Divulgaciones Matemáticas v. 6, No. 2 (1998), 149–164

Aggregation on a Nonlinear Parabolic Functional Differential Equation †

Agregación en una Ecuación Diferencial Funcional No Lineal

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Abstract

In this paper we study the equation

$$u_t = \Delta[\varphi(u(x,[t/\tau]\tau))u(x,t)]\,,\ x\in\Omega\,,t>0,$$

with homogeneous Neumann boundary conditions in a bounded domain in $\mathbb{R}^n$. We show existence and uniqueness for the initial value problem, and prove some results that show the aggregating behaviour exhibited by the solutions.
Key words and phrases: parabolic equation, functional differential equation, aggregating populations.

Resumen

En este artículo estudiamos la ecuación

$$u_t = \Delta[\varphi(u(x,[t/\tau]\tau))u(x,t)]\,,\ x\in\Omega\,,t>0,$$

con condiciones de frontera homogéneas de tipo Neumann en un dominio acotado en $\mathbb{R}^n$. Probamos la existencia y unicidad del

†Recibido 98/09/25. Aceptado 98/12/01.
MSC (1991): 35K99, 39B99, 92D25.

problema de valores iniciales y obtenemos algunos resultados que muestran el comportamiento de agregación que exhiben las soluciones.
Palabras y frases clave: ecuación parabólica, ecuación diferencial funcional, agregación en poblaciones.

Introduction

In this paper we study the equation

$$u_t = \Delta[\varphi(u(x,[t/\tau]\tau))u(x,t)]\,,\ x \in \Omega\,, t > 0 \tag{1}$$

with boundary conditions

$$\eta \cdot \nabla[\varphi(u(x,[t/\tau]\tau))u(x,t)] = 0\,,\ x \in \partial\Omega\,, t > 0 \tag{2}$$

and initial data

$$u(x,0) = u_0(x)\,,\ x \in \Omega. \tag{3}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\tau > 0$ is a constant, $[\theta]$ denotes the greatest integer less than or equal to θ (i.e. $[\theta]$ is an integer such that $[\theta] \leq \theta < [\theta] + 1$), and φ is a non-increasing function. This problem arises on a model for aggregating populations with migration rate φ and constant population. A first attempt to model aggregating behavior using partial differential equations conducts to the following equation (see D. Aronson [1]),

$$u_t = \Delta f(u), \tag{4}$$

where $f(u) = u\varphi(u)$, and φ is a non-increasing function of u. Nevertheless, since $f'(u)$ may be negative for positive values of u, the standard initial-boundary value problems for this equation are ill-posed.

Several models have been proposed to overcome this difficulty. These include models based on systems of difference-differential equations [7], on advection-diffusion equations [3], and on some type of regularization of equation (4) [5, 6, 8].

In this paper we assume that the density dependent dispersal coefficient $\varphi(u)$ gets actualized at certain predetermined intervals of time, letting us to consider the functional differential equation (1).

In Section 1 we prove existence and uniqueness of the solutions of (1)-(3). We also show some comparison results and study the asymptotic behavior of the solutions of a problem associated to (1)-(3).

In Section 2 we prove some results which show the aggregating behavior that the solutions of (1)-(3) exhibit.

We include an Appendix with the derivation of equation (1).

1 Existence and Uniqueness of Global Solutions

We will assume that the functions $\varphi(u)$ and $f(u) := u\varphi(u)$ satisfy the following hypothesis:

Hypothesis 1. *1. $\varphi : [0,\infty) \mapsto (0,\infty)$ is bounded and non-increasing.*

2. There exist constants α_1 and α_2 with $0 < \alpha_1 < \alpha_2 \leq \infty$ such that f is increasing for $u \in (0,\alpha_1)$ and f is decreasing for $u \in (\alpha_1, \alpha_2)$. If $\alpha_2 < \infty$, then f is nondecreasing for $u \in (\alpha_2, \infty)$.

For example, the following functions are admissible: $\varphi(u) = \exp(-u)$; $\varphi(u) = \frac{2}{3(1+u^{12})} + \frac{1}{3}$; $\varphi(u) = k_1$ for $0 \leq u \leq \alpha_1$, $\varphi(u) = k_1 + \frac{k_2-k_1}{\alpha_2-\alpha_1}(u-\alpha_1)$ for $\alpha_1 \leq u \leq \alpha_2$, and $\varphi(u) = k_2$ for $\alpha_2 \leq u < \infty$, where k_1 and k_2 are constants such that $0 < k_2 < k_1$ and $k_1\alpha_1 > k_2\alpha_2$.

In this section we will solve (1)-(3) by the method of steps, i.e., we integrate the equation inductively in $\Omega \times (k\tau, (k+1)\tau], fork = 0, 1, \ldots$. This leads us to solve the parabolic equation:

$$v_t \;=\; \triangle[a(x)v(x,t)],\;\; x \in \Omega\,,\;\; t \in (0,T], \tag{5}$$

with boundary conditions:

$$\eta \cdot \nabla[a(x)v(x,t)] \;=\; 0\,,\;\; x \in \partial\Omega\,,\; t > 0 \tag{6}$$

and initial data

$$v(x,0) = v_0(x)\,,\;\; x \in \Omega. \tag{7}$$

for any $T > 0$. We will solve (5)-(7) with the following assumptions about the data a and v_0:

A_1 $a \in L^\infty(\Omega)$ and $0 < \alpha \leq a(x) \leq \beta$ for a.e. $x \in \Omega$.
A_2 $v_0 \in L^\infty(\Omega)$ and $v_0(x) \geq 0$ for a.e. $x \in \Omega$
A_3 $av_0 \in W_2^1(\Omega)$

These will be called "Assumptions A".

Definition 1. *A solution of problem (5)-(7) on* $[0,T]$ *is a function* v *with the following properties:*

i) $v \in L^\infty(Q_T)$,

ii) $av \in C([0,T]\,;\,L_2(\Omega)) \cap W_2^{1,0}(Q_T)$,

iii) $\int_\Omega v(x,t)\psi(x,t)dx - \iint_{Q_T}[v(x,t)\psi_t(x,t) - \nabla(a(x)v(x,t))\cdot\nabla\psi]dxdt$
$= \int_\Omega v_0(x)\psi(x,0)dx$, *for all* $\psi \in W_2^1(Q_T)$ *and for all* $t \in (0,T]$.

A solution on $[0,\infty)$ means a solution on each $[0,T]$, and a sub-solution (super-solution) is defined by (i), (ii) and (iii) with equality replaced by $\leq$ $(\geq)$.
Here we are using the standard notation $Q_T := \Omega \times (0,T]$.
Next, we will obtain some comparison results for the solutions of (5)-(7).

Proposition 1. *Let* $\hat{v}$ *be a supersolution of problem (5)-(7) in* $[0,T]$ *with initial data* $\hat{v}_0$ *and let* v *be a sub-solution in* $[0,T]$ *with initial data* v_0. *Then, for all* $\lambda > 0$ *and* $0 \leq t \leq T$, *we have*

$$e^{\lambda t}\int_\Omega (v(x,t)-\hat{v}(x,t))^+ \quad\leq\quad \int_\Omega (v_0(x)-\hat{v}_0(x))^+ + \int_{Q_t}[\lambda(v-\hat{v})]^+e^{\lambda s}. \quad (8)$$

Proof: For any $\psi \in C^2(\overline{Q}_T)$ such that $\psi_x = 0$ for $(x,t) \in \partial\Omega \times [0,T]$, we have

$$\int_\Omega v\psi - \iint_{Q_t}(v\psi_t + av\psi_{xx}) \leq \int_\Omega v_0\psi(0)$$

and

$$-\int_\Omega \hat{v}\psi + \iint_{Q_t}(\hat{v}\psi_t + a\hat{v}\psi_{xx}) \leq -\int_\Omega \hat{v}_0\psi(0).$$

Adding term by term we obtain

$$\int_\Omega (v-\hat{v})\psi - \iint_{Q_t}(v-\hat{v})(\psi_t + a\psi_{xx}) \leq \int_\Omega (v_0-\hat{v}_0)\psi(0). \quad (9)$$

We now construct a special sequence of functions $\{\psi_n\}$ to use in (9). Fix $T > 0$ and choose a sequence $\{a_n\}$ of smooth functions such that

$$0 < \gamma \leq a_n \leq \|a\|_{L^\infty(\Omega)}$$

and

$$(a_n - a)/\sqrt{a_n} \longrightarrow 0 \ \text{ in } \ L^2(\Omega).$$

Since $||a_n^{-1/2}||_{L_\infty(\Omega)} < 1/\gamma$, for all n, it is enough to choose $\{a_n\}$ such that $(a_n - a) \to 0$ in $L^2(\Omega)$. □

Next, let $\chi \in C_0^\infty(\Omega)$ be such that $0 \le \chi \le 1$. Finally let ψ_n be the solution of the backward problem

$$\begin{array}{ll} \psi_{nt} + a_n \psi_{nxx} = \lambda \psi_n & \text{for } (x,t) \in \Omega \times [0,T) \\ \psi_{nx}(x,t) = 0 & (x,t) \in \partial\Omega \times [0,T) \\ \psi_n(x,T) = \chi(x) & x \in \Omega. \end{array}$$

This is a parabolic problem and has a unique solution $\psi_n \in C^\infty(\overline{Q}_T)$ that satisfies the properties stated in the following Lemma.

Lemma 1. *The function ψ_n has the following properties:*

(*i*) $0 \le \psi_n \le e^{\lambda^{(t-T)}}$ *in* $\overline{Q}_T$

(*ii*) $\displaystyle\iint_{Q_T} a_n (\psi_{nxx})^2 < c$

(*iii*) $\displaystyle\sup_{0 \le t \le T} \int_\Omega (\psi_{nx})^2 \ (t) < c$, *where the constant c depends only on χ.*

The proof of this Lemma is similar to the proof of Lemma 10 in D. Aronson, M. G. Crandall and L. A. Peletier [2] and it is omitted.
If we set $t = T$ and $\psi = \psi_n$ in (9) we obtain:

$$\begin{aligned} \int_\Omega (v - \hat{v})\chi \quad - & \iint_{Q_T} (v - \hat{v})(a - a_n)\psi_{nxx} \\ \le & \int_\Omega (v_0 - \hat{v}_0)\psi_n(0) + \iint_{Q_T} \lambda(v - \hat{v})\psi_n \\ \le & \int_\Omega (v_0 - \hat{v}_0)^+ e^{-\lambda T} + \iint_{Q_T} [\lambda(v - \hat{v})]^+ e^{\lambda(s-T)}. \end{aligned} \tag{10}$$

Since

$$\iint_{Q_T} |a - a_n||\psi_{nxx}| = \iint_{Q_T} \frac{|a - a_n|}{\sqrt{a_n}} (\sqrt{a_n}\ |\psi_{nxx}|),$$

we have, by Lemma 1 (*ii*),

$$\begin{aligned} \|(a - a_n)\psi_{nxx}\|_{L^1(Q_T)} \le & \ \|\frac{a - a_n}{\sqrt{a_n}}\|_{L^2(Q_T)} \ \|\sqrt{a_n}\ \psi_{nxx}\|_{L^2(Q_T)} \\ = & \ T^{1/2}\ \|\frac{a - a_n}{\sqrt{a_n}}\|_{L^2(\Omega)} \ \|\sqrt{a_n}\ \psi_{nxx}\|_{L^2(Q_T)} \\ \le & \ (cT)^{1/2}\ \|\frac{a - a_n}{\sqrt{a_n}}\|_{L^2(\Omega)}, \end{aligned}$$

which tends to zero as $n \to \infty$ by the choice of a_n. Thus, letting $n \to \infty$ in (10) we obtain

$$\int_\Omega (v(T) - \hat{v}(T))\chi \quad \leq \quad \int_\Omega (v_0 - \hat{v}_0)^+ e^{-\lambda T} + \iint_{Q_T} [\lambda(v - \hat{v})]^+ e^{\lambda(S-T)} \tag{11}$$

This inequality holds for any $\chi \in C_0^\infty(\Omega)$ with $0 \leq \chi \leq 1$. Hence, it continues to hold for $\chi(x) = 1$ on $\{x : v(T) > \hat{v}(T)\}$ and $\chi = 0$ otherwise (i.e., $\chi = \text{sign}(v(T) - \hat{v}(T))^+$). Here we have used the fact that $C_0^\infty(\Omega)$ is dense in $L^1(\Omega)$. Replacing T by any $t \leq T$ and applying the same argument we complete the proof of the Proposition.

□

Theorem 1. *(i) Let $v, \hat{v}$ be solutions problem (5)-(7) on $[0, T]$ with initial data v_0 and $\hat{v}_0$ respectively. Then*

$$\|v(t) - \hat{v}(t)\|_{L'(\Omega)} \leq \|v_0 - \hat{v}_0\|_{L'(\Omega)}$$

Thus, in particular, the solution of problem (5)-(7) is unique.
(ii) Let v be a sub-solution and $\hat{v}$ a super-solution of problem (5)-(7) with initial data v_0, and $\hat{v}_0$ respectively. Then if $v_0 \leq \hat{v}_0$ it follows that

$$v \leq \hat{v}$$

Proof: With the assumptions of (ii), Proposition 1 yields

$$e^{\lambda t} \int_\Omega (v(t) - \hat{v}(t))^+ \quad \leq \quad \int_\Omega (v_0 - \hat{v}_0)^+ + \int_0^t \int_\Omega e^{\lambda s} [\lambda(v - \hat{v})]^+. \tag{12}$$

Thus if we write

$$h(t) = e^{\lambda t} \int_\Omega (v(t) - \hat{v}(t))^+$$

(12) implies, by Gronwall's Lemma, that $h(t) \leq h(0)\, e^{\lambda t}$ or

$$\int_\Omega (v(t) - \hat{v}(t))^+ \leq \int_\Omega (v_0 - \hat{v}_0)^+.$$

This proves (ii). The assertion (i) follows by adding the corresponding inequality for $(\hat{v} - v)^+$.

□

Remark 1. *Since $v_0 \geq 0$ and zero is a solution of (5)-(7), we obtain that the solutions of (5)-(7) are non negative.*

Remark 2. *Since $v_0 \in L^\infty(\Omega)$, let K be a constant such that $v_0 \leq K$. Let $\hat{v}_0 = K$ then $\hat{v}(x,t) = e^{Mt}$ is a super-solution (in fact, a solution) of the problem (5)-(7). Then, by the theorem, $\hat{v}(x,t) \leq e^{Mt}$. In particular,*

$$v \in L^\infty(Q_T).$$

Now we proceed to the proof of the following theorem:

Theorem 2. *If the Assumptions A are fulfilled, then the problem (5)-(7) has a unique solution v un $[0,T]$ for any $T > 0$. Moreover, v satisfies the following energy relation*

$$\frac{1}{2}\int_\Omega av^2 \; + \; \int_{Q_t} (av)^2_{x_i} \;=\; \frac{1}{2}\int_\Omega (av_0)^2, \tag{13}$$

and the estimate

$$\text{ess} \sup_{0 \leq t \leq T} \|a(\cdot)v(\cdot,t)\|_{L^2(\Omega)} + \|\nabla(av)\|_{L^2(Q_T)} \leq C\|a(\cdot)v_0(\cdot)\|_{L^2(\Omega)}, \tag{14}$$

where $C = C(\alpha,\beta)$ is a constant independent of T.

Proof: The uniqueness is already given in Theorem 1 (i).
For the proof of solvability we make the change of variable $w(x,t) = a(x)v(x,t)$ and arrive to the following problem

$$\begin{cases} \tilde{a}w_t = \triangle w, & (x,t) \in Q_T \\ \eta \cdot \nabla w = 0, & (x,t) \in \partial\Omega \times (0,T] \\ w(x,0) = w_0(x) := a(x)v_0(x), & x \in \Omega, \end{cases} \tag{15}$$

where $\tilde{a} = 1/a$. It is clear that $\tilde{a} \in L^\infty(\Omega)$.
Now we take a fundamental system $\{\varphi_k(x)\}$ in $W_2^1(\Omega)$. Since $\tilde{a}(x) \geq 1/\beta > 0$, for a.e. $x \in \Omega$, we can choose $\varphi_k(x)$ such that $\int_\Omega \tilde{a}(x)\varphi_k(x)\varphi_l(x)\,dx = 0$ for $k \neq l$. We shall look for approximate solutions

$$w^N(x,t) = \sum_{k=1}^N C_k^N(t)\varphi_k(x)$$

from the relation

$$(\tilde{a}w_t^N, \varphi_l) + (w_{x_i}^N, \varphi_{lx_i}) \;\; = \;\; 0\,, \;\; l = 1, \ldots, N \tag{16}$$

and the equation

$$C_l^N(0) \quad = \quad (w_0, \varphi_l)\,, \;\; l = 1, \ldots, N, \tag{17}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. Here and in what follows the terms of the form $(w^N_{x_i}, \varphi_{lx_i})$ mean $\sum_{i=1}^N (w^N_{x_i}, \varphi_{lx_i})$.

The relation (16) is simply a system of N linear ordinary differential equations in the unknowns $C_l(t) \equiv C_l^N(t)$, $(l = 1, \ldots, N)$, whose principal terms are of the form $dC_l(t)/dt$, the coefficients of $C_k(t)$ being constant. By a well known theorem on the solvability of such systems, we see that (16) and (17) uniquely determine continuously differentiable functions $C_l^N(t)$ on $[0, T]$.

Now we shall obtain bounds for w^N which do not depend on N. To do this, let us multiply each equation of (16) by the appropriate C_l^N, add then up from 1 to N and then integrate the result with respect to t from 0 to $t \leq T$, to obtain:

$$\int_{Q_t} \tilde{a} w_t^N w^N + \int_{Q_t} (w^N_{x_i})^2 = 0.$$

From this we obtain

$$\frac{1}{2}\int_\Omega \tilde{a}(w^N)^2 \; + \; \int_{Q_t} (w^N_{x_i})^2 \; = \; \frac{1}{2}\int_\Omega \tilde{a}(w_0^N)^2, \tag{18}$$

where $w_o^N(x) = w^N(x,0) = \sum_1^N C_K^N(0)\psi_K(x) = \sum_1^N (w_o, \psi_K)\psi_K$. Now, since $1/\beta \leq \tilde{a} \leq 1/\alpha$, we obtain

$$\frac{1}{2\beta}\|w^N(\cdot,t)\|^2_\Omega \; + \; \|w^N_x\|^2_{Q_t} \; \leq \; \frac{1}{2\alpha}\|w^N(\cdot,0)\|^2_\Omega,$$

where

$$\|w_x\|_{Q_t} := \left(\int_{Q_t} \sum_{i=1}^n w^2_{x_i}\right)^{1/2}.$$

We replace $\|w_0^N\|^2_\Omega$ by $y(t)\|w_0^N\|_\Omega$, where $y(t) := \operatorname{ess\,sup}_{0\leq\tau\leq t} \|w^N(\cdot,t)\|_\Omega$. This gives the inequality

$$\|w^N(\cdot,t)\|^2_\Omega \; + \; \nu\|w_x\|^2_{Q_t} \; \leq \; \mu y(t)\|w_0^N\|_\Omega := j(t),$$

where $\mu = \frac{\beta}{2\alpha}$, $\nu = 2\beta$. From this the two inequalities

$$y^2(t) \leq j(t) \tag{19}$$

and

$$\|w_x^N\|_{Q_t}^2 \ \le \ \upsilon^{-1} j(t) \tag{20}$$

follow. We take the square root of each side of (19) and (20), add together the resulting inequalities, and then estimate the right-hand size in the following way:

$$\begin{aligned} |w^N|_{Q_t} \ &:= y(t) \ + \ \|w_x^N\|_{Q_t} \ \le \ (1+\upsilon^{-1/2}) j^{1/2}(t) \\ &\le \ (1+\nu^{-1/2})\mu^{1/2}\|w_0^N\|_\Omega^{1/2} \ |w^N|_{Q_t}^{1/2}. \end{aligned}$$

From this we obtain the following bound for $|w^N|_{Q_t}$:

$$|w^N|_{Q_t} \quad \le \quad (1+\nu^{-1/2})^2 \mu \|w_0^N\|_\Omega. \tag{21}$$

Thus, we obtain the inequality

$$|w^N|_{Q_t} \le c\|w_0^N\|_\Omega, \tag{22}$$

which holds for any t in $[0,T]$, with $c = c(\alpha,\beta)$ independent of t and T. But $\|w_0^N\|_\Omega \le \|w_0\|_\Omega$, so we have the bound

$$|w^N|_{Q_t} \le C_1, \tag{23}$$

with a constant C_1 independent of N. Because of (23), we can choose a subsequence $\{w^{N_k}\}$ $(k = 1,2,\dots)$ from the sequence $\{w^N\}$ $(N = 1,2,\dots)$ which converges weakly in $L_2(Q_T)$, together with the derivatives $w_{x_i}^{N_k}$, to some element $w \in W_2^1(Q_T)$ (as a result of subsequent arguments, we shall show that the entire sequence $\{w^N\}$ converges to w). This element $w(x,t)$ is the desired generalized solution of the problem (15).

Indeed, let us multiply (16) by an arbitrary absolutely continuous function $d_l(t)$ with $dd_l(t)/dt \in L_2(0,T)$, add up the equations thus obtained from 1 to N, and then integrate the result from 0 to $t \le T$. If we integrate the first term by parts with respect to t, we obtain an identity:

$$\int_\Omega \tilde{a} w^N \Phi dx - \int_{Q_t} [\tilde{a} w^N \Phi_t + w_{x_i}^N \Phi_{x_i}] dxdt \quad = \quad \int_\Omega \tilde{a} w_0^N \Phi(x,0) dx \tag{24}$$

in which $\Phi = \sum_{l=1}^N d_l(t)\varphi_l(x)$. Let us denote by $\mathcal{M}_N$ the set function Φ with $d_l(t)$ having the properties indicated above. The totality $\cup_{p=1}^\infty \mathcal{M}_p$ is dense in $W_2^1(Q_T)$.

For a fixed $\Phi \in \mathcal{M}_p$ in (24) we can take the limit of the subsequence $\{w^{N_k}\}$ chosen above, starting with $N_k \geq p$. As a result, we obtain (24) for w. But since $\cup_{p=1}^{\infty}\mathcal{M}_p$ is dense in $W_2^1(Q_T)$, it is not hard to obtain that w satisfies (ii) in the corresponding definition of solution of problem (15).

Finally it can be easily seen that the difference, $w^{N_k} - w^{N_l}$ satisfies the inequality (22):

$$|w^{N_k} - w^{N_l}|_{Q_T} \leq C(T)\|w_0^{N_k} - w_0^{N_l}\|_{\Omega}.$$

This implies that w^{N_k} converges to w in the norm $|\cdot|_{Q_T}$, showing that $w \in C([0,T\,; L^2(\Omega)) \cap W_2^{1,0}(Q_T)$. Now, applying (18) to the subsequence w^{N_k} and taking limits we obtain (13). From this, following the same argument that led to (22), we obtain (14). This finishes the proof of the Theorem. □

The following result is a consequence of the previous Theorem. The proof is given in [9].

Theorem 3. *Any solution $v(x,t)$ of (5)-(7) satisfies*

$$\lim_{t\to\infty} \|v(\cdot,t) - v_\infty\|_{L^2(\Omega)} = 0,$$

where $v_\infty := \frac{1}{a(x)}\left(\int_\Omega v_0\right)\left(\int_\Omega \frac{1}{a}\right)^{-1}$.

Now we are ready to prove the main result of this section. A *global solution* for the problem (1)-(3) is a function $u(x,t)$, $x \in \Omega$, $t > 0$, such that u is a solution of the problem (5)-(7) in $\Omega \times (k\tau, (k+1)\tau]$, $k = 0, 1, 2, ...$, with $a(x) = \varphi(u(x, [t/\tau]\tau))$.

Theorem 4. *If $u_0 \in L^\infty(\Omega)$ then the problem (1)-(3) has a unique global solution.*

Proof: The proof is by induction in k. The case $k = 0$ is obtained directly from Theorem 2. Assuming the case k and using the Remark 2 after the proof of Theorem 1 we obtain that $u(x,(k+1)\tau) \in L^\infty(\Omega)$ and $a(x)u(x,(k+1)\tau) \leq W_2^1(\Omega)$. From this it follows that we can apply Theorem 2 to solve (5)-(7) in $\Omega\times((k+1)\tau,(k+2)\tau]$ with $a(x) = \varphi(u(x,(k+1)\tau))$ and $u_0(x) = u(x,(k+1)\tau)$. This finishes the proof. □

2 Aggregation

In this section we consider some results that show the aggregating behavior that the solutions of (1)-(3) exhibit. The first result is a direct consequence of Theorem 3. For any $u_0 \in L^\infty(\Omega)$ and $\tau > 0$, let $u(x,t;u_0,\tau)$ denote the solution of (1)-(3).

Theorem 5. *Suppose that $\varphi(u)$ and $f(u) = u\varphi(u)$ satisfy Hypothesis 1. For any $\epsilon > 0$ there exists $\tau > 0$ such that*

$$\|u(\cdot, \tau; u_0, \tau) - u_\infty(\cdot)\|_{L^2(\Omega)} < \epsilon,$$

where $u_\infty := \frac{1}{\varphi(u_0(x))} \left(\int_\Omega u_0(x)\, dx\right) \left(\int_\Omega \frac{dx}{\varphi(u_0(x))}\right)^{-1}$.

Since $\varphi(u)$ is a non-increasing function this result states that, for large enough τ, the solutions of (1)-(3) concentrate its mass around the points of higher density of the initial data $u_0(x)$, thus showing the kind of aggregating behavior that we were expecting.

Another way to look at this result it is to notice that, by the change of variable $s = t/\tau$ and the definition $w(x, s) := u(x, s\tau)$, problem (1)-(3) is transformed into the equivalent problem

$$\begin{aligned} & w_s = \Delta[\tau\varphi(w(x, [s]))w(x, s)]\,,\ x \in \Omega\,, s > 0 \\ & \eta \cdot \nabla[\tau\varphi(w(x, [s]))w(x, s)] = 0\,,\ x \in \partial\Omega\,, s > 0 \\ & w(x, 0) = w_0(x) := u_0(x)\,,\ x \in \Omega. \end{aligned} \tag{25}$$

Hence, taking $\tau > 0$ big accounts for multiplying φ by a large constant. Therefore, Theorem 5 states that for any $\epsilon > 0$ we can choose $\tilde{\varphi} := \tau\varphi$, multiplying the original φ by a large constant τ, such that the solution w of (25) satisfies

$$\|w(\cdot, 1) - u_\infty(\cdot)\|_{L^2(\Omega)} < \epsilon.$$

That is, given an initial data u_0, we can generate aggregation around the points of higher density of u_0, at a prescribed time, by an adequate choice of φ.

Proof of Theorem 5: We consider the problem

$$\begin{aligned} & v_t = \Delta[a(x)v(x, t)]\,,\ x \in \Omega\,, t > 0 \\ & \eta \cdot \nabla[a(x)v(x, t)] = 0\,,\ x \in \partial\Omega\,, t > 0 \\ & v(x, 0) = v_0(x) := u_0(x)\,,\ x \in \Omega, \end{aligned}$$

with $a(x) := \varphi(u_0(x))$. By Theorem 3, for any $\epsilon > 0$ there exists $\tau > 0$ such that

$$\|v(\cdot, t) - v_\infty(\cdot)\|_{L^2(\Omega)} < \epsilon,$$

for any $t > \tau$, where $v_\infty := \frac{1}{a(x)}\left(\int_\Omega v_0(x)\,dx\right)\left(\int_\Omega \frac{dx}{a(x)}\right)^{-1}$. By uniqueness of the solutions of (1)-(3) it follows that $u(\cdot,\tau;u_0,\tau) = v(\cdot,\tau)$. This finishes the proof. □

In what follows we restrict ourselves to a more specific function φ. Let $\varphi(u)$ be a continuous function such that

$$\varphi(u) := \begin{cases} k_1, & 0 \le u \le \alpha_1 \\ \psi(u), & \alpha_1 \le u \le \alpha_2 \\ k_2, & \alpha_2 \le u \end{cases}$$

where $\psi(u)$ is a non-increasing function, k_1, k_2, α_1 and α_2 are positive constants such that $k_2 < k_1$, $\alpha_1 < \alpha_2$ and $k_2\alpha_2 < k_1\alpha_1$. For example, we can choose ψ to be linear, that is $\psi(u) = k_1 + \frac{k_2-k_1}{\alpha_2-\alpha_1}(u-\alpha_1)$.

The following result shows that, under certain restrictions on the initial data, the solutions of (1)-(3) converge to a steady state. It is not difficult to show that a function $u \in L^\infty(\Omega)$ is a steady state solution of (1)-(2) if and only if $f(u(x)) =$ constant for a.e. $x \in \Omega$. Let β_i be such that $\beta_2 < \alpha_1 < \alpha_2 < \beta_1$ and $f(\beta_i) = f(\alpha_i), i = 1,2$. That is, $k_1\beta_2 = k_2\alpha_2$ and $k_2\beta_1 = k_1\alpha_1$.

Theorem 6. *Let $\tilde\Omega \subset \Omega$ be such that both $\tilde\Omega$ and $\Omega \setminus \tilde\Omega$ have positive measure. Suppose that u_0 satisfies $\beta_2 \le u_0(x) \le \alpha_1$, for a.e. $x \in \tilde\Omega$ and $\alpha_2 \le u_0(x) \le \beta_1$ for a.e.$x \in \Omega \setminus \tilde\Omega$. Then, the solution $u(x,t)$ of (1)-(3) satisfies*

$$\lim_{t\to\infty} \|u(\cdot,t) - u_\infty\|_{L^2(\Omega)} = 0,$$

where u_∞ is a steady solution of (1)-(2). Moreover,

$$u_\infty = \begin{cases} \gamma_2, & x \in \tilde\Omega \\ \gamma_1, & x \in \Omega \setminus \tilde\Omega, \end{cases} \tag{26}$$

where

$$\gamma_i = \frac{k_i}{k_2|\tilde\Omega| + k_1|\Omega \setminus \tilde\Omega|}\int_\Omega u_0(x)\,dx, \quad i = 1,2$$

and $\beta_2 \le \gamma_2 \le \alpha_1 < \alpha_2 \le \gamma_1 \le \beta_1$.

Proof: First, we will show that $u(x,t)$ satisfies

$$\beta_2 \le u(x,t) \le \alpha_1,\ x \in \tilde\Omega,\ t \ge 0 \tag{27}$$

and

$$\alpha_2 \le u(x,t) \le \beta_1,\ x \in \Omega \setminus \tilde\Omega,\ t \ge 0. \tag{28}$$

Let

$$v_1(x) = \begin{cases} \alpha_1, & x \in \tilde{\Omega} \\ \beta_1, & x \in \Omega \setminus \tilde{\Omega}, \end{cases}$$

and

$$v_2(x) = \begin{cases} \beta_2, & x \in \tilde{\Omega} \\ \alpha_2, & x \in \Omega \setminus \tilde{\Omega}. \end{cases}$$

Then v_i $(i = 1, 2)$ are steady solutions of (1)-(2). For $0 \leq t \leq \tau$ let $a(x) := \varphi(u_0(x))$; then

$$a(x) = \begin{cases} k_1, & x \in \tilde{\Omega} \\ k_2, & x \in \Omega \setminus \tilde{\Omega}. \end{cases}$$

Then, $v(x,t) := u(x,t)$, $0 \leq t \leq \tau$, is the solution of (5)-(7) in $[0, \tau]$. Moreover, since

$$v_2(x) = \beta_2 \leq v_0(x) \leq \alpha_1 = v_1(x), \ x \in \tilde{\Omega}$$

and

$$v_2(x) = \alpha_2 \leq v_0(x) \leq \beta_1 = v_1(x), \ x \in \Omega \setminus \tilde{\Omega},$$

we have that

$$v_2(x) \leq v_0(x) \leq v_1(x)$$

for almost every $x \in \Omega$. Since v_i $(i = 1, 2)$ are steady solutions of (5)-(6), it follows from Theorem 1 that

$$v_2(x) \leq v(x,t) \leq v_1(x)$$

for almost every $x \in \Omega$ and $0 \leq t \leq \tau$. That is, (27) and (28) hold for $0 \leq t \leq \tau$. Repeating the same argument inductively we obtain that (27) and (28) hold for any $t \geq 0$, as we wanted to show.

This implies, in particular, that $u(x,t)$ is a solution of (5)-(7) on $[0, \infty)$ with $a(x) = \varphi(u_0(x))$. Therefore, it follows from Theorem 3 that

$$\lim_{t \to \infty} \|u(\cdot, t) - u_\infty\|_{L^2(\Omega)} = 0,$$

where

$$u_\infty := \frac{1}{a(x)} \left(\int_\Omega u_0 \right) \left(\int_\Omega \frac{1}{a} \right)^{-1}.$$

Then (26) follows by noticing that

$$\int_\Omega \frac{dx}{a(x)} = \frac{k_2|\tilde{\Omega}| + k_1|\Omega \setminus \tilde{\Omega}|}{k_1k_2}.$$

Now, by using the hypothesis that $\beta_2 \leq u_0(x) \leq \alpha_1$, for a.e. $x \in \tilde{\Omega}$ and $\alpha_2 \leq u_0(x) \leq \beta_1$ for $a.e. x \in \Omega \setminus \tilde{\Omega}$, we obtain

$$\beta_2|\tilde{\Omega}| + \alpha_2|\Omega \setminus \tilde{\Omega}| \leq \int_\Omega u_0 \leq \alpha_1|\tilde{\Omega}| + \beta_1|\Omega \setminus \tilde{\Omega}|.$$

Hence,

$$\beta_2 = \frac{k_2\beta_2|\tilde{\Omega}| + k_2\alpha_2|\Omega \setminus \tilde{\Omega}|}{k_2|\tilde{\Omega}| + k_1|\Omega \setminus \tilde{\Omega}|} \leq \gamma_2 \leq \frac{k_2\alpha_1|\tilde{\Omega}| + k_2\beta_1|\Omega \setminus \tilde{\Omega}|}{k_2|\tilde{\Omega}| + k_1|\Omega \setminus \tilde{\Omega}|}.$$

Here we have used the fact that $k_1\beta_2 = k_2\alpha_2$ and $k_2\beta_1 = k_1\alpha_1$. Therefore, $\beta_2 \leq \gamma \leq \alpha_1$. Similarly, we obtain that $\alpha_2 \leq \gamma_1 \leq \beta_1$. Hence, $f(\gamma_2) = k_1\gamma_2$ and $f(\gamma_1) = k_2\gamma_1$. Therefore, since $k_1\gamma_2 = k_2\gamma_1$, $f(\gamma_1) = f(\gamma_2)$. That is, u_∞ is a steady solution of (1)-(2). This finishes the proof.

□

Appendix

Following an approach as in M. E. Gurtin and R. C. MacCamy [4] we describe the dynamics of a biological species in a region $\Omega \subseteq \mathbb{R}^n$ by the following three functions of position $x \in \Omega$ and time t:

$u(x,t)$: the "population density",
$\varphi(x,t)$: the "migration rate",
$\gamma(x,t)$: the "rate of population supply".

The function $u(x,t)$ gives the number of individuals, per unit volume, at x at time t; its integral over any region R gives the total population of R at time t. The function $\varphi(x,t)$ gives the rate at which individuals migrate, per unit volume, from the point x at time t towards any of the coordinates directions $e_i := (0,\ldots,1,\ldots)$. The product $u(x,t)\varphi(x,t)$ gives the number of individuals that migrate from x at time t towards the direction e_i. The flow of population at the point x in the direction η is given by $\eta \cdot \nabla[u(x,t)\varphi(x,t)]$. Finally the function $\gamma(x,t)$ gives the rate at which individuals are supplied, per unit volume, directly at x by births and deaths. The product $u(x,t)\gamma(x,t)$ gives the number of individuals supplied at x.

The functions u, φ and γ must be consistent with the following "Law of population balance": *For every regular subregion R of Ω and for all t,*

$$\frac{d}{dt}\int_R u(x,t)dx = \int_{\partial R} \eta \cdot \nabla[u(x,t)\varphi(x,t)]ds_x + \int_R u(x,t)\gamma(x,t)dx,$$

where η is the outward unit normal to the boundary ∂R of R. This equation asserts that the rate of change of population of R must equal the rate at which individuals leave R across its boundary plus the rate at which individuals are supplied directly to R.
Using the well known Divergence Theorem we obtain

$$\frac{d}{dt}\int_R u(x,t)dx = \int_R \triangle[u(x,t)\varphi(x,t)]dx + \int_R u(x,t)\gamma(x,t)dx$$

Since R is an arbitrary region in Ω we obtain the following local counterpart

$$\frac{\partial u}{\partial t} = \triangle[\varphi(x,t)u(x,t)] + u(x,t)\gamma(x,t).$$

In this paper we are only concerned with migration mechanisms. Therefore we assume that φ is not explicitly dependent upon the position and time but on the population density u at times $t = k\tau$, $k = 0, 1, 2, \ldots$, for a given $\tau > 0$. that is, $\varphi(x,t) = \varphi(u(x,[t/\tau]\tau))$ where $[\theta]$ denotes the greatest integer less than or equal to θ.

Introducing this in the previous equation we arrive at the following nonlinear functional differential equation for the density u:

$$\frac{\partial u}{\partial t} = \triangle[\varphi(u(x,[t/\tau]\tau))u(x,t)] + \Gamma(u(x,[t/\tau]\tau))u(x,t)$$

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