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The Radon-Nikodym Theorem for Reflexive Banach Spaces

El Teorema de Radon-Nikodym para Espacios de Banach Reflexivos

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Abstract

In this short paper we prove the equivalence between the Radon-Nikodym Theorem for reflexive Banach spaces and the representability of weakly compact operators with domain $L^1(\mu)$.
Key words and phrases: Radon-Nikodym Theorem, factoring weakly compact operators.

Resumen

En este breve artículo demostramos la equivalencia entre el Teorema de Radon-Nikodym para espacios reflexivos y la representabilidad de operadores débilmente compactos con dominio $L^1(\mu)$.
Palabras y frases clave: Teorema de Radon-Nikodym, factorización de operadores débilmente compactos.

1 Introduction

In [2], for a probability space (Ω, Σ, μ), the following Theorems are stated:

Theorem 1.1. *A Banach space X has the Radon-Nikodym Property respect to μ if every bounded linear operator $T : L^1(\mu) \longrightarrow X$ is representable.*

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Theorem 1.2. *Let $T : L^1(\mu) \longrightarrow X$ be a bounded linear operator. For $E \in \Sigma$ define $G(E) = T(\chi_E)$. Then T is representable if and only if there exists $g \in L^1(\mu, X)$ such that*

$$G(E) = \int_E g \, d\mu$$

for all $E \in \Sigma$. In this case, the function $g \in L^\infty(\mu, X)$ and

$$T(f) = \int_\Omega fg \, d\mu.$$

Moreover,

$$\|g\|_\infty = \|T\|.$$

We recall that a Banach space X has the **Radon-Nikodym Property** respect to μ if for every bounded variation, countably additive μ-continuous vector measure $\nu : \Sigma \longrightarrow X$ there is a Bochner integrable function $g : \Omega \longrightarrow X$ such that $\nu(E) = \int_E g \, d\mu$, $\forall E \in \Sigma$, while a bounded linear operator $T : L^1(\mu) \longrightarrow X$ is ***representable*** if there is a function $g : \Omega \longrightarrow X$ strongly measurable and essentially bounded such that

$$T(f) = \int_\Omega fg \, d\mu, \quad \forall f \in L^1(\mu)$$

and

$$\|T\| = \|g\|_\infty.$$

It was proved in [4] that weakly compact operators $T : L^1(\mu) \longrightarrow X$ with separable range are representable and, soon after, Phillips [6] proved that weakly compact operators with domain $L^1(\mu)$ have separable range (see [5] for an alternative proof).

As a consequence, the Radon-Nikodym Theorem holds true for reflexive Banach spaces, regardless of the probability space (Ω, Σ, μ); indeed, the representability of weakly compact operators with domain $L^1(\mu)$ implies the Radon- Nikodym Theorem for reflexive Banach spaces.

It is the aim of this note to prove the following theorem:

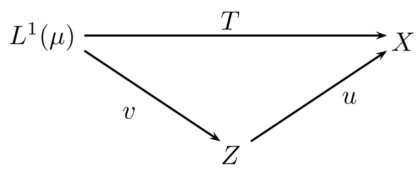
Theorem 1.3. *The Radon-Nikodym Theorem for reflexive Banach spaces implies the representability of weakly compact operators with domain $L^1(\mu)$.*

Our proof relies on the following result:

Theorem 1.4 ([1]). *Every weakly compact operator factorizes through a reflexive Banach space. Indeed, if X and Y are Banach spaces and $T : X \longrightarrow Y$ is a bounded weakly compact operator then there are a reflexive Banach space Z and two bounded linear operators $v : X \longrightarrow Z$ and $u : Z \longrightarrow Y$ such that $T = uv$.*

2 Proof of Theorem 1.3:

Let $T : L^1(\mu) \longrightarrow X$ be a weakly compact operator. Then, by Theorem 1.4 there are a reflexive Banach space Z and bounded linear operators $v : L^1(\mu) \longrightarrow Z$ and $u : Z \longrightarrow X$ such that the following diagram is commutative.



Since Z is a reflexive Banach space it has the Radon-Nikodym Property. Therefore the operator $v : L^1(\mu) \longrightarrow Z$ is representable. Hence there is $g \in L^\infty(\mu, X)$ such that

$$v(f) = \int_\Omega fg, \quad \forall f \in L^1(\mu).$$

Notice that, being $u \in L(Z, X)$, $u \circ g$ is defined from Ω to X and it belongs to $L^\infty(\mu, X)$ because $u \circ g$ is strongly measurable, since u is continuous, g measurable and $\|ug\|_\infty \leq \|u\|_{L(Z,X)} \|g\|_\infty$.

Furthermore, for E measurable,

$$\nu(E) = v(\chi_E) = \int_E g \, d\mu$$

defines a Z valued vector measure, so

$$\eta(E) = u \circ v(\chi_E) = u \int_E g \, d\mu = \int_E ug \, d\mu$$

defines an X valued vector measure. Since $T = u \circ v$ we have by Theorem 1.2 that T is representable and

$$T(f) = \int_\Omega fug \, d\mu, \quad \forall f \in L^1(\mu).$$

This finishes the proof.

In this way we have proved the equivalence between the Representability of weakly compact operators with domain $L^1(\mu)$ and the Radon-Nikodym Theorem for reflexive Banach spaces, since in [2] it is proved the other implication.

At this point we wonder if it is possible to find a proof of Radon-Nikodym Theorem for reflexive Banach spaces without using the Representation of weakly compact operators.

The answer is yes and it is found in [4], using the following ingredients:

Ingredient 1: *Separable dual Banach spaces have the Radon-Nikodym Property.*

Ingredient 2: *The Radon-Nikodym Property is separably determined; indeed a Banach space enjoys the Radon-Nikodym Property if and only if each of its separable subspaces does.*

Now we proceed to the proof of Radon-Nikodym Theorem for reflexive Banach spaces.

A Banach space X is reflexive if and only if every closed separable subspace of X is reflexive. Since reflexive Banach spaces are isomorphic to their second dual, they have the Radon Nikodym Property .

Remark 1. The separability of the range of T can be proved as follows ([2]):

If $T : L^1(\mu) \longrightarrow X$ is a bounded linear weakly compact operator, then it is representable. Therefore there is an essentially bounded strongly measurable function $g : \Omega \longrightarrow X$ such that

$$T(f) = \int_\Omega f g \, d\mu, \quad \forall f \in L^1(\mu).$$

This implies that there is a null set N such that the closed subspace Y generated by $g(\Omega \setminus N)$ is separable. Since $T(L^1(\mu)) \subset Y$, we obtain that T is separable valued.

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