Improvements of some Integral Inequalities of H. Gauchman involving Taylor's Remainder

Mejora de algunas Desigualdades Integrales de H. Gauchman que involucran el Resto de Taylor

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Abstract

In this paper we improve some integral inequalities recently obtained by H. Gauchman involving Taylor's remainder.
Key words and phrases: Taylor's remainder, Grüss' inequality, Inequality of Cheng-Sun, differentiable mappings.

Resumen

En este trabajo se mejoran algunas desigualdades integrales recientemente obtenidas por H. Gauchman que involucran restos de Taylor.
Palabras y frases clave: Resto de Taylor, desigualdad de Grüss, desigualdad de Cheng-Sun, aplicación diferenciable.

1 Introduction and recalls

This paper is a continuation of two recent works of H. Gauchman (see [5] and [6]). Its aim is to improve some integral inequalities obtained by H. Gauchman in [6] involving Taylor's remainder. Our method is based on the use of an inequality of Grüss type recently obtained by X. L. Cheng and J. Sun in [2].

In the following, n will be a non-negative integer. We denote by $R_{n,f}(c,x)$ the nth Taylor's remainder of function f with center c, that is

$$R_{n,f}(c,x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}}{k!}(x-c)^k.$$

We recall the following lemma established in [6].

Lemma 1. *Let f be a function dfined on $[a,b]$. Assume that $f \in C^{n+1}([a,b])$. Then*

$$\int_a^b \frac{(b-x)^{n+1}}{(n+1)!} f^{n+1}(x)\,dx = \int_a^b R_{n,f}(a,x)\,dx \tag{1}$$

$$\int_a^b \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(x)\,dx = (-1)^{n+1} \int_a^b R_{n,f}(b,x)\,dx \tag{2}$$

The following result contains an integral inequality which is well known in the literature as Grüss' inequality (cf., for example [8], p. 296),

Theorem 2. *Let I be an interval of the real line and let $F, G : I \to \mathbb{R}$ be two integrable functions such that $m \leq F(x) \leq M$ and $\varphi \leq G(x) \leq \Phi$ for all $x \in [a,b]$; m, M, φ and Φ are constants. Then we have the inequality*

$$\left| \int_a^b F(x)G(x)\,dx - \frac{1}{b-a} \int_a^b F(x)\,dx . \int_a^b G(x)\,dx \right| \leq \frac{b-a}{4}(M-m)(\Phi-\varphi) \tag{3}$$

and the inequality is sharp in the sense that the constant $\frac{1}{4}$ can not be replaced by a smaller one.

Using (3), H. Gauchman has proved (in [6]) the following result containing integral inequalities involving Taylor's remainder.

Theorem 3. *Let f be a function defined on $[a,b]$. Assume that $f \in C^{n+1}([a,b])$ and $m \leq f^{(n+1)} \leq M$ for each $x \in [a,b]$, where m and M are constants. Then*

$$\left| \int_a^b R_{n,f}(a,x)\,dx - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+2)!}(b-a)^{n+1} \right| \leq \frac{(b-a)^{n+2}}{4(n+1)!}(M-m), \tag{4}$$

$$\left| (-1)^{n+1} \int_a^b R_{n,f}(b,x)\,dx - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+2)!}(b-a)^{n+1} \right| \leq \frac{(b-a)^{n+2}}{4(n+1)!}(M-m). \tag{5}$$

The purpose of this paper is to provide some improvements to the inequalities (4) and (5) above.

2   The result

Before we give the main result of this paper we need to recall the following variant of the Grüss inequality which is recently obtained by X. L. Cheng and J. Sun (see [2]).

Theorem 4. *Let $F, G : [a, b] \to \mathbb{R}$ be two integrable functions such that $\varphi \leq G(x) \leq \Phi$ for some real constants φ, Φ and for all $x \in [a, b]$, then*

$$\left| \int_a^b F(x)G(x) \, dx - \frac{1}{b-a} \int_a^b F(x) \, dx \int_a^b G(x) \, dx \right|$$

$$\leq \ \frac{1}{2} \left(\int_a^b \left| F(x) - \frac{1}{b-a} \int_a^b F(y) \, dy \right| \, dx \right) (\Phi - \varphi) \tag{6}$$

The main result now follows.

Theorem 5. *Let f be a function defined on $[a, b]$. Assume that $f \in C^{n+1}([a, b])$ and $m \leq f^{(n+1)} \leq M$ for each $x \in [a, b]$, where m and M are constants. Then*

$$\left| \int_a^b R_{n,f}(a, x) \, dx - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+2)!}(b-a)^{n+1} \right| \leq \frac{(b-a)^{n+2}}{n!(n+2)^{\frac{2n+3}{n+1}}}(M - m),$$
$$\tag{7}$$

$$\left| (-1)^{n+1} \int_a^b R_{n,f}(b, x) \, dx - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+2)!}(b-a)^{n+1} \right|$$

$$\leq \frac{(b-a)^{n+2}}{n!(n+2)^{\frac{2n+3}{n+1}}}(M - m). \tag{8}$$

Proof. (i) For all $x, \in [a, b]$ we set $F(x) = \frac{(b-x)^{n+1}}{(n+1)!}$ and $G(x) = f^{(n+1)}(x)$. Then by assumption, F, G are integrable on $[a, b]$, with $m \leq G \leq M$. By using lemma 1 and Cheng-Sun inequality, we have

$$\left| \int_a^b R_{n,f}(a, x) \, dx - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+2)!}(b-a)^{n+1} \right|$$

$$= \left| \int_a^b \frac{(b-x)^{n+1}}{(n+1)!} f^{(n+1)}(x) \, dx - \frac{1}{b-a} \int_a^b f^{(n+1)}(x) \, dx \int_a^b \frac{(b-x)^{n+1}}{(n+1)!} \, dx \right|$$

$$\leq \frac{1}{2} \left(\int_a^b \left| \frac{(b-x)^{n+1}}{(n+1)!} - \frac{(b-a)^{n+1}}{(n+2)!} \right| dx \right) (M-m) \tag{9}$$

For all x in $[a,b]$, we set

$$\theta(x) = \frac{(b-x)^{n+1}}{(n+1)!} - \frac{(b-a)^{n+1}}{(n+2)!}.$$

It is easy to see that θ is a strictly decreasing function from $[a,b]$ onto $[\theta(b), \theta(a)]$, where $\theta(b) = -\frac{(b-a)^{n+1}}{(n+2)!}$ and $\theta(a) = \frac{(n+1)(b-a)^{n+1}}{(n+2)!}$. Let us set

$$x_n := b - \frac{b-a}{(n+2)^{\frac{1}{n+1}}}.$$

Then x_n is the unique point where θ vanishes and it is easy to show that θ is nonnegative on the interval $[a, x_n]$ and is negative on the interval $[x_n, b]$. Therefore, we have

$$\int_a^b |\theta(x)|\, dx = \int_a^{x_n} \theta(x)\, dx - \int_{x_n}^b \theta(x)\, dx := I_1 - I_2.$$

By easy computations, we get

$$I_1 - I_2 = \frac{(b-a)^{n+1}}{(n+2)!}(b-x_n) - \frac{(b-x_n)^{n+2}}{(n+2)!}. \tag{10}$$

However

$$b - x_n = \frac{b-a}{(n+2)^{\frac{1}{n+1}}} \quad and \quad (b-x_n)^{n+2} = \frac{(b-a)^{n+2}}{(n+2)^{\frac{n+2}{n+1}}}. \tag{11}$$

From (10) and (11), we deduce that

$$\int_a^b |\theta(x)|\, dx = \frac{2(b-a)^{n+2}}{(n+2)!(n+2)^{\frac{1}{n+1}}} \left(1 - \frac{1}{n+2} \right) = \frac{2(b-a)^{n+2}}{n!(n+2)^{\frac{2n+3}{n+1}}}. \tag{12}$$

From (9) and (12) we get the inequality (7).
(ii) In a similar manner, one could derive inequality (8). $\square$

Remark. (7) and (8) are actually improvements of (4) and (5) since for every natural number n, we have

$$\frac{n+1}{(n+2)^{\frac{2n+3}{n+1}}} < \frac{1}{4}.$$

Now we consider the cases when $n = 0$ or 1 in Theorem 5.

Corollary 6. *Let f be a function defined on $[a, b]$. Assume that $f \in C^2([a, b])$ and $m \le f'' \le M$ for each $x \in [a, b]$, where m and M are constants. Then*

$$\left| \int_a^b f(x)\, dx - f(a)(b - a) - \frac{2f'(a) + f'(b)}{6}(b - a)^2 \right| \le \frac{(b - a)^3}{9\sqrt{3}}(M - m),$$
(13)

$$\left| \int_a^b f(x)\, dx - f(b)(b - a) + \frac{2f'(b) + f'(a)}{6}(b - a)^2 \right| \le \frac{(b - a)^3}{9\sqrt{3}}(M - m),$$
(14)

$$\left| \int_a^b f(x)\, dx - \frac{f(a) + f(b)}{2}(b - a) + \frac{f'(b) - f'(a)}{12}(b - a)^2 \right| \le \frac{(b - a)^3}{9\sqrt{3}}(M - m).$$
(15)

Proof. To obtain (13) and (14) we take $n = 1$ in (7) and (8) of Theorem 5. (15) is obtained by taking half the sum of (13) and (14). □

Corollary 7. *Let f be a function defined on $[a, b]$. Assume that $f \in C^1([a, b])$ and $m \le f' \le M$ for each $x \in [a, b]$, where m and M are constants. Then*

$$\left| \int_a^b f(x)\, dx - \frac{f(a) + f(b)}{2}(b - a) \right| \le \frac{(b - a)^2}{8}(M - m).$$
(16)

Thus, we recapture the trapezoid inequality which has been obtained by several authors (see the papers [1,3,7]).

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