On Rational and Periodic Solutions of Stationary KdV Equations

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Abstract. Stationary solutions of higher order KdV equations play an important role for the study of the KdV equation itself. They give rise to the coefficients of the associated Lax pair (P, L) for which P and L have an algebraic relationship (and are therefore called algebro-geometric). This paper gives a sufficient condition for rational and simply periodic functions which are bounded at infinity to be algebro-geometric as those potentials of L for which $Ly = zy$ has only meromorphic solutions. It also gives a new elementary proof that this is a necessary condition for any meromorphic function to be algebro-geometric.

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1 Introduction

The collection of equations of the form

$$q_t = [P, L]$$

where $L = \partial^2/\partial x^2 + q$ and (P, L) is a Lax pair[2] is called the KdV hierarchy. Stationary solutions of equations in the KdV hierarchy are given as $[P, L] = 0$

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[2]That is, P is a monic odd-order differential expression whose coefficients are polynomials in q and its x-derivatives in such a way that the commutator $[P, L]$ is a multiplication operator, see Lax [13].

and are, according to a theorem of Burchnall and Chaundy [2], [3], related to a hyperelliptic curve. For this reason they are often called algebro-geometric potentials of L. In the case of continuous, real-valued, periodic potentials q Novikov [15] and Dubrovin [4] established the fact that q is algebro-geometric if and only if the spectrum of the associated $L^2(\mathbb{R})$-operator has a finite-band structure. Recently F. Gesztesy and myself [7] discovered that an elliptic potential is algebro-geometric if and only if, for every $z \in \mathbb{C}$, every solution of the equation $Ly = y'' + qy = zy$ is a meromorphic function of the independent variable. Our proof relied on a classical theorem of Picard [16], [17], [18] which states that a linear ordinary homogeneous differential equation with elliptic coefficients has always a solution which is elliptic of the second kind provided every solution of the equation is meromorphic. Note that in Picard's theorem the independent variable is considered to be a complex variable.
By extending this result to the AKNS hierarchy (cf. [8]) we proved that the connection between the algebro-geometric property and the existence of only meromorphic solutions is not restricted to the KdV case. For a review of these and related matters see [9].
The goal of this paper is to show with the aid of theorems of Halphen [10] and Floquet [6] that this characterization of elliptic algebro-geometric potentials may be carried over to the case of rational and simply periodic potentials. This covers the case of the famous N-soliton solutions of the KdV equation, which, when viewed as depending on a complex variable, are exponentially decaying along the real axis but are periodic with a purely imaginary period. Specifically, after giving a formal definition for the term "algebro-geometric" in Definition 1, necessary and sufficient conditions for a potential to be algebro-geometric will be provided in Theorems 1 and 2, respectively.

Definition 1. Let L be the differential expression $L = d^2/dx^2 + q$. A meromorphic function $q : \mathbb{C} \to \mathbb{C}_\infty$ will be called algebro-geometric (or an algebro-geometric potential of L) if there exists an ordinary differential expression P of odd order which commutes with L.

Note that by Theorem 6.10 of Segal and Wilson [19] any algebro-geometric potential which is smooth in some real interval may be extended to a meromorphic function on $\mathbb{C}$. The restriction to meromorphic functions in Definition 1 is made to provide a concise statement.

Theorem 1. *If q is an algebro-geometric potential then the following two statements hold:*

1. *Any pole of q is a regular singular point of the differential equation $y'' + qy = zy$. The principal part of the Laurent expansion of q near x_0 is given by $-k(k+1)/(x-x_0)^2$ for a suitable positive integer k. In particular, the residue of q at x_0 is equal to zero.*

2. *For all $z \in \mathbb{C}$ all solutions of $y'' + qy = zy$ are meromorphic functions of the independent variable.*

We prove this theorem in Section 4.
At this point it should be noted that, in the case when the curve associated with q is nondegenerate, the above result follows also from a theorem of Its and Matveev [12] published in 1975. In fact, Its and Matveev showed that, under the given circumstances, the potential q and a fundamental system of solutions of $y'' + qy = zy$ may be expressed in terms of Riemann's theta-function. From these expressions one can read off immediately the conclusions of Theorem 1.
In 1985 Segal and Wilson [19] looked at this type of questions from a very different perspective. They study the Gelfand-Dickey hierarchy (which contains the KdV hierarchy as a special case) employing loop group techniques. Instead of Riemann's theta-function they use an object called τ-function which is also an entire function of its arguments and this implies the validity of Theorem 1.
In justification of offering yet another proof of Theorem 1 let me remark that it will be completely elementary using only the well-known recursion formalism of the KdV-hierarchy.
We turn now our attention from necessary conditions for the algebro-geometric property to sufficient conditions. In Section 5 the following theorem will be proven.

Theorem 2. *Suppose that the function q satisfies one of the following three conditions:*

- *q is rational and bounded near infinity,*
- *q is simply periodic with period p and there exists a positive number R such that q is bounded in $\{x : |\operatorname{Im}(x/p)| \geq R\}$, or*
- *q is elliptic.*

Furthermore assume that, for infinitely many values of $z \in \mathbb{C}$, every solution of the differential equation $Ly = y'' + qy = zy$ is meromorphic. Then q is an algebro-geometric potential of L.

Note that, when q is elliptic, this result was proven in [7]. However, the proof given below will be new and much shorter than the one in [7].
In Section 2 the KdV hierarchy is formally introduced and some of its most important properties are collected. In Section 3 Frobenius' method of establishing series solutions of linear differential equations is used to prove two crucial lemmas. Section 4 is devoted to the proof of Theorem 1 while Section 5 furnishes the proof of Theorem 2.

2 The KdV hierarchy

Suppose q is a solution of some equation in the KdV hierarchy, i.e., there exists a positive integer g and a monic differential expression $\tilde{P}$ of order $2g+1$ such that $q_t = [\tilde{P}, L]$ where $L = \partial^2/\partial x^2 + q$. Since L commutes with its own powers we may add a polynomial $K(L)$ whose degree is at most g to $\tilde{P}$ and still have a monic differential expression of order $2g+1$ whose commutator with L equals q_t. It is well known that among all these expressions one can be written as

$$P = \sum_{j=0}^{g} \left[-\frac{1}{2} f'_{g-j}(x) + f_{g-j}(x) \frac{d}{dx} \right] L^j$$

where $f_0 = 1$ and, for $n \geq 1$, the f_n can be expressed as polynomials in q and its x-derivatives which obey the recursion relation

$$f'_{n+1}(x) = \frac{1}{4} f'''_n(x) + q(x) f'_n(x) + \frac{1}{2} q'(x) f_n(x). \tag{1}$$

In fact, since $[P, L] = f'_{g+1}$, the equation satisfied by q is $q_t = f'_{g+1}$. The condition that q be a stationary solution of some equation in the KdV hierarchy is therefore equivalent to the existence of an integer g such that

$$f'_{g+1}(x) = \frac{1}{4} f'''_g(x) + q(x) f'_g(x) + \frac{1}{2} q'(x) f_g(x) = 0. \tag{2}$$

Defining

$$F_g(z, x) = \sum_{j=0}^{g} f_{g-j}(x) z^j$$

and

$$R_{2g+1}(z) = (z - q(x)) F_g(z, x)^2 - \frac{1}{2} F''_g(z, x) F_g(z, x) + \frac{1}{4} F'_g(z, x)^2$$

one can show that in this case R_{2g+1} does not depend on x and that

$$P^2 = R_{2g+1}(L)$$

which defines the hyperelliptic curve mentioned in the introduction. Since it is also true that $[P, L] = 0$ if P and L satisfy the relationship $P^2 = R_{2g+1}(L)$ one has the following result which is a special case of a theorem of Burchnall and Chaundy [2], [3].

Theorem 3. *Let $L = d^2/dx^2 + q$ and suppose P is a monic differential expression of order $2g+1$. Then L and P are commutative if and only if there exist polynomials R and K of degree $2g+1$ and $k \leq g$, respectively, such that $(P + K(L))^2 = R(L)$.*

We need subsequently the following theorem which establishes a sufficient condition for the potential q to be algebro-geometric.

Theorem 4. *Let $y_1(z, \cdot)$ and $y_2(z, \cdot)$ be two solutions of $Ly = y'' + qy = zy$ which are linearly independent for all but at most countable many values of z. Define*

$$g(z, x) = y_1(z, x) y_2(z, x).$$

If

$$g(z,x) = \frac{F(z,x)}{\gamma(z)},$$

where γ is independent of x and $F(z,x)$ is a polynomial as a function of z and meromorphic as a function of x, then q is algebro-geometric.

Proof. A straightforward calculation[3] shows that the function $g(z,\cdot)$ satisfies the differential equation

$$4(z-q(x))g^2 - 2gg'' + g'^2 = W(y_1,y_2)(z)^2 \tag{3}$$

where $W(y_1,y_2)$ is the Wronskian determinant of y_1 and y_2 and where primes denote derivatives with respect to x. Hence

$$(z-q(x))F(z,x)^2 - \frac{1}{2}F(z,x)F''(z,x) + \frac{1}{4}F'(z,x)^2 = \gamma(z)W(y_1,y_2)(z)^2. \tag{4}$$

As a function of z the left hand side is a polynomial of degree $2g+1$ with leading coefficient $4f_0(x)^2$ when $F(\cdot,x)$ is of degree g and has leading coefficient $f_0(x)$. Since the right hand side does not depend on x we conclude that $f_0(x)$ is constant and we may assume without loss of generality that $f_0(x) = 1$. Equation (4) implies also that q is meromorphic. Therefore we may differentiate (4) with respect to x. Assuming that

$$F(z,x) = \sum_{n=0}^{g} f_n(x) z^{g-n}$$

and dropping a common factor $-2F(z,x)$ we obtain

$$\sum_{n=0}^{g-1} f'_{n+1}(x) z^{g-n} = \sum_{n=0}^{g} \left(\frac{1}{4} f'''_n(x) + \frac{1}{2} q'(x) f_n(x) + q f'_n(x) \right) z^{g-n}$$

since $f'_0 = 0$. This shows that the coefficients f_n satisfy the recursion relation (1) and that f_g satisfies (2). Hence, by the preceding considerations, q is algebro-geometric. □

3 Frobenius' Method

In this section we prove two results concerning the structure of solutions of the differential equation $y'' + qy = zy$. The first of these results is obtained from applying Frobenius' method of solving an ordinary linear differential equation by a power series to our particular case. A more general account can be found, for instance, in Ince [11], Chapter XVI. The proof of this standard result is only provided to facilitate references to it. The second result draws some further

[3] Apparently this observation was first made by Appell [1] in 1880.

conclusions in the presence of a spectral parameter and for the case when all solutions are meromorphic for infinitely many values of this spectral parameter. Suppose x_0 is a regular singular point of the equation $y'' + qy = 0$. Then q is meromorphic in a vicinity of x_0 and has, at worst, a second order pole there. Suppose

$$q(x) = \sum_{j=0}^{\infty} q_j (x - x_0)^{j-2}.$$

Then the indicial equation of the singularity is $r(r-1) + q_0 = 0$. The roots of this equation are called indices and since their sum must be equal to one we may denote them by $-k$ and $k+1$ where without loss of generality $\mathrm{Re}(-k) \leq \mathrm{Re}(k+1)$. Note that $q_0 = -k(k+1)$. Now introduce the series

$$w(\sigma, x) = \sum_{j=0}^{\infty} c_j(\sigma)(x - x_0)^{\sigma+j}.$$

Then

$$w'' + qw = \sum_{j=0}^{\infty} \left\{ (j+\sigma)(j+\sigma-1)c_j + q_0 c_j + \sum_{m=0}^{j-1} q_{j-m} c_m \right\} (x - x_0)^{j+\sigma-2}.$$

Define

$$f_0(\ell) = \ell(\ell - 1) + q_0 = (\ell + k)(\ell - k - 1)$$

and, recursively for $j \geq 1$,

$$c_j(\sigma) = \frac{-\sum_{m=0}^{j-1} q_{j-m} c_m(\sigma)}{f_0(\sigma + j)} \tag{5}$$

assuming that $f_0(\sigma + j) \neq 0$ for $j \geq 1$. Then

$$w'' + qw = c_0(\sigma) f_0(\sigma)(x - x_0)^{\sigma-2}.$$

Suppose first that $-k$ and $k+1$ do not differ by an integer. Then $f_0(\sigma) = 0$ but $f_0(\sigma + j) \neq 0$ for $j \geq 1$ and either choice of σ among the values $-k$ and $k+1$. Hence, choosing $c_0 = 1$, we find that $w(-k, \cdot)$ and $w(k+1, \cdot)$ are linearly independent solutions of $y'' + qy = 0$.

Next suppose that $2k+1$ is a nonnegative integer. Then $w(k+1, \cdot)$ is again a solution of $y'' + qy = 0$ but $w(-k, \cdot)$ becomes undefined since the requirement that $f_0(\sigma + j) \neq 0$ is not satisfied for $\sigma = -k$ and $j = 2k+1$. To obtain a second solution we choose $c_0 = \prod_{j=1}^{2k+1} f_0(\sigma + j)$. One shows then by induction that c_0, ..., c_{2k} are polynomials with simple zeros at $\sigma = -k$ while c_{2k+1} is a polynomial which may or may not have a zero at $-k$. Finally, c_{2k+2}, c_{2k+3}, ... are rational functions in σ which are analytic at $-k$. Now consider

$$v(\sigma, x) = \frac{\partial w}{\partial \sigma}(\sigma, x) = \sum_{j=0}^{\infty} \left(\frac{\partial c_j}{\partial \sigma} + c_j \log(x - x_0) \right) (x - x_0)^{\sigma+j}.$$

Since differentiation with respect to σ commutes with $d^2/dx^2 + q(x)$ we obtain that

$$v'' + qv = \left(\frac{\partial(c_0 f_0)}{\partial\sigma} + c_0 f_0 \log(x - x_0) \right) (x - x_0)^{\sigma-2}.$$

Since $c_0 f_0 = \prod_{j=0}^{2k+1} f_0(\sigma + j)$ has a double zero at $\sigma = -k$ we obtain that $v(-k, \cdot)$ is a solution of $y'' + qy = 0$ which is easily seen to be independent from $w(k + 1, \cdot)$. We may write

$$v(-k, x) = h_1(x) \log(x - x_0) + h_2(x)$$

where

$$h_1(x) = \sum_{j=2k+1}^{\infty} c_j(-k)(x - x_0)^{j-k} \quad \text{and} \quad h_2(z, x) = \sum_{j=0}^{\infty} \frac{\partial c_j}{\partial\sigma}(-k)(x - x_0)^{j-k}. \tag{6}$$

We collect these results for the particular case, when k is a positive integer in the following

Lemma 5. *Suppose q is meromorphic near x_0 with principal part*

$$-k(k + 1)/(x - x_0)^2 + q_1/(x - x_0)$$

where k is a positive integer. Then the differential equation $y'' + qy = zy$ has a solution w which is analytic at x_0 and a solution v defined by $v(x) = h_1(x) \log(x - x_0) + h_2(x)$ where h_1 is analytic at x_0 and h_2 is meromorphic at x_0.

This lemma and its proof are the main ingredients of the following one.

Lemma 6. *Let Z be the set of all values of $z \in \mathbb{C}$ such that $y'' + qy = zy$ has only meromorphic solutions. The following statements hold:*

1. *If Z is not empty then q is meromorphic and any pole of q is of the second order at most.*
2. *Z is either a finite set or equal to $\mathbb{C}$.*
3. *If $Z = \mathbb{C}$ and if x_0 is a pole of q then the principal part of the Laurent expansion of q about x_0 is given by $-k(k + 1)(x - x_0)^{-2}$ for some $k \in \mathbb{N}$, in particular, $\operatorname{res}_{x_0} q = 0$.*

Proof. The fact that $q = (y'' - zy)/y$ shows that q is meromorphic and has at most a double pole at any of its singular points even if $y'' + qy = zy$ has only one meromorphic solution for one value of z. This proves the first claim.
Hence, if $Z \neq \emptyset$, a pole x_0 of q is a regular singular point of $y'' + qy = zy$ and

$$q(x) = \sum_{j=0}^{\infty} q_j (x - x_0)^{j-2}$$

in a vicinity of x_0. The indices associated with x_0, which are given as the roots of $r(r-1)+q_0=0$ and hence are independent of z, must be distinct integers whose sum equals one. We denote them by $-k$ and $k+1$ where $k>0$ and note that $q_0=-k(k+1)$.

Note that replacing q by $q-z$ amounts to replacing q_2 by q_2-z in the Laurent expansion of q turning the recursion relation (5) into

$$c_j(\sigma,z)=\frac{-\sum_{m=0}^{j-1}(q_{j-m}-z\delta_{j-m,2})c_m(\sigma,z)}{f_0(\sigma+j)} \tag{7}$$

where $c_0=\prod_{j=1}^{2k+1} f_0(j+\sigma)$. The equation $y''+(q-z)y=0$ has a solution $v(z,x)=h_1(z,x)\log(x-x_0)+h_2(z,x)$ which is meromorphic at x_0 if and only if $h_1(z,\cdot)=0$. Recall that $c_1(-k,z)=...=c_{2k}(-k,z)=0$. Using this fact in the recursion relation (7) shows that the coefficients $c_j(-k,z)$ are zero for all j if and only if $c_{2k+1}(-k,z)=0$. Hence, because of (6), we have $h_1(z,\cdot)=0$ if and only if $c_{2k+1}(-k,z)=0$. The recursion relation (7) also implies immediately that the coefficients c_j are polynomials in their second variable. Hence $c_{2k+1}(-k,\cdot)$ has either finitely many zeros or else it is identically equal to zero. Therefore, if $c_{2k+1}(-k,\cdot)\neq 0$ for any singular point of the equation then Z is finite. However, if $c_{2k+1}(-k,\cdot)=0$ for all singular points of the equation then $Z=\mathbb{C}$. This proves the second claim.

To prove the third claim we need more detailed information about the leading coefficient of the polynomial $c_{2k+1}(-k,\cdot)$. We will show below that, if $q_1=\operatorname{res}_{x_0} q\neq 0$, then $c_{2k+1}(-k,\cdot)$ is a polynomial of degree k thus forcing Z to be a finite set and proving the last claim.

Suppose now that $q_1\neq 0$. Since $c_{2k+1}(\cdot,z)$ has a removable singularity at $-k$ we may determine $c_{2k+1}(-k,z)$ by computing $\lim_{\sigma\to -k} c_{2k+1}(\sigma,z)$ for $\sigma<-k$. Note that $c_0(\sigma,z)=\gamma_0(\sigma)$ and $c_1(\sigma,z)=-q_1\gamma_1(\sigma)$ where

$$-\gamma_0(\sigma)=-\prod_{j=1}^{2k+1} f_0(\sigma+j) \quad \text{and} \quad \gamma_1(\sigma)=\prod_{j=2}^{2k+1} f_0(\sigma+j).$$

The functions $-\gamma_0$ and γ_1 are positive in $(-k-1,-k)$ and have simple zeros at $-k$. Assume that $j\leq k$ and that $c_{2j-2}(\sigma,z)$ and $c_{2j-1}(\sigma,z)$ are polynomials in z of degree $j-1$ and that

$$c_{2j-2}(\sigma,z)=\gamma_{2j-2}(\sigma)z^{j-1}+O(z^{j-2}),$$
$$c_{2j-1}(\sigma,z)=-q_1\gamma_{2j-1}(\sigma)z^{j-1}+O(z^{j-2})$$

where $(-1)^j\gamma_{2j-2}$ and $(-1)^{j-1}\gamma_{2j-1}$ are positive in $(-k-1,-k)$ and have simple zeros at $-k$. Then, using the recursion relation (7), we obtain that $c_{2j}(\sigma,z)$ and $c_{2j+1}(\sigma,z)$ are polynomials in z of degree j and that, in particular,

$$c_{2j}(\sigma,z)=\frac{zc_{2j-2}}{f_0(\sigma+2j)}+O(z^{j-1})=\frac{\gamma_{2j-2}}{f_0(\sigma+2j)}z^j+O(z^{j-1}),$$
$$c_{2j+1}(\sigma,z)=\frac{zc_{2j-1}-q_1c_{2j}}{f_0(\sigma+2j+1)}+O(z^{j-1})=-q_1\frac{\gamma_{2j-1}+\gamma_{2j}}{f_0(\sigma+2j+1)}z^j+O(z^{j-1}).$$

Letting $\gamma_{2j} = \gamma_{2j-2}/f_0(\sigma+2j)$ and $\gamma_{2j+1} = (\gamma_{2j-1}+\gamma_{2j})/f_0(\sigma+2j+1)$ we find that $(-1)^{j+1}\gamma_{2j}$ and $(-1)^j\gamma_{2j+1}$ are positive in $(-k-1,-k)$ and that γ_{2j} has a simple zero at $-k$. If $j<k$ then γ_{2j+1} has a simple zero at $-k$, too, since γ_{2j-1} and γ_{2j} have the same sign in $(-k-1,-k)$. However, if $j=k$ then both the numerator and the denominator in $(\gamma_{2k-1}+\gamma_{2k})/f_0(\sigma+2k+1)$ have a simple zero at $-k$ proving that $\gamma_{2k+1}(-k)$ is different from zero. This, however, shows that $c_{2k+1}(-k,\cdot)$ is a polynomial of degree k which has at most k distinct zeros. However, $c_{2k+1}(-k,\cdot)$ must be zero for any value of z since $Z=\mathbb{C}$. This contradiction proves our assumption $q_1\neq 0$ wrong. □

4 Necessary Conditions

In this section we will prove Theorem 1 which gives conditions which must be satisfied for any algebro-geometric potential. We start with

Theorem 7. *If q is algebro-geometric then any of its poles is a regular singular point of the differential equation $Ly = y'' + qy = zy$. Moreover, when x_0 is a pole of q then the coefficient of $(x-x_0)^{-2}$ in the Laurent expansion of q about x_0 is equal to $-k(k+1)$ for some positive integer k.*

Proof. We show first that any pole of q is a regular singular point of $y''+qy = zy$, i.e., that its order is at most equal to two. Hence assume this were not the case. That is, suppose that x_0 which, without loss of generality, may be assumed to be equal to zero is a pole of q of order $k\geq 3$. Then q has a Laurent expansion $q = \alpha x^{-k} + ...$ where $\alpha\neq 0$. Consider the recursion relation (1). One shows by induction that the order of the pole $x_0=0$ of f_n''' is smaller than that of $qf_n' + q'f_n/2$ and that therefore

$$f_n'(x) = -nk\alpha^n x^{-nk-1}\prod_{j=1}^{n}\frac{2j-1}{2j} + O(x^{-nk}).$$

If q were algebro-geometric there would have to be an n such that $f_n'=0$. This contradiction shows that the order of the pole x_0 is at most two and that x_0 is a regular singular point of $y''+qy = zy$.

Next assume that $x_0 = 0$ is a pole of order one, i.e., $q = \alpha x^{-1} + O(1)$ with $\alpha\neq 0$. We prove, again by induction, that

$$f_n'(x) = -\frac{1}{2}\alpha x^{-2n}\prod_{j=1}^{n-1} j(j+1/2) + O(x^{-2n+1})$$

using that the order of the pole $x_0=0$ of f_n''' is larger than that of $qf_n'+q'f_n/2$. Hence, for no n is f_n' ever zero showing that x_0 must not be a first order pole if q is algebro-geometric.

Finally, suppose that $q = \alpha x^{-2} + ...$ for some α different from any number in $\{-k(k+1) : k \in \mathbb{N}\}$. Then another induction shows that

$$f_n'(x) = -2n \left[\prod_{j=1}^{n} \frac{2j-1}{2j} (\alpha + j(j-1)) \right] x^{-2n-1} + O(x^{-2n}).$$

Again $f_n' \neq 0$ for all $n \in \mathbb{N}$ contrary to the hypothesis. □

Theorem 8. *If q is algebro-geometric then every solution of $Ly = y'' + qy = zy$ is meromorphic for every $z \in \mathbb{C}$.*

Proof. Since q is algebro-geometric there exists a differential expression P of the form

$$P = \sum_{j=0}^{g} \left[-\frac{1}{2} f_{g-j}'(x) + f_{g-j}(x) \frac{d}{dx} \right] L^j$$

for which $[P, L] = 0$ and $P^2 = R_{2g+1}(L)$. In contradiction to what we want to prove assume that there exists a point z_0 such that $y'' + qy = z_0 y$ has a solution which is not meromorphic.

Let Z be the set of all values of $z \in \mathbb{C}$ such that $y'' + qy = zy$ has only meromorphic solutions. By Lemma 6 the set Z is closed and hence its complement is open. Therefore and because the zeros of R_{2g+1} are isolated there is no harm in assuming that $R_{2g+1}(z_0) \neq 0$.

Next denote the two-dimensional space of solutions of $Ly = z_0 y$ by $W(z_0)$. The restriction of P to the space $W(z_0)$ maps back into $W(z_0)$ since P and L commute. Note that

$$P|_{W(z_0)} = F_g(z_0, x) \frac{d}{dx} - \frac{1}{2} F_g'(z_0, x).$$

Introduce the basis $\{y_1, y_2\}$ of $W(z_0)$ which is defined by $y_j^{(\ell-1)}(x_0) = \delta_{j,\ell}$. In this basis the restriction of P to $W(z_0)$ is represented by the matrix

$$M = \frac{1}{2} \begin{pmatrix} -F_g'(z_0, x_0) & 2F_g(z_0, x_0) \\ 2(z_0 - q(x_0))F_g(z_0, x_0) - F_g''(z_0, x_0) & F_g'(z_0, x_0) \end{pmatrix}.$$

Note that $\operatorname{tr} M = 0$ and $\det M = -R_{2g+1}(z_0)$ regardless of x_0. Therefore M has distinct eigenvalues $\pm w_0 = \pm \sqrt{R_{2g+1}(z_0)}$. The associated eigenfunctions $\psi_\pm$ satisfy $P\psi_\pm = \pm w_0 \psi$ and $L\psi_\pm = z_0 \psi_\pm$. Define $\varphi_\pm = \psi_\pm' / \psi_\pm$ and note that

$$\pm w_0 = \frac{P\psi_\pm}{\psi_\pm} = F_g \varphi_\pm - \frac{1}{2} F_g'.$$

Hence

$$\varphi_\pm = \frac{\pm 2w_0 + F_g'}{2F_g}$$

are meromorphic functions on $\mathbb{C}$. Not both of the solutions $\psi_\pm$ can be meromorphic since they are linearly independent. Suppose ψ_+ is not meromorphic. Then, by Lemma 5, there is a constant γ such that ψ_+ (or an appropriate multiple) is given as

$$\psi_+(x) = h_1(x)\log(x-x_0) + h_2(x) + \gamma w(x) \tag{8}$$

where h_1, h_2, and w are functions which are meromorphic at x_0. Hence

$$\begin{aligned} &(x-x_0)(\varphi_+ h_1 - h_1')\log(x-x_0) \\ &= h_1 + (x-x_0)(h_2' + \gamma w') - (x-x_0)(h_2+\gamma w)\varphi_+ \end{aligned}$$

is meromorphic at x_0 and we conclude that $\varphi_+ h_1 - h_1' = 0$. This implies that $h_1 = c\psi_+$ for some constant c. If $c \neq 0$ we obtain from (8)

$$\log(x-x_0) = \left(\frac{1}{c}h_1(x) - h_2(x) - \gamma w(x)\right) h_1(x)^{-1}$$

which is impossible since the right hand side is meromorphic at x_0. Therefore $c = 0$, i.e., h_1 vanishes identically and ψ_+ is meromorphic at x_0 contrary to our assumption. □

We are now ready for the

Proof of Theorem 1. Theorem 7 proves that a pole of q is a regular singular point with principal part $-k(k+1)/(x-x_0)^2 + q_1/(x-x_0)$ for a suitable positive integer k and complex number q_1. Theorem 8 proves not only that all solutions of $y'' + qy = zy$ are meromorphic for all $z \in \mathbb{C}$ but also that the hypotheses of Lemma 6 are satisfied. This in turn shows then that $q_1 = 0$. □

5 Sufficient Conditions

In this section we will prove Theorem 2. As mentioned in the introduction the proofs rely on classical theorems by Halphen, Floquet, and Picard concerning the linear differential equation

$$q_0 y^{(n)} + q_1 y^{(n-1)} + \ldots + q_n y = 0. \tag{9}$$

While Floquet's theorem is well known (see e.g. Eastham [5] or Magnus and Winkler [14]) it is appropriate to repeat the theorems of Halphen and Picard. Halphen's theorem is concerned with the rational case. A proof is given by Ince [11] and this proof can be used to state the following version which is different from Ince's version.

Theorem 9. *Let the coefficients $q_0, \ldots, q_n$ in (9) be polynomials such that $\deg q_j \le \deg q_0 = s$ for $j = 1, \ldots, n$. For $j = 0, \ldots, n$ let A_j be the coefficient of x^s in q_j and let λ be a zero of $A_0\lambda^n + A_1\lambda^{n-1} + \ldots + A_n$. If the differential equation (9) has only meromorphic solutions then it has a solution $R(x)\exp(\lambda x)$ where R is a rational function.*

Picard's theorem is concerned with the elliptic case. It may also be found in [11].

Theorem 10. *Assume that the coefficients* $q_0, ..., q_n$ *in* (9) *are elliptic with common fundamental periods* $2\omega_1$ *and* $2\omega_2$ *and let* ρ_1 *be a Floquet multiplier with respect to the period* $2\omega_1$. *If the differential equation* (9) *has only meromorphic solutions then it has a solution which is elliptic of the second kind and satisfies* $y(x + 2\omega_1) = \rho_1 y(x)$.

5.1 Rational potentials

Suppose that q is rational and bounded at infinity. Let $z_0 = \lim_{x\to\infty} q(x)$. From Lemma 6 we know that $y'' + qy = zy$ has only meromorphic solutions for any value of z and from Halphen's theorem (Theorem 9) we obtain, for $z \neq z_0$, that there are linearly independent solutions

$$y_\pm(z, x) = R_\pm(z, x) \exp(\pm\sqrt{z - z_0}x)$$

where $R_\pm(z, \cdot)$ are rational functions. Also from Lemma 6 we obtain that

$$q = z_0 - \sum_{j=1}^{m} \frac{s_j(s_j + 1)}{(x - b_j)^2}$$

where $b_1, ..., b_m$ are distinct complex numbers and $s_1, ..., s_m$ are positive integers. The singular point b_j of $y'' + qy = zy$ has indices $-s_j$ and $s_j + 1$ and hence any pole of $y_\pm$ is located at one of the points b_j and has order s_j. Now define the function $g(z, x) = y_+(z, x)y_-(z, x)$. Letting $v(x) = \prod_{j=1}^{m}(x - b_j)^{s_j}$ we see that the functions $y_\pm v$ are entire as functions of x and hence $v^2 g(z, \cdot)$ is an entire rational function, i.e., a polynomial. Letting

$$v(x)^2 g(z, x) = \sum_{j=0}^{d} c_j x^j$$

the functions $v^2 g(z, x)$, $v^3 g'(z, x)$, $v^4 g''(z, x)$, and $v^5 g'''(z, x)$ are polynomials in x whose coefficients are homogeneous polynomials of degree one in $c_0, ..., c_d$. Since $v^2 q$ and $v^3 q'$ are polynomials we find that $v^5(g''' + 4(q - z)g' + 2q'g)$ is also a polynomial in x whose coefficients are homogeneous polynomials of degree one in $c_0, ..., c_d$. The coefficients of the c_ℓ in this last expression, in turn, are polynomials in z of degree at most one, i.e.,

$$v^5(g''' + 4(q - z)g' + 2q'g) = \sum_{j=0}^{N} \sum_{\ell=0}^{d} (\alpha_{j,\ell} + \beta_{j,\ell} z)c_\ell x^j \tag{10}$$

for suitable numbers N, $\alpha_{j,\ell}$, and $\beta_{j,\ell}$, which depend only on q. From Appell's equation (3) it follows upon differentiation that the expression (10) vanishes identically. This gives rise to a homogeneous system of $N+1$ linear equations for the c_ℓ of which we know that it has a nontrivial solution. Solving the system shows now that the coefficients c_ℓ are rational functions of z, i.e.,

$$c_\ell(z) = \frac{\tilde{c}_\ell(z)}{\gamma(z)}$$

where γ and $\tilde{c}_\ell$ are polynomials in z. Therefore

$$g(z,x) = \frac{\sum_{j=0}^d \tilde{c}_j(z)x^j}{\gamma(z)v(x)^2} = \frac{F(z,x)}{\gamma(z)}$$

where

$$F(z,x) = \frac{\sum_{j=0}^d \tilde{c}_j(z)x^j}{v(x)^2}$$

is a polynomial as function of z and a rational function as function of x. We have therefore proven that the hypotheses of Theorem 4 are satisfied and this shows that q is algebro-geometric.

5.2 Simply Periodic Potentials

Suppose q is meromorphic, simply periodic with period $p \in \mathbb{C}$, and bounded in $\{x : |\operatorname{Im}(x/p)| \geq R\}$ for some $R > 0$. Lemma 6 implies firstly that, for all values of z all solutions of $y'' + qy = zy$ are meromorphic. To simplify notation we assume without loss of generality that the fundamental period p of q is equal to 2π. Define $q^* : \mathbb{C} - \{0\} \to \mathbb{C}^\infty$ by $q^*(t) = q(-i\log t)$. Because of the periodicity of q the function q^* is well-defined and meromorphic. Since $q(x)$ remains bounded as $|\operatorname{Im}(x)|$ tends to infinity the points zero and infinity are removable singularities of q^* and hence q^* is a rational function which is bounded at infinity and zero. Denoting its poles by $t_1, ..., t_m$ we may write

$$q^*(t) = z_0 + \sum_{j=1}^m \sum_{k=1}^{N_j} \frac{t_j^k A_{j,k}}{(t-t_j)^k}$$

where $t_1, ..., t_m$ are distinct nonzero complex numbers. Let x_j be any complex number such that $\mathrm{e}^{ix_j} = t_j$. Then we obtain that

$$q(x) = q^*(\mathrm{e}^{ix}) = z_0 + \sum_{j=1}^m \sum_{k=1}^{N_j} \frac{A_{j,k}}{(\mathrm{e}^{i(x-x_j)} - 1)^k}.$$

Since

$$\mathrm{e}^{i(x-x_j)} - 1 = i(x-x_j)(1 + \frac{i}{2}(x-x_j) + O((x-x_j)^2))$$

we obtain from Lemma 6 that $N_j = 2$ and $A_{j,1} = A_{j,2} = s_j(s_j+1)$ with $s_j \in \mathbb{N}$. Hence

$$q^*(t) = z_0 + \sum_{j=1}^m s_j(s_j+1)\frac{tt_j}{(t-t_j)^2}.$$

In particular $q^*(0) = q^*(\infty) = z_0$.
From Floquet's theorem we know that there are solutions (called Floquet solutions) of $y'' + qy = zy$ of the form

$$\psi_\pm(z,x) = p_\pm(z,x)\mathrm{e}^{\pm i\lambda x}$$

where $p_\pm$ are periodic functions with period 2π and λ is a suitable complex number depending on z which is determined up to addition of an arbitrary integer. Unless 2λ is an integer which happens only for an isolated set of values of z the solutions $\psi_\pm$ are linearly independent.
The functions $\psi_\pm(z,\cdot)$ are meromorphic by assumption. Their poles are at the singularities of the differential equation, i.e., at the poles of q. Because the indices of the singularities $x_j = -i\log t_j$ are $-s_j$ and s_j+1 the functions given by

$$\psi_\pm(z,x)\mathrm{e}^{\mp i\lambda x}\prod_{j=1}^m(\mathrm{e}^{ix} - \mathrm{e}^{ix_j})^{s_j}$$

are entire and periodic functions of period 2π.
Define

$$v(x) = \prod_{j=1}^m(\mathrm{e}^{ix} - \mathrm{e}^{ix_j})^{s_j}.$$

The substitution $y = u\mathrm{e}^{i\lambda x}/v$ transforms $y'' + qy = zy$ into

$$v^2u'' + (2i\lambda v^2 - 2vv')u' + ((-\lambda^2 - z + q)v^2 - 2i\lambda vv' + 2v'^2 - vv'')u = 0 \tag{11}$$

which has entire solutions at least one of which is periodic with period 2π.
Next define $v^*(t) = v(-i\log t) = \prod_{j=1}^m(t - t_j)^{s_j}$ and substitute $u(x) = u^*(t)$ where $x = -i\log t$ in (11) to obtain $u'(x) = itu^{*\prime}(t)$, $u''(x) = -t^2u^{*\prime\prime}(t) - tu^{*\prime}(t)$ and hence

$$Q_0u^{*\prime\prime} + Q_1u^{*\prime} + Q_2u^* = 0 \tag{12}$$

where

$$\begin{aligned}
Q_0 &= t^2v^{*2},\\
Q_1 &= t((1+2\lambda)v^{*2} - 2tv^*v^{*\prime}),\\
Q_2 &= (z - q^* + \lambda^2)v^{*2} - (2\lambda+1)tv^*v^{*\prime} + 2t^2v^{*\prime 2} - t^2v^*v^{*\prime\prime}.
\end{aligned}$$

Because equation (11) has an entire 2π-periodic solution equation (12) has a solution which is analytic on $\mathbb{C} - \{0\}$, i.e., a solution for which zero and infinity are isolated singularities.
Since $v^*(0) \neq 0$ the point zero is a regular singular point of (12) with indicial equation

$$r^2 + 2\lambda r + z - z_0 + \lambda^2 = 0. \tag{13}$$

This equation must have at least one integer solution since otherwise no solution of (12) would be one-valued, i.e., zero would not be an isolated singularity. Thus suppose the solutions of (13) are m and $-2\lambda - m$ where $m \in \mathbb{Z}$. Then $-2\lambda m - m^2 = z - z_0 + \lambda^2$ which implies $\lambda = -m \pm i\sqrt{z - z_0}$. As we are free to change λ by adding an integer we may assume from now on that $\lambda^2 = z_0 - z$ and that the zeros of the indicial equation (13) are zero and -2λ.

Next turn to the point infinity. After introducing $1/t$ as independent variable it turns out that infinity is also a regular singular point with indicial equation

$$r^2 + (2S - 2\lambda)r + S^2 - 2\lambda S = 0 \tag{14}$$

where $S = \sum_{j=1}^m s_j = \deg v^*$. The solutions of (14) are $-S$ and $2\lambda - S$.

Now, if 2λ is not an integer then (11) has precisely one linearly independent 2π-periodic solution. Hence (12) has precisely one single-valued analytic solution in $\mathbb{C} - \{0\}$. This must therefore be the solution associated with the indices 0 and $-S$ at zero and infinity, respectively, i.e., this solution is a polynomial of degree S.

Repeating the above procedure after replacing λ by $-\lambda$ we now obtain that the differential equation $y'' + qy = zy$ has the solutions

$$y_\pm(z, x) = \frac{u^*_\pm(z, \mathrm{e}^{ix})}{v^*(\mathrm{e}^{ix})} \exp(\pm i\lambda x)$$

where $u^*_+(z, \cdot)$ and v^* are polynomials. These solutions are linearly independent except at an at most countable number of isolated points z.

Again define the function $g(z, x) = y_+(z, x)y_-(z, x)$. Then

$$v(x)^2 g(z, x) = u^*_+(z, \mathrm{e}^{ix})u^*_-(z, \mathrm{e}^{ix}) = \sum_{j=0}^{d} c_j(z)\mathrm{e}^{ijx}.$$

The functions $v^2 g(z, x)$, $v^3 g'(z, x)$, $v^4 g''(z, x)$, and $v^5 g'''(z, x)$ are now polynomials in e^{ix} whose coefficients are homogeneous polynomials of degree one in $c_0, ..., c_d$ and so is the function $v^5(g''' + 4(q - z)g' + 2q'g)$. Specifically,

$$v^5(g''' + 4(q - z)g' + 2q'g) = \sum_{j=0}^{N} \sum_{\ell=0}^{d} (\alpha_{j,\ell} + \beta_{j,\ell} z)c_\ell \mathrm{e}^{ijx}. \tag{15}$$

As the expression (15) must vanish identically we obtain again a system of linear equations which we use to show that the coefficients c_ℓ are rational functions of z. Therefore $g(z, x) = F(z, x)/\gamma(z)$ where $F(z, x)$ is a polynomial as function of z and a rational function as function of e^{ix}. Theorem 4 gives that q is algebro-geometric.

5.3 Elliptic potentials

Finally let q be elliptic with fundamental periods $2\omega_1$ and $2\omega_3$. Assume that none of the poles of q equals zero or a half-period (modulo the fundamental period parallelogram) which may always be achieved by a slight shift of the independent variable. Then, by Lemma 6 and general properties of elliptic functions,

$$q(x) = \frac{q_1(\wp(x)) + q_2(\wp(x))\wp'(x)}{\prod_{j=1}^{m}(\wp(x) - p_j)^2}$$

for suitable polynomials q_1 and q_2 and suitable numbers m and $p_1, ..., p_m$. Let

$$v(x) = \prod_{j=1}^{m}(\wp(x) - p_j)^{s_j}$$

where $-s_j < 0$ and $s_j + 1 > 0$ are the indices of the singularity x_j for which $\wp(x_j) = p_j$. Then $v^2 q$ and $v^3 q'$ are polynomials in $\wp(x)$ and $\wp'(x)$.
Picard's theorem guarantees the existence of two linearly independent solutions $y_\pm(z, \cdot)$ of $y'' + qy = zy$ which are elliptic of the second kind for all but an at most countable number of isolated points z since we then have different Floquet multipliers with respect to $2\omega_1$. ¿From Floquet theory we know that the product of these solutions must be doubly periodic since the product of Floquet multipliers with respect to any period is equal to one in our case. As all solutions are meromorphic (by Lemma 6) we have that $g(z, \cdot)$ the product of $y_+(z, \cdot)$ and $y_-(z, \cdot)$ is elliptic. Therefore and since the only poles of $g(z, \cdot)$ are at the points where $\wp(x) = p_j$ and have order at most $2s_j$ we get

$$g(z, x) = \frac{g_1(z, \wp(x)) + g_2(z, \wp(x))\wp'(x)}{v(x)^2}$$

where $g_1(z, \cdot)$ and $g_2(z, \cdot)$ are polynomials. Introduce the coefficients $c_0, ..., c_d$ by

$$g_1(z, t) = \sum_{j=0}^{\delta} c_j(z)t^j, \quad g_2(z, t) = \sum_{j=\delta+1}^{d} c_j(z)t^{j-\delta-1}.$$

Each of the functions $v^2 g$, $v^3 g'$, $v^4 g''$, and $v^5 g'''$ are now of the form $\phi_1(\wp) + \phi_2(\wp)\wp'$ where ϕ_1 and ϕ_2 represent various polynomials. The coefficients of these are homogeneous polynomials of degree one in $c_0, ..., c_d$. Therefore $v^5(g''' + 4(q - z)g' + 2q'g) = h_1(\wp(x)) + h_2(\wp(x))\wp'(x)$ where h_1 and h_2 are polynomials whose coefficients are polynomials in the variables $z, c_0, ..., c_d$ homogeneous of degree one with respect to $c_0, ..., c_d$ and of at most first order with respect to z. This implies just as before that the c_j are rational functions of z and proves that $g(z, x) = F(z, x)/\gamma(z)$ where $F(z, x)$ is a polynomial as function of z and a rational function as function of $\wp(x)$ and $\wp'(x)$. Theorem 4 gives that q is algebro-geometric.

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