Fuss-Catalan Numbers in Noncommutative Probability

Wojciech Młotkowski[1]

Received: January 3, 2010

Communicated by Friedrich Götze

Abstract. We prove that if $p, r \in \mathbb{R}$, $p \geq 1$ and $0 \leq r \leq p$ then the Fuss-Catalan sequence $\binom{mp+r}{m}\frac{r}{mp+r}$ is positive definite. We study the family of the corresponding probability measures $\mu(p, r)$ on $\mathbb{R}$ from the point of view of noncommutative probability. For example, we prove that if $0 \leq 2r \leq p$ and $r + 1 \leq p$ then $\mu(p, r)$ is $\boxplus$-infinitely divisible. As a by-product, we show that the sequence $\frac{m^m}{m!}$ is positive definite and the corresponding probability measure is $\boxtimes$-infinitely divisible.

2010 Mathematics Subject Classification: Primary 46L54; Secondary 44A60, 60C05

Keywords and Phrases: Fuss-Catalan numbers, free, boolean and monotonic convolution

1. Introduction

For natural numbers m, p, r let $A_m(p, r)$ denote the number of all sequences $(a_1, a_2, \ldots, a_{mp+r})$ such that: (1) $a_i \in \{1, 1-p\}$, (2) $a_1 + a_2 + \ldots + a_s > 0$ for all s such that $1 \leq s \leq mp + r$ and (3) $a_1 + a_2 + \ldots + a_{mp+r} = r$. It turns out that this is given by the two-parameter Fuss-Catalan numbers (2.1) (see [5, 13]). Note that the right hand side of (2.1) allows us to define $A_m(p, r)$ for all *real* parameters p and r. In particular, the *Catalan numbers* $A_m(2, 1)$ are known as moments of the Marchenko–Pastur distribution:

$$d\widetilde{\pi}(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}dx \qquad \text{on } [0, 4], \tag{1.1}$$

which in the free probability theory plays the role of the Poisson measure. In this paper we are going to study the question for which parameters $p, r \in \mathbb{R}$

[1]Research supported by MNiSW: N N201 364436, by ToK: MTKD-CT-2004-013389, by 7010 POLONIUM project: *"Non-Commutative Harmonic Analysis with Applications to Operator Spaces, Operator Algebras and Probability"* and by joint PAN-JSPS project: *"Noncommutative harmonic analysis on discrete structures with applications to quantum probability"*.

the sequence $\{A_m(p,r)\}_{m=0}^{\infty}$ is positive definite, i.e. is the moment sequence of some probability measure (which we will denote $\mu(p,r)$). Recently T. Banica, S. T. Belinschi, M. Capitaine and B. Collins [1] showed that if $p > 1$ then $\{A_m(p,1)\}_{m=0}^{\infty}$ is the moment sequence of a probability measure which can be expressed as the multiplicative free power $\widetilde{\pi}^{\boxtimes p-1}$.

We are going to prove that if $p, r \in \mathbb{R}$, $p \geq 1$ and $0 \leq r \leq p$ then $\{A_m(p,r)\}_{m=0}^{\infty}$ is the moment sequence of a unique probability measure $\mu(p,r)$ which has compact support contained in $[0,\infty)$. Moreover, if $0 \leq 2r \leq p$ and $r+1 \leq p$ then $\mu(p,r)$ is infinitely divisible with respect to the free convolution $\boxplus$. In some particular cases we are able to determine the multiplicative free convolution, the boolean power and the monotonic convolution of the measures $\mu(p,r)$. We will also prove that if $0 \leq r \leq p-1$ then the sequence $\left\{\binom{mp+r}{m}\right\}_{m=0}^{\infty}$ is positive definite and the corresponding probability measure can be expressed as $\mu(p-r,1)^{\uplus p} \rhd \mu(p,r)$, where $\uplus$ and $\rhd$ denote the boolean and the monotonic convolution, respectively.

The paper is organized as follows. In Section 2 we prove three combinatorial identities. Then we use them to derive some formulas for the generating functions. In Section 4 we regard the numbers $A_m(p,r)$ as moments of a *probability quasi-measure* $\mu(p,r)$ (by this we mean a linear functional $\mu : \mathbb{R}[x] \to \mathbb{R}$ satisfying $\mu(1) = 1$). On the class of probability quasi-measures one can introduce the free, boolean and monotonic convolutions in combinatorial way. The class of compactly supported probability measures on $\mathbb{R}$, regarded as a subclass of the former, is closed under these operations. We prove some formulas involving the probability quasi measures $\mu(p,r)$, for example we find the free R- and S-transforms (4.8), (4.11), the boolean powers $\mu(p,1)^{\uplus t}$ (4.18) and, in special cases, the multiplicative free (4.12), (4.13), (4.14) and the monotonic convolution (4.20) of the measures $\mu(p,r)$.

In Section 5 we prove that if $p \geq 1$ and $0 \leq r \leq p$ then $\mu(p,r)$ is a measure (we conjecture that this condition is also necessary for $p, r > 0$). The proof involves the multiplicative free convolution $\boxtimes$. Moreover, we show that if $0 \leq 2r \leq p$ and $r+1 \leq p$ then $\mu(p,r)$ is $\boxplus$-infinitely divisible.

In the final part we extend our results to the dilations of the measures $\mu(p,r)$, with parameter $h > 0$. Taking the limit with $h \to 0$ we prove in particular that the sequence $\left\{\frac{m^m}{m!}\right\}_{m=0}^{\infty}$ is positive definite and the corresponding probability measure ν_0 is $\boxtimes$-infinitely divisible.

2. Some combinatorial identities

We will work with the two-parameter Fuss-Catalan numbers (see [5, 13]) defined by: $A_0(p,r) := 1$ and

$$A_m(p,r) := \frac{r}{m!} \prod_{i=1}^{m-1} (mp + r - i) \tag{2.1}$$

for $m \geq 1$, where p, r are real parameters. Note that (2.1) can be written as $\binom{mp+r}{m}\frac{r}{mp+r}$, unless $mp + r = 0$. One can check that for $m \geq 0$

$$A_m(p, r) = A_m(p, r-1) + A_{m-1}(p, p+r-1), \tag{2.2}$$

under convention that $A_{-1}(p, r) := 0$, and

$$A_m(p, p) = A_{m+1}(p, 1). \tag{2.3}$$

It is also known (see [13]) that

$$\sum_{k=0}^{m} A_k(p, r) A_{m-k}(p, s) = A_m(p, r+s). \tag{2.4}$$

Now we are going to prove three identities, valid for $c, d, p, r, t \in \mathbb{R}$, which will be needed later on.

Proposition 2.1.

$$\sum_{k=0}^{m} A_k(p-r, c) A_{m-k}(p, kr+d) = A_m(p, c+d). \tag{2.5}$$

Proof. It is easy to check that the formula is true for $m = 0$ and $m = 1$. Denoting the left hand side by $S_m(p, r, c, d)$ we have from (2.2):

$$\begin{aligned}
S_m(p, r, c, d) &= \sum_{k=0}^{m} A_k(p-r, c) A_{m-k}(p, kr+d) \\
&= \sum_{k=0}^{m} \left[A_k(p-r, c-1) + A_{k-1}(p-r, p-r+c-1)\right] A_{m-k}(p, kr+d) \\
&= \sum_{k=0}^{m} A_k(p-r, c-1) A_{m-k}(p, kr+d) \\
&\qquad + \sum_{k=1}^{m} A_{k-1}(p-r, p-r+c-1) A_{m-k}(p, kr+d) \\
&= S_m(p, r, c-1, d) + \sum_{k=0}^{m-1} A_k(p-r, p-r+c-1) A_{m-1-k}(p, kr+r+d) \\
&= S_m(p, r, c-1, d) + S_{m-1}(p, r, p-r+c-1, r+d),
\end{aligned}$$

so that we have

$$S_m(p, r, c, d) = S_m(p, r, c-1, d) + S_{m-1}(p, r, p-r+c-1, r+d).$$

Fix m and assume that (2.5) holds for $m-1$. Now we prove that for m it holds for every natural c. Indeed, it holds for $c = 0$ and if it does for $c-1$ then, by assumption and by (2.2):

$$\begin{aligned}
S_m(p, r, c, d) &= S_m(p, r, c-1, d) + S_{m-1}(p, r, p-r+c-1, r+d) \\
&= A_m(p, c+d-1) + A_{m-1}(p, p+c+d-1) = A_m(p, c+d),
\end{aligned}$$

which proves that the statement is true for c. Therefore it holds for all natural c. Now we note that both sides of (2.5) are polynomials on c of order m, therefore the formula holds for all $c \in \mathbb{R}$, which completes the inductive step. □

PROPOSITION 2.2.

$$(1-t)\sum_{l=0}^{m} A_l\big(p,1\big)\sum_{j=0}^{m-l} A_{m-l-j}\big(p,j(p-1)+r\big)t^j + t\sum_{j=0}^{m} A_{m-j}\big(p,j(p-1)+r\big)t^j = A_m\big(p,r+1\big). \tag{2.6}$$

Proof. Using first (2.4) and then (2.2) we obtain:

$$\begin{aligned}
& t\sum_{j=0}^{m} A_{m-j}(p,j(p-1)+r)t^j \\
&\qquad + (1-t)\sum_{l=0}^{m} A_l\big(p,1\big)\sum_{j=0}^{m-l} A_{m-l-j}\big(p,j(p-1)+r\big)t^j \\
&= t\sum_{j=0}^{m} A_{m-j}(p,j(p-1)+r)t^j \\
&\qquad + (1-t)\sum_{j=0}^{m}\sum_{l=0}^{m-j} A_l(p,1)A_{m-j-l}(p,j(p-1)+r)t^j \\
&= t\sum_{j=0}^{m} A_{m-j}(p,j(p-1)+r)t^j \\
&\qquad + (1-t)\sum_{j=0}^{m} A_{m-j}(p,j(p-1)+r+1)t^j \\
&= \sum_{j=0}^{m} A_{m-j}(p,j(p-1)+r+1)t^j - \sum_{j=0}^{m-1} A_{m-j-1}(p,j(p-1)+r+p)t^{j+1} \\
&= A_m(p,r+1).
\end{aligned}$$

□

PROPOSITION 2.3.

$$\sum_{k=0}^{m} A_{m-k}(p,k(p-1)+r)p^k = \binom{mp+r}{m}. \tag{2.7}$$

Proof. Denoting the left hand side by $T_m(p,r)$ we use (2.2) and get

$$\begin{aligned}
T_m(p,r) &= \\
&= \sum_{k=0}^{m} A_{m-k}(p, k(p-1)+r)p^k \\
&= \sum_{k=0}^{m} \left[A_{m-k}(p, k(p-1)+r-1) + A_{m-1-k}(p, k(p-1)+p+r-1)\right] p^k \\
&= T_m(p, r-1) + T_{m-1}(p, p+r-1).
\end{aligned}$$

Now we proceed as in the proof of (2.5), using the binomial identity

$$\binom{mp+r}{m} = \binom{mp+r-1}{m} + \binom{mp+r-1}{m-1}. \qquad \square$$

3. Generating functions

In this part we are going to study the generating functions

$$\mathcal{B}_p(z) := \sum_{m=0}^{\infty} A_m(p,1) z^m, \tag{3.1}$$

which are convergent in some neighborhood of 0 (to observe this one can use the inequality

$$|A_m(p,r)| \le |r| \big[m(|p|+1) + |r|\big]^{m-1}/m!$$

and apply the Cauchy's radical test). From (2.4) and (2.3) we have

$$\mathcal{B}_p(z)^r = \sum_{m=0}^{\infty} A_m(p,r) z^m \tag{3.2}$$

and

$$\mathcal{B}_p(z) = 1 + z\mathcal{B}_p(z)^p. \tag{3.3}$$

Indeed, denoting the right hand side of (3.2) by $\mathcal{A}_{p,r}(z)$ we have $\mathcal{A}_{p,1}(z) = \mathcal{B}_p(z)$ and, by (2.4), $\mathcal{A}_{p,r}(z) \cdot \mathcal{A}_{p,s}(z) = \mathcal{A}_{p,r+s}(z)$, which implies that $\mathcal{A}_{p,r}(z) = \mathcal{B}_p(z)^r$. Taking $r = p$ and applying (2.3) we get (3.3).
Now we are going to interpret formulas (2.5), (2.6), (2.7) in terms of these generating functions.

Proposition 3.1. *For any real parameters p, r we have*

$$\mathcal{B}_{p-r}\left(z\mathcal{B}_p(z)^r\right) = \mathcal{B}_p(z). \tag{3.4}$$

Proof. First we note that if $A(z) = \sum_{m=0}^{\infty} a_m z^m$, $B(z) = \sum_{n=1}^{\infty} b_n z^n$ then

$$A(B(z)) = a_0 + \sum_{m=1}^{\infty} z^m \sum_{k=1}^{m} a_k \sum_{\substack{i_1,i_2,\ldots,i_k \ge 1 \\ i_1+i_2+\ldots+i_k=m}} b_{i_1} b_{i_2} \ldots b_{i_k}. \tag{3.5}$$

Putting $b_i := A_{i-1}(p,r)$ for fixed k,m we have:

$$\sum_{\substack{i_1,i_2,\ldots,i_k\geq 1\\ i_1+i_2+\ldots+i_k=m}} b_{i_1}b_{i_2}\ldots b_{i_k} = \sum_{\substack{j_1,j_2,\ldots,j_k\geq 0\\ j_1+j_2+\ldots+j_k=m-k}} A_{j_1}(p,r)A_{j_2}(p,r)\ldots A_{j_k}(p,r) = A_{m-k}(p,kr),$$

the coefficient of $\mathcal{B}_p(z)^{kr}$ at z^{m-k}. Now we put $a_k := A_k(p-r,1)$ and applying (2.5), with $c=1$, $d=0$, we get

$$\sum_{k=1}^{m} a_k \sum_{\substack{i_1,i_2,\ldots,i_k\geq 1\\ i_1+i_2+\ldots+i_k=m}} b_{i_1}b_{i_2}\ldots b_{i_k} = \sum_{k=0}^{m} A_k(p-r,1)A_{m-k}(p,kr) = A_m(p,1), \tag{3.6}$$

as $A_m(p,0)=0$ for $m\geq 1$, which completes the proof. □

Note that in the proof we applied (2.5) only with $c=1$ and $d=0$.
For $p,r,t\in\mathbb{R}$ we denote

$$\mathcal{D}_{p,r,t}(z) := \frac{\mathcal{B}_p(z)^{1+r}}{(1-t)\mathcal{B}_p(z)+t}. \tag{3.7}$$

PROPOSITION 3.2. *For $p,r,t\in\mathbb{R}$ we have*

$$\mathcal{D}_{p,r,t}(z) = \sum_{m=0}^{\infty} z^m \sum_{k=0}^{m} A_{m-k}(p,k(p-1)+r)t^k, \tag{3.8}$$

in particular:

$$\mathcal{D}_{p,r,p}(z) = \sum_{m=0}^{\infty}\binom{mp+r}{m} z^m. \tag{3.9}$$

Moreover

$$\mathcal{D}_{p-r,s,t}\left(z\mathcal{B}_p(z)^r\right)\mathcal{B}_p(z)^r = \mathcal{D}_{p,r+s,t}(z). \tag{3.10}$$

Proof. Using (2.6) we can verify that

$$\big[(1-t)\mathcal{B}_p(z)+t\big]\cdot\left[\sum_{m=0}^{\infty} z^m\sum_{k=0}^{m} A_{m-k}(p,k(p-1)+r)t^k\right] = \mathcal{B}_p(z)^{1+r}$$

which proves (3.8). Formulas (3.9) and (3.10) are consequences of (2.7) and (3.4). □

PROPOSITION 3.3. *In some neighborhood of* 0 *we have*

$$\mathcal{B}_p\big(z(1+z)^{-p}\big) = 1+z, \tag{3.11}$$

and more generally, for $r\neq 0$ we have

$$\mathcal{B}_p\left(\big((1+z)^{\frac{1}{r}}-1\big)(1+z)^{\frac{-p}{r}}\right)^r = 1+z. \tag{3.12}$$

Proof. Since we have $\mathcal{B}_p(0) = 1$ and $\mathcal{B}_p'(0) = 1$, there is a function f_p defined on a neighborhood of 0 such that $f_p(0) = 0$ and $\mathcal{B}(f_p(z)) = 1+z$. Substituting $z \mapsto f_p(z)$ in (3.3) we obtain $f_p(z) = z(1+z)^{-p}$. Now we put $z \mapsto (1+w)^{1/r} - 1$ to (3.11) and taking the r-th power we obtain (3.12). □

REMARK. Note that (3.11) leads to an analytic proof of (3.4). Namely, substituting in (3.4) $z \mapsto z(1+z)^{-p}$ we get

$$\begin{aligned}\mathcal{B}_{p-r}\left(z(1+z)^{-p}\mathcal{B}_p\left(z(1+z)^{-p}\right)^r\right) &= \mathcal{B}_{p-r}\left(z(1+z)^{-p}(1+z)^r\right)\\ &= 1+z = \mathcal{B}_p\left(z(1+z)^{-p}\right).\end{aligned}$$

Finally we note a symmetry possessed by our generating functions.

PROPOSITION 3.4. *For $p, r, t \in \mathbb{R}$ we have*

$$\mathcal{B}_p(-z)^r = \mathcal{B}_{1-p}(z)^{-r}, \tag{3.13}$$

$$\mathcal{D}_{p,r,t}(-z) = \mathcal{D}_{1-p,-1-r,1-t}(z). \tag{3.14}$$

Proof. One can check that $(-1)^m A_m(p,r) = A_m(1-p,-r)$, which proves (3.13), and by the definition (3.7), (3.13) implies (3.14). □

4. RELATIONS WITH NONCOMMUTATIVE PROBABILITY

By a *probability quasi-measure* we will mean a linear functional μ on the set $\mathbb{R}[x]$ of polynomials with real coefficients, satisfying $\mu(1) = 1$. In the case when μ is given by $\mu(P) = \int P(t)\,d\widetilde{\mu}(t)$ for some probability measure $\widetilde{\mu}$ on $\mathbb{R}$ we will identify μ with $\widetilde{\mu}$ and say that μ is *proper* or is just a *probability measure.* A probability quasi-measure μ is uniquely determined by its *moment sequence* $\{\mu(x^m)\}_{m=0}^{\infty}$. It is proper if and only if its moment sequence is *positive definite*, i.e. if

$$\sum_{i,j=0}^{\infty} \mu(x^{i+j})\alpha_i\alpha_j \geq 0$$

holds for every sequence $\{\alpha_i\}_{i=0}^{\infty}$ of real numbers, with only finitely many nonzero entries. All probability measures encountered in this paper are compactly supported and therefore uniquely determined by their moment sequences. For a probability quasi-measure μ we define its *moment generating function*, which is the (at least formal) power series

$$M_\mu(z) := \sum_{m=0}^{\infty} \mu(x^m) z^m$$

and its *reflection* $\widehat{\mu}$ by $\widehat{\mu}(x^m) := (-1)^m \mu(x^m)$ or, equivalently, $M_{\widehat{\mu}}(z) := M_\mu(-z)$. If μ is a probability measure then so is $\widehat{\mu}$ and then we have $\widehat{\mu}(X) = \mu(-X)$ for every Borel subset of $\mathbb{R}$.

For $p, r, t \in \mathbb{R}$ we define probability quasi-measures $\mu(p,r)$ and $\nu(p,r,t)$ by

$$\mu(p,r)(x^m) := A_m(p,r), \tag{4.1}$$

$$\nu(p,r,t)(x^m) := \sum_{k=0}^{m} A_{m-k}(p, k(p-1)+r)t^k, \tag{4.2}$$

in particular, by (2.7),

$$\nu(p,r,p)(x^m) = \binom{mp+r}{m}. \tag{4.3}$$

For example, $\mu(1,1) = \nu(1,0,1) = \delta_1$ and for every $p \in \mathbb{R}$ we have $\mu(p,0) = \nu(0,0,0) = \delta_0$. Note that $\nu(p,r,0) = \mu(p,r)$ so that the class of probability quasi-measures $\mu(p,r)$ is contained in that of $\nu(p,r,t)$, we will be interested however mainly in the former.

First we note that Proposition 3.4 leads to

Proposition 4.1.

$$\widehat{\mu(p,r)} = \mu(1-p,-r), \tag{4.4}$$

$$\widehat{\nu(p,r,t)} = \nu(1-p,-1-r,1-t). \qquad \square \tag{4.5}$$

There are several convolutions of probability quasi-measures, apart from the classical one: $(\mu * \nu)(x^n) := \sum_{k=0}^{n} \binom{n}{k} \mu(x^k)\nu(x^{n-k})$, which are related to various notions of independence (namely, the free, boolean and the monotonic independence) in noncommutative probability.

1. *Free convolution* (see [2, 15, 11]) is defined in the following way. For a probability quasi-measure μ we define its *free R-transform* (or the *additive free transform*) $R_\mu(z)$ by the formula:

$$M_\mu(z) = R_\mu(zM_\mu(z)) + 1. \tag{4.6}$$

The *free cumulants* $r_m(\mu)$ are defined as the coefficients of the Taylor expansion $R_\mu(z) = \sum_{k=1}^{\infty} r_k(\mu) z^k$ (combinatorial relation between moments and free cumulants is described in [11] and [4]). Then the free convolution $\mu \boxplus \nu$ can be defined as the unique probability quasi-measure which satisfies

$$R_{\mu\boxplus\nu}(z) = R_\mu(z) + R_\nu(z). \tag{4.7}$$

We also define *free power* $\mu^{\boxplus t}$, $t > 0$, by $R_{\mu^{\boxplus t}}(z) := tR_\mu(z)$.

As a consequence of (4.6) and (3.4) we obtain:

Proposition 4.2. *For the free additive transform of $\mu(p,r)$ we have*

$$R_{\mu(p,r)}(z) = \mathcal{B}_{p-r}(z)^r - 1 \tag{4.8}$$

so that for the free cumulants we have $r_m(\mu(p,r)) = A_m(p-r,r)$, $m \geq 1$. $\square$

The *free S-transform* (or the *free multiplicative transform*) of a quasi-measure μ, with $\mu(x^1) \neq 0$, is defined by the relation

$$R_\mu(zS_\mu(z)) = z \quad \text{or, equivalently,} \quad M_\mu\big(z(1+z)^{-1}S_\mu(z)\big) = 1+z. \tag{4.9}$$

Then the *multiplicative free convolution* $\mu_1 \boxtimes \mu_2$ and the *multiplicative free power* $\mu^{\boxtimes t}$, $t > 0$, are defined by

$$S_{\mu_1\boxtimes\mu_2}(z) := S_{\mu_1}(z)S_{\mu_2}(z) \quad \text{and} \quad S_{\mu^{\boxtimes t}}(z) := S_\mu(z)^t. \tag{4.10}$$

Proposition 4.3. *For $r \neq 0$ the S-transform of the measure $\mu(p,r)$ is equal to*

$$S_{\mu(p,r)}(z) = \frac{(1+z)^{\frac{1}{r}} - 1}{z}\big(1+z\big)^{\frac{r-p}{r}}. \tag{4.11}$$

Consequently

$$\mu(1+p_1,1) \boxtimes \mu(1+p_2,1) = \mu(1+p_1+p_2,1), \tag{4.12}$$

and more generally

$$\mu(p_1,r) \boxtimes \mu(1+p_2,1) = \mu(p_1+rp_2,r). \tag{4.13}$$

We have also

$$\mu(1+p,1)^{\boxtimes t} = \mu(1+tp,1). \tag{4.14}$$

Proof. Formula (4.11) is a consequence of (3.12). In particular

$$S_{\mu(1+p,1)}(z) = (1+z)^{-p} \tag{4.15}$$

which leads to (4.12), (4.13) and (4.14). □

2. The *boolean convolution* $\mu_1 \uplus \mu_2$ and the boolean power $\mu^{\uplus t}$, $t > 0$, (see [14, 3]) can be defined by putting

$$\frac{1}{M_{\mu_1\uplus\mu_2}(z)} = \frac{1}{M_{\mu_1}(z)} + \frac{1}{M_{\mu_2}(z)} - 1, \tag{4.16}$$

$$M_{\mu^{\uplus t}}(z) = \frac{M_\mu(z)}{(1-t)M_\mu(z)+t}. \tag{4.17}$$

Comparing this with definition (3.7) we get

Proposition 4.4. *For $p, t \in \mathbb{R}$ we have*

$$\mu(p,1)^{\uplus t} = \nu(p,0,t). \quad \square \tag{4.18}$$

3. *Monotonic convolution* (see [10]) is an associative, noncommuting operation $\rhd$ which is defined by: $\mu_1 \rhd \mu_2 = \mu$ iff

$$M_\mu(z) = M_{\mu_1}\big(zM_{\mu_2}(z)\big) \cdot M_{\mu_2}(z). \tag{4.19}$$

Then (3.4) and (3.10) yield

Proposition 4.5. *For any parameters $a, b, r, t \in \mathbb{R}$ we have*

$$\mu(a,b) \rhd \mu(a+r,r) = \mu(a+r,b+r), \tag{4.20}$$

$$\nu(a,b,t) \rhd \mu(a+r,r) = \nu(a+r,b+r,t). \quad \square \tag{4.21}$$

In the next section we are going to study which of the probability quasi-measures $\mu(p,r)$ and $\nu(p,r,t)$ are actually probability measures. For this purpose we will use some of the the following facts, which are available in literature (see [15, 11, 14, 10, 6, 7]): The class of all compactly supported probability measures on $\mathbb{R}$ is closed under the free, boolean, and monotonic convolution and also under taking the powers $\mu^{\boxplus s}$, $\mu^{\uplus t}$, for $s \geq 1$, $t > 0$. Moreover, the class of probability measures with compact support contained in $[0,\infty)$ is closed under the free, multiplicative free, boolean and monotonic convolution and also under taking the powers $\mu^{\boxplus s}$, $\mu^{\boxtimes s}$ and $\mu^{\uplus t}$ for $s \geq 1$ and $t > 0$.
A probability measure μ on $\mathbb{R}$ (resp. on $[0,\infty)$) is called $\boxplus$-*infinitely divisible* (resp. $\boxtimes$-*infinitely divisible*) if $\mu^{\boxplus t}$ (resp. $\mu^{\boxtimes t}$) is a probability measure for every $t > 0$. If μ has compact support and $r_m(\mu)$ are its free cumulants then μ is $\boxplus$-infinitely divisible if and only if the sequence $\{r_{m+2}(\mu)\}_{m=0}^{\infty}$ is positive definite.

5. Positivity

The aim of this section is to study which of the quasi measures $\mu(p,r)$ and $\nu(p,r,t)$ are actually measures, i.e. for which parameters $p, r, t \in \mathbb{R}$ the corresponding sequence is positive definite. We start with

Theorem 5.1. *If $p \geq 1$, $0 \leq r \leq p$ then $\{A_m(p,r)\}_{m=0}^{\infty}$ is the moment sequence of a probability measure $\mu(p,r)$ with a compact support contained in $[0,\infty)$. If $p \leq 0$, $p-1 \leq r \leq 0$ then $\mu(p,r)$ is a probability measure which is the reflection of $\mu(1-p,-r)$.*

Proof. We know already that $\widetilde{\pi} = \mu(2,1)$ is the free Poisson law (1.1). Then, as was noted in [1], $\widetilde{\pi}$ is $\boxtimes$-infinitely divisible and for $s > 0$ we have $\pi^{\boxtimes s} = \mu(1+s,1)$. Hence if $p \geq 1$ then $\mu(p,1)$ is a probability measure with a compact support contained in $[0,\infty)$. By (2.3) it implies that the sequence $A_m(p,p) = A_{m+1}(p,1)$ is also positive definite, namely we have

$$\int_{\mathbb{R}} f(x)\, d\mu(p,p)(x) = \int_{\mathbb{R}} f(x) x\, d\mu(p,1)(x)$$

for any continuous function f on $\mathbb{R}$. Hence $\mu(p,p)$, $p \geq 1$, is a probability measure with a compact support contained in $[0,\infty)$. For $1 \leq r \leq p$ we apply (4.13) to obtain:

$$\mu(p,r) = \mu(r,r) \boxtimes \mu(p/r,1),$$

which proves the first statement for the sector $1 \leq r \leq p$.
For $r \in (0,1)$ the measure $\mu(1,r)$ is related to the Euler beta function

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}\, dx. \tag{5.1}$$

We will use its well known properties: $B(a,1-a) = \frac{\pi}{\sin a\pi}$ and $B(a,b) = \frac{a-1}{a+b-1}B(a-1,b)$. If we define probability measure

$$\mu_r := \frac{\sin \pi r}{\pi} x^{r-1}(1-x)^{-r}\, dx \tag{5.2}$$

on $[0,1]$ then we have

$$\int_{\mathbb{R}} x^m\,d\mu_r(x) = \frac{\sin\pi r}{\pi} B(m+r,1-r) = \prod_{k=1}^{m} \frac{r+i-1}{i} = A_m(1,r).$$

which means that $\mu(1,r)=\mu_r$. Now for $s\geq 0$ we have

$$\mu(1+rs,r)=\mu(1,r)\boxtimes\mu(1+s,1),$$

which proves the first statement for $(p,r)\in[1,+\infty)\times(0,1)$. It remains to note that $\mu(p,0)=\delta_0$ for every $p\in\mathbb{R}$.
The second statement is a consequence of (4.4). □

We conjecture that the last theorem fully characterizes the set of parameters $p,r\in\mathbb{R}$ for which $\mu(p,r)$ is a measure (apart from the trivial case $\mu(p,0)=\delta_0$). It is easy to check that $A_0(p,r)A_2(p,r)-A_1(p,r)^2=r(2p-1-r)/2$, hence a necessary condition for positive definiteness of the sequence $A_m(p,r)$ is that $r(2p-1-r)\geq 0$.

Remark. According to Penson and Solomon [12]:

$$\mu(3,1)=\frac{\sqrt[6]{108}\left[2^{1/3}\left(27+3\sqrt{81-12x}\right)^{2/3}-6x^{1/3}\right]}{12\pi x^{2/3}(27+3\sqrt{81-12x})^{1/3}}\,dx \tag{5.3}$$

on $[0,27/4]$. More generally, for $\mu(p,1)$ with $p\in\mathbb{N}$ we refer to [8].

Corollary 5.1. *If either $0\leq 2r\leq p$, $r+1\leq p$ or $p\leq 2r+1$, $p\leq r\leq 0$ then $\mu(p,r)$ is $\boxplus$-infinitely divisible.*

Proof. By Theorem 13.16 in [11], a compactly supported probability measure μ, with free cumulants $r_m(\mu)$, is $\boxplus$-infinitely divisible if and only if the sequence $\{r_{m+2}(\mu)\}_{m=0}^{\infty}$ is positive definite. Then it is sufficient to refer to (4.8) and to note that the numbers $A_{m+2}(p-r,r)$ are the moments of the measure $x^2\,d\mu(p-r,r)(x)$. □

Corollary 5.2. *If $0\leq r\leq p-1$, $t>0$ then $\nu(p,r,t)$ is a probability measure with a compact support contained in $[0,+\infty)$. If $p\leq 1+r\leq 0$, $t<1$ then $\nu(p,r,t)$ is a probability measure which is the reflection of $\nu(1-p,-1-r,1-t)$. In particular, if either $0\leq r\leq p-1$ or $p\leq 1+r\leq 0$ then the sequence $\left\{\binom{mp+r}{m}\right\}_{m=0}^{\infty}$ is positive definite.*

Proof. For $0\leq r\leq p-1$, $t>0$ we apply (4.21) and (4.18):

$$\nu(p,r,t)=\nu(p-r,0,t)\rhd\mu(p,r)=\mu(p-r,1)^{\uplus t}\rhd\mu(p,r)$$

and Theorem 5.1. Then we use (4.5). □

A measure ν on $\mathbb{R}$ is called *symmetric* if $\widehat{\nu}=\nu$. For a probability quasi-measure μ define its *symmetrization* μ^{s} by $M_{\mu^{\mathrm{s}}}(z):=M_\mu\left(z^2\right)$. If μ is a probability measure with support contained in $[0,\infty)$ then μ^{s} is a symmetric measure on $\mathbb{R}$, which satisfies $\int_{\mathbb{R}} f(t^2)\,d\mu^{\mathrm{s}}(t)=\int_{\mathbb{R}} f(t)\,d\mu(t)$ for every compactly supported continuous function on $\mathbb{R}$. Denote by $\mu^{\mathrm{s}}(p,r)$ and $\nu^{\mathrm{s}}(p,r,t)$ the symmetrization of $\mu(p,r)$ and $\nu(p,r,t)$. Then, by (3.4) and (4.9), for the free additive transform we have

$$R_{\mu^{\mathrm{s}}(p,r)}(z) = \mathcal{B}_{p-2r}\big(z^2\big)^r - 1. \tag{5.4}$$

In the same way as Corollary 5.2 one can prove

COROLLARY 5.3. *If $p \geq 1$, $0 \leq r \leq p$ then $\mu^{\mathrm{s}}(p,r)$ is a symmetric probability measure on $\mathbb{R}$. Moreover, if $p-2r \geq 1$ and $0 \leq 3r \leq p$ then $\mu^{\mathrm{s}}(p,r)$ is $\boxplus$-infinitely divisible.* □

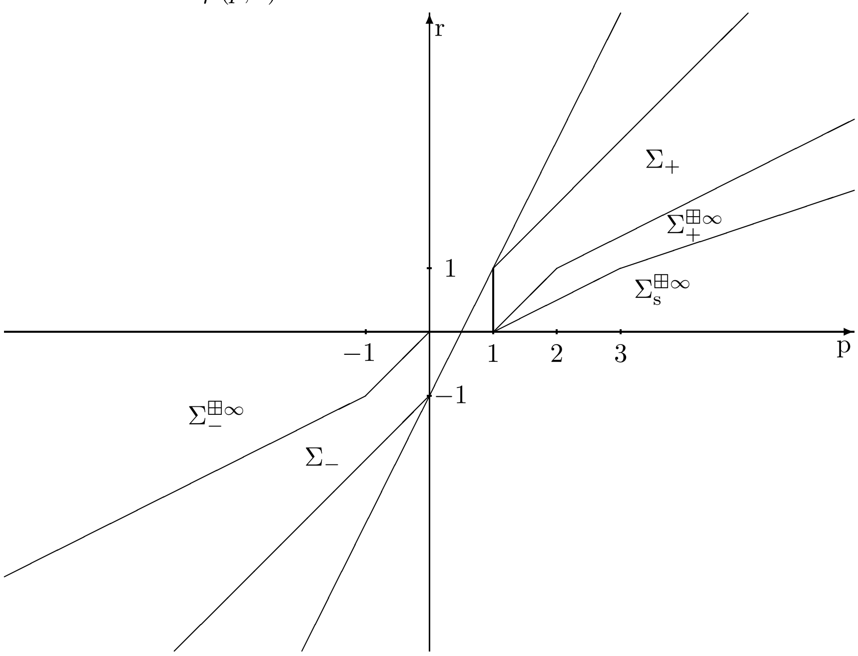
Let us record some formulas:

$$\mu^{\mathrm{s}}(p,1)^{\uplus t} = \nu^{\mathrm{s}}(p,0,t), \tag{5.5}$$

$$\mu^{\mathrm{s}}(a,b) \rhd \mu^{\mathrm{s}}(a+2r,r) = \mu^{\mathrm{s}}(a+2r,b+r), \tag{5.6}$$

$$\nu^{\mathrm{s}}(a,b,t) \rhd \mu^{\mathrm{s}}(a+2r,r) = \nu^{\mathrm{s}}(a+2r,b+r,t). \tag{5.7}$$

5.1. PICTURE FOR $\mu(p,r)$.



Here we illustrate the main results concerning the measures $\mu(p,r)$.

(1) If $\mu(p,r)$ is a probability measure then $r(2p-1-r) \geq 0$ (the right-top and left-bottom sector between the p-axis and the line $r = 2p-1$),
(2) Σ_+ (including $\Sigma_+^{\boxplus\infty}$ and $\Sigma_{\mathrm{s}}^{\boxplus\infty}$): $\mu(p,r)$ is a probability measure with a compact support contained in $[0,\infty)$,
(3) Σ_- (including $\Sigma_-^{\boxplus\infty}$): $\mu(p,r)$ is a probability measure, the reflection of $\mu(1-p,-r)$,
(4) $\Sigma_+^{\boxplus\infty} \cup \Sigma_-^{\boxplus\infty}$ (including $\Sigma_{\mathrm{s}}^{\boxplus\infty}$): $\mu(p,r)$ is $\boxplus$-infinitely divisible,
(5) $\Sigma_{\mathrm{s}}^{\boxplus\infty}$: the symmetrization of $\mu(p,r)$ is $\boxplus$-infinitely divisible.

6. Dilations

For a probability quasi-measure μ we define its *dilation with parameter* $c > 0$ by $(\mathrm{D}_c\mu)(x^m) := c^m\mu(x^m)$. Then for the moment generating function we have: $M_{\mathrm{D}_c\mu}(z) = M_\mu(cz)$ and similarly for the free R-transform: $R_{\mathrm{D}_c\mu}(z) = R_\mu(cz)$, while for the S-transform we have $S_{\mathrm{D}_c\mu}(z) = \frac{1}{c}S_\mu(z)$. If μ is proper then we have $(\mathrm{D}_c\mu)(X) = \mu\big(\frac{1}{c}X\big)$ for every Borel subset X of $\mathbb{R}$. In this part we are going to study dilations of the measures $\mu(p,r)$ and $\nu(p,r,t)$ and their limits as the parameter goes to 0.
For $h \geq 0$ and $a, p, r \in \mathbb{R}$ define sequences

$$\binom{a}{m}_h := \frac{1}{m!}\prod_{i=0}^{m-1}(a - ih), \tag{6.1}$$

$$A_m(p,r,h) := \frac{r}{m!}\prod_{i=1}^{m-1}(mp + r - ih), \tag{6.2}$$

with $A_0(p,r,h) := 1$. In particular $A_m(p,r,h) = \frac{r}{mp+r}\binom{mp+r}{m}_h$ whenever $mp + r \neq 0$. Then, for $h \geq 0$ and $p, r, t \in \mathbb{R}$ define probability quasi-measures:

$$\mu(p,r,h)(x^m) := A_m(p,r,h), \tag{6.3}$$

$$\nu(p,r,t,h)(x^m) := \sum_{k=0}^{m} A_{m-k}(p, k(p-h) + r, h)t^k. \tag{6.4}$$

and their moment generating functions $\mathcal{B}_{p,r,h}(z)$ and $\mathcal{D}_{p,r,t,h}(z)$ respectively. Note that if $h > 0$ then $A_m(p,r,h) = h^m A_m(p/h, r/h)$ and hence these probability quasi measures can be represented as dilations:

$$\mu(p,r,h) = \mathrm{D}_h\mu(p/h, r/h), \tag{6.5}$$

$$\nu(p,r,t,h) = \mathrm{D}_h\nu(p/h, r/h, t/h). \tag{6.6}$$

Therefore the corresponding moment generating functions are

$$\mathcal{B}_{p,r,h}(z) = \mathcal{B}_{p/h}(hz)^{r/h}, \tag{6.7}$$

$$\mathcal{D}_{p,r,t,h}(z) = \mathcal{D}_{p/h,r/h,t/h}(hz) = \frac{h\mathcal{B}_{p,h+r,h}(z)}{(h-t)\mathcal{B}_{p,h,h}(z) + t}. \tag{6.8}$$

These formulas allow us to derive properties of the probability quasi-measures $\mu(p,r,h)$ and $\nu(p,r,t,h)$ directly from our previous results when $h > 0$, and, after taking the limit with $h \to 0$, for $h = 0$.

Proposition 6.1. *For $h > 0$ and $p, r, t \in \mathbb{R}$*

$$\mathcal{B}_{p,h,h}(z) = 1 + zh\mathcal{B}_{p,p,h}(z), \tag{6.9}$$

$$\log\left(\mathcal{B}_{p,1,0}(z)\right) = z\mathcal{B}_{p,p,0}(z), \tag{6.10}$$

$$\mathcal{D}_{p,r,t,0}(z) = \frac{\mathcal{B}_{p,r,0}(z)}{1 - zt\mathcal{B}_{p,p,0}(z)}. \tag{6.11}$$

Proof. First formula is a consequence of (3.3) and (6.7). Then we have

$$\frac{\mathcal{B}_{p,1,h}(z)^h - 1}{h} = \frac{\mathcal{B}_{p,h,h}(z) - 1}{h} = z\mathcal{B}_{p,p,h}(z).$$

Taking the limit with $h \to 0$ we obtain (6.10).
For (6.11) we write use (6.8) and (6.9) to get

$$\frac{1}{h}\big[(h-t)\mathcal{B}_{p,h,h}(z) + t\big] = 1 - (t-h)\frac{\mathcal{B}_{p,h,h}(z) - 1}{h} = 1 - (t-h)z\mathcal{B}_{p,p,h}(z)$$

and then we take limit with $h \to 0$. □

PROPOSITION 6.2. *For $h \geq 0$ and $p, r, s \in \mathbb{R}$ we have*

$$\mathcal{B}_{p-r,s,h}\left(z\mathcal{B}_{p,r,h}(z)\right) = \mathcal{B}_{p,s,h}(z). \qquad \square \tag{6.12}$$

PROPOSITION 6.3. *For $h \geq 0$ and $p, r \in \mathbb{R}$ we have*

$$\nu(p, r, p, h)(x^m) = \binom{mp + r}{m}_h. \tag{6.13}$$

Proof. For $h > 0$ the formula is a consequence of (6.6). Then we take limit with $h \to 0$. □

PROPOSITION 6.4. *For $h \geq 0$ and $p, r, t \in \mathbb{R}$ we have*

$$\mu\widehat{(p, r, h)} = \mu(h - p, -r, h), \tag{6.14}$$

$$\nu\widehat{(p, r, t, h)} = \nu(h - p, -h - r, h - t, h). \tag{6.15}$$

Proof. First we note that $A_m(p, r, h)(-1)^m = A_m(h - p, -r, h)$ and then we apply (6.8) and (3.14). □

PROPOSITION 6.5. *For the free transforms we have*

$$R_{\mu(p,r,h)}(z) = \mathcal{B}_{p-r,r,h}(z) - 1 \tag{6.16}$$

$$S_{\mu(p,r,h)}(z) = \frac{(1+z)^{h/r} - 1}{hz}(1+z)^{(r-p)/r} \quad \textit{for } h > 0, \tag{6.17}$$

$$S_{\mu(p,r,0)}(z) = \frac{\log(1+z)}{rz}(1+z)^{(r-p)/r}, \tag{6.18}$$

$$S_{\nu(p,0,t,0)}(z) = \frac{1}{t}e^{\frac{-pz}{t(1+z)}}. \tag{6.19}$$

In particular $\nu(p, 0, t, 0) = \mathrm{D}_t\left(\nu(1, 0, 1, 0)^{\boxtimes p/t}\right)$.

Proof. Formulas (6.16), (6.17) are consequences of (6.7), (4.11) and (6.12). Therefore, for $h > 0$ we have

$$\mathcal{B}_{p,r,h}\left(\frac{(1+z)^{h/r} - 1}{h}(1+z)^{-p/r}\right) = 1 + z, \tag{6.20}$$

which leads to

$$\mathcal{B}_{p,r,0}\left(\frac{\log(1+z)}{r(1+z)^{p/r}}\right) = 1 + z \tag{6.21}$$

and to (6.18). In particular, substituting $(1+z)\mapsto e^{\frac{pz}{t(1+z)}}$, we have

$$\mathcal{B}_{p,p,0}\left(\frac{z}{t(1+z)}e^{\frac{-pz}{t(1+z)}}\right)=e^{\frac{pz}{t(1+z)}} \tag{6.22}$$

which, combined with (6.11) gives

$$\mathcal{D}_{p,0,t,0}\left(\frac{z}{t(1+z)}e^{\frac{-pz}{t(1+z)}}\right)=\frac{1}{1-\frac{z}{1+z}}=1+z. \tag{6.23}$$ □

Proposition 6.6. *For $h>0$ and $p,t\in\mathbb{R}$ we have*

$$\mu(p,h,h)^{\uplus t}=\nu(p,0,th,h), \tag{6.24}$$

$$\nu(p,0,1,0)^{\uplus t}=\nu(p,0,t,0). \tag{6.25}$$

Proof. Since $\mathcal{B}_{p,0,0}(z)=1$, formula (6.25) is a consequence of (6.11). □

Proposition 6.7. *For $h\geq 0$, $t>0$, $a,b\in\mathbb{R}$ we have*

$$\mu(a,b,h)\rhd\mu(a+r,r,h)=\mu(a+r,b+r,h), \tag{6.26}$$

$$\nu(a,b,t,h)\rhd\mu(a+r,r,h)=\nu(a+r,b+r,t,h). \tag{6.27}$$ □

Proposition 6.8. *Assume that $h\geq 0$.*
1. If $p\geq h$ and $0\leq r\leq p$ then $\mu(p,r,h)$ is a probability measure with support contained in $[0,\infty)$. If $p\leq 0$, $p-h\leq r\leq 0$ then $\mu(p,r,h)$ is a probability measure which is the reflection of $\mu(h-p,-r,h)$.
2. If either $0\leq 2r\leq p$, $r+h\leq p$ or $p\leq 2r+h$, $p\leq r\leq 0$ then $\mu(p,r,h)$ is $\boxplus$-infinitely divisible.
3. If $0\leq r\leq p-h$, $t>0$ then $\nu(p,r,t,h)$ is a probability measure with a compact support contained in $[0,+\infty)$. If $p\leq h+r\leq 0$, $t<h$ then $\nu(p,r,t,h)$ is a probability measure which is the reflection of $\nu(h-p,-h-r,h-t,h)$
In particular, if either $0\leq r\leq p-h$ or $p\leq h+r\leq 0$ then the sequence $\left\{\binom{mp+r}{m}_h\right\}_{m=0}^{\infty}$ is positive definite. □

We conclude with some remarks on the probability measure $\nu_0:=\nu(1,0,1,0)$, for which the moments are $\nu_0(x^m)=\binom{m}{m}_0=\frac{m^m}{m!}$. From (4.9), (6.19) we have

$$S_{\nu_0}(z)=e^{\frac{-z}{1+z}}, \tag{6.28}$$

$$R_{\nu_0}\big(ze^{\frac{-z}{1+z}}\big)=z, \tag{6.29}$$

$$M_{\nu_0}\Big(\frac{z}{1+z}e^{\frac{-z}{1+z}}\Big)=1+z. \tag{6.30}$$

Theorem 6.1. *The sequence $\left\{\frac{m^m}{m!}\right\}_{m=0}^{\infty}$ is positive definite and the corresponding probability measure ν_0 has compact support contained in $[0,e]$. Moreover, ν_0 is $\boxtimes$-infinitely divisible.*

Proof. First observe that $\lim_{m\to\infty}\sqrt[m]{\frac{m^m}{m!}}=e$, which implies that the support of ν_0 is contained in $[0,e]$. Now we recall (see Theorem 3.7.3 in [2]) that a probability measure μ with support contained in $[0,\infty)$ is $\boxtimes$-infinite divisible if and only if the function $\Sigma_\mu(z):=S_\mu\big(z(1-z)^{-1}\big)$ can be expressed as $\Sigma_\mu(z)=$

$e^{v(z)}$, where $v : \mathbb{C} \setminus [0, \infty) \mapsto \mathbb{C}$ is analytic, satisfies $v(\overline{z}) = \overline{v(z)}$ and maps the upper half-plane $\mathbb{C}^+$ into the lower half-plane $\mathbb{C}^-$. In our case $\Sigma_{\nu_0}(z) = e^{-z}$ and the function $v(z) = -z$ does satisfy these assumptions. □

Let us briefly reconstruct the way we have obtained the measure ν_0. We started with $\widetilde{\pi} = \mu(2, 1, 1)$, the free Poisson measure. Then

$$\mu(p, h, h) = \mathrm{D}_h \mu(p/h, 1, 1) = \mathrm{D}_h \left(\widetilde{\pi}^{\boxtimes \frac{p}{h} - 1} \right),$$

so putting $h = 1/n$, $p = 1$ and using (6.24) with $t = 1/h = n$ we have

$$\left(\mathrm{D}_{\frac{1}{n}} \left(\widetilde{\pi}^{\boxtimes n-1} \right) \right)^{\uplus n} \longrightarrow \nu_0, \qquad \text{with } n \to \infty, \tag{6.31}$$

where the convergence here means that the m-th moment of $\left(\mathrm{D}_{\frac{1}{n}} \left(\widetilde{\pi}^{\boxtimes n-1} \right) \right)^{\uplus n}$ tends to $\frac{m^m}{m!}$. Note also that from (6.29) one can calculate free cumulants of ν_0: $r_1 = 1$, $r_2 = 1$, $r_3 = \frac{1}{2}$, $r_4 = \frac{-1}{3}$. Since $r_4 < 0$, the measure ν_0 is not $\boxplus$-infinitely divisible.

References

[1] T. Banica, S. T. Belinschi, M. Capitaine, B. Collins, *Free Bessel laws,* to appear in Canadian J. Math.

[2] H. Bercovici, D. V. Voiculescu, *Free convolutions of measures with unbounded support,* Indiana Univ. Math. J. 42 (1993). 733–773.

[3] M. Bożejko, J. Wysoczański, *Remarks on t-transformations of measures and convolutions,* Annales Institute Henri Poincaré, 37/6 (2001) 737–761.

[4] D. Callan, *Sets, lists and noncrossing partitions,* Journal of Integer Sequences, 11 (2008), Art. 08.1.3.

[5] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete mathematics,* Addison-Wesley 1994.

[6] T. Hasebe, *On monotone convolution and momotone infinite divisibility,* arXiv:1002.3430v2.

[7] T. Hasebe, *Monotone convolution semigroups,* Studia Math. 200 (2010), 175–199.

[8] D.-Z. Liu, C. Song, Z.-D. Wang, *On explicit probability densities associated with Fuss-Catalan numbers,* arXiv:1008.0271v1.

[9] N. Muraki, *Monotonic convolution and monotonic Levy-Hincin formula,* preprint 2000.

[10] N. Muraki, *Monotonic independence, monotonic central limit theorem and monotonic law of small numbers,* Inf. Dim. Anal. Quantum Probab. Rel. Topics 4 (2001) 39–58.

[11] A. Nica, R. Speicher, *Lectures on the Combinatorics of Free Probability,* Cambridge University Press, 2006.

[12] K. A. Penson, A. I. Solomon, *Coherent states from combinatorial sequences,* Quantum theory and symmetries, Kraków 2001, World Sci.Publ., River Edge, NJ, 2002, 527–530.

[13] J. Riordan, *Combinatorial identities,* Willey 1968.

[14] R. Speicher, R. Woroudi, *Boolean convolution,* Fields Institute Communications 12 (1997), 267–279.

[15] D. V. Voiculescu, K. J. Dykema, A. Nica, *Free random variables*, CRM 1992.

Wojciech Młotkowski
Mathematical Institute
University of Wrocław
Pl. Grunwaldzki 2/4
50-384 Wrocław
Poland
mlotkow@math.uni.wroc.pl