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Second Method of Lyapunov and Existence of Periodic Solutions of Linear Impulsive Differential-Difference Equations

Segundo Método de Lyapunov y Existencia de Soluciones Periódicas de Ecuaciones Diferenciales en Diferencias con Impulsos

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Abstract

By means of piecewise continuous auxiliary functions which are analogues of the classical Lyapunov's functions, sufficient conditions are obtained for the existence of periodic solutions of a linear system of impulsive differential-difference equations with impulse effects at fixed moments.
Key words and phrases: impulsive differential-difference equation, periodic solution, Lyapunov's function.

Resumen

Por medio de funciones auxiliares continuas por partes que son análogas a las clásicas funciones de Lyapunov, se obtienen condiciones suficientes para la existencia de soluciones periódicas de un sistema lineal de ecuaciones diferenciales en diferencias con efectos de impulso en momentos fijos.
Palabras y frases clave: ecuación diferencial en diferencias con impulsos, solución periódica, función de Lyapunov.

1 Introduction

The impulsive differential-difference equations describe processes with aftereffect and state changing by jumps. These equations are an adequate mathematical apparatus for simulation in physics, chemistry, biology, population dynamics, biotechnologies, control theory, industrial robotics, economics, etc.

In spite of the great possibilities for application, the theory of the impulsive differential-difference equations is developing rather slowly [1], [2]. The investigations of the impulsive ordinary differential equations mark their beginning with the work of Mil'man and Myshkis [7]. The problem of existence of periodic solutions has been studied in many papers and monographs [3], [4], [5], [8].

In the present paper by means of piecewise continuous auxiliary functions which are analogues of the classical Lyapunov's functions, sufficient conditions are obtained for the existence of periodic solutions of a linear system of impulsive differential-difference equations. The impulses take place at fixed moments. The investigations are carried out by using minimal subsets of a suitable space of piecewise continuous functions, by the elements of which the derivatives of the piecewise continuous auxiliary functions are estimated [6].

2 Statement of the problem. Preliminary notes

Let $\mathbb{Z}$ be the set of all integers; $h > 0$; $\mathbb{R}^n$ be the n-dimensional euclidean space with elements $x = \mathrm{col}(x_1, \dots, x_n)$ and norm $|x| = (\sum_{k=1}^n x_k^2)^{1/2}$; $\mathbb{R}_+ = [0, \infty)$.

Consider the linear system of impulsive differential-difference equations

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-h), \quad t \neq t_i, \tag{1}$$

$$\Delta x(t_i) = x(t_i + 0) - x(t_i - 0) = C_i x(t_i), \tag{2}$$

where $x \in \mathbb{R}^n$, $A(\cdot)$ and $B(\cdot)$ are $(n \times n)$-matrix functions, C_i $(i \in \mathbb{Z})$ are matrices of type $(n \times n)$; $t_{i+1} > t_i$ $(i \in \mathbb{Z})$, $\lim_{i \to \pm\infty} t_i = \pm\infty$.

Let $\varphi_0 : [-h, 0] \to \mathbb{R}^n$ be a piecewise continuous function in $(-h, 0)$ with points of discontinuity of the first kind $t_i \in (-h, 0)$ at which it is continuous from the left.

Denote by $x(t) = x(t; 0, \varphi_0)$ the solution of system (1), (2) which satisfies the initial condition

$$x(t) = \varphi_0(t) \quad , \quad t \in (-h, 0) \tag{3}$$

The solution $x(t) = x(t; 0, \varphi_0)$ of problem (1), (2), (3) is a piecewise continuous function with points of discontinuity of the first kind t_i $(i \in \mathbb{Z})$, at which it is continuous from the left, i.e. at the moments of impulse effect t_i the following relations are valid

$$\begin{aligned} x(t_i - 0) &= x(t_i), \quad i \in \mathbb{Z} \\ x(t_i + 0) &= x(t_i) + C_i x(t_i), \quad t_i \notin (-h, 0) \\ \varphi_0(t_i + 0) &= \varphi_0(t_i) + C_i \varphi_0(t_i), \quad t_i \in (-h, 0). \end{aligned}$$

The function $x(t) = x(t; 0, \varphi_0)$ for $t \neq t_i$ $(i \in \mathbb{Z})$ satisfies equation (1) and equality (3), and for $t = t_i$ $(i \in \mathbb{Z})$ condition (2).

Introduce the following notation:

$|A| = \sup\{|Ax|/|x| : x \in \mathbb{R}^n \setminus 0\}$ is the norm of the $(n \times n)$-matrix A;

$PC[[0,T], \mathbb{R}^n] = \{x : [0,T] \to \mathbb{R}^n : x$ is piecewise continuous with points of discontinuity of the first kind $t_i \in (0,T)$ and $x(t_i - 0) = x(t_i)$, $T \geq h > 0\}$;

$\mathcal{V}_0 = \{V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+ : V(t,x)$ is continuous for $t \in \mathbb{R}$, $t \neq t_i$ $(i \in \mathbb{Z})$, $x \in \mathbb{R}^n$; periodic with respect to t with period T; for $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$ there exist the finite limits

$$V(t_i, x) = V(t_i - 0, x) = \lim{}_{t < t_i}^{t \to t_i} V(t,x)$$

and

$$V(t_i + 0, x) = \lim{}_{t > t_i}^{t \to t_i} V(t,x);$$

$$\begin{aligned} \Omega_0 = \{x \in PC[[0,T], \mathbb{R}^n] : V(s, x(s)) \leq L(V(t, x(t))),\ t - h \leq s \leq t, \\ t \in [0,T],\ V \in \mathcal{V}_0\}, \end{aligned}$$

where $L : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous in $\mathbb{R}_+$, nondecreasing and $L(u) > u$ for $u > 0$.

Let $V \in \mathcal{V}_0$, $t \in \mathbb{R}$, $t \neq t_i$ $(i \in \mathbb{Z})$ and $x \in PC[\mathbb{R}, \mathbb{R}^n]$.

Introduce the function

$$\dot{V}(t, x(t)) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x(t), x(t-h)).$$

We shall say that conditions (H) are satisfied if the following conditions hold:

(H1) The matrices $A(t)$ and $B(t)$ are of type $(n \times n)$, defined for $t \in \mathbb{R}$, continuous and T-periodic $(T \geq h > 0)$.

(H2) The matrices C_i $(i \in \mathbb{Z})$ are of type $(n \times n)$ with nonnegative entries.

(H3) There exists a positive integer p such that :
$t_{i+p} = t_i + T, \quad C_{i+p} = C_i$ for $i \in \mathbb{Z}$.

Remark 1. Without loss of generality we shall assume that

$$0 < t_1 < t_2 < \cdots < t_p < T.$$

3 Main comparison theorem

Theorem 1. *Let the following conditions hold:*

1. *Conditions* (H) *are met.*
2. $g \in PC[\mathbb{R} \times \mathbb{R}_+, \mathbb{R}_+]$ *and* $g(t,0) = 0$ *for* $t \in \mathbb{R}$.
3. $B_i \in C[\mathbb{R}_+, \mathbb{R}_+]$ *and* $B_i = 0$, $i = 1, \ldots, p$.
4. *There exists a solution* $u(t)$ *of the problem*

$$\begin{cases} \dot{u} = g(t,u), \quad t \neq t_i, \\ u(0) = u_0 > 0, \\ \Delta u(t_i) = B_i(u(t_i)), \ i = 1, \ldots, p. \end{cases} \tag{4}$$

which is defined in the interval $[0,T]$.

5. *The function* $V \in \mathcal{V}_0$ *is such that* $V(0, \varphi_0(0)) \leq u_0$ *and the inequalities*

$$\begin{aligned} \dot{V}(t, x(t)) =\leq g(t, V(t, x(t))), \quad t \neq t_i, \ i = 1, \ldots, p, \\ V(t_i + 0, x(t_i)) + C_i x(t_i)) \leq V(t_i, x(t_i)), \quad i = 1, \ldots, p \end{aligned} \tag{5}$$

are valid for $t \in [0,T]$, $x \in \Omega_0$.

Then

$$V(t, x(t; 0, \varphi_0)) \leq u(t), \quad t \in [0,T]. \tag{6}$$

Proof: The solution $u(t)$ of problem (4) defined by condition 4 of Theorem 1 satisfies the equality

$$u(t) = \begin{cases} u_0(t; 0, u_0^+), & 0 < t \leq t_1, \\ u_1(t; t_1, u_1^+), & t_1 < t \leq t_2, \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ u_i(t; t_i, u_i^+), & t_i < t \leq t_{i+1}, \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ u_p(t; t_p, u_p^+), & t_p < t \leq T, \end{cases}$$

where $u_i(t; t_i, u_i^+)$ is the solution of the equation without impulses $\dot{u} = g(t, u)$ in the interval (t_i, t_{i+1}), $i = 1, \ldots, p$, for which $u_i^+ = u_{i-1}(t_i; t_{i-1}, u_{i-1}^+)$, $i = 1, \ldots, p$, and $u_i(t; 0, u_0^+)$ is the solution of $\dot{u} = g(t, u)$ in the interval $[0, t_1]$, $u_0^+ = u_0$.

Let $t \in [0, t_1]$. Then from the respective comparison theorem for the continuous case [6] it follows that

$$V(t, x(t; 0, \varphi_0)) \leq u(t),$$

i.e. inequality (6) is valid for $t \in [0, t_1]$.

Suppose that (6) is satisfied for $t \in (t_{i-1}, t_i] \cap [0, T]$, $i > 1$. Then using (5) we obtain

$$\begin{aligned} V(t_i + 0, x(t_i + 0; 0, \varphi_0)) &\leq V(t_i, x(t_i; 0, \varphi_0)) \\ &\leq u(t_i) = u_{i-1}(t_i; t_{i-1}, u_{i-1}^+) = u_i^+. \end{aligned}$$

We again apply (6) for $t \in (t_{i-1}, t_i] \cap [0, T]$ and obtain

$$V(t, x(t; 0, \varphi_0)) \leq u_i(t; t_i, u_i^+) = u(t),$$

i.e. inequality (6) is valid for $t \in (t_{i-1}, t_i] \cap [0, T]$.

The proof is completed by induction. □

4 Main results

Theorem 2. *Let the following conditions hold:*

1. *Conditions* (H) *are satisfied.*
2. *There exists a continuous real $(n \times n)$-matrix function $D(t)$, $t \in \mathbb{R}$, which is T-periodic, symmetric, positively definite, differentiable for $t \neq t_i$ and such that*

$$x^T [A^T(t) D(t) + D(t) A(t) + \dot{D}(t)] x \leq -a(t) |x|^2, \; x \in \mathbb{R}^n, \; t \neq t_i, \quad (7)$$

$$x^T [C_i^T D(t_i) + D(t_i) C_i + C_i^T D(t_i) C_i] x \leq 0, \; i \in \mathbb{Z}, \quad (8)$$

   where $a(t) > 0$ is a continuous and T-periodic function.

3. *There exists a continuous, T-periodic function $\wp : \mathbb{R} \to \mathbb{R}$ such that*

$$\int_0^T \wp(s)\, ds \geq 0, \quad (9)$$

$$b(t) = a(t) - \max\{\alpha(t)\wp(t), \beta(t)\wp(t)\} \geq 0, \tag{10}$$

$$\frac{2\rho\beta^{\frac{1}{2}}(t)}{\alpha^{\frac{1}{2}}(t-h)}|D(t)B(t)| \leq b(t) \tag{11}$$

where $\alpha(t)$ and $\beta(t)$ are respectively the smallest and the greatest eigenvalues of $D(t)$ and

$$\rho = \sup_{t\in[0,T]} \exp(\frac{1}{2}\int_0^t \wp(s)\,ds).$$

Then system (1) has a T-periodic solution ($T \geq h \geq 0$).

Proof: Define the function $V(t,x) = x^T D(t)x$. From the fact that $D(t)$ is real, symmetric and positively definite it follows that for $x \in \mathbb{R}^n$, $x \neq 0$ the following inequalities are valid

$$\alpha(t)|x|^2 \leq x^T D(t)x \leq \beta(t)|x|^2. \tag{12}$$

It is easily verified that $V \in \mathcal{V}_0$.

Define the function $L(u) = \rho^2 u$. Then the set Ω_0 is defined by the equality

$$\begin{aligned} \Omega_0 \quad = \quad & \{x \in PC[[0,T],\mathbb{R}^n] : x^T(s)D(s)x(s) \leq \rho^2 x^T(t)D(t)x(t), \\ & t-h \leq s \leq t,\ t \in [0,T]\}. \end{aligned}$$

For $t \in [0,T]$ and $x \in \Omega_0$ the following inequalities are valid

$$\begin{aligned} \alpha(t-h)|x(t-h)|^2 &\leq x^T(t-h)D(t-h)x(t-h) \\ &\leq \rho^2 x^T(tx^T(t)D(t)x(t) \leq \rho^2\beta(t)|x(t)|^2. \end{aligned}$$

From the above inequalities there follows the estimate

$$|x(t-h)|^2 \leq \frac{\rho\beta^{\frac{1}{2}}(t)}{\alpha^{\frac{1}{2}}(t-h)}|x(t)|,\ t \in [0,T],\ x \in \Omega_0. \tag{13}$$

We estimate $\dot{V}(t,x(t))$ for $t \in [0,T]$, $t \neq t_i$ and $x \in \Omega_0$. From (7), (10), (11) and (13) we obtain

$$\begin{aligned} \dot{V}(t,x(t)) &\leq -a(t)|x(t)|^2 + 2|D(t)B(t)||x(t)||x(t-h)| \\ &\leq -\left[a(t) - \frac{2\rho\beta^{\frac{1}{2}}(t)}{\alpha^{\frac{1}{2}}(t-h)}\right]|x(t)|^2 \leq -[a(t)-b(t)]|x(t)|^2 \\ &\leq -\wp(t)V(t,x(t)) \end{aligned}$$

Let $t = t_i$. Using (8) we obtain

$$\begin{aligned} V(t_i + 0, x + C_i x) &= (x^T + x^T C_i^T) D(t_i)(x + C_i x) \\ &= x^T D(t_i) x + x^T [C_i^T D(t_i) + D(t_i) C_i + C_i^T D(t_i) C_i] \\ &\leq V(t_i, x), \quad x \in \mathbb{R}^n. \end{aligned}$$

Consider the equation without impulses

$$\dot{u} = -\wp(t) u \tag{14}$$

(i.e. $\Delta u(t_i) = 0$, $i = 1, \dots, p$). The solution of equation (14) which satisfies the initial condition $u(0) = u_0 \geq V(0, \varphi_0(0)) > 0$ is defined by the equality

$$u(t) = u_0 e^{-\int_0^t \wp(s)\, ds}, \quad t \in [0, T].$$

Then the conditions of Theorem 1 are satisfied, hence

$$V(t, x(t; 0, \varphi_0) \leq u(t), \quad t \in [0, T]. \tag{15}$$

Denote by $\mathcal{J}^+(0, \varphi_0)$ the maximal interval of type $[0, \omega)$ in which the solution $x(t; 0, \varphi_0)$ of problem (1), (2), (3) is defined. We shall show that the following inclusion is valid

$$[0, T] \subset \mathcal{J}^+(0, \varphi_0).$$

Suppose that this is not true, i.e. there exists $\sigma \in (0, T]$ such that

$$\lim_{\tau \to \sigma} |x(\tau; 0, \varphi_0)| = \infty.$$

Then from inequalities (12) and (15) it follows that $\lim_{\tau \to \sigma} u(\tau) = \infty$ which contradicts the condition that $u(t)$ is defined for $t \in [0, T]$. Hence $[0, T] \subset \mathcal{J}^+(0, \varphi_0)$.

Consider the set

$$S = \{x \in PC[[t - h, t], \mathbb{R}^n] : V(0, x(t)) \leq u_0, \ t \in \mathbb{R}\}.$$

The function $\varphi_0 \in S$. From inequalities (15), (9) and the T-periodicity of the function $V(t, x) = x^T D(t) x$ it follows that

$$V(0, x(T; 0, \varphi_0)) = V(T, x(T; 0, \varphi_0)) \leq u(t) \leq u_0.$$

From the above inequalities it follows that the operator $Q : \varphi_0 \to x(t; 0, \varphi_0)$, $T - h \leq t \leq T$ maps the set S into itself. From conditions (2) and (3) of Theorem 2 it follows that S is a non-empty, closed, bounded and convex set in $PC[[t - h, t], \mathbb{R}^n]$, $t \in \mathbb{R}$. Hence the operator $Q : S \to S$ has a fixed point in S. □

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