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Estimating Powers with Base Close to Unity and Large Exponents

Estimacón de Potencias con Base Cercana a la Unidad y Grandes Exponentes

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Abstract

In this paper we derive the relation

$$\exp\left(ht - \frac{h^2 t}{2(1-\varepsilon)}\right) < (1+h)^t < \exp\left(ht - \frac{h^2 t}{2(1+\varepsilon)}\right),$$

valid for $\varepsilon \in (0,1)$, $t > 0$ and $0 < |h| \leq \varepsilon$. These inequalities estimate the rates of convergence of

$$\lim_{|t|\to\infty} \left(1 + \frac{x}{t}\right)^t = e^x \quad \text{and} \quad \lim_{t\to\infty} \left[t \cdot \left(e^x - \left(1 + \frac{x}{t}\right)^t\right)\right] = \frac{e^x x^2}{2}$$

and enable numerical computation of a power with a base close to unity and large exponent.

Key words and phrases: approximation of powers, asymptotic inequalities, computation of powers with large exponent and base close to unity, exponential function, estimation of powers.

Resumen

En este artículo se obtiene la relación

$$\exp\left(ht - \frac{h^2 t}{2(1-\varepsilon)}\right) < (1+h)^t < \exp\left(ht - \frac{h^2 t}{2(1+\varepsilon)}\right),$$

válida para $\varepsilon \in (0,1)$, $t > 0$ y $0 < |h| \leq \varepsilon$. Estas desigualdades estiman las tasas de convergencia de

$$\lim_{|t|\to\infty} \left(1 + \frac{x}{t}\right)^t = e^x \quad \text{y} \quad \lim_{t\to\infty} \left[t \cdot \left(e^x - \left(1 + \frac{x}{t}\right)^t\right)\right] = \frac{e^x x^2}{2}$$

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y permiten el cálculo numérico de una potencia de base cercana a la unidad y gran exponente.
Palabras y frases clave: aproximación de potencias, desigualdades asintóticas, cálculo de una potencia de base cercana a la unidad y gran exponente, función exponencial, estimación de potencias.

1 Introduction

How to compute a^t for a close to 1 and t being very large? Such question occurs when we want to obtain numerical value of the solution $x_n = x_0(1+h)^n = x_0\left(1+\frac{nh}{n}\right)^n$ of difference equation $x_n - x_{n-1} = hx_n$, ($h = const. \approx 0$), which is frequently replaced by its continuous version, namely the differential equation, $\frac{dx}{dt} = hx$, having solution $x = x(0)e^{ht}$. Although the computation of a^t is usually an easy task, especially in the age of computers, the question is not as simple as it seems. For example, using calculators to compute "singular" powers such as $\alpha = (1-10^{-59.1})^{10^{58.6}}$ and $\beta = (1+10^{-58.6})^{10^{59.1}}$ we do not obtain correct result due to overflow problems. For the same reason, the computation of numbers α and β above is not quite an easy task, even for powerful math software, like Mathematica [5], for example. Moreover, if powers α and β are substituted by "more singular" powers, even Mathematica does not give a useful result.

However, it is well known that $\left(1+\frac{x}{t}\right)^t \approx e^x$ for t large, according to convergence $\lim\limits_{t\to\infty}\left(1+\frac{x}{t}\right)^t = e^x$. But, to use this approximation for numerical computation of powers like α and β, we need simple bounds for the errors, i.e. we need simple functions $A(t,x)$ and $B(t,x)$, close to 1 for t large, such that $A(t,x)\cdot e^x \le \left(1+\frac{x}{t}\right)^t \le B(t,x)\cdot e^x$ for t large. To this effect let us go back to the definition of a power to find such functions $A(t,x)$ and $B(t,x)$.

Several authors introduce power with positive base and real exponent by allowing for exponent first positive integer values and then generalize the notion of the power from the case when the exponent is negative integer to the cases when the exponent is rational and real. This requires a lot of time and a fair of effort to prove the additivity and differentiability properties of real exponential function. Hence, we not recommend this approach to powers, not even from only the theoretical point of view. In addition, this way is not productive for our purpose as well. Fortunately, there exists an easier method based on the definite integral. Choosing this method, the logarithmic and exponential functions can be introduced easily, and their fundamental properties can be derived in a simple manner (see for example [**1,** p. 409] or

[**4,** p. 117]).

To sum up, the logarithmic function $\ln : \mathbb{R}^+ \to \mathbb{R}$, $\ln(x) := \int_1^x \frac{1}{t}\, dt$, is differentiable, strictly monotonically increasing bijection with derivative $\ln'(x) = \frac{1}{x}$ and with the additive property $\ln(x_1\, x_2) = \ln(x_1) + \ln(x_2)$, obtained by substituting the integration variable with a new one. Its inverse function, $\exp := \ln^{-1} : \mathbb{R} \to \mathbb{R}^+$, called the exponential function, is consequently also differentiable, strictly monotonically increasing bijection with derivative $\exp'(x) = \exp(x)$ and with the additive property $\exp(x_1 + x_2) = \exp(x_1)\exp(x_2)$.

The power of a positive real number a is defined by

$$a^x := \exp(x \ln(a)), \quad x \in \mathbb{R}.$$

Powers, defined this way, have all the usual properties, which are easily verified. With the Euler number $e := \exp(1)$, the identity $\exp(x) = e^x$ holds for every real x. By showing that the derivative of quotient $q(x) := e^{-x} f(x)$ is identically equal to 0, provided that function $f : \mathbb{R} \to \mathbb{R}^+$ is differentiable, its derivative coincides with itself and $f(0) = 1$, it becomes clear that the exponent function is the unique differentiable function, whose derivative coincides with itself and takes the value 1 at the point 0.

Taylor's formula and the equality $\exp'(x) = \exp(x)$ enable us to make some initial numerically useful approximations for the exponential function. Using Taylor's formula, we can also approximate logarithms. Therefore, computation of "regular" powers is not a hard work. On the other hand, computation of "singular" powers, as has been mentioned above, could be rather problematic. We would like to find a way how to compute such "singular" powers, as well as we would like to estimate the expression $\left(1 + \frac{x}{t}\right)^t$ for t large.

2 Monotonous convergence

Our main concern is the function

$$t \longmapsto \left(1 + \frac{x}{t}\right)^t =: E(x,t) \tag{1}$$

defined on the intervals $I_x^- := (-\infty, -\max\{0,x\})$ and $I_x^+ := (-\min\{0,x\}, \infty)$ for any real x. For nonzero x we have

$$\begin{aligned}
\lim_{|t|\to\infty} \ln E(x,t) &= \lim_{|t|\to\infty} \left[x \cdot \frac{t}{x} \ln\left(1+\frac{x}{t}\right)\right] \\
&= x \cdot \lim_{\tau\to 0} \frac{\ln(1+\tau)-\ln(1)}{\tau} \\
&= x \cdot \ln'(1) = x.
\end{aligned}$$

This means, due to continuity (differentiability) of logarithmic function, that $\lim\limits_{|t|\to\infty} E(x,t)$ exists and is equal to e^x for every real x:

$$\lim_{|t|\to\infty} \left(1+\frac{x}{t}\right)^t = e^x. \tag{2}$$

For every $t \in I_x^- \cup I_x^+$ we have

$$\frac{d}{dt} E(x,t) = L(x,t) \cdot E(x,t), \tag{3}$$

where

$$L(x,t) := \ln\left(1+\frac{x}{t}\right) - \frac{x}{t+x}. \tag{3a}$$

The function $t \longmapsto L(x,t)$ has derivative

$$\frac{d}{dt} L(x,t) = \frac{1}{1+\frac{x}{t}} \left(-\frac{x}{t^2}\right) + \frac{x}{(t+x)^2} = -\frac{x^2}{t(t+x)^2}\,.$$

Consequently, it is strictly monotonically increasing on the interval I_x^- and decreasing on the interval I_x^+ for any $x \neq 0$. Therefore, $L(x,t) > 0$ for $x \neq 0$ and $t \in I_x^- \cup I_x^+$, because $\lim_{|t|\to\infty} L(x,t) = 0$ for any x. Furthermore, since $E(x,t) > 0$, we conclude from (3) that the function $t \mapsto E(x,t)$ increases strictly monotonously on both intervals I_x^- and I_x^+ for every $x \neq 0$. Hence, for any $x \neq 0$, the convergence

$$\lim_{t\to\infty} \left(1+\frac{x}{t}\right)^t = e^x = \lim_{t\to\infty} \left(1-\frac{x}{t}\right)^{-t} \tag{4}$$

is strictly monotonically increasing or decreasing, respectively. Figure 1 illustrates this dynamics.

3 The rate of convergence $\lim_{t\to\infty} \left(1+\frac{x}{t}\right)^t$

For any $\varepsilon \in [0,1)$ the function $f_\varepsilon : (-1,\infty) \to \mathbb{R}$, defined by

$$f_\varepsilon(\tau) := \ln(1+\tau) - \tau + \frac{\tau^2}{2(1-\varepsilon)}\,,$$

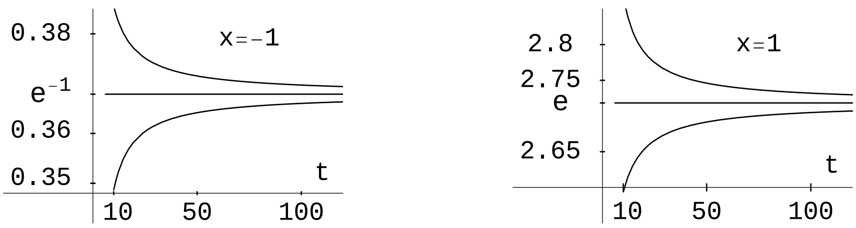

Figure 1: $\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t = e^x = \lim_{t\to\infty}\left(1-\frac{x}{t}\right)^{-t}$.

has derivative

$$f'_\varepsilon(\tau) = \frac{1}{1+\tau} - 1 + \frac{\tau}{1-\varepsilon} = \frac{(\tau+\varepsilon)\tau}{(1-\varepsilon)(1+\tau)}\,.$$

Therefore

$$\min_{\tau\geq-\varepsilon} f_\varepsilon(\tau) = f_\varepsilon(0) = 0$$

and consequently $f_\varepsilon(\tau) > 0$ for $\tau \in [-\varepsilon,\infty)\backslash\{0\}$. That is

$$e^{\tau - \frac{\tau^2}{2(1-\varepsilon)}} < 1+\tau \tag{5}$$

for any $\varepsilon \in [0,1)$ and for every nonzero $\tau \geq -\varepsilon$.

Similarly, for any $\varepsilon \in [0,1)$, we treat the function $g_\varepsilon : (-1,\infty) \to \mathbb{R}$ defined by

$$g_\varepsilon(\tau) := \tau - \frac{\tau^2}{2(1+\varepsilon)} - \ln(1+\tau)\,.$$

Its derivative,

$$g'_\varepsilon(\tau) = 1 - \frac{\tau}{1+\varepsilon} - \frac{1}{1+\tau} = -\frac{(\tau-\varepsilon)\cdot\tau}{(1+\varepsilon)(1+\tau)}\,,$$

shows that

$$\min_{-1<\tau\leq\varepsilon} g_\varepsilon(\tau) = g_\varepsilon(0) = 0\,.$$

Hence, $g_\varepsilon(\tau) > 0$ for $\tau \in (-1,\varepsilon]\backslash\{0\}$; thus

$$1+\tau < e^{\tau - \frac{\tau^2}{2(1+\varepsilon)}} \tag{5a}$$

for every $\varepsilon \in [0,1)$ and for every nonzero τ such that $-1 < \tau \leq \varepsilon$.

Let us exploit the above relations (5) and (5a). Indeed, for any real $x \neq 0$, $\varepsilon \in (0,1)$, and $t \geq |x|\varepsilon$ the number $\tau := \frac{x}{t} \neq 0$ lies on the interval $[-\varepsilon,\varepsilon]$. Therefore, according to (5) and (5a), the following relation holds

$$e^{\frac{x}{t} - \frac{x^2}{2(1-\varepsilon)t^2}} < 1 + \frac{x}{t} < e^{\frac{x}{t} - \frac{x^2}{2(1+\varepsilon)t^2}}\,.$$

Taking the powers we obtain the main estimate

$$e^{x - \frac{x^2}{2(1-\varepsilon)t}} < \left(1+\frac{x}{t}\right)^t < e^{x - \frac{x^2}{2(1+\varepsilon)t}}\,, \tag{6}$$

valid for every real $x \neq 0$, $\varepsilon \in (0,1)$ and $t \geq |x|/\varepsilon$. From these inequalities we obtain, taking $h = \frac{x}{t}$, the asymptotic estimate

$$\exp\left(ht - \frac{h^2 t}{2(1-\varepsilon)}\right) < (1+h)^t < \exp\left(ht - \frac{h^2 t}{2(1+\varepsilon)}\right)\,, \tag{6a}$$

true for $\varepsilon \in (0,1)$, $t > 0$ and $0 < |h| \leq \varepsilon$.
Figure 2 illustrates the estimate (6) for $x = 1$ and $x = -1$, and for $\varepsilon = \frac{1}{10}$, where dashed curves represent lower and upper bounds. Considering the

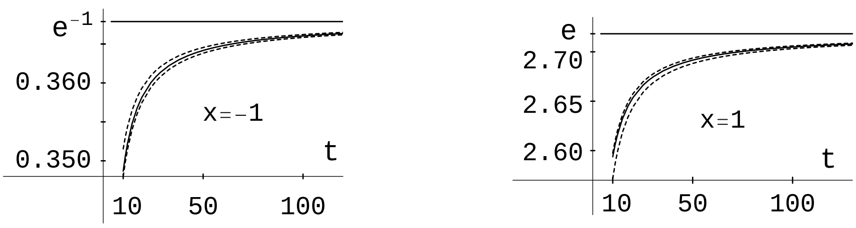

Figure 2: Lower and upper bounds (6) for the function $t \mapsto \left(1+\frac{x}{t}\right)^t$.

approximation

$$\left(1+\frac{x}{t}\right)^t \approx e^x, \tag{7}$$

we obtain from (6) the asymptotic estimate

$$1 - e^{-\frac{x^2}{2(1+\varepsilon)t}} < r(x,t) < 1 - e^{-\frac{x^2}{2(1-\varepsilon)t}} \tag{8}$$

for the relative error

$$r(x,t) := \frac{e^x - \left(1 + \frac{x}{t}\right)^t}{e^x}, \tag{8a}$$

which holds with the same conditions as were quoted for (6).
Figure 3 illustrates the estimate (8) for $x = \pm 1$ and $x = \pm 2$ at $\varepsilon = \frac{1}{10}$.

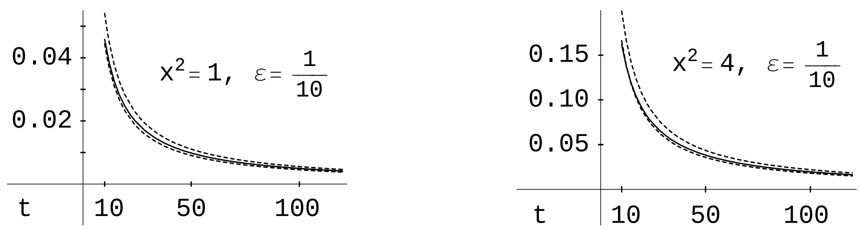

Figure 3: Estimate (8) of the convergence $r(x,t) \to 0$ as $t \to \infty$.

From (8) we can get an additional, less accurate, but simpler estimate for $r(x)$. To this effect we observe that the function $\varphi : \mathbb{R} \to \mathbb{R}$, defined by $\varphi(\tau) := \tau + e^{-\tau} - 1$, has a positive derivative for $\tau > 0$. This means $\varphi(\tau) > 0$, i.e. there holds the estimate

$$1 - e^{-\tau} < \tau \tag{9}$$

for $\tau > 0$; consequently the function $\psi : \mathbb{R} \to \mathbb{R}$, where

$$\psi(\tau) := 1 - e^{-\tau} - \tau + \tau^2/2, \tag{10}$$

has a derivative

$$\psi'(\tau) = e^{-\tau} - 1 + \tau > 0$$

for $\tau > 0$. Hence, we have found that $\psi(\tau)$ is positive for τ positive. Therefore, by definition (10), we have

$$1 - e^{-\tau} > \tau - \frac{\tau^2}{2} = \tau(1 - \frac{\tau}{2}) \tag{11}$$

for $\tau > 0$. Combining (8) with (9) and (11), we obtain the estimate

$$\frac{x^2}{2(1+\varepsilon)} \left(1 - \frac{x^2}{4(1+\varepsilon)t}\right) \cdot \frac{1}{t} < r(x,t) < \frac{x^2}{2(1-\varepsilon)} \cdot \frac{1}{t}\,, \tag{12}$$

which holds for every real $x \neq 0$, $\varepsilon \in (0,1)$ and $t \geq |x|/\varepsilon$. Referring to (8a), the left part of this relation is obviously interesting only in case $t > \frac{x^2}{4(1+\varepsilon)}$, since $r(x,t) > 0$ for $x \neq 0$ and $t > 0$, due to the fact that $\left(1+\frac{x}{t}\right)^t$ converges monotonously from below towards e^x as $t \to \infty$, see §2. Bounds presented in (12) are close to the bounds in (8). In Figure 4 we illustrate this fact for $t \in [10,30]$ by plotting graphs of differences between lower (left) and upper (right) bounds $d_l(\varepsilon,x,t)$ and $d_u(\varepsilon,x,t)$

$$d_l(\varepsilon,x,t) := \left[1-\exp\left(-\frac{x^2}{2(1+\varepsilon)t}\right)\right] - \left[\frac{x^2}{2(1+\varepsilon)t}\left(1-\frac{x^2}{4(1+\varepsilon)t}\right)\right]$$

and

$$d_u(\varepsilon,x,t) := \frac{x^2}{2(1-\varepsilon)t} - \left[1-\exp\left(-\frac{x^2}{2(1-\varepsilon)t}\right)\right].$$

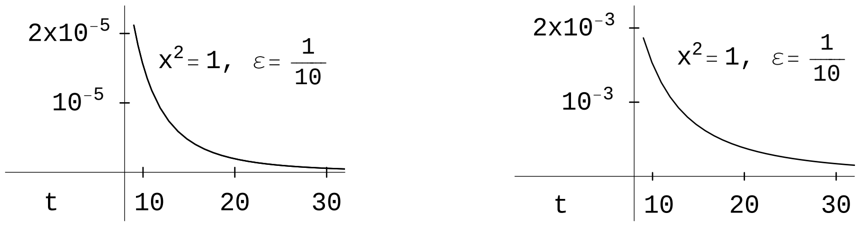

Figure 4: Graphs of differences d_l (left) and d_u (right).

According to definition (8a) we find, from relation (12) above, the estimate

$$\frac{e^x x^2}{2(1+\varepsilon)}\left(1-\frac{x^2}{4(1+\varepsilon)t}\right)\cdot\frac{1}{t} < e^x - \left(1+\frac{x}{t}\right)^t < \frac{e^x x^2}{2(1-\varepsilon)}\cdot\frac{1}{t}, \qquad \text{(12a)}$$

valid under the same conditions as were stated for (12). We also note an obvious and useful fact that the function

$$t \mapsto 1-\frac{x^2}{4(1+\varepsilon)t}$$

increases monotonously on the interval $(0,\infty)$, while the functions

$$\varepsilon \mapsto \frac{1}{1+\varepsilon} \quad \text{and} \quad \varepsilon \mapsto \frac{1}{1-\varepsilon}$$

respectively, decrease and increase monotonously on the interval $(0,1)$.

Setting $\varepsilon = \frac{1}{2}$ in (12) we deduce the estimate

$$\frac{x^2}{3}\left(1-\frac{x^2}{6t}\right)\cdot\frac{1}{t} < r(x,t) < x^2\cdot\frac{1}{t}\,, \tag{12b}$$

which is valid for $x \neq 0$ and $t \geq 2\,|x|$. For the same reasons as were stated in comment to (12), the left part of this estimate is interesting only if $t \geq x^2/6$.

From (12b) we can extract the relation

$$\left(1-\frac{x^2}{t}\right)\cdot e^x < \left(1+\frac{x}{t}\right)^t < \left[1-\frac{x^2}{3t}\left(1-\frac{x^2}{6t}\right)\right]\cdot e^x, \tag{13}$$

which holds for $x \neq 0$ and $t \geq 2\,|x|$. Considering the remark above, the left part of this relation is obviously interesting only for $t > \max\{2|x|, x^2\}$ and the right part for $t > \max\{2|x|, x^2/6\}$. Putting $h = \frac{x}{t}$ into (13), we obtain the estimate

$$\left(1-h^2t\right)\cdot e^{ht} \;<\; (1+h)^t \;<\; \left[1-\frac{h^2t}{3}\left(1-\frac{h^2t}{6}\right)\right]\cdot e^{ht}, \tag{14}$$

valid for $t > 0$ and $0 < |h| \leq 1/2$. Having t positive, the left side of (14) is obviously interesting only for $0 < |h| < \min\{1/2, 1/\sqrt{t}\}$ and the right side for $0 < |h| < \min\{1/2, \sqrt{6/t}\}$.

Taking $x \neq 0$ and $t > |x|$, and putting $\varepsilon := \frac{|x|}{t}$ in (12a), we obtain the estimate

$$\frac{e^x x^2}{2}\cdot\frac{t}{t+|x|}\left(1-\frac{x^2}{4(t+|x|)}\right) < t\left[e^x-\left(1+\frac{x}{t}\right)^t\right] < \frac{e^x x^2}{2}\cdot\frac{t}{t-|x|}\,, \tag{15}$$

valid for $x \neq 0$ and $t > |x|$. Letting t to approach infinity in (15), we obtain the next result

$$\lim_{t\to\infty}\left[t\cdot\left(e^x-\left(1+\frac{x}{t}\right)^t\right)\right] = \frac{e^x x^2}{2}. \tag{16}$$

Figure 5 illustrates estimate (15) for $t \in [10, 100]$.

4 Examples

4.1 Let us take $x = 1$. Choosing $\varepsilon = \frac{1}{2}$, the relation (12a) can be applied to those t which fulfil the condition $t \geq |x|/\varepsilon = 2$. Hence, for all $t \geq 2$, the following estimate holds

$$\frac{e}{2\times\frac{3}{2}}\left(1-\frac{1}{4\times\frac{3}{2}\times 2}\right)\cdot\frac{1}{t} \;<\; e-\left(1+\frac{1}{t}\right)^t \;<\; \frac{e}{2\times\frac{1}{2}}\cdot\frac{1}{t}$$

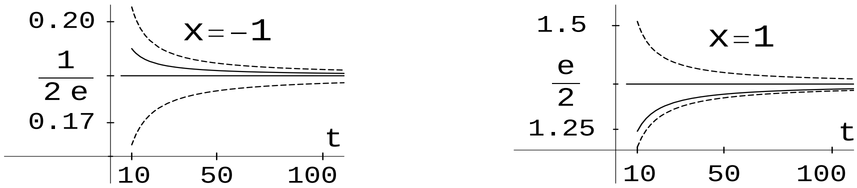

Figure 5: Bounds (15) for the convergence (16).

i.e.

$$\frac{2.7}{3}\frac{11}{12}\cdot\frac{1}{t} \;<\; e-\left(1+\frac{1}{t}\right)^{t} \;<\; 2.8\;\cdot\frac{1}{t}$$

or

$$\frac{0.8}{t} \;<\; e-\left(1+\frac{1}{t}\right)^{t} \;<\; \frac{2.8}{t}\,.$$

If we take $\varepsilon = 0.1$ in (12a), then, for $t \geq 10$, we obtain the relation

$$\frac{e}{2\times 1.1}\left(1-\frac{1}{4\times 1.1\times 10}\right)\cdot\frac{1}{t} \;<\; e-\left(1+\frac{1}{t}\right)^{t} \;<\; \frac{e}{2\times 0.9}\cdot\frac{1}{t}\,.$$

Thus, we have more accurate estimate

$$\frac{1.20}{t} \;<\; e-\left(1+\frac{1}{t}\right)^{t} \;<\; \frac{1.51}{t},$$

true for $t \geq 10$.

Taking $\varepsilon = 0.01$ and $t \geq 100$ in (12a), we obtain the estimate

$$\frac{e}{2\times 1.01}\left(1-\frac{1}{4\times 1.01\times 100}\right)\cdot\frac{1}{t} < e-\left(1+\frac{1}{t}\right)^{t} < \frac{e}{2\times 0.99}\cdot\frac{1}{t},$$

which amounts to an even more accurate relation

$$\frac{1.34}{t} \;<\; e-\left(1+\frac{1}{t}\right)^{t} \;<\; \frac{1.38}{t}, \tag{17}$$

valid for $t \geq 100$. For ε still closer to 0, we would obtain from (12a) further more accurate estimates, which are certainly true for larger values of t.

4.2 Setting $x = -1$, $\varepsilon = 0.01$ and $t \geq 100$ in (12a) we obtain

$$\frac{1}{e\cdot 2\cdot 1.01}\left(1-\frac{1}{4\cdot 1.01\cdot 100}\right)\cdot\frac{1}{t} \;<\; \frac{1}{e} - \left(1-\frac{1}{t}\right)^{t} \;<\; \frac{1}{e\cdot 2\cdot 0.99}\cdot\frac{1}{t}\,,$$

i.e.

$$\frac{0.181}{t} \quad < \quad \frac{1}{e} - \left(1 - \frac{1}{t}\right)^t \quad < \quad \frac{0.186}{t} \tag{18}$$

for $t \geq 100$.

4.3 To determine power $\alpha := \left(1 - 10^{-59.1}\right)^{10^{58.6}}$ we use (14), setting $h = -10^{-59.1}$ and $t = 10^{58.6}$. Since $ht = -10^{-0.5}$ and $h^2 t = 10^{-59.6}$ we estimate

$$\begin{aligned}
\alpha \quad &> \quad \left(1 - 10^{-59.6}\right) \cdot e^{-10^{-0.5}} \quad > \quad e^{-10^{-0.5}} - (10^{-0.6} e^{-10^{-0.5}}) \cdot 10^{-59} \\
&> \quad e^{-10^{-0.5}} - 0.2 \cdot 10^{-59} \quad > \quad e^{-10^{-0.5}} - 10^{-59}
\end{aligned}$$

and

$$\begin{aligned}
\alpha \quad &< \quad \left[1 - \frac{10^{-59.6}}{3}\left(1 - \frac{10^{-59.6}}{6}\right)\right] \cdot e^{-10^{-0.5}} \\
&< \quad \left[1 - 0.3 \cdot 10^{-59.6}\left(1 - 10^{-1}\right)\right] \cdot e^{-10^{-0.5}} \\
&= \quad \left(1 - 2.7 \cdot 10^{-60.6}\right) \cdot e^{-10^{-0.5}} \\
&= \quad e^{-10^{-0.5}} - \left(2.7 \cdot 10^{-0.6} \cdot e^{-10^{-0.5}}\right) \cdot 10^{-60} \\
&< \quad e^{-10^{-0.5}} - 0.4 \cdot 10^{-60} < e^{-10^{-0.5}} - 10^{-61}.
\end{aligned}$$

Hence,

$$\exp\left(\frac{-1}{\sqrt{10}}\right) - 10^{-59} \ < \ \alpha \ < \ \exp\left(\frac{-1}{\sqrt{10}}\right) - 10^{-61}$$

or numerically $\alpha = 0.728\,893\,414\,110\,024\,601\,973\ldots$, where all 21 decimal places are correct.

4.4 To compute power $\beta := \left(1 + 10^{-58.6}\right)^{10^{59.1}}$ we put $h = 10^{-58.6}$ and $t = 10^{59.1}$. Since $ht = 10^{0.5}$ and $h^2 t = 10^{-58.1}$ we are estimating, according to (14), as follows:

$$\begin{aligned}
\beta \quad &> \quad \left(1 - 10^{-58.1}\right) \cdot e^{10^{0.5}} > e^{10^{0.5}} - (10^{-0.1} e^{10^{0.5}}) 10^{-58} \\
&> \quad e^{10^{0.5}} - 20 \cdot 10^{-58} \quad > \quad e^{10^{0.5}} - 10^{-56}
\end{aligned}$$

and

$$\begin{aligned}
\beta \quad &< \quad \left[1 - \frac{10^{-58.1}}{3}\left(1 - \frac{10^{-58.1}}{6}\right)\right] \cdot e^{10^{0.5}} \\
&< \quad \left[1 - 0.3 \cdot 10^{-58.1}\left(1 - 10^{-1}\right)\right] \cdot e^{10^{0.5}} \\
&= \quad \left(1 - 2.7 \cdot 10^{-59.1}\right) \cdot e^{10^{0.5}} = e^{10^{0.5}} - \left(2.7 \cdot 10^{-0.1} \cdot e^{10^{0.5}}\right) \cdot 10^{-59} \\
&< \quad e^{10^{0.5}} - 50 \cdot 10^{-59} < e^{10^{0.5}} - 10^{-58},
\end{aligned}$$

thus

$$\exp\left(\sqrt{10}\right) - 10^{-56} < \beta < \exp\left(\sqrt{10}\right) - 10^{-58}$$

or numerically $\beta = 23.\,624\,342\,922\,017\,801\,092\ldots$, where all 18 decimal places are correct.

5 Remarks

5.1 Using a slightly different techniques as those that have been applied deriving (17), we can obtain an estimate, similar to (17). Namely, putting $\varepsilon = 0$ in (5) we get, for $\tau > 0$, the estimate

$$\exp\left(\tau - \frac{\tau^2}{2}\right) < 1 + \tau$$

or

$$1 - \frac{\tau}{2} < \ln(1+\tau)^{1/\tau}$$

i.e.

$$e - (1+\tau)^{1/\tau} < e - e^{1-\frac{\tau}{2}}. \tag{19}$$

To get an opposite inequality, we consider the function

$$F(\tau) := \tau - \frac{\tau^2}{2} + \frac{\tau^3}{3} - \ln(1+\tau),$$

having the derivative

$$F'(\tau) = \frac{\tau^3}{1+\tau} > 0$$

for $\tau > 0$. Hence, $F(\tau) > F(0) = 0$ for $\tau > 0$, that is

$$\tau - \frac{\tau^2}{2} + \frac{\tau^3}{3} > \ln(1+\tau)$$

or

$$1 - \frac{\tau}{2} + \frac{\tau^2}{3} > \ln(1+\tau)^{1/\tau},$$

i.e.

$$e - (1+\tau)^{1/\tau} > e - e^{1-\frac{\tau}{2}+\frac{\tau^2}{3}}$$

at any $\tau > 0$. Combining this relation with (19), we find that the estimate

$$e \cdot \left(1 - e^{-\left(\frac{\tau}{2} - \frac{\tau^2}{3}\right)}\right) < e - (1+\tau)^{1/\tau} < e \cdot \left(1 - e^{-\frac{\tau}{2}}\right) \tag{20}$$

holds for every $\tau > 0$.

For $\tau \in \left(0, \frac{3}{2}\right)$, the number

$$d(\tau) := \frac{\tau}{2} - \frac{\tau^2}{3} = \tau\left(\frac{1}{2} - \frac{\tau}{3}\right)$$

is lying on the interval $\left(0, \frac{3}{16}\right) \subset (0,1)$. But, the function $G\colon d \mapsto 1-e^{-d}-e^{-1}d$ strictly increases on interval $(0,1)$, due to its positive derivative. Therefore, for $d \in (0,1)$, we have $G(d) > G(0) = 0$, i.e. $1 - e^{-d} > d/e$. Hence, according to (9), we get

$$\frac{d}{e} < 1 - e^{-d} < d$$

for every $d \in (0,1)$. With this in mind, according to (20), we conclude with relation

$$e \cdot \frac{1}{e}\left(\frac{\tau}{2} - \frac{\tau^2}{3}\right) < e - (1+\tau)^{1/\tau} < e \cdot \frac{\tau}{2},$$

valid for every $\tau \in \left(0, \frac{3}{2}\right)$. Consequently, setting $\tau = \frac{1}{t}$, we obtain the estimate

$$\frac{1}{2t} - \frac{1}{3t^2} \;<\; e - \left(1 + \frac{1}{t}\right)^t \;<\; \frac{e}{2t}, \tag{21}$$

true for $t > 2/3$.

5.2 Inequalities (6) have been obtained already in **[2]**, but using an integral.

6 Questions

6.1 Prove or disprove the equality

$$\lim_{t\to\infty}\left\{t\left[\frac{e^x x^2}{2} - t\left(e^x - \left(1 + \frac{x}{t}\right)^t\right)\right]\right\} = \frac{e^x x^3}{24}(3x+8)$$

and find further "nested limits", together with suitable estimates.

6.2 How to estimate the norm $\|\left(1 + \frac{x}{n}\right)^n - e^x\|$ from below and from above for $n \in \mathbb{N}$ and $x \in A$, A being real or complex unital Banach algebra, possibly B^* algebra or only the field $\mathbb{C}$ or matrix algebra $\mathbb{C}_{n\times n}$?

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