

Let $G \in \mathcal{G}(\Omega)$. The complement of the largest open set of Ω in which G is equal to the zero generalized function is called the support of G, $\operatorname{supp}_g G$.

The space of generalized functions with compact supports in the interior of Ω is denoted by $\mathcal{G}_c(\Omega)$.

$\mathcal{G}^\infty(\Omega)$ (cf. [6]) is the space of all generalized functions which have a representative in $\mathcal{E}_M^\infty$. It is a subalgebra of $\mathcal{G}(\Omega)$ and

$$\mathcal{G}^\infty(\Omega) \cap \mathcal{D}'(\Omega) = C^\infty(\Omega) \text{ (see [6]).}$$

The space of tempered Colombeau's generalized functions $\mathcal{G}_{\mathbf{t}}(\mathbb{R}^n)$ is defined to be $\mathcal{E}_{\mathbf{t}}(\mathbb{R}^n)/\mathcal{E}_{0\mathbf{t}}(\mathbb{R}^n)$, where $\mathcal{E}_{\mathbf{t}}(\mathbb{R}^n)$ is the set of all $G_{\phi,\varepsilon} \in \mathcal{E}$ such that for every $\alpha \in \mathbb{N}_0^n$ there exist $\gamma > 0$, $N \in \mathbb{N}$ and $r \in \mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}^n} |\partial^\alpha G_{\phi,\varepsilon}(x)|/(1+|x|)^\gamma = \mathcal{O}(\varepsilon^r),\ \varepsilon \to 0 \text{ for every } \phi \in \mathcal{A}_N, \tag{2}$$

and $\mathcal{E}_{0\mathbf{t}}(\mathbb{R}^n)$ is the space of all $G_{\phi,\varepsilon} \in \mathcal{E}_{\mathbf{t}}$ with the property that for every $\alpha \in \mathbb{N}_0^n$ there exists $\gamma \in \mathbb{R}$ such that for every $r \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that (2) holds for every $\phi \in \mathcal{A}_N$.

Note that $\mathcal{G}_{\mathbf{t}}(\mathbb{R}^n)$ is not a subspace of $\mathcal{G}(\mathbb{R}^n)$, but there is a canonical map $\mathcal{G}_{\mathbf{t}}(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n)$.

Let $G \in \mathcal{G}(\Omega)$. The complement of the largest open set of Ω in which G is in $\mathcal{G}^\infty(\Omega)$ is called the singular support of G. It is denoted by $\operatorname{singsupp}_g G$.

The equality in $\mathcal{G}$ is often too strong for applications, so we shall use a notion of equality in generalized distribution (resp. generalized tempered distribution) sense $\overset{g.d.}{=}$ (resp. $\overset{g.t.d.}{=}$) which is defined by:

$G_1 \overset{g.d.}{=} G_2$ (resp. $G_1 \overset{g.t.d.}{=} G_2$ for tempered generalized functions) if for every $\psi \in \mathcal{D}$ (resp. $\psi \in \mathcal{S}$), $\langle G_1, \psi\rangle = \langle G_2, \psi\rangle$ in $\overline{\mathbb{C}}$, where $\langle G, \psi\rangle$ means $\int G(x)\psi(x)dx$.

3. *Hypoellipticity*

Following [8], we define polynomials in n real variables as elements of the ring $\overline{\mathbb{C}}[x_1, \dots, x_n]$. A generalized polynomial function is a tempered generalized function of the form

$$\sum_{|\alpha|\le m} a_\alpha x^\alpha,\ x \in \mathbb{R},\ a_\alpha = [a_{\alpha,\phi,\varepsilon}] \in \overline{\mathbb{C}},\ \alpha \in \mathbb{N}_0^n.$$

It is of degree m if $a_\alpha = 0$ for $|\alpha| > m$ and there exists β, $|\beta| = m$ such that $a_\beta \neq 0$.

If $[H_{\phi,\varepsilon}(x)] = \sum_{|\alpha|\le m}[a_{\alpha,\phi,\varepsilon}]x^\alpha$ is such a generalized function, then it can be written only in one way as a polynomial. In fact, if $\sum_{|\beta|\le m} b_{\beta,\phi,\varepsilon}x^\beta = N_{\phi,\varepsilon}(x) \in \mathcal{E}_{0\mathbf{t}}(\mathbb{R}^n)$, then by making successive derivations and by putting $x = 0$ it follows $b_{\beta,\phi,\varepsilon} \in \mathbb{C}_0$, $|\beta| \le m$ ([8]).

Let us remind that in the classical distribution theory a fundamental solution of a differential operator is a distribution E such that $P(D)E = \delta$.

Let

$$P(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha = [P_{\phi,\varepsilon}(i\frac{\partial}{\partial x})]\ (D^\alpha = i^{|\alpha|}\partial^\alpha), \tag{3}$$

where $\sum_{|\alpha|\le m} a_\alpha x^\alpha$ is a polynomial in $\mathcal{G}$. In Colombeau's theory, the fundamental solution of P is a generalized function $E \in \mathcal{G}$ satisfying $P(D)E = [\delta_{\phi,\varepsilon}]$. This means that its representatives $E_{\phi,\varepsilon} \in \mathcal{E}_M$ satisfy

$$\sum a_{\alpha,\phi,\varepsilon} D^\alpha E_{\phi,\varepsilon}(x) = \delta_{\phi,\varepsilon}(x) + N_{\phi,\varepsilon}(x),\ x \in \mathbb{R}^n,$$

for some $N_{\phi,\varepsilon} \in \mathcal{E}_0$.

This fundamental solution allows us to solve the equation $P(D)U \overset{g.d.}{=} G$ for $G \in \mathcal{G}$, because $G * [\delta_{\phi,\varepsilon}] \overset{g.d.}{=} G$.

Proposition 5 ([5]) *Let $P(D)$ be a generalized differential operator of the form (4) with coefficients in $\overline{\mathbb{C}}$ of degree m such that for some* $(c_1, c_2, \dots, c_n) \in$

$\mathbb{R}^n$ there exist $r > 0$ and $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0$ and $\eta > 0$ such that

$$|\sum_{|\alpha|=m} a_{\alpha,\phi,\varepsilon} c^{\alpha}| \geq C\varepsilon^{r},\ \varepsilon \in (0,\eta). \tag{4}$$

Then, $P(D)$ admits a generalized fundamental solution.

In non-standard models of Colombeau's theory this hypothesis can be replaced by $\sum_{|\alpha|=m} a_{\alpha,\phi,\varepsilon} c^{\alpha} \neq 0$, since $\overline{\mathbb{C}}$ is a field in such models (cf. Li Bang He, [4] and Oberguggenberger [7]).

Solution to

$$\sum_{|\alpha|\leq m} a_{\alpha,\phi,\varepsilon} D^{\alpha} G_{\phi,\varepsilon} = F_{\phi,\varepsilon},\ F_{\phi,\varepsilon} \in \mathcal{E}_M,\ a_{\alpha,\phi,\varepsilon} \in \mathbb{C}_M, \tag{5}$$

in $\mathcal{E}_M$ are constructed in [8], in a simplified version of Colombeau's theory, by adapting the classical distributional method of solving a constant coefficients partial differential equation. Problems which are specific for (5) are connected with the growth rate of solutions with respect to ε which implies that the main assertions in [8] and in this paper are non-trivial generalization of the corresponding ones in the space of distributions.

In this paper we investigate the hypoellipticity of the families of equations (5). This family is hypoelliptic if $F_{\phi,\varepsilon} \in \mathcal{E}_M^{\infty}$ implies $G_{\phi,\varepsilon} * \delta_{\phi,\varepsilon} \in \mathcal{E}_M^{\infty}$.

Now we give the definition of hypoellipticity in the framework of Colombeau generalized functions. Let $[P_{\phi,\varepsilon}(D)]$ be of the form (3) and suppose that (4) holds for some $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$, $C_1 > 0$, $r > 0$, $N \in \mathbb{N}$ and $\eta > 0$. This operator is called hypoelliptic if for every open $\Omega \subset \mathbb{R}^n$ and every solution $G \in \mathcal{G}(\Omega)$ to

$$P(D)G = 0, \tag{6}$$

the generalized function $G * [\delta_{\phi,\varepsilon}]$ is in $\mathcal{G}^{\infty}(\Omega)$.

Proposition 6 ([5])
a) $P(D)$ is hypoelliptic if it admits a fundamental solution E with

$$\operatorname*{singsupp}_{g} E = \{0\}. \tag{7}$$

b) Let $[P_{\phi,\varepsilon}(D)]$ be hypoelliptic. Then, for every open set Ω and $G \in \mathcal{G}(\Omega)$, $P(D)G \in \mathcal{G}^{\infty}(\Omega)$ implies $G * [\delta_{\phi,\varepsilon}] \in \mathcal{G}^{\infty}(\Omega)$.

Let $P_{\phi,\varepsilon}$ be a hypoelliptic differential operator on Ω, Ω_0, W, and O are the same as in the proof of assertion a) in Proposition 6, G a solution of (6) and $W \pm O \subset \Omega_0$. Then we have the following

Proposition 7 ([5]) *There exists $N \in \mathbb{N}$ such that for every $q \in \mathbf{N}_0^n$ there exist $C > 0$ and $\eta > 0$ such that*

$$\max_{x \in W} |D^q G_{\phi,\varepsilon} * \delta_{\phi,\varepsilon}(x)| \le C \max_{x \in \Omega_0} |G_{\phi,\varepsilon}(x)| \varepsilon^{-N},\ \varepsilon < \eta.$$

4. *Main assertion*

The main prerequisite, Lemma 1 of [5], has the same proof as in [5].

Lemma 1 *Let N, A, η and B be the same as in (8) below. Assume that a), b) and c) hold for $\varepsilon < \varepsilon_0$, $\phi \in \mathcal{A}_{q_0}$ where*

a) $(\sigma_1, \sigma_2) \in \mathbb{R}^2$ *such that* $A \log |(\sigma_1, \sigma_2)| + N \log \varepsilon \ge B + 1$.

b) $\tau_1 \in \mathbb{R}$, $|\tau_1| \le (A \log |(\sigma_1, \sigma_2)| + N \log \varepsilon - B)/2$.

c) $(\bar{\sigma}_1 + i\bar{\tau}_1) \in V(P_{\phi,\varepsilon})$ *and* $|\bar{\tau}_1| \ge (A \log |(\bar{\sigma}_1, \sigma_2)| + N \log \varepsilon - B)$.

Then, for every fixed $\varepsilon < \eta$, $|\bar{\sigma}_1 - \sigma_1| \ge \varepsilon^{N+1}$ or $|\bar{\tau}_1 - \tau_1| \ge \varepsilon^{N+1}$.

Theorem 1 *The operator $P(D)$ is hypoelliptic if and only if there exist $N > 0$ and $q \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_q$ and $A > 0$ there exist $\eta > 0$ and $B > 0$ such that*

$$|\tau| \ge A \log |\sigma| + N \log \varepsilon - B,\ \sigma + i\tau \in V(P_{\phi,\varepsilon}),\ \varepsilon \in (0, \eta). \tag{8}$$

Proof of sufficiency Without a loss of generality one can assume that $\eta < \varepsilon_0$ and $q > q_0$, where η and q are used in the estimates bellow, while ε_0 and q_0 are from Lemma 1. The change of constant A bellow will cause an appropriate decrease of η, but we shall use the same letter η in all cases.

One needs to prove that (8) implies (5). We will assume, without loss of generality, that $P_{\phi,\varepsilon}$ is of the form $P_{\phi,\varepsilon}(s) = a_{m,\phi,\varepsilon} s_1^m +$ lower order terms. The fundamental solution for $P_{\phi,\varepsilon}$ is given by

$$\begin{aligned} E_{\phi,\varepsilon}(x) &= (2\pi)^{-n} \int_{T_{\phi,\varepsilon}} \frac{e^{-i\langle x,s\rangle} \hat{\phi}(\varepsilon s)}{P_{\phi,\varepsilon}(s)} ds \\ &= (2\pi)^{-n} \sum_{k=1}^{\infty} \int_{\mathbb{R}} d\sigma_1 \int_{\Gamma_{k,\phi,\varepsilon}} \frac{e^{-i\langle x,(\sigma_1 + i\tau_1, \sigma')\rangle} \hat{\phi}(\varepsilon(\sigma_1 + i\tau_1, \sigma'))}{P_{\phi,\varepsilon}((\sigma_1 + i\tau_1, \sigma'))} d\sigma', \end{aligned}$$

where $T_{\phi,\varepsilon} = \bigcup_{j=1}^{\infty}\{(\sigma_1 + i\tau_1, \sigma');\ \sigma_1 \in \mathbb{R},\ \tau_1 = k_j \in \{0, \ldots, m+1\},\ \sigma' = \sigma_2 \in \Gamma_{j,\phi,\varepsilon}\}$ and $\Gamma_{j,\phi,\varepsilon}$ are bounded closed domains of $\mathbb{R}^{n-1}$ such that for $\phi \in \mathcal{A}_q$, $P_{\phi,\varepsilon}(\xi) > C_P\varepsilon^r$, $\xi \in T_{\phi,\varepsilon}$, $\varepsilon < \eta$.

Only the proof for $n = 2$ is given bellow. Higher dimensions can be handled in a similar way.

Let the ball $B_h = B((x_1, x_2), h), h > 0$, does not contain the point 0 and $a = \operatorname{dist}(0, B_h)$. We will prove that $E_{\phi,\varepsilon}$ represents an element in $\mathcal{G}^\infty(B_h)$.

Divide (σ_1, σ_2)-plane into nine regions Ω_j, $j = 1, \ldots, 9$ by the lines $\sigma_1 = \pm\mu$, $\sigma_2 = \pm\mu$ and denote them by $\Omega_1 = \{|\sigma_1| \le \mu, |\sigma_2| \le \mu\}$, $\Omega_2 = \{\sigma_1 \ge \mu, |\sigma_2| \le \mu\}$, $\Omega_3 = \{\sigma_1 \ge \mu, \sigma_2 \ge \mu\}$, $\Omega_4 = \{|\sigma_1| \le \mu, \sigma_2 \ge \mu\}$, …. Now, choose $\mu > 0$ such that

$$A\log\mu + N\log\varepsilon - B = \max\{\max_{s\in T_{\phi,\varepsilon}} |\tau|, 1\} = C_0,\ \text{ i.e. }\ \mu = e^{B + C_0/A}\varepsilon^{-N/A},\ \varepsilon < \eta.$$

Without a loss of generality one can assume that $B((x_1, x_2), h) \subset [0,\infty)\times[0,\infty)$. Denote by $T_{j,\phi,\varepsilon}$ the projection of Ω_j on $T_{\phi,\varepsilon}$,

$$T_{j,\phi,\varepsilon} = \{(\sigma_1 + i\tau_1, \sigma_2)|\ \sigma_1 \in \mathbb{R}, \tau_1 \in \{0, \ldots, m+1\}, \sigma_2 \in \Gamma_{k,\phi,\varepsilon} \cap \Omega_j,\ k \in \mathbb{N}\}.$$

Then $E_{\phi,\varepsilon} = \sum_{j=0}^{9} E_{j,\phi,\varepsilon}$, where

$$E_{j,\phi,\varepsilon}(x) = (2\pi)^{-n}\int_{T_{j,\phi,\varepsilon}} \frac{e^{-i\langle(x_1,x_2),(s_1,s_2)\rangle}\hat{\phi}(\varepsilon s)}{P_{\phi,\varepsilon}(s)}ds,\ j = 1, \ldots, 9,\ x \in \mathbb{R}^2.$$

We will show that there exist $\tilde{N}$ and $\tilde{q}$ such that for every $\phi \in \mathcal{A}_{\tilde{q}}$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ there exist $C_\alpha > 0$ and η_α such that

$$\sup_{x\in B_h} |\partial^\alpha E_{j,\phi,\varepsilon}(x)| \le C_\alpha\varepsilon^{\tilde{N}},\ \varepsilon < \eta_\alpha,\ j = 1, \ldots, 9, \tag{9}$$

We always take $\tilde{q} = q$ and $\phi \in \mathcal{A}_q$.

Let $j = 1$ and $A > 3|\alpha|$. Put $\eta_\alpha = \eta$ and take any $\varepsilon < \eta$. Using that $\operatorname{mes}(T_{1,\phi,\varepsilon}) = (2\mu)^2 = 4(e^{(B+C_0)/A}\varepsilon^{-N/A})^2$, $P_{\phi,\varepsilon}(s) \ge C_P\varepsilon^r$, $s \in T_{1,\phi,\varepsilon}$ and $|s_1^{\alpha_1}s_2^{\alpha_2}| \le \tilde{C}_\alpha\varepsilon^{-(N/A)|\alpha|}$ $(\varepsilon < \eta)$, we have

$$\begin{aligned}\sup_{x\in B_h} |\partial^\alpha E_{1,\phi,\varepsilon}(x)| &= (2\pi)^{-n}\left|\int_{T_{1,\phi,\varepsilon}} \frac{(-i)^{|\alpha|}s_1^{\alpha_1}s_2^{\alpha_2}\hat{\phi}(\varepsilon s)}{P_{\phi,\varepsilon}(s)}ds\right| &(10)\\ &\le C\varepsilon^{-3N|\alpha|/A - r}, &(11)\end{aligned}$$

for some $C > 0$. This proves (9) for $j = 1$.

Consider $E_{2,\phi,\varepsilon}, \varepsilon < \eta$. The integration over the contour $\sigma_1 + i\tau_1$, $\mu \leq \sigma_1 \leq \nu$, $\tau_1 \in \{0, \dots, m+1\}$ is changed by integration over the contour

$$\overline{Q(\mu)Q_1(\mu)} \cup \overline{Q_1(\mu)Q_1(\nu)} \cup -\overline{Q(\nu)Q_1(\nu)},$$

where $\overline{Q(\mu)Q_1(\mu)} = \{\mu + it|\ 0 \leq t \leq \frac{1}{2}(A\log\mu + N\log\varepsilon - B)\}$, $\overline{Q_1(\mu)Q_1(\nu)} = \{\sigma_1 + i\tau_1|\ \tau_1 = \frac{1}{2}(A\log|\sigma_1| + N\log\varepsilon - B),\ \sigma_1 \in [\mu, \nu]\}$, $\overline{Q(\nu)Q_1(\nu)} = \{\nu + it|\ 0 \leq t \leq \frac{1}{2}(A\log\nu + N\log\varepsilon - B)\}$. We have

$$\begin{aligned}\partial^\alpha E_{2,\phi,\varepsilon}(x) &= (2\pi)^{-n}\int_{-\mu}^{\mu}\Big(\int_{\overline{Q(\mu)Q_1(\mu)}} - \int_{\overline{Q(\nu)Q_1(\nu)}} + \int_{\overline{Q_1(\mu)Q_1(\nu)}}\Big) \\ &\qquad \frac{s_1^{\alpha_1}s_2^{\alpha_2}e^{-i\langle(x_1,x_2),(s_1,s_2)\rangle}\hat{\phi}(\varepsilon s)}{P_{\phi,\varepsilon}(s)}ds = I_{1\varepsilon} + I_{2\varepsilon} + I_{3\varepsilon},\ x \in B_h.\end{aligned}$$

Since

$$|P_{\phi,\varepsilon}(s_1, \sigma_2)| = |a_{m,\phi,\varepsilon}|\prod_{j=1}^{m}(|\sigma_1 - \overline{\sigma}_1|^2 + |\tau_1 - \overline{\tau}_1|^2)^{1/2},$$

where σ_2 is fixed, $\overline{\sigma}_1 + i\overline{\tau}_1 \in V(P_{\phi,\varepsilon})$ and $s_1 = \sigma_1 + i\tau_1$ belongs to any of the quoted contours, Lemma 1 implies that $|P_{\phi,\varepsilon}(s)| \geq C_P\varepsilon^r$ on these contours (for $\varepsilon < \eta$). Now, one can prove that $I_{1\varepsilon} \leq C\varepsilon^{-r-2}\varepsilon^{-N/A}$, $\varepsilon < \eta$ for some $C > 0$.

For every $k > 0$ there exists $C_k > 0$ such that

$$\begin{aligned}&|\nu + i\tau|^{\alpha_1}|\sigma_2|^{\alpha_2}|\hat{\phi}(\varepsilon(\nu + i\tau, \sigma_2))| \\ \leq\ &|\nu + i\tau|^{\alpha_1}|\mu|^{\alpha_2}\frac{C_k e^{\varepsilon|\tau|}}{(1 + \varepsilon(\nu^2 + \tau^2 + |\sigma_2|^2)^{1/2})^k} \text{ on } \overline{Q(\nu)Q_1(\nu)}\end{aligned}$$

(see (1.4) in [2],Ch.2, Sec.2). Choosing $\nu = \varepsilon^{-2}$ and k large enough one gets $I_{2\varepsilon} \leq C\varepsilon^{-r-1}\varepsilon^{-N/A}$, for some $C > 0$.

Consider $I_{3\varepsilon}$. Again,

$$\begin{aligned}&|\hat{\phi}(\varepsilon(\sigma_1 + i(\log|\sigma_1|^A\varepsilon^N - B)/2, \sigma_2))| \\ \leq\ &\frac{C_k e^{\varepsilon(\log|\sigma_1|^A\varepsilon^N)/2}}{(1 + \varepsilon(\sigma_1^2 + (\log|\sigma_1|^A\varepsilon^N - B)^2/4 + |\sigma_2|^2)^{1/2})^k} \leq C|\sigma_1|^{A\varepsilon/2},\end{aligned}$$

on $\overline{Q_1(\mu)Q_1(\nu)}$ for some $C > 0$. Thus, with suitable constants, we have

$$|I_{3\varepsilon}| \leq C_1\int_{-\mu}^{\mu}\int_{\mu}^{\infty}\frac{|\sigma_1 + i(\log(|\sigma_1|^A\varepsilon^N) - B)/2|^{\alpha_1}|\sigma_2|^{\alpha_2}e^{-x_1(\log(|\sigma_1|^A\varepsilon^N)-B)/2}}{P_{\phi,\varepsilon}(s)}$$

$$\hat{\phi}(\varepsilon s)(1+\frac{A}{2|\sigma_1|})d\sigma_1 d\sigma_2$$
$$\leq C_2\varepsilon^{-r}(2\mu)|\mu|^{\alpha_2}\varepsilon^{-aN/2}\int_\mu^\infty |\sigma_1|^{A\varepsilon/2-aA/2}(|\sigma_1|^{\alpha_1}+\log^{\alpha_1}|\sigma_1|^A\varepsilon^N)d\sigma_1.$$

Taking A so large that $-aA/4$ dominates all exponents of σ_1 under the integral sign and $\sigma_1 > \mu$, we have (with suitable constants)

$$|I_{3\varepsilon}| \leq C_3\varepsilon^{-r-aN/2}|\mu|^{\alpha_2+1-aA/4+A\varepsilon/2+\alpha_1}\log^{\alpha_1}|\mu|^A\int_1^\infty|\tilde{\sigma}|^{-aN/4}d\tilde{\sigma}$$
$$\leq C_4\varepsilon^{-r-aN/2}(\varepsilon^{-N/A})^{\alpha_2+2-aA/4+A\varepsilon/2+\alpha_1}.$$

Now, it is easy to see that (4) holds in the case $j=2$ if one takes $A > a(|\alpha|+2)/4$.

One can give similar estimate for $E_{3,\phi,\varepsilon}, \varepsilon < \eta$ with a change of the integration over $T_{3,\phi,\varepsilon}$ (for $\mu \leq \sigma_1$ and $\mu \leq \sigma_2$) by the path consisting of lines connecting the boundary points of $T_{3,\phi,\varepsilon}$ and the points

$$((\sigma_1+i(A\log|\sigma_1|+N\log\varepsilon-B)/2),\sigma_2)\text{and}$$
$$((\sigma_1+i(A\log|\sigma_1|+N\log\varepsilon-B)/2),\sigma_2+i(A\log|\sigma_2|+N\log\varepsilon-B)/2)).$$

The proof is based on the estimate on $\hat{\phi}$ as above.

The proof for each of $E_{k,\phi,\varepsilon},\ j=4,\dots,9$ is the same as for $j=2$ or $j=3$. □

The change of condition (8) implies the change of Theorem 2 in [5]. Again its proof has the same idea as in [5].

Theorem 2 *An operator $[P_{\phi,\varepsilon}(D)]$ is hypoelliptic if and only if there exist $N>0$ and $q\in\mathbb{N}$ such that for every $\phi\in\mathcal{A}_q$ and every $A>0$ there exist $h>0$, $\eta>0$ and $b\in\mathbb{R}$ such that*

$$\sigma+i\tau\in V(P_{\phi,\varepsilon}) \Rightarrow |\tau|\geq\varepsilon^{hN}|\sigma|^{hA}-b,\ \varepsilon\in(0,\eta).$$

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