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DISTRIBUTION SEMIGROUPS ON $\mathcal{K}_1$

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A b s t r a c t. Distribution semigroup in the sense of Wang and Kunstmana and the properties of infinitesimal generator are considered with in exponentially bounded distributions. Results are applied on a class of equations of the form $\frac{\partial}{\partial t}4 - An = f,\ f \in \mathcal{K}_1^+(L(e))$*, where* $D(A) \subset L^\infty(\mathbb{R})$ *or* $D(A) \subset E = C_b(\mathbb{R})$.

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0. *Introduction*

Distribution semigroups of Lions [12] and, later introduced, n-times integrated semigroups of Arendt [2], have been studied by many authors. The aim has been applications to Cauchy problems with the luck of regularity conditions or with non-densely defined infinitesimal generators. The references contain enough informations in these sense. Wang [23] and Kunstmann [11] introduced quasi-distribution semigroups and exponentially bounded distribution semigroups which we call (DS) and (EDS).

In this paper we analyze the properties of the infinitesimal generator of such a semigroup within distribution theory. As a basic space we use the test function space $\mathcal{K}_1$. This is a natural framework for exponentially bounded distributions. The density of $D(A)$ in E and of a set $\{S(\varphi, x);\ x \in D(A),\ \varphi \in \mathcal{D}_0\}$ in $D(A)$, where S is a (EDS) with the infinitesimal generator A, is used in solving equations of the form $\frac{\partial}{\partial t}u - Au = f$, where $f \in \mathcal{K}_1'(D(A)),\ A = \sum_{j=0}^{k} a_j \frac{\partial^j}{\partial x^j},\ \sup Re(p(x)) < \infty,\ p(x) = \sum_{j=0}^{k} a_j (ix)^j$ and $D(A) \subset E = L^\infty(\mathbb{R})$ or $D(A) \subset E = C_b(\mathbb{R})$.

1. *Preliminaries*

We denote by E a Banach space with the norm $\|\cdot\|$; $L(E) = L(E, E)$ is the space of bounded linear operators from E into E and $C(\mathbb{R}, L(E))$ is the space continuous mappings from $\mathbb{R}$ into $L(E)$. We refer to [18-19] and [21] for the definitions of spaces $\mathcal{D}(\mathbb{R}),\ \mathcal{E}(\mathbb{R}),\ \mathcal{S}(\mathbb{R})$, their strong duals and $\mathcal{S}'(E) = L(\mathcal{S}(\mathbb{R}), E)$. Moreover, we refer to [21] for the space

$$\mathcal{S}_+ = \{\varphi;\ |t^k \varphi^{(\nu)}(t)| < C_{k,\nu},\ t \in [0, \infty),\ k, \nu \in \mathbb{N}_0\}\ (\mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

and its dual $\mathcal{S}_+'$, which consists of tempered distributions supported by $[0, \infty)$. Recall ([7]), the space of exponentially decreasing test functions on the real line $\mathbb{R}$ is defined by $\mathcal{K}_1(\mathbb{R}) = \{\varphi; e^{k|t|}|\varphi^{(\nu)}(t)| < C_{k,\nu},\ t \in \mathbb{R},\ k, \nu \in \mathbb{N}_0\}$. This space has the same topological properties as $\mathcal{S}(\mathbb{R})$. The space $\mathcal{K}_1(\mathbb{R}^2)$ is defined in an appropriate way. The strong dual of $\mathcal{K}_1(\mathbb{R}),\ \mathcal{K}_1'(\mathbb{R})$, is the space of exponential distributions. The space $\mathcal{K}_{1+}' \subset \mathcal{K}_1'(\mathbb{R})$ consists of distributions supported by $[0, \infty)$. It is the dual space to $\mathcal{K}_{1+} = \{\varphi; e^{k|t|}|\varphi^{(\nu)}(t)| < C_{k,\nu}, t \in [0, \infty),\ k, \nu \in \mathbb{N}_0\}$ which has the same topological properties as $\mathcal{S}_+$. Spaces $\mathcal{K}_1'(E),\ \mathcal{K}_{1+}'(E)$ are defined in an appropriate way. Their properties, similar to $\mathcal{S}'(E)$ and $\mathcal{S}_+'(E)$, are given in [15]. Note,

$$f \in \mathcal{K}_1'(\mathbb{R}) \text{ if and only if } e^{-r|x|} f \in \mathcal{S}'(\mathbb{R}) \text{ for some } r \in \mathbb{R}. \tag{1}$$

Let $S : [0, \infty) \to L(E)$ be strongly continuous. Then it is exponentially bounded at infinity if there exist $M \geq 0$ and $\omega \geq 0$ such that

$$\|S(t)\| \leq M e^{\omega t}, \quad t \geq 0. \tag{2}$$

In this case $\varphi \mapsto \int_0^\infty S(t)\varphi(t)dt,\ \varphi \in \mathcal{K}_1(\mathbb{R})$, defines an element of $\mathcal{K}_{1+}'(L(E))$.

We need a representation for elements of $\mathcal{K}'_{1+}(L(E))$. This is given in part a) of the next theorem.

Theorem 1. *Let* $S \in \mathcal{K}'_{1+}(L(E))$.

a) There exists $n_0 \in \mathbb{N}$ *such that for every* $n \geq n_0$ *there exist a strongly continuous function* $F_n : \mathbb{R} \to L(E)$, $\operatorname{supp} F_n \subset [0, \infty)$ *and positive constants* m_n *and* C_n, *such that*

$\|F_n(t)\| \leq C_n e^{m_n t}$, $t \geq 0$, $S = F_n^{(n)}$ *($^{(n)}$ is the distributional n-th derivative).*

b) Let $\psi, \varphi \in \mathcal{K}_1(\mathbb{R})$. *Then*

$$\begin{aligned}&\langle S(t, \langle S(s,x), \psi(s)\rangle),\ \varphi(t)\rangle \\ &= \int S_{n_0}(t, S_{n_0}(s,x))\psi^{(n_0)}(s)\varphi^{(n_0)}(t)ds\,dt.\end{aligned} \tag{3}$$

c) Let $\varphi(t,s) \in \mathcal{K}_1(\mathbb{R}^2)$ *and* $\varphi_\nu(t)$, $\psi_\nu(s)$ *be sequences in* $\mathcal{D}(\mathbb{R})$ *such that the product sequence* $\varphi_\nu(t) \cdot \psi_\nu(s)$ *converge to* $\varphi(t,s)$ *in* $\mathcal{K}_1(\mathbb{R}^2)$ *as* $\nu \to \infty$. *Then the limit*

$$\lim_{\nu\to\infty} \langle S(t, \langle S(s,x), \psi_\nu(s)\rangle), \varphi_\nu(t)\rangle,\ \varphi \in \mathcal{K}_1(\mathbb{R}^2)$$

exists and defines an element of $\mathcal{K}'_1(\mathbb{R}^2)$ *which we denote by* $S(t, S(s,x))$ *i.e.,*

$$\langle S(t,S(s,x)), \varphi(t,s)\rangle = \lim_{\nu\to\infty} \langle S(t, \langle S(s,x), \psi_\nu(s)\rangle), \varphi_\nu(t)\rangle,\ \varphi \in \mathcal{K}_1(\mathbb{R}^2). \tag{4}$$

d) Let $\varphi \in \mathcal{K}_1(\mathbb{R}^2)$ *and* $r, p \in \mathbb{N}$. *We have*

$$\Big\langle \frac{\partial^r}{\partial t^r} S(t,S(s,x)), \varphi(t,s)\Big\rangle = (-1)^r \langle S(t,S(s,x)),\ \frac{\partial^r}{\partial t^r}\varphi(t,s)\langle; \tag{i}$$

$$\begin{aligned}\Big\langle \frac{\partial^p}{\partial s^p} S(t,S(s,x)), \varphi(t,s)\Big\rangle &= \langle S(t, \frac{\partial^p}{\partial s^p} S(s,x)), \varphi(t,s)\rangle \\ &= (-1)^p \langle S(t,S(s,x)),\ \frac{\partial^p}{\partial s^p}\varphi(t,s)\rangle.\end{aligned} \tag{ii}$$

P r o o f. Part a) can be proved in the same way as in the case of scalar valued distributions.

Parts b), c) and d) are consequences of the continuity and linearity of $S \in \mathcal{K}'_{1+}(L(E))$, more precisely of the generalized Fubini-type theorem.

Using (1) one can prove easily:

$$f \in \mathcal{K}'_{1+}(L(E)) \text{ if and only if } e^{-r|x|}f \in \mathcal{S}'_{+}(L(E)) \text{ for some } r \geq 0. \tag{5}$$

Let f satisfy (5). Then the Laplace transformation of f is defined by

$$\mathcal{L}(f)(\lambda) = \widehat{f}(\lambda) = \left\langle f(t), e^{-\lambda t}\eta(t)\right\rangle,\ Re\lambda > r,$$

where $\eta \in \mathbb{C}^{\infty}(\mathbb{R})$, supp$\eta = [-\varepsilon, \infty)$, $\varepsilon > 0$ and $\eta \equiv 1$ on $[0,\infty)$. As in the case of tempered distributions, one can easily show that this definition does not depend on η (cf.[21]).If $f \in L^1([0,\infty), E)$ (which means $\left\| \int\limits_0^\infty f(t)dt \right\|_E < \infty$, then

$$\widehat{f}(\lambda) = \int\limits_0^\infty e^{-\lambda t} f(t)dt = \langle f(t), e^{-\lambda t}\rangle,\ \ Re\lambda > 0,$$

where integral is taken in Bochner's sense.

The convolution of $f \in \mathcal{K}'_{1+}(E)$ and $g \in \mathcal{K}'_{1+}(\mathbb{R})$ is defined by $\langle f * g, \varphi\rangle = \langle f, \check{g} * \varphi\rangle$, $\varphi \in \mathcal{K}_1(\mathbb{R}), (\check{g}(t) = g(-t))$. One can prove easily that $f * g = g * f \in \mathcal{K}'_{1+}(E)$.

In the sequal, we will use the family of distributions

$$f_\alpha(t) = \begin{cases} \dfrac{H(t)t^{\alpha-1}}{\Gamma(\alpha)}, & t \in \mathbb{R},\ \alpha > 0, \\ \\ f^{(n)}_{\alpha+n}(t), & t \in \mathbb{R},\ \alpha \leq 0,\ \alpha + n > 0, n > 0, \end{cases}$$

where H is Heaviside's function. Note $f_{-1} = \delta'$.

Let $S \in \mathcal{K}'_{1+}(L(E))$ and $R(\lambda) = \mathcal{L}(S)(\lambda)$, $Re\lambda > \omega$ (cf . [2]). Then $(R(\lambda))_{Re\lambda\geq\omega}$ is a pseudoresolvent if and only if there exists $n_0 \in \mathbb{N}$ such that $S_{n_0}(t) = (S * f_{n_0})(t)$, $t \in \mathbb{R}$, is continuous, supp$S_{n_0} \subset [0,\infty)$ and, for $\varphi, \psi \in \mathcal{K}_1$,

$$\langle S(t, S(s,x)), \varphi(t)\psi(s)\rangle = \langle (S_{n_0}(t, S_{n_0}(s,x)))^{(n_0,n_0)}, \varphi(t)\psi(s)\rangle$$

$$= \left\langle \frac{1}{(n_0-1)!}\left(\int\limits_t^{t+s} (t+s-r)^{n_0-1} S_{n_0}(r,x)dr \right.\right. \tag{6}$$

$$\left.\left. - \int\limits_0^s (t+s-r)^{n_0-1} S_{n_0}(r,x)dr \right)^{(n_0,n_0)}, \varphi(t)\psi(s) \right\rangle,$$

The next definition is equivalent to the one given by Kunstmann and Wang, with $\mathcal{D}$ instead of $\mathcal{K}_1$.

Definition 1. *Let $S \in \mathcal{K}'_{1+}(L(E))$. Then S is called exponentially bounded distribution semigroup (EDS), in short, if there exists $n_0 \in \mathbb{N}$, such that $S_{n_0} = S * f_{n_0}$ is continuous on $\mathbb{R}$, supported by $[0, \infty)$, exponentially bounded, satisfies (6) and it is non-degenerate: $\langle S(t,x), \varphi(t)\rangle = 0$ for all $\varphi \in \mathcal{K}_1$, implies $x = 0$.*

We will also use the notation $(S(t))_{t\geq 0}$ for an (EDS). If (6) holds for $\psi \in \mathcal{D}(-\infty, a)$ for some $a > 0$, $(S(t))_{t\geq 0}$ is called a local distribution semigroup. This definition coincides with the definition of (DS) of Kunstmann and Wang but with $\mathcal{D}$ instead of $\mathcal{K}_1$.

Also the next definition is equivalent to the known one of cited authors.

Definition 2. *A closed operator A is the generator of an (EDS) $(S(t))_{t\geq 0}$ if $(a, \infty) \subset \rho(A)$ for some $a \in \mathbb{R}$ so that $(\lambda I - A)^{-1} = \mathcal{L}(S)(\lambda)$, $Re\lambda > a$ holds and $\lambda \mapsto (\lambda I - A)^{-1}$ is injective, where the Laplace transformation is understood in the sense of distribution theory.*

As in case with $\mathcal{D}$ instead of $\mathcal{K}_1$, one can simply prove the next theorem.

Theorem 2. *Let A be a generator of a (EDS) $(S(t))_{t\geq 0}$. Then, for all $\varphi \in \mathcal{K}_1$, we have*

a) $A\langle S(t,x), \varphi(t)\rangle = \langle S(t, Ax), \varphi(t)\rangle, \;\; x \in D(A).$

b) $\langle S(t,x), \varphi(t)\rangle \in D(A), \; x \in E.$

c) $\langle S(t,x), \varphi(t)\rangle = \langle f_1(t,x), \varphi(t)\rangle + \langle (f_1 * S)(t, Ax), \varphi(t)\rangle, \; x \in D(A)$ *and*

$$A\langle (f_1 * S)(t,x), \varphi(t)\rangle = \langle S(t,x), \varphi(t)\rangle - \langle f_1(t,x), \varphi(t)\rangle, \; x \in E.$$

In particular

$$A\langle S(t,x), \varphi(t)\rangle = -\langle S(t,x), \varphi'(t)\rangle - \varphi(0)x, \;\; x \in E.$$

We refer to Definition 6.1 in [12] for a distribution semigroup, (DS-L) in short and exponentially distribution semigroups (EDS-L) in short in the sense of Lions. If $D(A)$ is dense in E, then these notions coincide with (DS) and (EDS).

2. *Comments on generators*

Let $(S(t))_{t\geq 0}$ be a (DS) or (EDS). Recall $S(T,\cdot),\ T\in\mathcal{E}'(\mathbb{R})$ is defined as follows:

$y=S(T,x)$, if $S(T*\psi,x)=S(\phi,y),\ \phi\in\mathcal{D}_0$. The set of $x\in E$ for which this holds is denoted by $D(T)$. It follows that the domain of $S(-\delta',\cdot)$ is $D(A)$ and $S(-\delta',x)=Ax,\ x\in D(A)$.

Let $S_n(\cdot,x)=S(\cdot,x)*f_n,\ x\in E$ be an n-times integrated semigroup determined by the (EDS), $(S(t))_{t\geq 0}$ with the infinitesimal generator A.

One can simply prove

$$S_n(t,x)=\lim_{\nu\to\infty}\langle S_n(s,x),\rho_\nu(t-s)\rangle,\ \ t\geq 0,$$

$$S_n(\varphi^{(n)},x)=(-1)^nS(\varphi,x),\ \varphi\in\mathcal{D}_0,\ x\in E,$$

where $\{\rho_\nu\}$ is δ sequences in $\mathcal{D}_0$, $(\rho_\nu\to\delta,\ \nu\to\infty)$.

Theorem 3. *Let $(S(t))_{t\geq 0}$ be an (EDS) and $(S_n(t)=(S*f_n(t))_{t\geq 0}$ be an n-times integrated exponentially bounded semigroup, $n\in\mathbb{N}_0$ with the infinitesimal generator A. Then*

a) $D(S(f))=D(S_n(f^{(n)})),\ \ f\in\mathcal{E}'(\mathbb{R}),\ \ \mathrm{supp}f\subset[0,\infty)$ and

$$S_n(f^{(n)},x)=(-1)^nS(f,x),\ \ x\in D(S(f)),$$

$$S_n(h,x)=S_n(\delta(t-h),x),\ \ x\in E,\ \ h>0,$$

In particular

$$(-1)^nS_n(\delta^{(n)},x)=x,\ \ x\in E,$$

$$(-1)^nS_n(-\delta^{(n+1)},x)=Ax,\ \ x\in D(A).$$

b)

$$Ax=(n+1)!\lim_{h\downarrow 0}\frac{S_n(h)x-\frac{h^n}{n!}x}{h^{n+1}},\ \ x\in D(A). \tag{7}$$

P r o o f. We will prove only b). Let $\varphi\in\mathcal{D}$. Since,

$$\frac{(n+1)!}{h^{n+1}}\Big(\varphi(h)-\frac{h^n}{n!}\varphi^{(n)}(0)\Big)\to\varphi^{(n+1)}(0),\quad\text{as}\ \ h\to 0^+,$$

it follows

$$\frac{(n+1)!}{h^{n+1}}\left\langle\delta(t-h)-\frac{h^n}{n!}(-1)^n\delta^{(n)}(t),\varphi(t)\right\rangle\to$$

$$\to (-1)^{n+1}\Big\langle \delta^{(n+1)}(t), \varphi(t)\Big\rangle, \quad \varphi \in \mathcal{D} \ \text{ as } \ h \to 0^+.$$

Then, for $x \in D(A)$ we have,

$$(n+1)! \lim_{h\downarrow 0} \frac{S_n(h,x) - \frac{h^n}{n!}x}{h^{n+1}} = (n+1)! \lim_{h\downarrow 0} \frac{S_n(\delta(t-h),x) - S_n\Big(\frac{h^n}{n!}(-1)^n\delta^{(n)}(t),x\Big)}{h^{n+1}}$$

$$= (n+1)! \lim_{h\downarrow 0} \frac{S_n\Big(\delta(t-h) - \frac{h^n}{n!}(-1)^n\delta^{(n)}(t),x\Big)}{h^{n+1}}$$

$$= S_n\Big((-1)^{n+1}\delta^{(n+1)}(t),x\Big) = S(-\delta',x) = Ax.$$

Theorem 4. *Let $(S(t))_{t\geq 0}$ be an (EDS) with the infinitesimal generator A and $F = \{S(\varphi,x),\ x \in D(A), \varphi \in \mathcal{D}_0\}$. Then F is dense in $D(A)$.*

P r o o f. Let $x \in D(A)$. Since

$$S(-\delta'_\nu, x) = \lim_{h\downarrow 0} \left\langle \frac{S(t+h,x) - S(t,x)}{h},\, \delta_\nu(t) \right\rangle,$$

there exists a sequence $(h_\nu)_{\nu\in\mathbb{N}},\ h_\nu \to 0^+$, such that

$$\left\langle \frac{S(t+h_\nu,x) - S(t,x)}{h_\nu},\, \delta_\nu(t) \right\rangle \to 0, \ \ \nu \to \infty$$

and therefore

$$\langle S(t+h_\nu,x) - S(t,x),\, \delta_\nu(t)\rangle \to 0, \ \ \nu \to \infty.$$

But we have $\langle S(t,x), \delta_\nu(t)\rangle \to x,\ \nu \to \infty$, which implies that

$$\Big(\langle S(t,x), \delta_\nu(t-h)\rangle\Big)_{\nu\in\mathbb{N}}$$

is a sequence in F which converges to x. Thus F is dense in $D(A)$.

Theorem 4 implies that there exists a closed subspace E_0 of E such that $(S(t))_{t\geq 0}$ is an (EDS -L) on E_0, where E_0 is the closure in E of the set $F = \{S(\varphi,x);\ x \in D(A),\ \varphi \in \mathcal{D}_0\}$.

The following theorem is proved by Wang and Kunstmann. We reformulate it with $\mathcal{K}_1$ instead of $\mathcal{D}$.

Theorem 5. *Let $(S(t))_{t\geq 0}$ be an (EDS) with the infinitesimal generator A. Then the restriction of $(S(t))_{t\geq 0}$ on $E_0\times\mathcal{K}_1, (S_{|E_0\times\mathcal{K}_1})$, is an $(EDS-L)$.*

3. *Applications*

Example. Let $E = C_b(\mathbb{R})$, or $L^\infty(\mathbb{R})$ and A be defined by $Af = \sum_{j=0}^k \alpha_j D^j f$, where $D^j = \frac{\partial^j}{\partial x^j}, \alpha_0, ..., \alpha_k \in \mathbb{C}, p(x) = \sum\limits_{j=0}^k \alpha_j (ix)^j, k \geq 1, \alpha_k \neq 0, \sup_{x\in\mathbb{R}} Re(p(x)) < \infty$, where $D(A) = \{f \in E, \sum_{j=0}^k \alpha_j D^j f \in E$, distributionally $\}$. It is known that $D(A)$ is not dense in E (cf.[10]).

Let $S_t(f) = \dfrac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}(e^{p(x)t}) * f$. Here $\mathcal{F}$ denotes the Fourier transformation and $\mathcal{F}^{-1}$ denotes is inverse; $\mathcal{F}(f)(\lambda) = \int\limits_{\mathbb{R}} e^{-i\lambda t} f(t)dt,\ \lambda \in \mathbb{R}$. Then it is an (EDS) because $S_t(f) = \dfrac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}(\int\limits_0^1 e^{p(x)s}ds) * f, \in D(A)$, is 1-time integrated semigroup. Moreover, since the set $\{\langle S(t,f), \varphi(t)\rangle,\ f \in D(A),\ \varphi \in \mathcal{D}_0\}$ is dense in $\overline{D(A)}$ it follows that S_t is (EDS-L) on the subspace $E_0 = \{\overline{\langle S(t,f), \varphi(t)\rangle, f \in D(A), \varphi \in \mathcal{K}_0}\}$. Note ([16]), A generates a norm continuous α-times integrated semigroup for $\alpha \in \Big(\dfrac{1}{2}, 1\Big]$ equal to $S_t * f_\alpha, t \geq 0$.

Recall, for $U \in \mathcal{K}'_{1+}(L(E, D(A))), V \in \mathcal{K}'_{1+}(L(D(A), E))$ and $\operatorname{supp} U \subset [a, \infty), \operatorname{supp} V \subset [b, \infty), a, b \in \mathbb{R}$. Then $U * V$ and $V * U$ are defined as in [19]. Moreover, they are elements of $\mathcal{K}'_{1+}(L(D(A)))$ and $\mathcal{K}'_{1+}(L(E))$ respectively and their supports are bounded on the left.

Now we apply our results to equation

$$u' = Au + T, \quad T \in \mathcal{K}_{1+}(L(E_0)).$$

We refer to this equations in the case $T \in \mathcal{S}'_{1+}(L(E_0))$ to ([12],[15]).

Theorem 6. *Let $(S(t))_{t\geq 0}$ be an (EDS-L) with the infinitesimal generator A. Then*

$$\Big(-A + \frac{\partial}{\partial t}\Big) * S = I_{E_0}, S * \Big(-A + \frac{\partial}{\partial t}\Big) = I_{D(A)}, \qquad a)$$

where

$$-A + \frac{\partial}{\partial t} = \delta \otimes A + \delta' \otimes I.$$

b) $u = S * T$ in $\mathcal{K}'_{1+}(L(E_0))$ is a unique solution of (11)

Remark. In particular, with the notation given above this theorem gives the unique solution to $\frac{\partial}{\partial t}u(t,x) - \sum_{j=0}^k a_j \frac{\partial^j}{\partial x^j} u(t,x) = f, f \in \mathcal{K}'_{1+}(L(E_0))$ in $\mathcal{K}'_{1+}(L(E_0))$.

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