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On zero free sets

Sobre los conjuntos libres de ceros

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Abstract

Let G be a finite abelian group and let $ZFS_s(G)$ and $\mu_s(G)$ be respectively, the set of zero free sets and the set of minimal zero sets of G. The Olson constant, $O(G)$, is $1 + max\{|S| : S \in ZFS_s(G)\}$ and the strong Davenport constant, $SD(G)$, is $max\{|S| : S \in \mu_s(G)\}$. We show that there exists a very large class of groups G for which $SD(G) = O(G)$. Then we give new values of $SD(G)$.
Key words and phrases: zero sets, minimal zero sets, Davenport constant, Olson constant, strong Davenport constant.

Resumen

Sea G un grupo abeliano finito. Sean $ZFS_s(G)$ y $\mu_s(G)$ respectivamente, el conjunto de los conjuntos libres de ceros y el conjunto de los conjuntos minimales de suma cero de G. La constante de Olson, $O(G)$, es $1 + max\{|S| : S \in ZFS_s(G)\}$ y la constante fuerte de Davenport, $SD(G)$, es $max\{|S| : S \in \mu_s(G)\}$. Mostramos que existe una clase bastante grande de grupos G para los cuales se tiene $SD(G) = O(G)$. En consecuencia es posible establecer nuevos valores para $SD(G)$.
Palabras y frases clave: conjuntos de suma cero, conjuntos minimales de suma cero, constante de Davenport, constante de Olson, constante fuerte de Davenport.

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1 Introduction

Let G be a finite abelian group. Then $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, $1 < n_1 | \cdots | n_r$, where $n_r = \exp(G)$ is the *exponent* of G and r is the *rank* of G. Let $M(G) = \sum_{i=1}^{r}(n_i - 1) + 1$. In this paper, we denote by p a prime number.

Definition 1. Let G be a finite abelian group. The *Davenport constant* $D(G)$ is the least positive integer d such that every sequence of length d in G contains a non-empty subsequence with zero-sum.

It is well known that $M(G) \leq D(G) \leq |G|$ [12]. Moreover if G is the cyclic group of order n then $D(G) = n$; for noncyclic groups we have:

Theorem 1 ([19])**.** *Let G be a finite noncyclic group of order n then $D(G) \leq \lceil \frac{n+1}{2} \rceil$, where $\lceil x \rceil$ denotes the smallest integer not less than x.*

The following lemma is used:

Lemma 1 ([18])**.** *Let $G = \mathbb{Z}_{p^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_k}}$ be a p-group. Then we have $D(G) = M(G)$.*

A zero sequence in G without zero subsequences is called a minimal zero sequence. Let $ZFS(G)$ be the set of zero free sequences in G. Let $\mu(G)$ be the set of all minimal zero sequences. The number of distinct elements of a sequence S is denoted by $C(S)$ and its length by $|S|$.

It is clear that

$$D(G) = max\{|S| : S \in \mu(G)\} = 1 + max\{|S| : S \in ZFS(G)\}.$$

Let $\sigma(S)$ be the sum of elements of S and set

$$\sum S = \{\sigma(T) : T \text{ is a non-empty subsequence of } S\}.$$

Theorem 2 ([12])**.** *Let G be a finite abelian group. Then for every zero free sequence S in G with $|S| = D(G) - 1$ we have $\sum S \cup \{0\} = G$.*

A set S is zero free if it contains no zero subsets. Let $ZFS_s(G)$ be the set of zero free sets in G. A zero-sum set in G without zero-sum subsets is called minimal zero set. Let $\mu_s(G)$ be the set of minimal zero sets.

Definition 2 ([5],[6],[12],[21])**.** Let G be a finite abelian group. The *Olson constant*, denoted $O(G)$, is the least positive integer d such that every subset $A \subseteq G$, with $|A| = d$ contains a non-empty subset with zero-sum.

It is clear that $O(G) \leq D(G)$ and moreover we have:

$$O(G) = 1 + max\{|S| : S \in ZFS_s(G)\}.$$

Related to the Olson constant are the works of Erdős and Heilbronn in [8], Szemerédi in [22], Erdős in [9], Olson in [16, 17], Hamidoune and Zémor in [15] Dias da Silva and Hamidoune in [7], where the existence conditions of sets, in an abelian finite group G, with zero-sum are established. For example from Hamidoune and Zémor works we can deduce that $O(\mathbb{Z}_p) \leq \lceil\sqrt{2p}+5\ln(p)\rceil$ and for an arbitrary abelian group G, they proved that $O(G) \leq \lceil\sqrt{2|G|} + \varepsilon(|G|)\rceil$ where $\varepsilon(n) = \mathcal{O}(\sqrt[3]{n}\ln n)$. Moreover from Dias da Silva and Hamidoune results we have $O(\mathbb{Z}_p) \leq \lfloor\sqrt{4p-7}\rfloor$.

In what follows we denote by $v_g(S)$ the multiplicity of g in a given sequence S. The following result was proved by Bovey, Erdős and Niven.

Theorem 3 ([3]). *Let S be a zero free sequence in $\mathbb{Z}_n$ with $|S| \geq \frac{n+1}{2}$ and $n \geq 3$. Then there exists some $g \in \mathbb{Z}_n$ such that $v_g(S) \geq 2|S| - n + 1$.*

Corollary 1. $O(\mathbb{Z}_n) \leq \lceil \frac{n+1}{2} \rceil$ *for* $n \geq 3$.

Proof. Directly from Theorem 3.

□

However, the following result due to Olson improves Corollary 1 for $n \geq 34$.

Theorem 4 ([16, Corollary 3.2.1]). *Let G be a finite abelian group. Then $O(G) \leq 3\sqrt{|G|}$.*

Definition 3 ([4]). Let G be a finite abelian group. The strong Davenport constant, denoted $SD(G)$, is defined by

$$SD(G) = max\{C(S) : S \in \mu(G)\}.$$

The next result shows that $SD(G)$ is witnessed by minimal zero sets.

Theorem 5 ([4]). *Let G be a finite abelian group of order $n \geq 3$. Then there exists a minimal zero sequence S such that $C(S) = |S| = SD(G)$.*

Remark 1. For some finite abelian group G of order $n \geq 3$, there exists $S \in \mu(G)$ with $|S| = SD(G)$ and $S \notin \mu_s(G)$. Let S be the sequence in $\mathbb{Z}_p$ of length $d = SD(\mathbb{Z}_p)$ consisting of $d-1$ instances of the elements 1 and then the element $p - d - 1$. It is clear that $S \in \mu(\mathbb{Z}_p)$ with length $SD(\mathbb{Z}_p)$, but $S \notin \mu_s(\mathbb{Z}_p)$.

We have the following corollary:

Corollary 2. *Let G be a finite abelian group of order $n \geq 3$. Then we have:*

$$SD(G) = max\{|S| : S \in \mu_s(G)\}.$$

Proof. Directly from Theorem 5.

□

The Olson constant is defined in [5] and denoted by $O(G)$ in honor to the Olson works. In [6], [12] and [21] it is denoted by $SD(G)$, $D_s(G)$ and $Ol(G)$ respectively. In [1] Baginski noted that the constants $O(G)$ and $SD(G)$ were different. He shows that $SD(G) \leq O(G) \leq SD(G) + 1$. For example $SD(\mathbb{Z}_3) = O(\mathbb{Z}_3) = 2$, however $O(\mathbb{Z}_4) = 3$, $SD(\mathbb{Z}_4) = 2$ and $O(\mathbb{Z}_2) = 2$, $SD(\mathbb{Z}_2) = 1$. Moreover Baginski poses the following problem:

Problem 1 ([1])**.** Determine for which finite abelian groups G of order ≥ 3 one has $O(G) = SD(G)$.

The main goal of this paper is to show that there exists a very large class of groups which have $SD(G) = O(G)$.

Remark 2. The controversy between the constants $O(G)$ and $SD(G)$ is for the construction of the minimal zero sets. The construction of the minimal zero sequences is clear. If S is a zero free sequence then $S \circ -\sigma(S) \in \mu(G)$ where $\circ$ denotes the sequence concatenation operation. In the construction of minimal zero sets from a zero free set S, we must check whether $S \circ -\sigma(S) \in \mu(G)$ is still a set. For example $\{1, 2\} \in ZFS_s(\mathbb{Z}_5)$ and $1, 2, 2 \notin \mu_s(\mathbb{Z}_5)$.

Let G be a finite abelian group. The minimal zero sequences S with $|S| = SD(G)$ are studied by Baginski in [1], where they are called Freeze sequences. Nice properties of the groups and Freeze sequences are given when $SD(G) = O(G)$.

Problem 2. Many authors have studied the zero free sequences structure, in an abelian finite group G, with length $D(G) - 1$. See for example: [2], [10], [11], [12], [13], [14] and [20]. However there are few results on zero free sets with cardinality $O(G) - 1$. A natural question is to ask about the structure of $S \in ZFS_s(G)$ with maximal cardinality in groups G such that $SD(G) = O(G) = D(G) = M(G)$ or $O(G) = D(G) = M(G)$.

This paper contains two main sections. In Section 1 a family of groups G with $O(G) = SD(G)$ is given. In Section 2, some reflections on the properties of $S \in ZFS_s(\mathbb{Z}_p^s)$ of maximal cardinality are pointed out.

2 Baginski Problem

The following proposition is due to Baginski. In order to be self-contained we give its proof.

Proposition 1 ([1]). *Let G be an abelian group. Then we have:*

$$SD(G) \leq O(G) \leq SD(G) + 1.$$

Proof. Let $A \subseteq G$ be with $|A| = O(G) - 1$ and $A \in ZFS_s(G)$. If $|G| \geq 2$ then $O(G) \geq 2$ and then $A \neq \emptyset$. So that the sequence $A \circ -\sigma(A) \in \mu(G)$ and it contains at least $|A|$ different elements. Therefore $O(G) - 1 \leq SD(G)$. Moreover, since each minimal zero sequences contains at most $O(G)$ different elements, we have $SD(G) \leq O(G)$. □

Remark 3. Since $SD(G) \leq O(G)$ then the upper bounds on $O(G)$ are also valid for $SD(G)$.

We use the following theorem and its corollary:

Theorem 6 ([12]). *Let $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r} \oplus \mathbb{Z}_n^{s+1}$ with $r \geq 0$, $s \geq 0$, $1 < n_1 | \cdots | n_r | n$ and $n_r \neq n$. If $r + \frac{s}{2} \geq n$, then there exists a minimal zero set S in G such that $|S| = M(G)$.*

Corollary 3 ([12]). *Let $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r} \oplus \mathbb{Z}_n^{s+1}$ with $r \geq 0$, $s \geq 0$, $1 < n_1 | \cdots | n_r | n$ and $n_r \neq n$. If G is a p-group and $r + \frac{s}{2} \geq n$, then $O(G) = M(G) = D(G)$.*

The following theorem gives a very large class of groups which have $SD(G) = O(G)$.

Theorem 7. *Let $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r} \oplus \mathbb{Z}_n^{s+1}$ with $r \geq 0$, $s \geq 0$, $1 < n_1 | \cdots | n_r | n$ and $n_r \neq n$. If G is a p-group and $r + \frac{s}{2} \geq n$, then $SD(G) = O(G) = M(G) = D(G)$.*

Proof. By Theorem 6 we have $M(G) \leq SD(G)$. By Proposition 1 and Corollary 3 we have $M(G) \leq SD(G) \leq O(G) = M(G) = D(G)$. Therefore $SD(G) = O(G) = M(G) = D(G)$.

□

Corollary 4. *Let $\mathbb{Z}_p^s$ be an elementary p-group with $s \geq 2p + 1$. Then $SD(\mathbb{Z}_p^s) = O(\mathbb{Z}_p^s) = D(\mathbb{Z}_p^s) = s(p - 1) + 1$.*

Proof. Directly from Theorem 7.

□

Problem 3. Does it exist $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r} \oplus \mathbb{Z}_n$ with $r \geq 0$, $1 < n_1 | \cdots | n_r | n$, $n_r \neq n$ and $r \geq n$, different from the p-groups, such that $D(G) = M(G)$? In the affirmative case we can also conclude, as in Theorem 7, that $SD(G) = O(G) = M(G) = D(G)$.

3 Zero free sets in $\mathbb{Z}_p^s$

Elementary p-groups $\mathbb{Z}_p^s$ are vector spaces of dimension s over the finite field $\mathbb{Z}_p$. In this section we deal with the property of set $S \in ZFS_s(\mathbb{Z}_p^s)$ with $|S| = s(p-1)$. In particular when $p = 2, 3$.

We use the following proposition:

Proposition 2. *For any zero free set in $\mathbb{Z}_p^s$ with $|S| = s(p-1)$, we have $\sum S \cup \{0\} = Z_p^s$. Moreover $\{e_1, \ldots, e_s\} \subseteq S$, where $e_1, \ldots, e_s$ is a basis of vector space $\mathbb{Z}_p^s$.*

Proof. Directly from Theorem 2 and the fact that $D(\mathbb{Z}_p^s) = s(p-1) + 1$. □

We have also the theorem:

Theorem 8 ([12])**.** *Let S be a zero free sequence in $\mathbb{Z}_p^s$ be with $|S| = D(\mathbb{Z}_p^s) - 1 = s(p-1)$. Then each two distinct elements in S are linearly independent.*

Corollary 5. *Let S be a zero free set in $\mathbb{Z}_p^s$ with $|S| = s(p-1)$. Then each two elements in S are linearly independent.*

Gao and Geroldinger also give the following proposition:

Proposition 3 ([12])**.** *Let S be a sequence in $\mathbb{Z}_2^s$ with $s \geq 1$. Then S is a zero free sequence if and only if $S = \{e_1, \ldots, e_k\}$ where $e_1, \ldots, e_k$ are linearly independent over $\mathbb{Z}_2$*

Corollary 6. *The zero free sets S in $\mathbb{Z}_2^s$ are of the form $\{e_1, \ldots, e_k\}$ where $e_1, \ldots, e_k$ are linearly independent over $\mathbb{Z}_2$.*

We use the following theorem:

Theorem 9 ([1, 6])**.** $O(\mathbb{Z}_2^s) = s + 1$ *for* $s \geq 1$.

The following result is cited in [1]. Here we give a proof.

Corollary 7. $SD(\mathbb{Z}_2^s) = O(\mathbb{Z}_2^s) = D(\mathbb{Z}_2^s) = s + 1$ *for* $s \geq 2$.

Proof. By Proposition 1 and Theorem 9, we have $SD(\mathbb{Z}_2^s) \leq O(\mathbb{Z}_2^s) = D(\mathbb{Z}_2^s)$. The set $S = \{e_1, e_2, \dots, e_s, e_1 + \cdots + e_s\}$, where $\{e_i\}_{i=1}^s$ is a basis of the vector space $\mathbb{Z}_2^s$, is a minimal zero set with $|S| = D(\mathbb{Z}_2^s) = s+1$. Therefore by Corollary 2 we have $D(\mathbb{Z}_2^s) \leq SD(\mathbb{Z}_2^s)$. Hence $SD(\mathbb{Z}_2^s) = O(\mathbb{Z}_2^s) = D(\mathbb{Z}_2^s) = s+1$ for $s \geq 2$. Note that for $s \geq 5$ the result follows from Corollary 4. □

Problem 4. Describe the structure of zero free sets S in $\mathbb{Z}_p^s$ with $|S| = s(p-1)$ and $s \geq 2p+1$.

We have the following theorem:

Theorem 10 ([6, 21])**.** *$O(\mathbb{Z}_3^s) = D(\mathbb{Z}_3^s) = 2s+1$ for $s \geq 3$.*

In what follows we give some zero free sets S with $|S| = O(\mathbb{Z}_3^s) - 1 = 2s$. Moreover two lemmas are given in order to derive zero-sum sets from the other one. We will denote by $\{e_i\}_{i=1}^s$ the canonical basis of $\mathbb{Z}_p^s$, i.e., e_i is the s-tuple with entry 1 at position i and 0 elsewhere.

Example 1. Let $S = \{e_1, e_2, e_3, e_4, e_1+e_2, e_1+e_3, e_1+e_4, e_1+e_2+e_3+e_4\} \in ZFS_s(Z_3^4)$. This set contains vectors with only coordinates equal 0 or 1.

In [21] Subocz gives the following zero free sets.

Example 2. Let $S = \{e_i : 1 \leq i \leq s\} \cup \{e_1 + e_i, 2 \leq i \leq s\} \cup \{2e_1 + e_2 + e_3\} \in ZFS_s(\mathbb{Z}_3^s)$, $s \geq 3$ and $|S| = 2s$.

Example 3. Let (ij) denote the vector $e_i + e_j$ in Z_3^8, $1 \leq i, j \leq 8$. Let $S = \{(12), (13), (14), (15), (16), (17), (18), (23), (24), (25), (26), (37), (47), (58), (68), (78)\} \subseteq Z_3^8$. Then $|S| = 16$ and S is a zero free sets.

In this set, each vector contains exactly two coordinates equal to 1 and the remaining coordinates are equal to 0. Each 8 elements in S constitutes a basis of Z_3^8. Moreover S can be set in the following form:

Set $e_1^* = (12) = (1,1,0,0,0,0,0,0)$, $e_2^* = (13) = (1,0,1,0,0,0,0,0)$, $e_3^* = (14) = (1,0,0,1,0,0,0,0)$, $e_4^* = (15) = (1,0,0,0,1,0,0,0)$, $e_5^* = (16) = (1,0,0,0,0,1,0,0)$, $e_6^* = (17) = (1,0,0,0,0,0,1,0)$, $e_7^* = (18) = (1,0,0,0,0,0,0,1)$, $e_8^* = (23) = (0,1,1,0,0,0,0,0)$, the basis chosen for Z_3^8. Then for the other elements in S we have:

$f_9 = (24) = 2e_2^* + e_3^* + e_8^*$, $f_{10} = (25) = 2e_2^* + e_4^* + e_8^*$, $f_{11} = (26) = 2e_2^* + e_5^* + e_8^*$, $f_{12} = (37) = 2e_1^* + e_6^* + e_8^*$, $f_{13} = (47) = 2e_1^* + 2e_2^* + e_3^* + e_6^* + e_8^*$, $f_{14} = (58) = 2e_1^* + 2e_2^* + e_4^* + e_7^* + e_8^*$,$f_{15} = (68) = 2e_1^* + 2e_2^* + e_5^* + e_7^* + e_8^*$. $f_{16} = (78) = 2e_1^* + 2e_2^* + e_6^* + e_7^* + e_8^*$.

The following two lemmas can be used to build inductively zero free sets:

Lemma 2 ([21])**.** *Let S be a zero free set in $\mathbb{Z}_3^s$ with $s \geq 3$ and $|S| = 2s$. Then $S \cup \{e_{s+1}, e_1 + e_{s+1}\}$ is a zero free set in $\mathbb{Z}_3^{s+1}$.*

Lemma 3 ([21])**.** *Let S be a zero free set in $\mathbb{Z}_3^s$ with $s \geq 3$ and $|S| = 2s$. Suppose that each vector in S has two coordinates equal to 1 and all other coordinates equal to 0. Then $S \cup \{e_1 + e_{s+1}, e_2 + e_{s+1}\}$ is a zero free set in $\mathbb{Z}_3^{s+1}$.*

Finally the following conjecture due to Subocz remains open.

Conjecture 1 ([21])**.** *Let G be a finite abelian group of order n, then $O(G) \leq O(\mathbb{Z}_n)$.*

The Conjecture 1, appears analogous to the following conjecture due to Ponomarenko.

Conjecture 2 ([10])**.** *Let G and H be finite abelian groups of the same order and $rank(G) \leq rank(H)$. Then $|\mu(G)| \geq |\mu(H)|$.*

Moreover Ponomarenko in personal comunication, gives the following generalization of Conjecture 1:

Conjecture 3. *Let G and H be finite abelian groups of the same order and $rank(G) \leq rank(H)$. Then $O(G) \leq O(H)$.*

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