Hicas of Length ≤ 4

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Abstract. A *hica* is a highest weight, homogeneous, indecomposable, Calabi-Yau category of dimension 0. A hica has length l if its objects have Loewy length l and smaller. We classify hicas of length ≤ 4, up to equivalence, and study their properties. Over a fixed field F, we prove that hicas of length 4 are in one-one correspondence with bipartite graphs. We prove that an algebra A_Γ controlling the hica associated to a bipartite graph Γ is Koszul, if and only if Γ is not a simply laced Dynkin graph, if and only if the quadratic dual of A_Γ is Calabi-Yau of dimension 3.

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1. Introduction

Once a mathematical definition has been made, the theory surrounding that definition usually begins with a study of small examples. A striking violation of this principle occurred at the birth of category theory, where early theory was concerned with establishing results valid for large and floppy mathematical structures like the category of sets, or the category of groups, or the category of topological spaces. But time has passed, categories have begun to be taken seriously, and they are now objects of detailed study. Since categories are often large and floppy, the 2-category of all categories is very large and very floppy. To prove theorems about categories, it is necessary to make strong restrictions on their structure. To prove classification theorems for categories, it is necessary to make very strong restrictions on their structure.

There is by now an extensive collection of categorical classification theorems. A category with one object and invertible morphisms is a group, and there are many examples of classification theorems in group theory. Rings are endowed with various categories, like their module categories. Classification theorems for commutative rings can be thought of as classification theorems in algebraic geometry. There are a number of classification theorems for rings of finite homological dimension, to which the term noncommutative geometry is applied. For example, hereditary algebras over an algebraically closed field can be parametrised by quivers. Calabi-Yau algebras of dimensions 2 and 3 can be loosely parametrised by quivers with a superpotential [2], [5], [8]. Categorical classification theorems also appear in the representation theory of 2-categories: irreducible integrable representations of 2-Kac-Moody Lie algebras can be parametrised by integral dominant weights [18].

Our paper runs in this vein. A *hica* is a highest weight, homogeneous, indecomposable, Calabi-Yau category of dimension 0. Here, we say a highest weight category is homogeneous if its standard objects all have the same Loewy length, and its costandard objects all have the same Loewy length. We say a hica has length l if its projective objects have Loewy length l and smaller. We classify hicas of length ≤ 4 up to equivalence.

Hicas show up naturally in group representation theory and in the theory of tilings [20, 3, 14, 15]. A multitude of examples of hicas were constructed by Mazorchuk and Miemietz [13]. Every hica can be realised as the module category of some symmetric quasi-hereditary algebra. If the hica is not semisimple, the corresponding algebra is necessarily infinite dimensional, noncommutative, of infinite homological dimension.

Let us fix a field F, and consider hicas over F, up to equivalence. The only hica of length 1 is the category of vector spaces over F. There are no hicas of length 2. There is a unique hica of length 3, which is the module category of the Brauer tree algebra on a bi-infinite line. Our first main result is

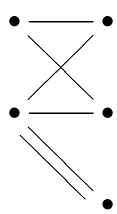
Theorem 1. *There is a natural one-one correspondence*

$$\{\textit{bipartite graphs}\} \leftrightarrow \{\textit{hicas of length } 4\}.$$

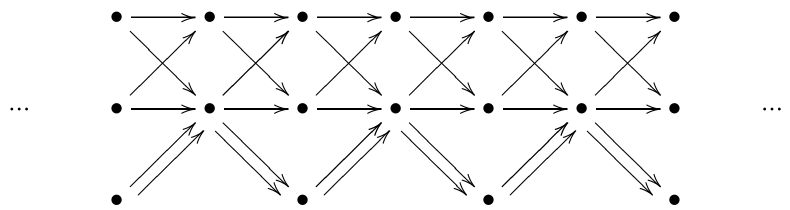
Here, and throughout this paper, a bipartite graph will by definition be connected.

The one-one correspondence of Theorem 1 is obtained from a sequence of three one-one correspondences: a one-one correspondence between bipartite graphs and topsy-turvy quivers; a one-one correspondence between topsy-turvy quivers and basic indecomposable self-injective directed algebras of Loewy length 3; and a one-one correspondence between basic self-injective directed algebras of Loewy length 3 and hicas of length 4.

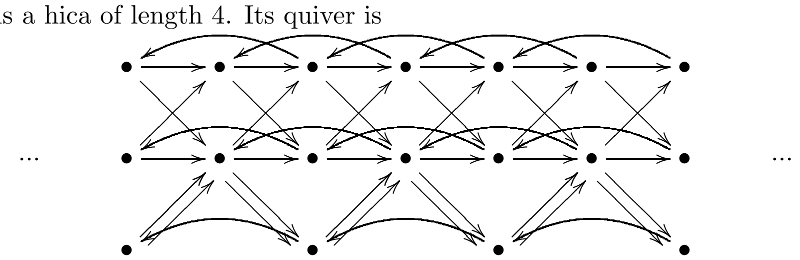
Let us describe here the construction of a hica $\mathcal{C}_\Gamma$ of length 4 from the following bipartite graph Γ:



First, we construct a quiver Q_Γ, by consecutively gluing together opposite orientations of this bipartite graph, one next to the other:



This quiver has an automorphism ϕ which shifts a vertex to a vertex which is two steps horizontally to its right. We take the path algebra of this quiver. We now construct a self-injective directed algebra B_Γ of Loewy length 3, factoring the path algebra by relations which insist that all squares commute, and that paths $u \to v \to w$ of length 2 are zero, unless $w = \phi(u)$. We define A_Γ to be the trivial extension $B_\Gamma \oplus B_\Gamma^*$ of B_Γ by its dual. The module category $\mathcal{C}_\Gamma$ of A_Γ is a hica of length 4. Its quiver is



The relations for A_Γ are those for B_Γ, along with relations which insist that the product of two leftwards pointing arrows is zero whilst squares involving a pair of parallel leftwards pointing arrows commute. The algebra A_Γ has some pleasing properties. It admits a derived self-equivalence ψ_γ for every vertex γ of Γ. It also admits a number of $\mathbb{Z}_+^3$-gradings, one for each orientation of the graph Γ. It is Koszul and its quadratic dual $A_\Gamma^!$ is Calabi-Yau of dimension 3. More generally, we have the following theorem.

Theorem 2. *Suppose Γ is a connected bipartite graph, and $\mathcal{C}_\Gamma = A_\Gamma$-mod the associated hica of length 4. The following are equivalent:*

1. *Γ is not a simply laced Dynkin graph.*
2. *A_Γ is Koszul.*
3. *The quadratic dual of A_Γ is Calabi-Yau of dimension 3.*

The way this paper evolved was surprising to us. We began with the problem of classifying small hicas, categories whose structural features (Calabi-Yau 0, highest weight) were motivated by exposure to group representation theory. We ended having made contact with mathematics of different kin: bipartite graphs, Calabi-Yau 3s, and Dynkin classifications. The hica restrictions indeed capture some features of Lie theoretic representation theory, but they can also be thought of as noncommutative geometric restrictions: highest weight categories were invented to capture stratification properties appearing in algebraic geometry, whilst 0-Calabi-Yau categories are categories possessing a homological duality with trivial Serre functor.

2. Preliminaries

Our main objects of study, *hicas*, are a species of abelian categories. As we study them, we will use freely the languages of abelian categories, algebras, and triangulated categories. Here we give a short phrasebook for these languages.
Let F be a field. The collection of F-algebras is a 2-category, whose arrows are bimodules ${}_AM_B$ which are flat on the right, and 2-arrows are bimodule homomorphisms. We have a 2-functor

$$\mathfrak{Algebra} \to \mathfrak{Abelian}$$

from the 2-category $\mathfrak{Algebra}$ of F-algebras to the 2-category $\mathfrak{Abelian}$ of abelian categories. This 2-functor takes an algebra A to its module category, a bimodule ${}_AM_B$ to the functor $M \otimes_B -$, and a bimodule homomorphism to a natural transformation. We have a 2-functor

$$\mathfrak{Abelian} \to \mathfrak{Triangulated}$$

taking values in the 2-category of triangulated categories, which takes an abelian category $\mathcal{A}$ to its derived category $D(\mathcal{A})$.
If X is an object of an abelian category of finite composition length, we define the *Loewy length* of X (or *length* of X, or $l(X)$) to be the smallest number l for which there exists a filtration of X with l nonzero sections, all of which are semisimple. We define the *head*, or *top* of X to be the maximal semisimple quotient of X, and the *socle* of X to be the maximal semisimple submodule. If $\mathcal{A}$ is an abelian category, we define the *length* of $\mathcal{A}$ to be the supremum over all lengths of objects in $\mathcal{A}$. If A is an algebra, we define the *length* of A to be the length of the abelian category A-mod of A-modules.
Given a finite dimensional F-vector space V, we denote by V^* the dual $Hom_F(V, F)$ of V. We call an object X of a triangulated category *compact* if $\mathrm{Hom}(X, -)$ commutes with infinite direct sums. We say an F-linear triangulated category $\mathcal{T}$ is *Calabi-Yau of dimension* d if $\mathrm{Hom}_{\mathcal{T}}(P, X)$ is finite dimensional for objects $X \in \mathcal{T}$, and compact objects $P \in \mathcal{T}$, and

$$\mathrm{Hom}_{\mathcal{T}}(P, X) \cong \mathrm{Hom}_{\mathcal{T}}(X, P[d])^*$$

naturally in objects $X \in \mathcal{T}$, and compact objects $P \in \mathcal{T}$. For background, we recommend a survey article of B. Keller concerning Calabi-Yau triangulated categories [8]. To avoid confusion here, let us emphasise that the definition of a Calabi-Yau triangulated category Keller uses is slightly different from this one since he makes no compactness assumption on P.
We say an F-linear abelian category $\mathcal{A}$ is Calabi-Yau of dimension d if its derived category $D(\mathcal{A})$ is Calabi-Yau of dimension d. We say an F-algebra A is Calabi-Yau of dimension d if its module category A-mod is Calabi-Yau of dimension d.

Suppose A is a basic (not necessarily unital) F-algebra satisfying the following assumptions:

(i) A has a countable set $\{e_x \mid x \in \Lambda\}$ of orthogonal primitive idempotents, such that $A = \oplus_{x,y} e_x A e_y$;
(ii) for any $x, y \in \Lambda$ the F-vector space $e_x A e_y$ is finite dimensional;
(iii) for any $x \in \Lambda$ there exist only finitely many $y \in \Lambda$ such that $e_x A e_y \neq 0$;
(iv) for any $x \in \Lambda$ there exist only finitely many $y \in \Lambda$ such that $e_y A e_x \neq 0$.

Under these assumptions all indecomposable projective A-modules Ae_x and all injective A-modules $\mathrm{Hom}_F(e_x A, F)$ are finite-dimensional. A-modules $M = {}_A M$ will be left A-modules unless they carry a right subscript as in M_A in which case they will be right A-modules. We denote by A-mod the collection of all finite-dimensional left A-modules and by mod-A the collection of all finite-dimensional right A-modules. We denote by A-perf the subcategory of the derived category of A-mod consisting of perfect complexes, that is the smallest thick subcategory of the derived category of A-mod containing all projective objects of A-mod, or equivalently the subcategory of compact objects in the derived category of A. We define A^* to be the A-A-bimodule $\bigoplus_{x \in \Lambda} \mathrm{Hom}_F(Ae_x, F)$.
We say A is a *symmetric algebra* if $A \cong A^*$ as A-A-bimodules. Then A is symmetric if and only if A-mod is Calabi-Yau of dimension 0 (cf. [17], Theorem 3.1).
Suppose A is an algebra satisfying the above conditions, and Λ is ordered. For $\lambda \in \Lambda$, let $J_{\geq \lambda} = \sum_{\mu \geq \lambda} Ae_\mu A$ and $J_{>\lambda} = \sum_{\mu > \lambda} Ae_\mu A$. Let $J_\lambda = J_{\geq \lambda}/J_{>\lambda}$. We say A is *quasi-hereditary* if the product map $J_\lambda e_\lambda \otimes_F e_\lambda J_\lambda \to J_\lambda$ is an isomorphism for every $\lambda \in \Lambda$ [4].

Now suppose $\mathcal{A}$ is an abelian category over F, with enough projective objects, enough injective objects, and a countable set Λ indexing the isomorphism classes of simple objects of $\mathcal{A}$, such that all objects of $\mathcal{A}$ have a finite composition series with sections in Λ. Abusing notation, an element λ of Λ we sometimes take to represent an index, sometimes an isomorphism class of irreducible object, and sometimes a representative of the latter. We denote by $P(\lambda)$ a minimal projective cover of λ in $\mathcal{A}$. Such exist, since we have enough projectives, and finite composition series.
We call $\mathcal{A}$ a *highest weight category* [4] if there is an ordering $<$ on Λ, and a collection of objects $\Delta(\lambda)$, for $\lambda \in \Lambda$, such that

(i) there is an epimorphism $\Delta(\lambda) \twoheadrightarrow \lambda$ whose kernel $X(\lambda)$ has composition factors $\mu < \lambda$;
(ii) $P(\lambda)$ has a filtration with a single section isomorphic to $\Delta(\lambda)$ and every other section isomorphic to $\Delta(\mu)$, for $\mu > \lambda$.

If A is quasi-hereditary, then A-mod is a highest weight category, with standard objects ${}_A\Delta(\lambda) = J_\lambda e_\lambda$, and mod-$A$ is a highest weight category with standard modules $\Delta_A(\lambda) = e_\lambda J_\lambda$. Thus A has a filtration by ideals, whose sections are isomorphic to

$$ {}_A\Delta(\lambda) \otimes_F \Delta_A(\lambda). $$

Conversely, if $\mathcal{A}$ is a highest weight category, then $A = \oplus_{\lambda,\mu\in\Lambda} Hom(P(\lambda), P(\mu))$ is a quasi-hereditary algebra.
The left and right costandard modules ${}_A\nabla(\lambda)$, $\nabla(\lambda)_A$ of A are defined to be the duals of the right and left standard modules $\nabla(\lambda)_A$, ${}_A\nabla(\lambda)$ of A. We write $\Delta = \oplus_\lambda \Delta(\lambda)$ and $\nabla = \oplus_\lambda \nabla(\lambda)$.

Lemma 3. *Let A be a selfinjective quasi-hereditary algebra. If A is not semisimple, then A is infinite dimensional.*

Proof. Nonsemisimple selfinjective algebras have infinite homological dimension, since Heller translation is invertible. Finite dimensional quasi-hereditary algebras have finite homological dimension. □

We say a highest weight category $\mathcal{C}$ is homogeneous if its standard objects all have the same Loewy length, and its costandard objects all have the same Loewy length. Equivalently, $\mathcal{C} = A$-mod, where A is a quasi-hereditary algebra whose left standard modules all have the same Loewy length, and whose right standard modules all have the same Loewy length.

Definition 4. *A* hica *is is a <u>hi</u>ghest weight, <u>h</u>omogeneous, <u>i</u>ndecomposable <u>Ca</u>labi-Yau <u>ca</u>tegory of dimension* 0.

The collection $\mathfrak{Hica}$ of hicas forms a 2-category (arrows are exact functors, 2-arrows are natural transformations). We denote by $\mathfrak{Hica}_l$ the 2-category of hicas of length l.

Lemma 5. *The* 2*-functor*

$$ \{\textit{symmetric, homogeneous, quasihereditary basic algebras}\} \twoheadrightarrow \mathfrak{Hica} $$

which takes an algebra to its module category is essentially bijective on objects.

Proof. We must define a correspondence between objects of our 2-categories, under which isomorphic algebras correspond to equivalent categories, and vice versa. If A is a symmetric, Δ-homogeneous quasihereditary algebra then A-mod is a hica ([4], [17], Theorem 3.1). If $\mathcal{C}$ is a hica, then $A = \oplus_{\lambda\in\Lambda} \operatorname{Hom}(P(\lambda), P(\mu))$ is an algebra such that A-mod $= \mathcal{C}$. □

A highest weight category $\mathcal{C}$ has a collection of indecomposable tilting modules $T(\lambda)$ indexed by Λ, characterised as indecomposable objects with a Δ-filtration and a ∇-filtration. The *Ringel dual* $\mathcal{C}'$ of $\mathcal{C}$ is the module category A'-mod of the algebra

$$ A' = \oplus_{\lambda,\mu} \operatorname{Hom}_{\mathcal{C}}(T(\lambda), T(\mu)). $$

The Ringel dual $\mathcal{C}'$ of $\mathcal{C}$ is a highest weight category. If $\mathcal{C} = A$-mod, we call A' the Ringel dual of A. If $\mathcal{C} \cong \mathcal{C}'$ then we say $\mathcal{C}$ and A are *Ringel self-dual*.

Lemma 6. *Suppose $\mathcal{C} = A$*-mod *is a hica. Then*

$$ l(A) = l({}_A\Delta) + l(\Delta_A) - 1. $$

Proof. The length of A is the least number l such that the product of any l elements of the radical of A is zero. This can be otherwise defined as the radical length of the $A \otimes A^{op}$-module A. Since A is quasi-hereditary, ${}_AA_A$ has a bimodule filtration with sections ${}_A\Delta(\lambda) \otimes_F \Delta_A(\lambda)$. These sections have radical length $l({}_A\Delta) + l(\Delta_A) - 1$, as $A \otimes A^{op}$-modules. Therefore the Loewy length of A is at least $l({}_A\Delta) + l(\Delta_A) - 1$.
The tops of all of these sections lie in the top of ${}_AA_A$. Since A is symmetric, every irreducible lies in the socle of A. Since A is also quasi-hereditary, every irreducible lies in the socle of some standard object Δ. Given $\lambda \in \Lambda$, the socle Fx_λ of Ae_λ is generated by $\mathrm{soc}({}_A\Delta(\nu)) \otimes_F \mathrm{soc}(\Delta_A(\nu))$, for suitable ν, modulo lower terms in the filtration. The lower terms in the filtration have zero intersection with Fx_λ, since this space is one dimensional. Therefore, lifting an element of $\mathrm{soc}({}_A\Delta(\nu)) \otimes \mathrm{soc}(\Delta_A(\nu))$ to an element of radical length $l({}_A\Delta) + l(\Delta_A) - 1$ in A, we obtain an element of Fx_λ of radical length $l({}_A\Delta) + l(\Delta_A) - 1$. It follows that the Loewy length of A is at most $l({}_A\Delta) + l(\Delta_A) - 1$. □

We also wish to consider graded algebras, which may satisfy weaker assumptions than those given above. If G is a group, and A an algebra, then a G-grading of A is a decomposition $A = \oplus_{g\in G} A^g$, such that $A^g.A^h \subset A^{gh}$. A graded A-module is an A module with a decomposition $M = \oplus_{g\in G} M^g$, such that $A^g.M^h \subset M^{gh}$; a homomorphism $\phi : M \to N$ of graded modules is an A-module homomorphism sending M^g to N^g, for $g \in G$.
We say A is $\mathbb{Z}_+$-graded if it is $\mathbb{Z}$-graded, with $A^i = 0$ for $i < 0$. Suppose A a $\mathbb{Z}$-graded algebra, whose degree 0 part A^0 satisfies the conditions (i)-(iv) above. Then we denote by A-mod the abelian subcategory of the category of all A-modules generated by A^0-mod, and by A-gr the abelian subcategory of the category of all graded A-modules generated by the category of finite dimensional A^0-mod$\langle i \rangle$, for $i \in \mathbb{Z}$. We denote by A-grperf the thick subcategory of the the derived category of graded A-modules generated by objects of the form $A \otimes_{A^0} X\langle i \rangle$, where $X \in A^0$-mod and $i \in \mathbb{Z}$.

3. Elementary constructions

Let us give some elementary constructions of symmetric algebras.
Suppose B is an algebra. Let $A = T(B)$ denote the trivial extension of B by B^*. Then A is symmetric, and A-mod is Calabi-Yau of dimension 0.
Suppose B is an algebra and M is a B-B-bimodule such that $e_\lambda M e_\mu$ is finite-dimensional for every $\lambda, \mu \in \Lambda$ and such that for every λ only finitely many of $e_\lambda M e_\mu$ and $e_\mu M e_\lambda$ are non-zero. Define $M^* := \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_F(Me_\lambda, F)$ and assume we have a fixed bimodule isomorphism $M \cong M^*$. Then we have a

sequence of bimodule homomorphisms

$$\begin{aligned} B \to \mathrm{Hom}_B(M,M) \cong \mathrm{Hom}_B(M,M^*) &= \mathrm{Hom}_B(M, \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}(Me_\lambda, F)) \\ &\cong \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_B(M, \mathrm{Hom}(Me_\lambda, F)) \\ &\cong \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_F(M \otimes_B Me_\lambda, F) \\ &= \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_F((M \otimes_B M)e_\lambda, F) \\ &= (M \otimes_B M)^*, \end{aligned}$$

noting that $M \otimes_B M$ satisfies that $e_\lambda M \otimes_B Me_\mu = \bigoplus_{\nu\in\Lambda} e_\lambda Me_\nu \otimes_B e_\nu Me_\mu$ is finite-dimensional (finitely many finite-dimensional direct summands) for all λ, μ, and for every λ only finitely many of $e_\lambda M \otimes_B Me_\mu$ and $e_\mu M \otimes_B Me_\lambda$ are nonzero. The obtained bimodule homomorphisms compose to give a bimodule homomorphism $B \to (M \otimes_B M)^*$. Let $\mu : M \otimes_B M \to B^*$ denote the dual map.

Associated to the data (B, M), we have a $\mathbb{Z}$-graded algebra $U = U(B, M)$ concentrated in degrees 0,1, and 2 whose degree 0,1,2 part is B,M, B^* respectively. The product map $U^0 \otimes U^i \to U^i$ is given by the left action of B on the bimodule U^i, for $i = 0, 1, 2$. The product map $U^i \otimes U^0 \to U^i$ is given by the right action of B on the bimodule U^i. We define the product $U^1 \otimes_{U^0} U^1 \to U^2$ to be given by μ. The product is associative since the product of three components $U^i \otimes U^j \otimes U^k$ is non-zero if and only if $i + j + k \leq 2$, in which case associativity is clearly visible.

Lemma 7. *$U(B, M)$-mod is Calabi-Yau of dimension zero.*

Proof. We have a bimodule isomorphism $U \cong U^*$ which exchanges U^0 and U^2, and sends U^1 to U^{1*} via the fixed isomorphism $M \cong M^*$. □

4. Topsy-turvy quivers

Given a vertex w in a quiver Q, let $\mathcal{P}(w)$ denote the collection of vertices v of Q for which there is an arrow pointing from v to w (the *past* of w), counted with multiplicity. Let $\mathcal{F}(u)$ denote the collection of vertices v of Q for which there is an arrow pointing from u to v (the *future* of u), counted with multiplicity.

Definition 8. *A connected quiver is* topsy-turvy *if it contains at least one arrow, and there is an automorphism ϕ of the vertices of Q such that $\mathcal{F}(u) = \mathcal{P}(u^\phi)$ for every vertex u of Q.*

For any topsy-turvy quiver, the automorphism ϕ extends to a quiver automorphism, since arrows from x to y can be placed in bijection with arrows from y to x^ϕ, which can be placed in bijection with arrows from x^ϕ to y^ϕ.

Lemma 9. *If Q is a topsy-turvy quiver, then $\mathcal{PF}(w) = \mathcal{FP}(w)$ for all vertices w of Q.*

Proof. Any x in $\mathcal{FP}(w)$ lies in the future of some u in the past of w, and therefore lies in the past of u^ϕ; since Q is topsy-turvy, u^ϕ also lies in the future of w and x lies in $\mathcal{PF}(w)$. By symmetry, if x lies in $\mathcal{PF}(w)$ then x also lies in $\mathcal{FP}(w)$. □

A directed topsy-turvy quiver Q can be $\mathbb{Z}$-graded in the following way: take an arbitrary vertex u of Q and place it in degree 0. We say another vertex v in Q is in degree k if there exist $i_1, \dots, i_r$ and $j_1, \dots, j_r$ such that $v \in \mathcal{P}^{i_1}\mathcal{F}^{j_1} \dots \mathcal{P}^{i_r}\mathcal{F}^{j_r}(u)$ and $\sum_{1\leq s\leq r} j_s - \sum_{1\leq s\leq r} i_s = k$. This is well-defined since $\mathcal{PF}(w) = \mathcal{FP}(w)$. It follows that all arrows in Q point from degree i to degree $i+1$ and that ϕ has degree 2.
A *bipartite graph* is a countable connected graph Γ whose set V of vertices decomposes into two nonempty subsets $V = V_l \cup \Gamma_r$ such that no edges of Γ connect V_l to V_l, or V_r to V_r. Note that we do *not* call the graph with one vertex and no arrows bipartite.
Given a graph Γ with a bipartite decomposition of vertices $V = V_l \cup V_r$, we have an associated directed topsy-turvy quiver Q_Γ, obtained by orienting $\mathbb{Z}$ copies of Γ, identifying, for i even, the r-vertices of i^{th} copy of Γ with the r-vertices of the $i+1^{th}$ copy of Γ, the l-vertices of i^{th} copy of Γ with the l-vertices of the $i-1^{th}$ copy of Γ, and insisting that arrows in the i^{th} copy of Γ point from the $i-1^{th}$ copy to the $i+1^{th}$ copy, for $i \in \mathbb{Z}$. Note that if we label our bipartite decomposition with the opposite orientation, we obtain an isomorphic topsy-turvy quiver.

Lemma 10. *We have a one-one correspondence $\Gamma \leftrightarrow Q_\Gamma$ between bipartite graphs and directed topsy-turvy quivers.*

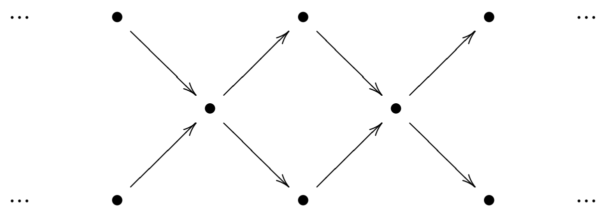
Proof. Given a directed topsy-turvy quiver, we have a $\mathbb{Z}$-grading of the set of vertices $V = \amalg_{i\in\mathbb{Z}} V_i$, see above. Let A_i denote the set of arrows from V_i to V_{i+1}. The set of arrows of our quiver is graded $A = \amalg_{i\in\mathbb{Z}} A_i$. The automorphism ϕ defines isomorphisms between V_i and V_j and between A_i and A_j when i and j are both even, or when i and j are both odd. We can thus identify the V_i for i even with a single vertex set V_{even}, the V_i for i odd with a single vertex set V_{odd}, the A_i for i even with a single arrow set A_{eo} from V_{even} to V_{odd}, the A_i for i odd with a single arrow set A_{oe} from V_{odd} to V_{even}. The topsy-turviness of the quiver means precisely that A_{eo} is the opposite of A_{oe}. We thus obtain a graph with vertices $V_{even} \cup V_{odd}$, and with edges between V_{even} and V_{odd}, such that directing edges from V_{even} to V_{odd} gives us A_{eo} and directing edges from V_{odd} to V_{even} gives us A_{oe}. This is a bipartite graph, by definition.
Reversing the above argument, from any bipartite quiver, we obtain a directed topsy-turvy quiver. □

Example 11 The bipartite graph $\bullet \!-\!\!\!-\!\!\!-\, \bullet$ with two vertices and a single edge results in a topsy-turvy quiver which can be depicted as an oriented line:

$$\cdots \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad \cdots$$

The bipartite graph

results in a topsy-turvy quiver which can be depicted as a directed square lattice in $\mathbb{R}^2$:



The bipartite graph whose vertices are elements of the square lattice lattice in $\mathbb{R}^2$ results in a topsy-turvy quiver whose arrows can be thought of as the diagonals of a face-centred cubic lattice in $\mathbb{R}^3$.

5. Self-injective directed algebras of length ≤ 3

Throughout the following, let B be an indecomposable self-injective directed algebra. Here self-injective means that $B \cong \bigoplus_{x\in\Lambda} \operatorname{Hom}_F(Ae_x, F)$ as left B-modules or, equivalently, that all projective B-modules are also injective, and vice versa. Directed is understood to mean that the Ext^1-quiver of B is a directed quiver.
Note that such an algebra is necessarily infinite-dimensional, since directed implies quasi-hereditary which, in the finite-dimensional case, implies finite global dimension, contradicting self-injectivity.

Lemma 12. *If B is radical-graded, all projective B-modules have the same Loewy length.*

Proof. For finite-dimensional algebras, this was shown in [12, Theorem 3.3]. We remark that the same proof holds for algebras in our setup, as the comparisons of Loewy length only need to be done using neighbouring projectives in the Ext-quiver. □

Let us now assume that B be an indecomposable self-injective algebra of Loewy length ≤ 3.

Lemma 13. *B is radical-graded.*

Proof. Set $A_0 := \bigoplus_{x\in\Lambda} Fe_x \cong A/\operatorname{Rad} A$ realized by the semisimple algebra generated by the idempotents, this is obviously a subalgebra. It acts naturally on the bimodule $A_1 \cong \operatorname{Rad} A/\operatorname{Rad}^2 A$ given by the arrows in the Ext-quiver and on $A_2 := \operatorname{Rad}^2 A$. Obviously the multiplication maps $A_1 \otimes A_1$ to A_2, so A is radical-graded. □

Corollary 14. *All projectives of B have the same Loewy length.*

Lemma 15. *The quiver of B is a topsy-turvy quiver.*

Proof. A projective indecomposable B-module $P(\lambda)$ can be identified with an injective indecomposable B-module $I(\lambda^\phi)$. Here ϕ is a quiver isomorphism, corresponding to the Nakayama automorphism of B. Since B is selfinjective Loewy length 3, elements of $\mathcal{F}(\lambda)$ correspond to composition factors in the heart of $P(\lambda) =_B Be_\lambda$. Switching from left action to right action, we find elements of $\mathcal{P}(\lambda^\phi)$ correspond to composition factors in the heart of $e_\lambda B_B$. Taking duals, we find elements of $\mathcal{P}(\lambda^\phi)$ correspond to composition factors in the heart of $I(\lambda^\phi)$. Since $P(\lambda) = I(\lambda^\phi)$, we conclude $\mathcal{F}(\lambda) = \mathcal{P}(\lambda^\phi)$. Thus B has a topsy-turvy quiver, as required. □

To any topsy-turvy quiver Q, we can associate a self-injective algebra $R(Q)$ of Loewy length 3 by factoring out relations from the path algebra as follows: make products of arrows of Q which do not lie in some $\mathcal{F}(u) \cup \mathcal{P}(u^\phi)$ equal to zero; make squares in $\mathcal{F}(u) \cup \mathcal{P}(u^\phi)$ commute.
Let us now assume B is directed.

Lemma 16. (a) *If B has Loewy length 2, it is isomorphic to the FQ/I, where Q is the infinite quiver*

$$\cdots \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad \cdots$$

and I is the quadratic ideal generated by all paths of length two.

(b) *If B has Loewy length 3, it is given by $R(Q)$, where Q is a directed, topsy-turvy quiver.*

Proof. (a) Obvious.
(b) Since projectives are injectives, both have irreducible head and socle. Since B is directed, projectives have structure

$$\begin{matrix} \lambda \\ \mu_1 \oplus \ldots \oplus \mu_n \\ \nu, \end{matrix}$$

where $\nu < \mu_i < \lambda$ all i. We only have to worry about the nonzero relations. These take the form $ac = \xi bd$, for $\xi \in F^\times$, where a, b are arrows in $F(u)$ and c, d are arrows in $P(u^\phi)$ for some u. We want to remove the scalars ξ from this description.
Let us write $B = FQ/I$. Then Q is topsy-turvy with ϕ described by the Nakayama automorphism of B. Since Q is directed as well, we can give the collection of vertices of our quiver a $\mathbb{Z}$-grading, so that arrows have degree 1, and ϕ has degree 2. We now alter scalars inductively. Arrows from vertices of degree 0 to vertices of degree 1 we leave alone. An arrow a from degree 1 to degree 2 lies in $P(t(a))$, and in no other $P(w)$. Therefore, multiplying arrows between vertices of degree 1 and degree 2 by nonzero scalars if necessary, we can force squares in quiver degree $0, 1, 2$ to commute. Similarly, multiplying arrows in degree $2, 3$ by scalars, we can force squares in quiver degree $1, 2, 3$

to commute. And so on. Working backwards, make squares in degree $-1, 0, 1$ commute and so on. □

Suppose Γ is a bipartite graph. The double quiver of Γ is the quiver which has vertices as Γ and a pair of opposing arrows running along each edge of Γ.

Definition 17. *Let B_Γ denote the self-injective directed algebra $R(Q_\Gamma)$. Let A_Γ denote the trivial extension $T(B_\Gamma)$ of B_Γ. Let $\mathcal{C}_\Gamma$ denote the category A_Γ-mod.*

We define Z_Γ to be the zigzag algebra associated to Γ [7]. It is the path algebra of the double quiver associated to Γ modulo relations insisting that all quadratic paths based at a single vertex are equal, whilst all other quadratic relations are zero. Since the relations are homogeneous, Z_Γ is a $\mathbb{Z}_+$-graded algebra with homogeneous elements graded by path length.

Lemma 18. *The category Z_Γ-mod is Calabi-Yau of dimension 0. We have an equivalence*

$$Z_\Gamma\text{-gr} \simeq B_\Gamma\text{-mod}^{\oplus 2}$$

between the category Z_Γ-gr of graded modules of Z_Γ, taken with respect to the $\mathbb{Z}_+$-grading by path length, and the direct sum of two copies of B_Γ-mod.
Under this equivalence, twisting by the automorphism ϕ of Q_Γ corresponds to a degree shift by 2 in Z_Γ-gr.

Proof. The irreducible objects of Z_Γ-gr are $S\langle i\rangle$, where S is an irreducible Z_Γ-module concentrated in degree 0. There are homomorphisms in Z_Γ-gr between $S\langle i\rangle$ and $T\langle j\rangle$ precisely when $S = T$ and $i = j$. There is an extension in Z_Γ-gr of $S\langle i\rangle$ by $T\langle j\rangle$ precisely when there is an extension between S by T in Z_Γ-mod and $j = i+1$. In particular when there exists such an extension, S corresponds to a vertex in V_l and T corresponds to a vertex in V_r. We thus have two blocks in Z_Γ-gr: one block is generated by $S\langle i\rangle$ where S lies in V_l and i is even or S lies in V_r and i is odd; the other block is generated by $S\langle i\rangle$ where S lies in V_r and i is even or S lies in V_l and i is odd. It is not difficult to see that each block is isomorphic to B_Γ-mod so that the automorphism ϕ corresponds to a degree shift $\langle 2\rangle$. □

For a quiver Q, we define P_Q to be the path algebra of Q, modulo the ideal of all paths of length ≥ 2.

Lemma 19. *For every orientation $\overrightarrow{\Gamma}$ of the bipartite graph Γ, we have an isomorphism*

$$Z_\Gamma \cong T(P_{\overrightarrow{\Gamma}})$$

between Z_Γ and the trivial extension algebra $T(P_{\overrightarrow{\Gamma}})$ of $P_{\overrightarrow{\Gamma}}$ by its dual.

Proof. Projectives for $P_{\overrightarrow{\Gamma}}$ take two shapes: they are either of Loewy length two, hence have a simple top with a certain number of extensions, or they are simple. Similarly injectives are simple in the first case or of length two with a simple socle and a certain number of simples in the top in the second case. Projectives for $T(P_{\overrightarrow{\Gamma}})$ are extensions of projectives for $P_{\overrightarrow{\Gamma}}$ by injectives for the same algebra, hence either of a module of Loewy length two with a certain number of simples in the socle by a simple or of a simple by a module of Loewy length two with a simple socle and some composition factors in the top. In both cases top and socle of the resulting extension have to be simple which forces, in the first case, all of the simples in the socle of the $P_{\vec{\Gamma}}$-projective to extend the simple $P_{\vec{\Gamma}}$-injective, and in the second case, the simple $P_{\vec{\Gamma}}$-projective to extend all the simples in the top of the $P_{\vec{\Gamma}}$-injective. This is the same as saying that for every arrow in $\vec{\Gamma}$ the quiver for $T(P_{\vec{\Gamma}})$ has an arrow in the opposite direction as well, and that all quadratic paths based at a single vertex are the same (we can easily get rid of scalars by rescaling the arrows) while all other quadratic relations are zero. This exactly describes the algebra Z_Γ. □

In this way, every orientation $\vec{\Gamma}$ of the graph Γ defines a $\mathbb{Z}_+^{\{f,a\}}$-grading on Z_Γ, whose f component corresponds to the $\mathbb{Z}_+$-grading of $P_{\vec{\Gamma}}$ by path length, and whose a component corresponds to the $\mathbb{Z}_+$-grading of $T(P_{\vec{\Gamma}})$ which puts $P_{\vec{\Gamma}}$ in degree 0 and its dual in degree 1.

Correspondingly, the orientation $\vec{\Gamma}$ of Γ gives rise to a $\mathbb{Z}_+^{\{f,a\}}$-grading of the associated selfinjective directed algebra B_Γ as follows: define a bigrading of the corresponding topsy-turvy quiver by grading arrows with an f if they run with the orientation $\vec{\Gamma}$ of Γ, and grading them a if they run against the orientation. This grading extends to a $\mathbb{Z}_+^{\{f,a\}}$-grading of B_Γ.

6. HICAS OF LENGTH ≤ 4

The following is a classical statement which holds for any quasi-hereditary algebra:

LEMMA 20. (a) ${}_A\Delta \cong (\nabla_A)^*$
(b) ${}_A\nabla \cong (\Delta_A)^*$

LEMMA 21. *Suppose $\mathcal{C} = A$-mod is a highest weight category which is Calabi-Yau of dimension 0, and Ringel self-dual. Then A is quasi-hereditary with respect to two orders, denoted $\blacktriangle$ and $\blacktriangledown$, and we have*

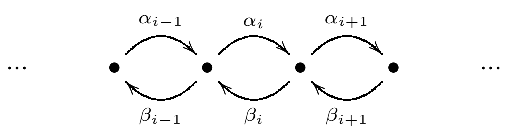
(a) ${}_A\Delta^{\blacktriangle} \cong {}_A\nabla^{\blacktriangledown}$
(b) ${}_A\Delta^{\blacktriangledown} \cong {}_A\nabla^{\blacktriangle}$

Proof. Let us suppose the quasi-hereditary structure on A is given by the partial order $\blacktriangle$, and the one induced by Ringel duality is $\blacktriangledown$. Since A is Ringel self-dual, we have an isomorphism $A \cong A'$. Say that under this homomorphism the right projective e_xA corresponding to $x \in \Lambda$ goes to the right projective e'_yA' for some $y \in \Lambda$. Then by $\mathrm{Hom}_A(Ae_x, A) \cong e_xA \cong e'_yA' = \mathrm{Hom}_A(T(y), A)$ for $T(y)$ the tilting module for y and the fact that any projective for A is also injective and therefore tilting, it follows that $T(y) = P(x)$. So all tilting modules are projective A-modules. So, there is a 1-1-correspondence between tilting modules and projective modules for A, say it is, in the above scenario given by $y = \sharp x$. In particular this gives a one-to-one correspondence between standard modules and their socles $x = \operatorname{soc} \Delta^{\blacktriangle}(\sharp x)$. This makes the definition $\Delta^{\blacktriangledown}(x) := \nabla^{\blacktriangle}(\sharp x)$ well-defined. Filtrations of projectives by $\Delta^{\blacktriangledown}$s as well as the respective ordering conditions follow immediately from the dual statments for injectives (=projectives) and $\nabla^{\blacktriangle}$s. □

We wish to classify hicas of length ≤ 4. To warm up, let us classify hicas of length ≤ 3.

Lemma 22. *Hicas of length* 1 *are semisimple. There are no hicas of length* 2*. There is a unique hica of length* 3*, which is the module category of the Brauer tree algebra associated to a bi-infinite line.*

Proof. Length 1 hicas are trivially semisimple.
Suppose $\mathcal{C} = A$-mod is a hica of length 2. Standard objects in $\mathcal{C}$ must have length 2, since $\mathcal{C}$ is indecomposable, but not semisimple. Since $\mathcal{C}$ itself has length 2, all projective objects in $\mathcal{C}$ also have length 2. Thus standard objects are projective, and the socle of a projective indecomposable object Ae_x has irreducible summands indexed by elements y of Λ with $y < x$. Since A is a symmetric algebra, the top and socle of Ae_x are equal, which is a contradiction. Therefore there are no hicas of length 2.
Suppose $\mathcal{C} = A$-mod is a hica of length 3. Then $l({}_A\Delta) + l(\Delta_A) = 4$, by Lemma 6. We have $1 \leq l({}_A\Delta), l(\Delta_A) \leq 3$ since $\mathcal{C}$ has length 3. It is impossible that $l({}_A\Delta) = 3$, since this would imply standard objects are projective, leading to a contradiction as in the case when $\mathcal{C}$ is a hica of length 2. It is dually impossible that $l(\Delta_A) = 3$. Therefore $l({}_A\Delta) = l(\Delta_A) = 2$. The next step is to show our hica $\mathcal{C}$ of length 3 is Ringel self-dual. This follows just in the proof of Ringel duality for hicas of length 4 in Lemma 25 below: it is only necessary to replace the numbers 4 and 3 by the numbers 3 and 2. Since a standard object $\Delta(x)$ is a costandard object for some other ordering, by Lemma 21, $\Delta(x)$ must have an irreducible socle x_{-1}, as well as an irreducible top $x = x_0$. Likewise, x is the socle of some standard object $\Delta(x_1)$, for some $x_1 > x$. The projective Ae_x has a filtration whose sections are $\Delta(x_1)$ and $\Delta(x_0)$; it is not possible there are any other standard objects in a Δ-filtration since the existence of such would imply either the socle or top of Ae_x was not irreducible. We conclude Ae_x has top and socle isomorphic to x_i, and top modulo socle isomorphic to $x_{-1} \oplus x_1$. Inductively, we find $x_i \in \Lambda$, for $i \in \mathbb{Z}$, such that Ae_{x_i} has a filtration whose top and socle are isomorphic to x_i, and top modulo socle isomorphic to $x_{i-1} \oplus x_{i+1}$. It follows A is isomorphic to the path algebra of the quiver



modulo relations $\alpha_{i+1}\alpha_i = \beta_i\beta_{i+1} = 0$, and relations $\alpha_i\beta_i - \lambda_i\beta_{i+1}\alpha_{i+1} = 0$, for some nonzero $\lambda_i \in k$. Rescaling the generators if necessary, we may take all $\lambda_i = 0$. Thus A is isomorphic to the Brauer tree algebra associated to a bi-infinite line. □

Let us now assume $\mathcal{C}$ is a hica of length 4. Thus $\mathcal{C} = A$-mod for a symmetric quasi-hereditary Δ-homogeneous algebra A of Loewy length 4.

LEMMA 23. *The endomorphism ring of a projective indecomposable object in $\mathcal{C}$ is isomorphic to $F[d]/d^2$.*

Proof. The top and socle of a projective indecomposable are isomorphic, and such a simple cannot appear in either of the middle radical layers as this would imply a self-extension of the simple, contradicting quasi-heredity. □

LEMMA 24. *Either ${}_A\Delta$ has length 3 and Δ_A has length 2, or else ${}_A\Delta$ has length 2 and Δ_A has length 3.*

Proof. Since A is a hica, we have

$$l({}_A\Delta) + l(\Delta_A) = 5.$$

It is impossible that $l({}_A\Delta) = 1$ since this would imply that $A_A = \Delta_A$, which contradicts Lemma 23. Likewise it is impossible that $l(\Delta_A) = 1$. It follows that $\{l({}_A\Delta), l(\Delta_A)\} = \{2, 3\}$, as required. □

We use $<$ to mean "less than, in the order $\blacktriangle$".

LEMMA 25. *$\mathcal{C}$ is Ringel self-dual.*

Proof. To say that $\mathcal{C}$ is Ringel self-dual is to say that ${}_AA$ is a full tilting module for A. This is equivalent to saying that A_A is a full tilting module for A (consider finite dimensional quotients/subalgebras, and pass to a limit). In other words, A is left Ringel self-dual if and only if A is right Ringel self-dual. To establish the Ringel self-duality of $\mathcal{C}$, we may therefore assume that ${}_A\Delta$ has length 3, by Lemma 24.
Suppose $\mathcal{C}$ is not Ringel self-dual. Then we have a nonprojective indecomposable tilting module $T(\lambda)$, which has a filtration with sections

$$\Delta(\lambda), \Delta(\lambda_2), ..., \Delta(\lambda_n).$$

Note that $\Delta(\lambda)$ is the bottom section, and up to scalar we have a unique homomorphism from $P(\lambda)$ to T whose image is $\Delta(\lambda)$ (reference Ringel). Since T is nonprojective, it has length < 4. Since the sections all have length 3, the tilting module has length 3, and the tops of the sections all lie in the top of T. The module T also has a ∇-filtration since it is tilting. Any simple in the top of T must lie in the top of some ∇ of length 2. In particular, λ itself must lie in the top of some $\nabla(\mu)$ of length 2. The resulting homomorphism $P(\lambda) \to \nabla(\mu)$ must lift to a homomorphism $P(\lambda) \to T$. Up to scalar, there is a unique such homomorphism whose image is $\Delta(\lambda)$, implying that μ is a factor of $\Delta(\lambda)$. Thus, λ is a factor of $\nabla(\mu)$ and μ is a factor of $\Delta(\lambda)$. Thus $\lambda > \mu > \lambda$ which is a contradiction. □

LEMMA 26. *Standard modules for A have irreducible head and socle.*

Proof. A standard module in one ordering is isomorphic to a costandard module in another ordering, by Lemmas 21 and 25. □

If there is a nonsplit extension of λ by μ then either $\lambda > \mu$ or $\lambda < \mu$. We define *Rel* to be the set of relations $\lambda > \mu$ or $\lambda < \mu$ of this kind. We define $\uparrow$ to be the partial order on Λ generated by *Rel*. The order $\blacktriangle$ is a refinement of $\uparrow$.
We define $\downarrow$ to be the ordering on Λ which is Ringel dual to $\uparrow$.

LEMMA 27. *$\mathcal{C}$ is a highest weight category with respect to the partial order $\uparrow$ on Λ.*

Proof. $\mathcal{C}$ has length 4, which implies that either left or right standard modules have length two and, by Lemma 26, are therefore uniserial. The quasi-hereditary structure induced by $\blacktriangle$ is already determined by these non-split extensions and therefore the order $\uparrow$ already induces the same quasi-hereditary structure as its refinement $\blacktriangle$. □

From now on, whenever we refer to standard or costandard modules, or to orderings, without specifying the order, we mean the order $\uparrow$.
We say an A-module M is *directed*, if given a subquotient of M which is a non-split extension of a simple module λ by a simple module μ, λ is greater than μ.

LEMMA 28. *Standard A-modules are directed.*

Proof. We want that all standard modules are directed, which means for any subquotient of a standard module which is a non-split extension of simple modules λ by μ, λ is greater than μ. This is trivial for a standard module of Loewy length 2.
Let $\Delta(x)$ be a standard module of Loewy length 3. It must have an irreducible socle y by Lemma 26. Thus $\Delta(x)$ appears in a Δ-filtration of $P(y)$. $\Delta(y)$ appears as the top factor of a Δ-filtration of $P(y)$. Indeed, since $P(y)$ has length 4 with irreducible top and socle, a Δ-filtration of $P(y)$ has precisely two factors, namely $\Delta(x)$ and $\Delta(y)$.
The module $\Delta(y)$ must have irreducible socle z, where $y > z$, by Lemma 26. Since $P(y)$ has length 4 and $\Delta(y)$ has length 3, we conclude there is an extension of z by y. Since $\nabla(y)$ is dual to a Δ which has length 2, $\nabla(y)$ itself has length 2, and it must in fact be this extension of z by y.
For any other nonsplit extension of an irreducible modules w by y, we must have $w > y$ by Lemma 27. These are precisely the extensions of w by y contained in $\Delta(x)$. The extensions of x by w contained in $\Delta(x)$ imply $x > w$ by definition of a standard module. Thus any extension of λ by μ in $\Delta(x)$ implies $\lambda > \mu$ as required. □

COROLLARY 29. *The orders $\uparrow$ and $\downarrow$ on Λ are opposite.*

Proof. Just as standard modules are directed in the $\uparrow$ ordering, costandard modules are directed in the $\downarrow$ ordering. But standard modules in the $\uparrow$ ordering are equal to costandard modules in the $\downarrow$ ordering by Lemmas 21 and 25. Therefore $\uparrow$ and $\downarrow$ orderings are opposite, as required. □

REMARK 30 If a finite-dimensional algebra is quasi-hereditary with respect to two opposing orders then it must be directed, in which case the standard modules are projectives in one ordering, and simples in the opposite ordering. This can easily be proved by induction on the size of the indexing set. Symmetric quasi-hereditary algebras are never directed, since their projective indecomposable objects have isomorphic head and socle.

REMARK 31 It is not necessarily the case that a Ringel self-dual hica is a highest weight category with respect to two opposing orderings. Examples of length 5 are found amongst module categories of rhombal algebras [3].

Let $X(\lambda)$ denote the kernel of the surjective homomorphism $\Delta(\lambda) \twoheadrightarrow \lambda$, for $\lambda \in \Lambda$.

DEFINITION 32. *The* Δ-quiver *of A is the quiver with vertices indexed by Λ, and with arrows $\lambda \to \mu$ corresponding to simple composition factors μ in the top of $X(\lambda)$.*

LEMMA 33. *Components of the Δ-quiver of length 2 are directed lines. Components of the Δ-quiver of length 3 are directed topsy-turvy quivers.*

Proof. The length 2 case is easy.
In length 3, we have a permutation $\phi \circlearrowleft \Lambda$ which takes λ to the socle of $\Delta(\lambda)$. We prove that $\mathcal{F}(\lambda) = \mathcal{P}(\lambda^\phi)$ via a sequence of correspondences: arrows emanating from λ in the Δ-quiver are in correspondence with simple composition factors μ in the top of $X(\lambda)$; simple composition factors μ in the top of $X(\lambda)$ are in correspondence with extensions of λ by μ such that $\lambda > \mu$; since $\Delta^\uparrow(\lambda) = \nabla^\downarrow(\lambda^\phi)$, whilst $\uparrow$ and $\downarrow$ are opposites, extensions of λ by μ such that $\lambda > \mu$ are in one-one correspondence with extensions of μ by λ^ϕ such that $\mu > \lambda^\phi$; extensions of μ by λ^ϕ such that $\mu > \lambda^\phi$ are in correspondence with simple composition factors λ^ϕ in the top of $X(\mu)$; simple composition factors λ^ϕ in the top of $X(\mu)$ are in one-one correspondence with arrows into λ^ϕ in the Δ-quiver.
Since standard modules are directed, the Δ-quivers are also directed (ie they generate a poset). $\square$

We next find Δ-subalgebra of A, in the sense of S. Koenig [10].

LEMMA 34. *A has a Δ-subalgebra B.*

Proof. We want to find B such that ${}_B\Delta \cong {}_BB$. Let us write $A = FQ/I$ as the path algebra of Q modulo relations, where Q is the Ext^1-quiver of A. If there is a *positive* arrow $x \to y$ in Q, that is to say an arrow $x \to y$ in Q such that $x > y$, then x and y lie in the same component of the Δ-quiver. Since all standard modules are directed, the connected component of the quiver generated by these arrows are the components of the Δ-quiver.
Let B be the subalgebra of A generated by arrows $x \to y$ in Q such that $x > y$. Since all standard modules are directed, composing the natural maps

$Be_\lambda \to Ae_\lambda \to \Delta(\lambda)$ gives us a surjection $Be_\lambda \twoheadrightarrow \Delta(\lambda)$. To establish this composition map is an isomorphism, we have to worry about its kernel, which must lie in $\mathrm{Rad}^2(B)e_\lambda$, which is the socle of B since A has length 4. Assume there is a simple S in the kernel. Then S would have to be a factor of $\Delta(\mu)$ in a Δ-filtration of Ae_λ; restrictions imply S would lie in the socle of some $\Delta(\mu)$ of length 2, where $\mu > \lambda$ (otherwise if $\Delta(\mu)$ has length 3 then λ lies in the socle of $\Delta(\mu)$ so $S > \lambda$, $\lambda > S$ since S appears in Be_λ, contradiction). Since S lies in $\mathrm{Rad}^2(B)e_\lambda$, we have positive arrows $\lambda \to \nu \to S$, for some ν, so S must lie in $\Delta(\nu)$, and there is an arrow $\lambda \to \nu$ in the Δ-component of λ. There are now two possibilities. Either $\Delta(\lambda)$ has length 3, implying S lies in a Δ-quiver component of length 2 (for μ), and a Δ-quiver component of length 3 (for λ)- contradiction. Else $\Delta(\lambda)$ has length 2, which implies we have a Δ-quiver component of length 2 containing the quiver $\mu \to S \leftarrow \nu$ - contradiction (the structure of any length 2 Δ-quiver component is an oriented line by Lemma 33). We conclude that the map $B \twoheadrightarrow \Delta$ must in fact have zero kernel, ie B is a Δ-subalgebra.

□

Let B be the Δ-subalgebra of A.

LEMMA 35. *Suppose B has length 3. Then the algebra homomorphism $B \to A$ splits.*

Proof. Let I denote the ideal of A which is a sum of spaces AaA where a is a *negative* arrow in the quiver Q of A. Then the kernel J of the A-module homomorphism $A \to {}_A\Delta$ is contained in I, since A has length 4 and Δs have length 3, implying J is generated in the top of the radical of A. Also, J contains I since I is generated as a vector space by products of 1, 2, or 3 arrows in the quiver, at least one of which lies in I, and these products all lie in J since all Δs are directed. Thus the kernel of $A \to {}_A\Delta$ is equal to I. By symmetry, the homomorphism of right A-modules $A \to \Delta_A$ also has kernel I. Therefore $B \oplus I \to A$ is an isomorphism of B-B-bimodules, and the algebra homomorphisms

$$B \to A \to A/I$$

compose to give an algebra isomorphism $B \cong A/I$. Therefore the homomorphism $B \to A$ splits as required. □

LEMMA 36. *B is self-injective.*

Proof. We write ${}^\uparrow B$ for the ${}_A\Delta$-subalgebra taken with respect to the $\uparrow$ ordering, and $B^\downarrow$ the Δ_A-subalgebra taken with respect to the $\downarrow$ ordering. We know that

$$B = {}^\uparrow B \cong \bigoplus_{x\in\Lambda} {}_A\Delta^\uparrow(x) \cong \bigoplus_{x\in\Lambda} {}_A\nabla^\downarrow(x) \cong \bigoplus_{x\in\Lambda} (\Delta_A^\downarrow(x))^* \cong (B^\downarrow)^*,$$

where $B^\downarrow$ is also a Δ-subalgebra of A. Thus ${}^\uparrow B \cong (B^\downarrow)^*$ as A-modules, and therefore as ${}^\uparrow B$-modules. To prove ${}^\uparrow B$ is self-injective we must show that ${}^\uparrow B \cong B^\downarrow$. Indeed, ${}^\uparrow B$ is defined to be the subalgebra generated by left positive

$\uparrow$-arrows, whilst $B^{\downarrow}$ is defined to be the subalgebra generated by right positive $\downarrow$-arrows. Passing from the left regular action of an algebra on itself to the right regular action reverses arrow orientation. Therefore left positive $\uparrow$-arrows are equal to right negative $\uparrow$-arrows which are equal to right positive $\downarrow$-arrows. Thus ${}^{\uparrow}B \cong B^{\downarrow}$ as required. $\square$

Lemma 37. *If B has Loewy length 3, then A is isomorphic to $T(B)$, the trivial extension algebra of B by its dual.*
If B has Loewy length 2, then A is isomorphic to $U(B, M)$ where M is a self-dual B-B-bimodule.

Proof. We may assume $B = B^{\uparrow}$ has Loewy length 3, in which case $B^{\downarrow}$ has Loewy length 2. We have a surjection of algebras $A \twoheadrightarrow B$ which splits, via an algebra embedding $B \hookrightarrow A$. Dually, we have an embedding of A-A-bimodules $B^* \hookrightarrow A^*$. Since $A \cong A^*$ as bimodules, we have a homomorphism of A-A-bimodules $B^* \hookrightarrow A$. Taking the sum of our two embeddings gives us a homomorphism of B-B-bimodules,

$$B \oplus B^* \to A.$$

This homomorphism is a bimodule isomorphism, because every projective indecomposable A-module has a canonical Δ-filtration featuring precisely two $\Delta(\lambda)$s, one of which is a summand of B, and the other of which is a summand of B^*. We can thus identify the image of B^* in A with the kernel of the algebra homomorphism $A \to B$. The image of B^* in A multiplies to zero, because the map $B^* \to A$ is a homomorphism of A-A-bimodules, on which the kernel of the surjection $A \twoheadrightarrow B$ acts trivially. The image of B in A multiplies via according to multiplication in B. In other words, the map $T(B) = B \oplus B^* \to A$ is an algebra isomorphism, as required.
The algebra A has a $\mathbb{Z}_+^2$-grading whose first component comes from the radical grading on $B^{\uparrow}$, and whose second component comes from the trivial extension grading, with $B^{\uparrow}$ in degree 0 and its dual in degree 1. In other words, the degree $(*, 0)$ part of A is $B^{\uparrow}$. We can then identify the degree $(0, *)$ part of A with $B^{\downarrow}$, which is self-injective of Loewy length 2. The degree $(2, *)$ part of A is then isomorphic to $B^{\downarrow *}$, and we define M to be the degree $(1, *)$ part of A. The isomorphism $A \cong A^*$ exchanges the $B^{\downarrow}$-$B^{\downarrow}$-bimodules $B^{\downarrow}$ and $B^{\downarrow *}$, whilst it defines an isomorphism $M \cong M^*$. This way, we obtain the algebra isomorphism $A \cong U(B^{\downarrow}, M)$. $\square$

Let $\mathfrak{Bip}$ denote the 2-category whose objects are bipartite graphs; whose arrows $\Gamma \to \Gamma'$ are given by sequences $(\gamma_1, ..., \gamma_n)$ of distinct vertices of Γ, such that $\Gamma' = \Gamma \backslash \{\gamma_1, ..., \gamma_n\}$; whose 2-arrows are given by permutations of such sequences.
The following result is a refinement of Theorem 1.

Theorem 38. *The correspondence $\Gamma \mapsto \mathcal{C}_\Gamma$ extends to a 2-functor*

$$\mathfrak{Bip} \to \mathfrak{Hica}_4$$

which is essentially bijective on objects.

Proof. The correspondence $\Gamma \mapsto A_\Gamma$ -mod is essentially bijective on objects, by Lemmas 16, 34, 36, and 37.
We have to associate functors and natural transformations in $\mathfrak{Hica}_4$ to arrows and 2-arrows in $\mathfrak{Bip}$. Suppose $\gamma \in \Gamma$ is a vertex of a bipartite graph, and $\Gamma' = \Gamma \backslash \gamma$. We have an isomorphism $A_{\Gamma'} \cong e_{\Gamma'} A_\Gamma e_{\Gamma'}$, and therefore an exact functor

$$F_\gamma = e_{\Gamma'} A_\Gamma \otimes_{A_\Gamma} : A_\Gamma\text{-mod} \to A_{\Gamma'}\text{-mod}$$

which sends the irreducible corresponding to a vertex v to the irreducible corresponding to a vertex v, if $v \neq \gamma$ and to zero if $v = \gamma$. To a sequence $(\gamma_1, ..., \gamma_n)$ we associate the composition functor $F_{\gamma_n} ... F_{\gamma_1}$. There are natural isomorphisms between various functors corresponding to isomorphisms of bimodules. □

Let $B^\uparrow = FQ^\uparrow / R^\uparrow$, $B^\downarrow = FQ^\downarrow / R^\downarrow$ be minimal presentations of $B^\uparrow$ and $B^\downarrow$ by quiver and relations.
Let Q be the union of $Q^\uparrow$ and $Q^\downarrow$ in which we identify the vertices of these quivers if they represent the same irreducible A-module. Let R be the union of $R^\uparrow$, $R^\downarrow$ and $R^\updownarrow$. Let $R^\updownarrow$ denote the set of relations which insist that squares in Q involving two arrows of $Q^\uparrow$ and two arrows of $Q^\downarrow$ commute.

Lemma 39. *$A = FQ/R$ is a minimal presentation of A by quiver and relations.*

Proof. We have a surjective map $FQ \twoheadrightarrow A$. It is not difficult to see this must factor through a map $FQ/R \twoheadrightarrow A$. We now want to bound the dimension of a projective of FQ/R. Without loss of generality assum that $B = B^\uparrow$ has Loewy length 3 and $B^\downarrow$ therefore has Loewy length 2. So $Q^\uparrow$ is a topsy-turvy quiver and $Q^\downarrow$ is linear. We claim that a spanning set of $(FQ/R)e_x$ is given by abe_x where $b \in B$ and a is either an idempotent or an arrow from $Q^\downarrow$. Without a doubt a spanning set is given by the union of all elements of the form $a_1 b_1 \cdots a_r b_r e_x$ where a_i are either idempotents or arrows in $Q^\downarrow$ and $b_i \in B$. However, if we have an arrow a in $Q^\downarrow$ (say with source y and target $\phi^{-1}(y)$) and and arrow $b \in Q^\uparrow$ starting in $\phi^{-1}(y)$, the product $bae_y = be_{\phi^{-1}y} ae_y$ equals $a'b'e_y$ where $b' = \phi(b)$ and a' is the unique arrow starting at the end vertex of $b'e_y$. Indeed, $Q^\uparrow$ being topsy turvy implies the existence of b' and in $Q^\downarrow$ there is an arrow from x to $\phi^{-1}x$ for every x. So denoting by z the end vertex of b, there is a square

$$\begin{array}{ccc} y & \xrightarrow{a} & \phi^{-1}(y) \\ \downarrow{\scriptstyle \phi(b)} & & \downarrow{\scriptstyle b} \\ z & \xrightarrow{a'} & \phi^{-1}(z) \end{array}.$$

By the required relations this has to commute and we obtain $bae_y = a'b'e_y$. Hence the path $a_1 b_1 \cdots a_r b_r e_x$ is equivalent modulo R to a path $a'_1 \cdots a'_r b'_1 \cdots b'_r = a'_1 \cdots a'_r b'$. However, by the relations in $B^\downarrow$, any product of arrows in $Q^\downarrow$ is zero, so we obtain the claim that $(FQ/R)e_x$ is spanned by abe_x where $b \in B$ and a is either an idempotent or an arrow from $Q^\downarrow$.

This implies that $\dim(FQ/R)e_x \leq 2\dim Be_x = \dim(B+B^*)e_x = \dim Ae_x$, the equality $\dim Be_x = \dim(B+B^*)e_x$ coming from the fact that B is self-injective. Combining the above surjection $FQ/R \twoheadrightarrow A$ and this inequality, we obtain the statment of the lemma. □

7. KOSZULITY

For an algebra C we denote by $C^!$ the quadratic dual of C.

THEOREM 40. *The following are equivalent:*

1. Γ *is not a simply laced Dynkin graph.*
2. Z_Γ *is Koszul.*
3. B_Γ *is Koszul.*
4. A_Γ *is Koszul.*
5. $A_\Gamma^!$-mod *is Calabi-Yau of dimension* 3.

The length of the proof of this result is the length of the section.

1 is equivalent to 2, by a theorem of Martínez-Villa [11].

2 is equivalent to 3, since B_Γ-mod$^{\oplus 2}$ is equivalent to Z_Γ-gr by Lemma 18. The implication $3 \Rightarrow 4$ follows from the following lemma, in case $A = A_\Gamma$, and $B = B_\Gamma$.

LEMMA 41. *If B is a self-injective Koszul algebra of length n, the trivial extension algebra $A = B \oplus B^*\langle n\rangle$ is Koszul.*

Proof. Since B is selfinjective, we have an isomorphism $B \cong B^*$ of B-modules. The algebra A is a trivial extension $A = B \oplus B^*$, and we thus have a map $A \to A$ of B-modules extending to a map of A-modules whose kernel is B^* and whose cokernel is B. Stringing these together gives us a projective resolution

$$... \to A \to A \to B$$

of B as a left A-module. Since B is self-injective and radical graded, every injective B-module has length n, and consequently this is a linear resolution of B as a left A-module. Taking summands, we find that every projective B-module has a linear resolution as a left A-module.
If B is Koszul, then B^0 has a linear resolution by projective B-modules. Thus B^0 is quasi-isomorphic to a linear complex of projective B-modules. Since projective B-modules are quasi-isomorphic to a linear complex of projective A-modules, we deduce B^0 is isomorphic to a linear complex of projective A-modules. That is to say, $A^0 = B^0$ has a linear resolution by projective A-modules. In other words, A is Koszul. □

The implication $4 \Rightarrow 3$ follows from the following lemma, in case $A = A_\Gamma$, and $B = B_\Gamma$.

LEMMA 42. *If B is a radical-graded selfinjective algebra of length n, such that $A = B \oplus B^*\langle n\rangle$ is Koszul, then B is Koszul.*

Proof. We have a $\mathbb{Z}_+ \times \mathbb{Z}_+$-grading on A in which B lies in degree $(?, 0)$, the dual of B lies in degree $(?, 1)$, and in which the inherent $\mathbb{Z}_+$-grading on B is the radical grading. This corresponds to the action of a two-dimensional torus $\mathbb{T}$ on A. Thus $\mathbb{T}$ acts on A^1 and we have an exact sequence

$$0 \to R \to A^1 \otimes_{A^0} A^1 \to A^2 \to 0$$

of $\mathbb{T}$-modules, where R denotes the relations for A and A^j refers to the jth component in the total grading, whose dual

$$0 \leftarrow A^{!2} \leftarrow A^{!1} \otimes_{A^{!0}} A^{!1} \leftarrow R^! \leftarrow 0$$

is also an exact sequence of $\mathbb{T}$-modules. Since $A^!$ is quadratic by definition, with relations $R^!$, we have an action of $\mathbb{T}$ on $A^!$, which gives a $\mathbb{Z}_+ \times \mathbb{Z}_+$-grading on $A^!$. We have a linear resolution of $W = A^{(0,0)}$, given by the Koszul complex

$$A \otimes_W A^{!*}$$

of A ([1], 2.8). Here $A^{!*}$ denotes the graded dual of $A^!$. The differential on the Koszul complex respects the $\mathbb{Z}_+ \times \mathbb{Z}_+$-grading on A and $A^!$ (see [1], 2.6). In other words, it sends terms involving arrows in $A^{(0,1)}$ or $A^{!(0,1)*}$ to terms involving arrows in $A^{(0,1)}$ or $A^{!(0,1)*}$, and terms not involving arrows in $A^{(0,1)}$ or $A^{!(0,1)*}$ to terms not involving arrows in $A^{(0,1)}$ or $A^{!(0,1)*}$. Consequently the subcomplex $A^{(?,0)} \otimes_R A^{!(-,0)*})$ is a direct summand of the Koszul complex regarded as a complex of B-modules. Taking this component gives us a linear resolution of $R = B^0$ as a B-module. Therefore B is Koszul. □

If C is a graded algebra and C-mod is Calabi-Yau of dimension n, then $Ext^*_C(C^0, C^0)$ is a super-symmetric algebra concentrated in degrees $0, 1, ..., n$, by Van den Bergh A.5.2 [2]. We have a converse which applies for Koszul algebras:

Theorem 43. *Suppose K is a Koszul algebra such that $K^!$ is super-symmetric of length $n+1$, then K-mod is Calabi-Yau of dimension n.*

Proof. There is an equivalence between derived categories of graded modules for $K^!$ and K via the Koszul complex. Since $K^!$ is locally finite dimensional, this restricts an equivalence of bounded derived categories, by a theorem of Beilinson, Ginzburg, and Soergel ([1], Theorem 2.12.6). Under this equivalence, simple $K^!$-modules correspond to projective indecomposable K-modules. Since $K^!$ is locally finite-dimensional The equivalence therefore restricts to an equivalence between $D^b(K^!\text{-gr})$ and $D^b(K\text{-grperf})$. Also under this equivalence, injective $K^!$-modules correspond to simple K-modules, whilst shifts $\langle i \rangle$ in $D^b(K^!\text{-gr})$ correspond to shifts in degree $\langle -i \rangle[-i]$ in $D^b(K\text{-grperf})$. This homological shift in degree means that the Calabi-Yau-n property for K-mod is equivalent to the super-Calabi-Yau-0 property for $K^!$-perf, thanks to Van den Bergh's calculation A.5.2 [2]. To prove the super-Calabi-Yau-0 property for $K^!$-perf, it is enough to check that $K^!$ is a super-symmetric algebra (cf [17], Theorem 3.1). □

Assume 4. Then the Koszul dual $A^!$ of A is Calabi-Yau of dimension 3. The Koszul dual of a supersymmetric algebra of length $n+1$ is Calabi-Yau of dimenion n by Theorem 43. The trivial extension algebra $A = B + B^*\langle 3\rangle$ is super-symmetric with B concentrated in degrees $0, 1, 2$, with $B^*\langle 3\rangle$ concentrated in degrees $3, 2, 1$, and with bilinear form pairing B^i and $(B^i)^*\langle 3\rangle$ via

$$<\beta, b>=\beta(b) \qquad <b, \beta>=(-1)^{i(3-i)}\beta(b),$$

for $b \in B^i$, $\beta \in (B^i)^*$. Thus 4 implies 5.

Assume 5. Since $A^!$ is Calabi-Yau of dimension 3, its relations are the derivatives of a superpotential, and its degree 0 part has a 4-term resolution, its Jacobi resolution [2]. The superpotential must be cubic, since $A^!$ is quadratic. This implies further that the Jacobi resolution of $A^{!0}$ is linear, so $A^!$ must be Koszul. Thus 5 implies 4.

We have now shown that $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$, completing the proof of Theorem 40.

Remark 44 If Γ is a bipartite graph, then an orientation of Γ gives rise to a $\mathbb{Z}_+^3$-grading on A_Γ. If every vertex of Γ is attached to at least two vertices, then this leads to a $\mathbb{Z}_+^3$-grading of the Calabi-Yau algebra $A^!$ of dimension 3, which can otherwise be thought of as the action of a 3-dimensional torus on $A^!$. The algebra $A^!$ has homological dimension 3, and admits the action of a 3-dimensional torus. It thus belongs to the realm of 3-dimensional noncommutative toric geometry.

Example 45 If Γ is given by tiling of a bi-infinite line

$$\cdots \qquad \bullet \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \bullet \qquad \cdots$$

then the Calabi-Yau algebra of dimension 3 we obtain is familiar from toric geometry. It is the algebra associated to the brane tiling of the plane by hexagons [6]. Its quiver can be thought of as an orientation of the A_2-lattice (for a picture, see section 8, assumption 3). If we give Γ an alternating orientation,

$$\cdots \qquad \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \qquad \cdots$$

then in the resulting grading on $A^!$, the three copies of $\mathbb{Z}_+$ correspond to the three directions of arrows in the A_2-lattice.

8. Relaxing the assumptions

We have given a combinatorial classification of hicas of length ≤ 4 by bipartite graphs. Here we show that the relaxation of any of the homological assumptions on our categories would necessarily introduce further combinatorial complexity into the classification.

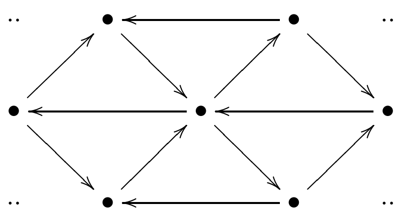
Assumption 1: highest weight structure.

The existence of a highest weight structure on a category is a strong assumption, and the assumptions of Ringel self-duality and homogeneity of standard modules require the existence of a highest weight structure on the category. It is a cinch to give examples of indecomposable Calabi-Yau 0 categories of length 4 which are not highest weight categories, such as the module category of the local symmetric algebra $F[x]/x^4$.

Assumption 2: Calabi-Yau 0 *property.*
The Calabi-Yau property is another strong homological restriction on a category. An example of a length 4 highest weight category which is indecomposable and Ringel self-dual, and whose standard modules are homogeneous, is the path algebra of the linear quiver

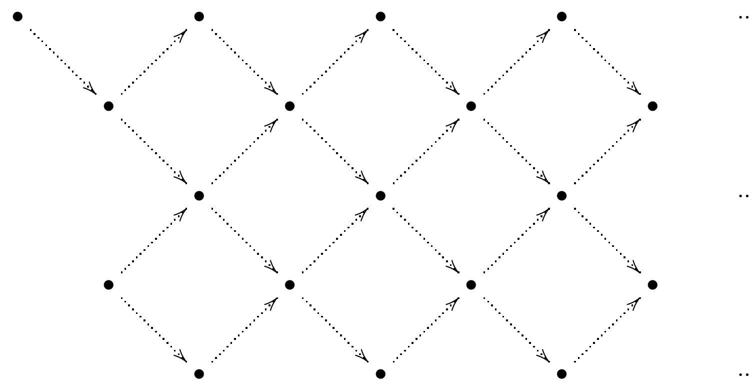
modulo all relations of degree ≥ 4.

Assumption 3: homogeneity.
The homogeneity restriction on a hica is fairly natural, since the known examples of highest weight Calabi-Yau 0 categories arising in group representation theory and the theory of tilings satisfy this restriction. However, some interesting combinatorics arise in length 4 if the condition is dropped.
For example, let $\mathcal{C}_\Gamma$ be the hica associated to a bi-infinite line Γ, whose quiver is an orientation

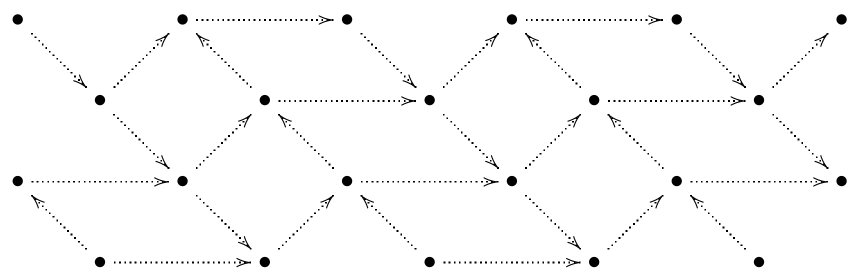


of the A_2 lattice, by example 45, and by construction comes with a (horizontal) projection π onto Γ.
There are two natural ways to obtain highest weight indecomposable Calabi-Yau 0 categories which are not homogeneous from $\mathcal{C}_\Gamma$. The first is by choosing a section of π, that is a path in the A_2 lattice which projects homeomorphically onto Γ via π. The elements of Λ to the right of the path form a coideal in the poset. Truncating $\mathcal{C}_\Gamma$ at this coideal gives us a highest weight category of length 4 which is CY-0, but not homogeneous (cf. [4], 3.5(b)). Such a truncation is not Ringel self-dual. Beneath is a portion of such a truncated poset, whose left edge defines an orientation of Γ. We use dotted arrows to represent directions in a partial order on the vertices of the lattice, rather than solid arrows which

represent arrows in a quiver:



The second way to obtain an inhomogeneous highest weight indecomposable Calabi-Yau 0 category from $\mathcal{C}_\Gamma$ is by merely altering the ordering of the vertices. Certain orderings of the vertices of the A_2 lattice give A_Γ-mod is an inhomogeneous highest weight category, which is Ringel self-dual. Here is an example of a portion of such a partial ordering:



Assumption 4: length ≤ 4.
We have studied hicas of length 4, since 4 is the shortest length in which a nontrivial classification is possible. There are two kinds of hicas of length 5: those whose left and right standard modules have length 4 and 2, and those whose left and right standard modules have length 3 and 3.
The category of graded modules over a radical-graded symmetric algebra of length 4 is equivalent to the category of modules over a directed self-injective algebras of length 4. Trivial extensions of directed self-injective algebras of length 4 by their duals give examples of hicas of length 5 whose left and right standard modules have length 4 and 2.
Michael Peach's rhombal algebras give examples of hicas of length 5 associated to rhombic tilings of the plane whose left and right standard modules have length 3 and 3.
We would be interested to learn more about hicas of length 5.

9. Tilting

There are natural self-equivalences of the derived categories of $\mathcal{C}_\Gamma$, which are obtained from a standard tilting procedure for symmetric algebras:

Lemma 46. *Suppose A is a symmetric algebra, and suppose that the endomorphism ring $e_\lambda A e_\lambda$ is isomorphic to the dual numbers $F[d]/d^2$. Then we have an exact self-equivalence ψ_λ of the derived category of A given by tensoring with the two-term complex*

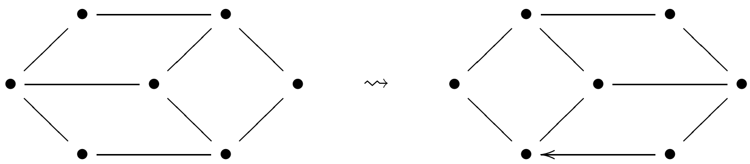
$$Ae_\lambda \otimes_F e_\lambda A \to A,$$

whose arrow is the multiplication map.

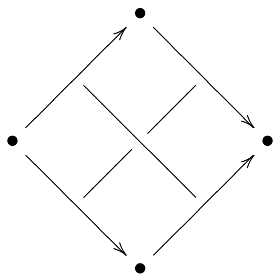
Proof. This functor is obviously exact. It fixes all simple modules with the exception of the simple top λ of Ae_λ, which it sends to $\Omega(\lambda)$. The module $\Omega(\lambda)$ has simple socle λ since A is symmetric, and other composition factors different from λ since $e_\lambda A e_\lambda$ is isomorphic to the dual numbers. Therefore collection of simples $\mu \neq \lambda$, along with Ω generate $D^b(A\text{-mod})$, and ψ_λ is an equivalence. □

The self-equivalence ψ_λ is called a spherical twist, because the cohomology ring of the sphere can be identified with the dual numbers (cf. [19]).
One way to obtain self-equivalences of $D^b(\mathcal{C}_\Gamma)$ from spherical twists is by lifting self-equivalences of the derived category of the zigzag algebra Z_Γ, whose projective indecomposable modules for the algebra Z_Γ all have an endomorphism ring isomorphic to the algebra of dual numbers. A second way to obtain self-equivalences of $D^b(\mathcal{C}_\Gamma)$ is to apply spherical twists directly to $\mathcal{C}_\Gamma$, whose projective indecomposable objects also have endomorphism rings isomorphic to the dual numbers.
Let us consider the first case. The projective indecomposable modules for the algebra Z_Γ all have an endomorphism ring isomorphic to the algebra of dual numbers. Standard tilts generate an action of a 2-category $\mathcal{T}_\Gamma$ on $D^b(Z_\Gamma\text{-gr})$ which lifts to an action of $\mathcal{T}_\Gamma$ on $D^b(\mathcal{C}_\Gamma)$, by a result of Rickard [16, Thm 3.1]. A second way to obtain self-equivalences of $D^b(\mathcal{C}_\Gamma)$ is to apply Seidel-Thomas twists directly to A_Γ, whose projective indecomposable modules have endomorphism rings isomorphic to the dual numbers. Standard tilts generate the action of a 2-category $\mathcal{U}_\Gamma$ on $D^b(\mathcal{C}_\Gamma)$, whose combinatorics is rather different from that of $\mathcal{T}_\Gamma$.

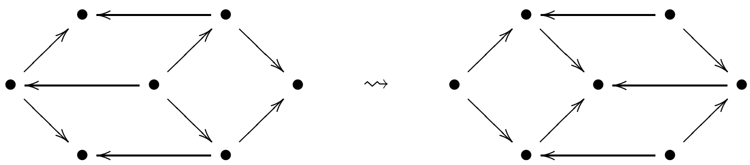
Example 47 When Γ is a bi-infinite line, we have an action of the braid 2-category $\mathcal{BC}_\infty$ on a bi-infinite line on the derived category of Z_Γ, by a theorem of Seidel and Thomas [9]. The action of $\mathcal{BC}_\infty$ on $D^b(Z_\Gamma\text{-mod})$ lifts to an action of $\mathcal{BC}_\infty$ on $\mathcal{C}_\Gamma$. Arrows in $\mathcal{BC}_\infty$ are braids with an infinite number of strands, and 2-arrows are braid cobordisms, such as Reidemeister moves pictured as follows:



In this picture, a rhombus represents a pair of braids running parallel to the two sides and crossing in the middle.



We have a natural 2-functor $\mathcal{BC}_\infty \to \mathcal{T}_\Gamma$. The Reidemeister move depicted above therefore corresponds in a natural way to a 2-*arrow* in $\mathcal{T}_\Gamma$. However, this Reidemeister move also corresponds naturally to an *arrow* in $\mathcal{S}_\Gamma$. Let us explain how. Suppose we remove edges of the A_2-lattice to give a rhombic tiling T of the plane, whose edges lie in the quiver Q of A_Γ. We have a grading of A_Γ which places arrows in Q which are edges of T in degree 0 and arrows in Q which are not edges of T in degree 1. Let us denote by D_T the degree 0 part of A taken with respect to this tiling. The algebra A_Γ is a trivial extension of D_T by D_T^*. If T' is obtained from T by a Reidemeister move centred on the vertex λ,



then $D_{T'}$ is derived equivalent to D_T, because the complex of D_T-modules given by the sum of $D_T e_\lambda \otimes e_\lambda D_T \to D_T$ and $D_T e_\lambda \to 0$ is a tilting complex whose derived endomorphism ring is isomorphic to $D_{T'}$. This derived equivalence between D_T and $D_{T'}$ lifts to an equivalence of trivial extensions, that is to say a self-equivalence of $D^b(A_\Gamma\text{-mod}) = D^b(\mathcal{C}_\Gamma)$; this self-equivalence of $D^b(\mathcal{C}_\Gamma)$ is precisely the spherical twist ψ_λ.

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