Ergodic Properties and KMS Conditions on C^*-Symbolic Dynamical Systems

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Abstract. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ consists of a unital C^*-algebra $\mathcal{A}$ and a finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms ρ_α of $\mathcal{A}$ indexed by symbols α of Σ satisfying some conditions. The endomorphisms $\rho_\alpha, \alpha \in \Sigma$ yield both a subshift Λ_ρ and a C^*-algebra $\mathcal{O}_\rho$. We will study ergodic properties of the positive operator $\lambda_\rho = \sum_{\alpha\in\Sigma} \rho_\alpha$ on $\mathcal{A}$. We will next introduce KMS conditions for continuous linear functionals on $\mathcal{O}_\rho$ under gauge action at inverse temperature taking its value in complex numbers. We will study relationships among the eigenvectors of λ_ρ in $\mathcal{A}^*$, the continuous linear functionals on $\mathcal{O}_\rho$ satisfying KMS conditions and the invariant measures on the associated one-sided shifts. We will finally present several examples of continuous linear functionals satisfying KMS conditions.

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1. Introduction

D. Olesen and G. K. Pedersen [37] have shown that the C^*-dynamical system $(\mathcal{O}_N, \alpha, \mathbf{R})$ for the Cuntz algebra $\mathcal{O}_N$ with gauge action α admits a KMS state at the inverse temperature γ if and only if $\gamma = \log N$, and the admitted KMS state is unique. By Enomoto-Fujii-Watatani [9], the result has been generalized to the Cuntz-Krieger algebras $\mathcal{O}_A$ as $\gamma = \log r_A$, where r_A is the Perron-Frobenius eigenvalue for the irreducible matrix A with entries in $\{0, 1\}$. These results are generalized to several classes of C^*-algebras having gauge actions (cf. [7], [10], [11], [15], [17], [18], [27], [35], [36], [41], etc.).

Cuntz-Krieger algebras are considered to be constructed by finite directed graphs which yield an important class of symbolic dynamics called shifts of finite type. In [29], the author has generalized the notion of finite directed graphs to a notion of labeled Bratteli diagrams having shift like maps, which we call λ-graph systems. A λ-graph system $\mathfrak{L}$ gives rise to both a subshift $\Lambda_{\mathfrak{L}}$ and a C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ with gauge action. Some topological conjugacy invariants of subshifts have been studied through the C^*-algebras constructed from λ-graph systems ([30]).

A *C^*-symbolic dynamical system* is a generalization of both a λ-graph system and an automorphism of a unital C^*-algebra ([31]). It is a finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms indexed by a finite set Σ of a unital C^*-algebra $\mathcal{A}$ such that $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}$ for $\alpha \in \Sigma$ and $\sum_{\alpha\in\Sigma} \rho_\alpha(1) \geq 1$ where $Z_{\mathcal{A}}$ denotes the center of $\mathcal{A}$. A finite directed labeled graph $\mathcal{G}$ gives rise to a C^*-symbolic dynamical system $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$ such that $\mathcal{A}_{\mathcal{G}} = \mathbf{C}^N$ for some $N \in \mathbf{N}$. A λ-graph system $\mathfrak{L}$ also gives rise to a C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ such that $\mathcal{A}_{\mathfrak{L}}$ is $C(\Omega_{\mathfrak{L}})$ for some compact Hausdorff space $\Omega_{\mathfrak{L}}$ with $\dim \Omega_{\mathfrak{L}} = 0$. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a subshift denoted by Λ_ρ over Σ and a Hilbert C^*-bimodule $(\phi_\rho, \mathcal{H}^\rho_{\mathcal{A}})$ over $\mathcal{A}$. By using general construction of C^*-algebras from Hilbert C^*-bimodules established by M. Pimsner [40], a C^*-algebra denoted by $\mathcal{O}_\rho$ from $(\phi_\rho, \mathcal{H}^\rho_{\mathcal{A}})$ has been introduced in [31]. The C^*-algebra $\mathcal{O}_\rho$ is realized as the universal C^*-algebra generated by partial isometries $S_\alpha, \alpha \in \Sigma$ and $x \in \mathcal{A}$ subject to the relations:

$$\sum_{\gamma\in\Sigma} S_\gamma S_\gamma^* = 1, \qquad S_\alpha S_\alpha^* x = x S_\alpha S_\alpha^*, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$. We call the algebra $\mathcal{O}_\rho$ *the C^*-symbolic crossed product of $\mathcal{A}$ by the subshift Λ_ρ*. The gauge action on $\mathcal{O}_\rho$ denoted by $\hat\rho$ is defined by

$$\hat\rho_z(x) = x, \quad x \in \mathcal{A} \quad \text{and} \quad \hat\rho_z(S_\alpha) = zS_\alpha, \quad \alpha \in \Sigma$$

for $z \in \mathbf{C}, |z| = 1$. If $\mathcal{A} = C(X)$ with $\dim X = 0$, there exists a λ-graph system $\mathfrak{L}$ such that Λ_ρ is the subshift presented by $\mathfrak{L}$ and $\mathcal{O}_\rho$ is the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with $\mathfrak{L}$. If in particular, $\mathcal{A} = \mathbf{C}^N$, the subshift Λ_ρ is a sofic shift and $\mathcal{O}_\rho$ is a Cuntz-Krieger algebra. If $\Sigma = \{\alpha\}$ an automorphism α of a unital C^*-algebra $\mathcal{A}$, the C^*-algebra $\mathcal{O}_\rho$ is the ordinary C^*-crossed product $\mathcal{A} \times_\alpha \mathbf{Z}$. Throughout the paper, we will assume that the C^*-algebra $\mathcal{A}$ is commutative. For a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, define the positive operator λ_ρ on $\mathcal{A}$ by

$$\lambda_\rho(x) = \sum_{\alpha\in\Sigma} \rho_\alpha(x), \qquad x \in \mathcal{A}.$$

We set for a complex number $\beta \in \mathbf{C}$ the eigenvector space of λ_ρ

$$\mathcal{E}_\beta(\rho) = \{\varphi \in \mathcal{A}^* \mid \varphi \circ \lambda_\rho = \beta\varphi\}. \tag{1.1}$$

Let $Sp(\rho)$ be the set of eigenvalues of λ_ρ defined by

$$Sp(\rho) = \{\beta \in \mathbf{C} \mid \mathcal{E}_\beta(\rho) \neq \{0\}\}. \tag{1.2}$$

Let r_ρ denote the spectral radius of λ_ρ on $\mathcal{A}$. We set $T_\rho = \frac{1}{r_\rho}\lambda_\rho$. $(\mathcal{A}, \rho, \Sigma)$ is said to be *power-bounded* if the sequence $\|T_\rho^k\|, k \in \mathbf{N}$ is bounded. A state φ on $\mathcal{A}$ is said to be invariant if $\varphi \circ T_\rho = \varphi$. If an invariant state is unique, $(\mathcal{A}, \rho, \Sigma)$ is said to be *uniquely ergodic*. If $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} T_\rho^k(a)$ exists in $\mathcal{A}$ for $a \in \mathcal{A}$, $(\mathcal{A}, \rho, \Sigma)$ is said to be *mean ergodic*. If there exists no nontrivial ideal of $\mathcal{A}$ invariant under λ_ρ, $(\mathcal{A}, \rho, \Sigma)$ is said to be *irreducible*. It will be proved that a mean ergodic and irreducible $(\mathcal{A}, \rho, \Sigma)$ is uniquely ergodic and power-bounded (Theorem 3.12).

Let $A = [A(i,j)]_{i,j=1}^N$ be an irreducible matrix with entries in $\{0,1\}$, and $S_i, i = 1,\dots,N$ be the canonical generating family of partial isometries of the Cuntz-Krieger algebra $\mathcal{O}_A$. Let $\mathcal{A}_A$ be the C^*-subalgebra of $\mathcal{O}_A$ generated by the projections $S_jS_j^*, j = 1,\dots,N$. Put $\Sigma = \{1,\dots,N\}$ and $\rho_i^A(x) = S_i^* x S_i, x \in \mathcal{A}_A, i \in \Sigma$. Then the triplet $(\mathcal{A}_A, \rho^A, \Sigma)$ yields an example of C^*-symbolic dynamical system such that its C^*-symbolic crossed product $\mathcal{O}_{\rho^A}$ is the Cuntz-Krieger algebra $\mathcal{O}_A$. The above space $\mathcal{E}_\beta(\rho)$ is identified with the eigenvector space of the matrix A for an eigenvalue β. By Enomoto-Fujii-Watatani [9], a tracial state $\varphi \in \mathcal{E}_\beta(\rho^A)$ on $\mathcal{A}_A$ extends to a KMS state for gauge action on $\mathcal{O}_A$ if and only if $\beta = r_A$ the Perron-Frobenius eigenvalue, and its inverse temperature is $\log r_A$. The admitted KMS state is unique.

In this paper, we will study the space $\mathcal{E}_\beta(\rho)$ of a general C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ for a general eigenvalue β in $\mathbf{C}$ not necessarily maximum eigenvalue and then introduce KMS condition for inverse temperature taking its value in complex numbers. In this generalization, we will study possibility of extension of a continuous linear functional on $\mathcal{A}$ belonging to the eigenvector space $\mathcal{E}_\beta(\rho)$ to the whole algebra $\mathcal{O}_\rho$ as a continuous linear functional satisfying KMS condition. For a C^*-algebra with a continuous action of the one-dimensional torus group $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ and a complex number $\beta \in \mathbf{C}$, we will introduce KMS condition for a continuous linear functional without assuming its positivity at inverse temperature $\mathrm{Log}\beta$. Let $\mathcal{B}$ be a C^*-algebra and $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ be a continuous action of $\mathbf{T}$ to the automorphism group $\mathrm{Aut}(\mathcal{B})$. We write a complex number β with $|\beta| > 1$ as $\beta = re^{i\theta}$ where $r > 1, \theta \in \mathbf{R}$. Denote by $\mathcal{B}^*$ the Banach space of all complex valued continuous linear functionals on $\mathcal{B}$.

Definition. A continuous linear functional $\varphi \in \mathcal{B}^*$ is said to satisfy *KMS condition* at $\mathrm{Log}\beta$ if φ satisfies the condition

$$\varphi(y\alpha_{i\log r}(x)) = \varphi(\alpha_\theta(x)y), \qquad x \in \mathcal{B}^a, y \in \mathcal{B}, \tag{1.3}$$

where $\mathcal{B}^a$ is the set of analytic elements of the action $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ (cf.[3]). We will prove

Theorem 1.1. *Let $(\mathcal{A}, \rho, \Sigma)$ be an irreducible and power-bounded C^*-symbolic dynamical system. Let $\beta \in \mathbf{C}$ be a complex number with $|\beta| > 1$.*

(i) *If $\beta \in Sp(\rho)$ and $|\beta| = r_\rho$ the spectral radius of the positive operator $\lambda_\rho : \mathcal{A} \longrightarrow \mathcal{A}$, then there exists a nonzero continuous linear functional* *on $\mathcal{O}_\rho$ satisfying KMS condition at* $\mathrm{Log}\beta$ *under gauge action. The converse implication holds if* $(\mathcal{A}, \rho, \Sigma)$ *is mean ergodic.*

(ii) *Under the condition* $|\beta| = r_\rho$*, there exists a linear isomorphism between the space* $\mathcal{E}_\beta(\rho)$ *of eigenvectors of continuous linear functionals on* $\mathcal{A}$ *and the space* $KMS_\beta(\mathcal{O}_\rho)$ *of continuous linear functionals on* $\mathcal{O}_\rho$ *satisfying KMS condition at* $\mathrm{Log}\beta$.

(iii) *If* $(\mathcal{A}, \rho, \Sigma)$ *is uniquely ergodic, there uniquely exists a state on* $\mathcal{O}_\rho$ *satisfying KMS condition at* $\log r_\rho$.

(iv) *If in particular* $(\mathcal{A}, \rho, \Sigma)$ *is mean ergodic, then* $\dim \mathcal{E}_\beta(\rho) \leq 1$ *for all* $\beta \in \mathbf{C}$.

In the proof of the above theorem, a Perron-Frobenius type theorem is proved (Theorem 3.13).
Let $\mathcal{D}_\rho$ be the C^*-subalgebra of $\mathcal{O}_\rho$ generated by all elements of the form: $S_{\alpha_1} \cdots S_{\alpha_k} x S^*_{\alpha_k} \cdots S^*_{\alpha_1}$ for $x \in \mathcal{A}, \alpha_1, \dots, \alpha_k \in \Sigma$. Let ϕ_ρ be the endomorphism on $\mathcal{D}_\rho$ defined by $\phi_\rho(y) = \sum_{\alpha\in\Sigma} S_\alpha y S^*_\alpha, y \in \mathcal{D}_\rho$, which comes from the left-shift on the underlying shift spase Λ_ρ. Suppose that $(\mathcal{A}, \rho, \Sigma)$ is uniquely ergodic. The restriction of the unique KMS state on $\mathcal{O}_\rho$ is not necessarily a ϕ_ρ-invariant state. We will clarify a relationship between KMS states on $\mathcal{O}_\rho$ and ϕ_ρ-invariant states on $\mathcal{D}_\rho$ as in the following way:

Theorem 1.2. *Assume that* $(\mathcal{A}, \rho, \Sigma)$ *is irreducible and mean ergodic. Let* τ *be the restriction to* $\mathcal{D}_\rho$ *of the unique KMS state on* $\mathcal{O}_\rho$ *at* $\log r_\rho$ *and* x_ρ *be a positive element of* $\mathcal{A}$ *defined by the limit of the mean*

$$\lim_{n\to\infty} \frac{1}{n}(1 + T_\rho(1) + \cdots + T_\rho^{n-1}(1)).$$

Let μ_ρ *be a linear functional on* $\mathcal{D}_\rho$ *defined by*

$$\mu_\rho(y) = \tau(y x_\rho), \qquad y \in \mathcal{D}_\rho.$$

(i) μ_ρ *is a faithful,* ϕ_ρ*-invariant and ergodic state on* $\mathcal{D}_\rho$ *in the sense that the formula*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu_\rho(\phi_\rho^k(y)x) = \mu_\rho(y)\mu_\rho(x), \qquad x, y \in \mathcal{D}_\rho$$

holds.

(ii) μ_ρ *gives rise to a unique* ϕ_ρ*-invariant probability measure absolutely continuous with respect to the probability measure for the state* τ.

(iii) μ_ρ *is equivalent to the state* τ *as a measure on* $\mathcal{D}_\rho$.

For a C^*-symbolic dynamical system $(\mathcal{A}_A, \rho^A, \Sigma)$ coming from an irreducible matrix $A = [A(i,j)]_{i,j=1}^N$ with entries in $\{0, 1\}$, the subalgebra $\mathcal{D}_{\rho^A}$ is nothing but the commutative C^*-algebra $C(X_A)$ of all continuous functions on the right one-sided topological Markov shift X_A. As ϕ_{ρ^A} corresponds to the left-shift σ_A on X_A, the above unique ϕ_{ρ^A} -invariant state τ is the Parry measure on X_A. The positive element x_{ρ^A} is given by the positive Perron eigenvector

$x_{\rho^A} = [x_j]_{j=1}^N$ of the transpose A^t of A satisfying $\sum_{j=1}^N \tau(S_j S_j^*) x_j = 1$, where $[\tau(S_j S_j^*)]_{j=1}^N$ is the normalized Perron eigenvector of A.
This paper is organized as follows: In Section 2, we will briefly review C^*-symbolic dynamical systems and its C^*-algebras $\mathcal{O}_\rho$. In Section 3, we will study ergodic properties of the operator $T_\rho : \mathcal{A} \longrightarrow \mathcal{A}$ and the eigenspace $\mathcal{E}_\beta(\rho)$. In Section 4, we will study extendability of a linear functional belonging to $\mathcal{E}_\beta(\rho)$ to the subalgebra $\mathcal{D}_\rho$ of $\mathcal{O}_\rho$, which will extend to $\mathcal{O}_\rho$. In Section 5, we will prove Theorem 1.1. In Section 6, we will study a relationship between KMS states and ϕ_ρ-invariant states on $\mathcal{D}_\rho$ to prove Theorem 1.2. In Section 7, we will present several examples of continuous linear functionals on $\mathcal{O}_\rho$ satisfying KMS conditions.

2. C^*-symbolic dynamical systems and their crossed products

Let $\mathcal{A}$ be a unital C^*-algebra. In what follows, an endomorphism of $\mathcal{A}$ means a $*$-endomorphism of $\mathcal{A}$ that does not necessarily preserve the unit 1 of $\mathcal{A}$. Denote by $Z_\mathcal{A}$ the center $\{x \in \mathcal{A} \mid ax = xa \text{ for all } a \in \mathcal{A}\}$ of $\mathcal{A}$. Let Σ be a finite set. A finite family of nonzero endomorphisms $\rho_\alpha, \alpha \in \Sigma$ of $\mathcal{A}$ indexed by elements of Σ is said to be *essential* if $\rho_\alpha(Z_\mathcal{A}) \subset Z_\mathcal{A}$ for $\alpha \in \Sigma$ and $\sum_{\alpha\in\Sigma} \rho_\alpha(1) \geq 1$. If in particular, $\mathcal{A}$ is commutative, the family $\rho_\alpha, \alpha \in \Sigma$ is essential if and only if $\sum_{\alpha\in\Sigma} \rho_\alpha(1) \geq 1$. We remark that the definition in [31] of "essential" for $\rho_\alpha, \alpha \in \Sigma$ is weaker than the above dfinition. It is said to be *faithful* if for any nonzero $x \in \mathcal{A}$ there exists a symbol $\alpha \in \Sigma$ such that $\rho_\alpha(x) \neq 0$.
Definition ([31]). A *C^*-symbolic dynamical system* is a triplet $(\mathcal{A}, \rho, \Sigma)$ consisting of a unital C^*-algebra $\mathcal{A}$ and an essential, faithful finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms of $\mathcal{A}$.
Two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are said to be isomorphic if there exist an isomorphism $\Phi : \mathcal{A} \to \mathcal{A}'$ and a bijection $\pi : \Sigma \to \Sigma'$ such that $\Phi \circ \rho_\alpha = \rho'_{\pi(\alpha)} \circ \Phi$ for all $\alpha \in \Sigma$. For an automorphism α of a unital C^*-algebra $\mathcal{A}$, by setting $\Sigma = \{\alpha\}, \rho_\alpha = \alpha$ the triplet $(\mathcal{A}, \rho, \Sigma)$ becomes a C^*-symbolic dynamical system. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a subshift Λ_ρ over Σ such that a word $\alpha_1 \cdots \alpha_k$ of Σ is admissible for Λ_ρ if and only if $\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1} \neq 0$ ([31, Proposition 2.1]). We say that *a subshift Λ acts on a C^*-algebra $\mathcal{A}$* if there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift Λ_ρ is Λ.
For a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ the C^*-algebra $\mathcal{O}_\rho$ has been originally constructed in [31] as a C^*-algebra from a Hilbert C^*-bimodule by using a Pimsner's general construction of Hilbert C^*-bimodule algebras [40] (cf. [16] etc.). It is called *the C^*-symbolic crossed product of $\mathcal{A}$ by the subshift* Λ_ρ, and realized as the universal C^*-algebra $C^*(x, S_\alpha; x \in \mathcal{A}, \alpha \in \Sigma)$ generated by $x \in \mathcal{A}$ and partial isometries $S_\alpha, \alpha \in \Sigma$ subject to the following relations called (ρ):

$$\sum_{\gamma\in\Sigma} S_\gamma S_\gamma^* = 1, \qquad S_\alpha S_\alpha^* x = x S_\alpha S_\alpha^*, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$.

Let $\mathcal{G} = (G, \lambda)$ be a left-resolving finite labeled graph with underlying finite directed graph $G = (V, E)$ and labeling map $\lambda : E \to \Sigma$ (see [28, p.76]). Denote by $v_1, \dots, v_N$ the vertex set V. Assume that every vertex has both an incoming edge and an outgoing edge. Consider the N-dimensional commutative C^*-algebra $\mathcal{A}_{\mathcal{G}} = \mathbf{C}E_1 \oplus \cdots \oplus \mathbf{C}E_N$ where each minimal projection E_i corresponds to the vertex v_i for $i = 1, \dots, N$. Define an $N \times N$-matrix for $\alpha \in \Sigma$ by

$$A^{\mathcal{G}}(i, \alpha, j) = \begin{cases} 1 & \text{if there exists an edge } e \text{ from } v_i \text{ to } v_j \text{ with } \lambda(e) = \alpha, \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

for $i, j = 1, \dots, N$. We set $\rho^{\mathcal{G}}_\alpha(E_i) = \sum_{j=1}^N A^{\mathcal{G}}(i, \alpha, j) E_j$ for $i = 1, \dots, N$. Then $\rho^{\mathcal{G}}_\alpha, \alpha \in \Sigma$ define endomorphisms of $\mathcal{A}_{\mathcal{G}}$ such that $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$ is a C^*-symbolic dynamical system for which the subshift $\Lambda_{\rho^{\mathcal{G}}}$ is the sofic shift $\Lambda_{\mathcal{G}}$ presented by $\mathcal{G}$. Conversely, for a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, if $\mathcal{A}$ is $\mathbf{C}^N$, there exists a left-resolving labeled graph $\mathcal{G}$ such that $\mathcal{A} = \mathcal{A}_{\mathcal{G}}$ and $\Lambda_\rho = \Lambda_{\mathcal{G}}$ the sofic shift presented by $\mathcal{G}$ ([31, Proposition 2.2]). Put $A_{\mathcal{G}}(i, j) = \sum_{\alpha \in \Sigma} A^{\mathcal{G}}(i, \alpha, j), i, j = 1, \dots, N$. The $N \times N$ matrix $A_{\mathcal{G}} = [A_{\mathcal{G}}(i, j)]_{i,j=1,\dots,N}$ is called the underlying nonnegative matrix for $\mathcal{G}$. Consider the matrix $A^{[2]}_{\mathcal{G}} = [A^{[2]}_{\mathcal{G}}(e, f)]_{e,f \in E}$ indexed by edges E whose entries are in $\{0, 1\}$ by setting

$$A^{[2]}_{\mathcal{G}}(e, f) = \begin{cases} 1 & \text{if } f \text{ follows } e, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The C^*-algebra $\mathcal{O}_{\rho^{\mathcal{G}}}$ for the C^*-symbolic dynamical system $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$ is the Cuntz-Krieger algebra $\mathcal{O}_{A^{[2]}_{\mathcal{G}}}$ (cf. [30, Proposition 7.1], [1]).

More generally let $\mathfrak{L}$ be a λ-graph system (V, E, λ, ι) over Σ. We equip each vertex set V_l with discrete topology. We denote by $\Omega_{\mathfrak{L}}$ the compact Hausdorff space with $\dim \Omega_{\mathfrak{L}} = 0$ of the projective limit $V_0 \xleftarrow{\iota} V_1 \xleftarrow{\iota} V_2 \xleftarrow{\iota} \cdots$ as in [30, Section 2]. Since the algebra $C(V_l)$ denoted by $\mathcal{A}_{\mathfrak{L},l}$ of all continuous functions on V_l is the commutative finite dimensional algebra, the commutative C^*-algebra $C(\Omega_{\mathfrak{L}})$ is an AF-algebra, that is denoted by $\mathcal{A}_{\mathfrak{L}}$. We then have a C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ such that the subshift $\Lambda_{\rho^{\mathfrak{L}}}$ coincides with the subshift $\Lambda_{\mathfrak{L}}$ presented by $\mathfrak{L}$. Conversely, for a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, if the algebra $\mathcal{A}$ is $C(X)$ with $\dim X = 0$, there exists a λ-graph system $\mathfrak{L}$ over Σ such that the associated C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ is isomorphic to $(\mathcal{A}, \rho, \Sigma)$ ([31, Theorem 2.4]). The C^*-algebra $\mathcal{O}_{\rho^{\mathfrak{L}}}$ is the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph system $\mathfrak{L}$.

Let α be an automorphism of a unital C^*-algebra $\mathcal{A}$. Put $\Sigma = \{\alpha\}$ and $\rho_\alpha = \alpha$. The C^*-algebra $\mathcal{O}_\rho$ for the C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is the ordinary C^*-crossed product $\mathcal{A} \times_\alpha \mathbf{Z}$.

In what follows, for a subset F of a C^*-algebra $\mathcal{B}$, we will denote by $C^*(F)$ the C^*-subalgebra of $\mathcal{B}$ generated by F.

Let $(\mathcal{A}, \rho, \Sigma)$ be a C^*-symbolic dynamical system over Σ and Λ the associated subshift Λ_ρ. We denote by $B_k(\Lambda)$ the set of admissible words μ of Λ with length

$|\mu| = k$. Put $B_*(\Lambda) = \cup_{k=0}^{\infty} B_k(\Lambda)$, where $B_0(\Lambda)$ consists of the empty word. Let $S_\alpha, \alpha \in \Sigma$ be the partial isometries in $\mathcal{O}_\rho$ satisfying the relation (ρ). For $\mu = (\mu_1, \dots, \mu_k) \in B_k(\Lambda)$, we put $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$ and $\rho_\mu = \rho_{\mu_k} \circ \cdots \circ \rho_{\mu_1}$. In the algebra $\mathcal{O}_\rho$, we set for $k \in \mathbf{Z}_+$,

$$\begin{aligned} \mathcal{D}_\rho^k &= C^*(S_\mu x S_\mu^* : \mu \in B_k(\Lambda), x \in \mathcal{A}), \\ \mathcal{D}_\rho &= C^*(S_\mu x S_\mu^* : \mu \in B_*(\Lambda), x \in \mathcal{A}), \\ \mathcal{F}_\rho^k &= C^*(S_\mu x S_\nu^* : \mu, \nu \in B_k(\Lambda), x \in \mathcal{A}) \quad \text{and} \\ \mathcal{F}_\rho &= C^*(S_\mu x S_\nu^* : \mu, \nu \in B_*(\Lambda), |\mu| = |\nu|, x \in \mathcal{A}). \end{aligned}$$

The identity $S_\mu x S_\nu^* = \sum_{\alpha \in \Sigma} S_{\mu\alpha} \rho_\alpha(x) S_{\nu\alpha}^*$ for $x \in \mathcal{A}$, $\mu, \nu \in B_k(\Lambda)$ holds so that the algebra $\mathcal{F}_\rho^k$ is embedded into the algebra $\mathcal{F}_\rho^{k+1}$ such that $\cup_{k \in \mathbf{Z}_+} \mathcal{F}_\rho^k$ is dense in $\mathcal{F}_\rho$. Similarly $\mathcal{D}_\rho^k$ is embedded into the algebra $\mathcal{D}_\rho^{k+1}$ such that $\cup_{k \in \mathbf{Z}_+} \mathcal{D}_\rho^k$ is dense in $\mathcal{D}_\rho$. The gauge action $\hat{\rho}$ of the one-dimensional torus group $\mathbf{T} = \{z \in \mathbf{C} \mid |z| = 1\}$ on $\mathcal{O}_\rho$ is defined by $\hat{\rho}_z(x) = x$ for $x \in \mathcal{A}$ and $\hat{\rho}_z(S_\alpha) = z S_\alpha$ for $\alpha \in \Sigma$. The fixed point algebra of $\mathcal{O}_\rho$ under $\hat{\rho}$ is denoted by $(\mathcal{O}_\rho)^{\hat{\rho}}$. Let $E_\rho : \mathcal{O}_\rho \longrightarrow (\mathcal{O}_\rho)^{\hat{\rho}}$ be the conditional expectaton defined by

$$E_\rho(X) = \int_{z \in \mathbf{T}} \hat{\rho}_z(X) dz, \qquad X \in \mathcal{O}_\rho. \tag{2.3}$$

It is routine to check that $(\mathcal{O}_\rho)^{\hat{\rho}} = \mathcal{F}_\rho$.

Definition ([33]). A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ satisfies *condition* (I) if there exists a unital increasing sequence

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$$

of C^*-subalgebras of $\mathcal{A}$ such that $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$ for all $l \in \mathbf{Z}_+, \alpha \in \Sigma$, the union $\cup_{l \in \mathbf{Z}_+} \mathcal{A}_l$ is dense in $\mathcal{A}$ and for $\epsilon > 0$, $k, l \in \mathbf{N}$ with $k \le l$ and $X_0 \in \mathcal{F}_{\rho,l}^k = C^*(S_\mu x S_\nu^* : \mu, \nu \in B_k(\Lambda), x \in \mathcal{A}_l)$, there exists an element $g \in \mathcal{D}_\rho \cap \mathcal{A}_l{}' (= \{y \in \mathcal{D}_\rho \mid ya = ay \text{ for } a \in \mathcal{A}_l\})$ with $0 \le g \le 1$ such that

(i) $\|X_0 \phi_\rho^k(g)\| \ge \|X_0\| - \epsilon$,

(ii) $g \phi_\rho^m(g) = 0$ for all $m = 1, 2, \dots, k$, where $\phi_\rho^m(X) = \sum_{\mu \in B_m(\Lambda)} S_\mu X S_\mu^*$.

As the element g belongs to the diagonal subalgebra $\mathcal{D}_\rho$ of $\mathcal{F}_\rho$, the condition (I) is intrinsically determined by $(\mathcal{A}, \rho, \Sigma)$ by virtue of [31, Lemma 4.1]. The condition (I) for $(\mathcal{A}, \rho, \Sigma)$ yields the uniqueness of the C^*-algebra $\mathcal{O}_\rho$ under the relations (ρ) ([33]).

If a λ-graph system $\mathfrak{L}$ over Σ satisfies condition (I), then $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ satisfies condition (I) (cf. [30, Lemma 4.1]).

Recall that the positive operator $\lambda_\rho : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by $\lambda_\rho(x) = \sum_{\alpha \in \Sigma} \rho_\alpha(x)$, $x \in \Sigma$. Then a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *irreducible*, if there exists no nontrivial ideal of $\mathcal{A}$ invariant under λ_ρ. It has been shown in [31] that if $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I) and is irreducible, then the C^*-algebra $\mathcal{O}_\rho$ is simple.

Interesting examples of $(\mathcal{A}, \rho, \Sigma)$ in [31], [34] which we have seen from the view point of symbolic dynamics come from ones for which $\mathcal{A}$ is commutative. Hence we assume that the algebra $\mathcal{A}$ is commutative so that $\mathcal{A}$ is written as

$C(\Omega)$ for some compact Hausdorff space Ω henceforth. For the cases that $\mathcal{A}$ is noncommutative, our discussions in this paper well work by considering tracial states on $\mathcal{A}$ in stead of states on $\mathcal{A}$ under slight modifications.

3. Ergodicity and Perron-Frobenius type theorem

In this section, we will study ergodic properties of a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ and prove a Perron-Frobenius type theorem.
Let $\mathcal{A}^*$ denote the Banach space of all complex valued continuous linear functionals on $\mathcal{A}$. For $\beta \in \mathbf{C}$ with $\beta \neq 0$, set

$$\mathcal{E}_\beta(\rho) = \{\varphi \in \mathcal{A}^* \mid \varphi \circ \lambda_\rho(a) = \beta\varphi(a) \text{ for all } a \in \mathcal{A}\}.$$

It is possible that $\mathcal{E}_\beta(\rho)$ is $\{0\}$. A nonzero continuous linear functional φ in $\mathcal{E}_\beta(\rho)$ is called an eigenvector of the operator λ_ρ^* with respect to the eigenvalue β. Let r_ρ be the spectral radius of the positive operator $\lambda_\rho : \mathcal{A} \longrightarrow \mathcal{A}$. Since $\lambda_\rho^k(1) \geq 1, k \in \mathbf{N}$, one sees that $r_\rho \geq 1$. As $\mathrm{Sp}(\lambda_\rho) = \mathrm{Sp}(\lambda_\rho^*)$ (cf. [8, VI. 2.7]), we note $r_\rho = r(\lambda_\rho^*)$. Let $\mathcal{S}(\mathcal{A})$ denote the state space of $\mathcal{A}$.

Lemma 3.1. *$(\mathcal{A}, \rho, \Sigma)$ is irreducible if and only if for a state φ on $\mathcal{A}$ and a nonzero element $x \in \mathcal{A}$, there exists a natural number n such that $\varphi(\lambda_\rho^n(x^*x)) > 0$.*

Proof. Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. For a state φ on $\mathcal{A}$, put

$$I_\varphi = \{x \in \mathcal{A} | \varphi(\lambda_\rho^n(x^*x)) = 0 \text{ for all } n \in \mathbf{N}\}$$

which is an ideal of $\mathcal{A}$ because $\mathcal{A}$ is commutative. The Schwarz type inequality

$$\lambda_\rho^n(\lambda_\rho(x)^*\lambda_\rho(x)) \leq \|\lambda_\rho\|\lambda_\rho^{n+1}(x^*x) \qquad \text{for} \quad x \in \mathcal{A}$$

implies that I_φ is λ_ρ-invariant. Hence I_φ is trivial.
Conversely, let I be an ideal of $\mathcal{A}$ invariant under λ_ρ. Put $\mathcal{B} = \mathcal{A}/I$. Denote by $q : \mathcal{A} \longrightarrow \mathcal{B}$ the quotient map. Take $\psi \in \mathcal{S}(\mathcal{B})$ a state. Put $\varphi = \psi \circ q$. For $y \in I$, as $\varphi(\lambda_\rho^n(y^*y)) = 0, n \in \mathbf{N}$, one sees that $y = 0$ and hence $I = \{0\}$ by the hypothesis. Hence $(\mathcal{A}, \rho, \Sigma)$ is irreducible. $\square$

We denote by $T_\rho : \mathcal{A} \longrightarrow \mathcal{A}$ the positive operator $\frac{1}{r_\rho}\lambda_\rho$. The spectral radius of T_ρ is 1. A state τ on $\mathcal{A}$ is called an *invariant state* if $\tau \circ T_\rho = \tau$ on $\mathcal{A}$, equivalently $\tau \in \mathcal{E}_{r_\rho}(\rho)$.

Corollary 3.2. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. Then any positive eigenvector of λ_ρ^* for a nonzero eigenvalue is faithful.*

Proof. Let $\varphi \in \mathcal{E}_\beta(\rho)$ be a positive linear functional for some nonzero $\beta \in \mathbf{C}$. Since $\varphi(\lambda_\rho(1)) = \beta\varphi(1)$, one has $\beta > 0$. By the preceding lemma, one has $\varphi(x^*x) > 0$ for nonzero $x \in \mathcal{A}$. $\square$

Yasuo Watatani has kindly informed to the author that the lemma below, which is seen from [41, Theorem 2.5], is needed in the proof of Lemma 3.4. In our restrictive situation, we may directly prove it as in the following way.

Lemma 3.3. *The spectral radius r_ρ of the operator λ_ρ is contained in the spectrum $Sp(\lambda_\rho)$ of λ_ρ.*

Proof. The resolvent $R(z) = (z - \lambda_\rho)^{-1}$ for λ_ρ has the expansion $R(z) = \sum_{n=0}^{\infty} \frac{{\lambda_\rho}^n}{z^{n+1}}$ for $z \in \mathbf{C}, |z| > r_\rho$ which converges in norm. We note that the family $\{R(z)\}_{|z|>r_\rho}$ is not uniformly bounded. Otherwise, there exists a constant $M > 0$ such that $\|R(z)\| < M$ for $z \in \mathbf{C}, |z| > r_\rho$. By the compactness of $\mathrm{Sp}(\lambda_\rho)$, we may find $z_\circ \in \mathrm{Sp}(\lambda_\rho)$ with $|z_\circ| = r_\rho$. Take $z_n \notin \mathrm{Sp}(\lambda_\rho)$ satisfying $\lim_{n\to\infty} z_n = z_\circ$ and $|z_n| > r_\rho$. The resolvent equation $R(z_n) - R(z_m) = (z_n - z_m)R(z_n)R(z_m)$ implies the inequality $\|R(z_n) - R(z_m)\| \le |z_n - z_m|M^2$ so that there exists a bounded linear operator $R_\circ = \lim_{n\to\infty} R(z_n)$ on $\mathcal{A}$. The equality $(z_n - \lambda_\rho)R(z_n)x = x, x \in \mathcal{A}$ implies $(z_\circ - \lambda_\rho)R_\circ x = x, x \in \mathcal{A}$ and hence $z_\circ \notin \mathrm{Sp}(\lambda_\rho)$ a contradiction. Thus there exists $r_n \in \mathbf{C}$ such that $|r_n| \notin \mathrm{Sp}(\lambda_\rho)$ and $|r_n| \downarrow r_\rho$ and $\lim_{n\to\infty} \|R(r_n)f\| = \infty$ for some $f \in \mathcal{A}$. We may assume that $f \ge 0$. For a state φ on $\mathcal{A}$, one has

$$|\varphi(R(r_n)f)| \le \sum_{k=0}^{\infty} \frac{\varphi({\lambda_\rho}^k(f))}{|r_n|^{k+1}} = \varphi(R(|r_n|)f).$$

Denote by $w(y)$ the numerical radius of an element $y \in \mathcal{A}$, which is defined by

$$w(y) = \sup\{\varphi(y) \mid \varphi \in \mathcal{S}(\mathcal{A})\}.$$

As the inequalities $\frac{1}{2}\|y\| \le w(y) \le \|y\|$ always hold (cf. [13, p.95]), one sees

$$\frac{1}{2}\|R(r_n)f\| \le w(R(r_n)f) \le w(R(|r_n|)f) \le \|R(|r_n|)f\|$$

so that

$$\lim_{n\to\infty} \|R(|r_n|)f\| = \infty.$$

If $r_\rho \notin \mathrm{Sp}(\lambda_\rho)$, the condition $|r_n| \notin \mathrm{Sp}(\lambda_\rho)$ means that $R(|r_n|) \uparrow R(r_\rho)$ because $R(z)$ increases for $z \downarrow r_\rho$. Hence $R(|r_n|)f \uparrow R(r_\rho)f$ and $\lim_{n\to\infty} \|R(|r_n|)f\| = \|R(r_\rho)f\| < \infty$, a contradiction. Therefore we conclude $r_\rho \in \mathrm{Sp}(\lambda_\rho)$. □

The following lemma is crucial.

Lemma 3.4. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. Then there exists a faithful invariant state on $\mathcal{A}$.*

Proof. We denote by $R^*(t)$ the resolvent of $\lambda_\rho^* : \mathcal{A}^* \to \mathcal{A}^*$ defined by $R^*(t)\varphi = (t - \lambda_\rho^*)^{-1}\varphi$ for $\varphi \in \mathcal{A}^*, t > r(\lambda_\rho^*)$. As $r_\rho = r(\lambda_\rho^*)$, there exists $\varphi_0 \in \mathcal{A}^*$ such that $\|R^*(t)\varphi_0\|$ is unbounded for $t \downarrow r_\rho$ by Lemma 3.3. We may assume that φ_0 is a state on $\mathcal{A}$. Put

$$\varphi_n = \frac{R^*(r_\rho + \frac{1}{n})\varphi_0}{\|R^*(r_\rho + \frac{1}{n})\varphi_0\|} \qquad \text{for} \quad n = 1, 2, ...$$

Since $R^*(t)$ is positive for $t > r_\rho$, each φ_n is a state on $\mathcal{A}$ so that there exists a weak* cluster point $\varphi_\infty \in \mathcal{S}(\mathcal{A})$ of the sequence $\{\varphi_n\}$ in $\mathcal{S}(\mathcal{A})$. As we see

$$(r_\rho - \lambda_\rho^*)\varphi_n = -\frac{1}{n}\varphi_n + \frac{\varphi_0}{\|R^*(r_\rho + \frac{1}{n})\varphi_0\|},$$

we get $r_\rho \varphi_\infty = \lambda_\rho^* \varphi_\infty$ so that $\varphi_\infty \in \mathcal{E}_{r_\rho}(\rho)$. By Corollary 3.2, one knows that φ_∞ is faithful on $\mathcal{A}$. □

DEFINITION. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *uniquely ergodic* if there exists a unique invariant state on $\mathcal{A}$. Denote by τ the unique invariant state.

If $(\mathcal{A}, \rho, \Sigma)$ is irreducible and uniquely ergodic, the unique invariant state τ is automatically faithful because any invariant state is faithful.

There is an example of a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ for which a unique invariant state is not faithful, unless $(\mathcal{A}, \rho, \Sigma)$ is irreducible. Let $\mathcal{A} = \mathbf{C} \oplus \mathbf{C}$, $\Sigma = \{1, 2\}$ and $\rho_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \rho_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\lambda_\rho = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $r_\rho = 2$ and $T_\rho = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a unique invariant state on $\mathcal{A}$, that is not faithful.

We will see, in Section 7, that the C^*-symbolic dynamical system $(\mathcal{A}_\mathcal{G}, \rho^\mathcal{G}, \Sigma)$ for a finite labeled graph $\mathcal{G}$ is uniquely ergodic if and only if the underlying nonnegative matrix $A_\mathcal{G}$ is irreducible.

We will next consider the eigenvector space of the operator λ_ρ on $\mathcal{A}$. We are assuming that the algebra $\mathcal{A}$ is commutative so that $\mathcal{A}$ is written as $C(\Omega)$ for some compact Hausdorff space Ω.

LEMMA 3.5. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible.*

(i) *If T_ρ has a nonzero fixed element in $\mathcal{A}$, then T_ρ has a nonzero positive fixed element in $\mathcal{A}$,*

(ii) *A nonzero positive fixed element by T_ρ in $\mathcal{A}$ must be strictly positive.*

(iii) *If there exist two nonzero positive fixed elements by T_ρ in $\mathcal{A}$, then one is a scalar multiple of the other.*

(iv) *The dimension of the space consisting of the fixed elements by T_ρ is at most one.*

Proof. (i) Let $y \in \mathcal{A}$ be a nonzero fixed element by T_ρ. Since y^* is also fixed by T_ρ, we may assume that $y = y^*$. Denote by $y = y^+ - y^-$ with $y^+, y^- \geq 0$ the Jordan decomposition of y. We have $y^+ \geq y$ and hence $T_\rho(y^+) \geq T_\rho(y) = y$. As $T_\rho(y^+) \geq 0$, one sees that $T_\rho(y^+) \geq y^+$. Now $(\mathcal{A}, \rho, \Sigma)$ is irreducible so that there exists a faithful invariant state τ on $\mathcal{A}$. Since $\tau(T_\rho(y^+) - y^+) = 0$, one has $T_\rho(y^+) = y^+$. Similarly we have $T_\rho(y^-) = y^-$. As $y \neq 0$, either y^+ or y^- is not zero.

(ii) Let $y \in \mathcal{A}$ be a nonzero fixed positive element by T_ρ. Suppose that there exists $\omega_0 \in \Omega$ such that $y(\omega_0) = 0$. Let I_y be the closed ideal of $\mathcal{A}$ generated by y. For a nonzero positive element $f \in \mathcal{A}$ we have

$$T_\rho(fy) \leq \|f\| T_\rho(y) = \|f\| y$$

so that $T_\rho(fy)$ belongs to I_y. As the ideal I_y is approximated by linear combinations of the elements of the form $fy, f \in \mathcal{A}, f \geq 0$, the ideal I_y is invariant under T_ρ. Now $(\mathcal{A}, \rho, \Sigma)$ is irreducible so that $I_y = \mathcal{A}$. As any element of I_y vanishes at ω_0, a contradiction.

(iii) Let $x, y \in \mathcal{A}$ be nonzero positive fixed elements by T_ρ. By the above discussions, they are strictly positive. Set $c_0 = \min\{\frac{x(\omega)}{y(\omega)} \mid \omega \in \Omega\}$. The function $x - c_0 y$ is positive element but not strictly positive, so that it must be zero.
(iv) Let $y \in \mathcal{A}$ be a fixed element under T_ρ, which is written as the Jordan decomposition $y = y_1 - y_2 + i(y_3 - y_4)$ for some positive elements $y_i, i = 1, 2, 3, 4$ in $\mathcal{A}$. By the above discussions, all the elements $y_i, i = 1, 2, 3, 4$ are fixed under T_ρ and they are strictly positive if it is nonzero. Hence (iii) implies the desired assertion. □

Definition. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to satisfy *(FP)* if there exists a nonzero fixed element in $\mathcal{A}$ under T_ρ.
If in particular, $(\mathcal{A}, \rho, \Sigma)$ is irreducible, a nonzero fixed element can be taken as a strictly positive element in $\mathcal{A}$ by the previous lemma.

Lemma 3.6. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and satisfies (FP).*
(i) *If there exists a state in $\mathcal{E}_\beta(\rho)$ for some $\beta \in \mathbf{C}$ with $\beta \neq 0$, then we have $\beta = r_\rho$.*
(ii) *If in particular, $(\mathcal{A}, \rho, \Sigma)$ is uniquely ergodic, the eigenspace $\mathcal{E}_{r_\rho}(\rho)$ is of one-dimensional.*

Proof. (i) Suppose that there exists a state ψ in $\mathcal{E}_\beta(\rho)$ for some $\beta \in \mathbf{C}$ with $\beta \neq 0$. Let $x_0 \in \mathcal{A}$ be a nonzero fixed element by T_ρ. One may take it to be strictly positive by the preceding lemma. Since $\lambda_\rho(x_0) = r_\rho x_0$, one has

$$\beta\psi(x_0) = \psi(\lambda_\rho(x_0)) = r_\rho\psi(x_0).$$

By Corollary 3.2, one has $\psi(x_0) > 0$ so that $\beta = r_\rho$.
(ii) Take an arbitrary $\varphi \in \mathcal{E}_{r_\rho}(\rho)$. Put $\varphi^*(x) = \overline{\varphi(x^*)}, x \in \mathcal{A}$ and hence $\varphi^* \in \mathcal{E}_{r_\rho}(\rho)$. Both of the continuous linear functionals $\varphi_{Re} = \frac{1}{2}(\varphi + \varphi^*)$ and $\varphi_{Im} = \frac{1}{2i}(\varphi - \varphi^*)$ belong to $\mathcal{E}_{r_\rho}(\rho)$ which come from real valued measures on Ω. Put $\psi = \varphi_{Re}$. Let $\psi = \psi_+ - \psi_-$ be the Jordan decomposition of ψ, where ψ_+, ψ_- are positive linear functionals on $\mathcal{A}$. Since $\psi_+ \geq \psi$, one has $T_\rho^*\psi_+ \geq T_\rho^*\psi = \psi$. As $T_\rho^*\psi_+$ is positive, one has $T_\rho^*\psi_+ \geq \psi_+$. Now $(\mathcal{A}, \rho, \Sigma)$ is irreducible and satisfies (FP) so that one finds a strictly positive element $x_0 \in \mathcal{A}$ fixed by T_ρ. Then $\tilde{\psi} = T_\rho^*\psi_+ - \psi_+$ is a positive linear functional satisfying $\tilde{\psi}(x_0) = 0$. It follows that $\tilde{\psi} = 0$ so that $T_\rho^*\psi_+ = \psi_+$. Similarly we have $T_\rho^*\psi_- = \psi_-$. As both ψ_+, ψ_- are positive linear functionals on $\mathcal{A}$, the unique ergodicity asserts that there exist $0 \leq c_+, c_- \in \mathbf{R}$ such that $\psi_+ = c_+\tau, \psi_- = c_-\tau$. By putting $c_{Re} = c_+ - c_-$, one has $\varphi_{Re} = c_{Re}\tau$ and similarly $\varphi_{Im} = c_{Im}\tau$ for some real number c_{Im}. Therefore we have

$$\varphi = (c_{Re} + ic_{Im})\tau.$$

Hence any continuous linear functional fixed by T_ρ is a scalar multiple of τ, so that

$$\dim \mathcal{E}_{r_\rho}(\rho) = 1.$$

□

A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *power-bounded* if the sequence $\{\|T_\rho^k\| \mid k \in \mathbf{N}\}$ is bounded. As $T_\rho^k : \mathcal{A} \longrightarrow \mathcal{A}$ is completely positive, the equalities $\|T_\rho^k\| = \|T_\rho^k(1)\| = \|\frac{1}{{r_\rho}^k} \sum_{\mu \in B_k(\Lambda)} \rho_\mu(1)\|$ hold. We remark that for an irreducible matrix $A = [A(i,j)]_{i,j=1}^N$ with entries in $\{0,1\}$, the associated C^*-symbolic dynamical system $(\mathcal{A}_A, \rho^A, \Sigma)$ defined in the Cuntz-Krieger algebra $\mathcal{O}_A$ is power-bounded. One indeed sees that there is a constant $d > 0$ such that

$$\sum_{i,j=1}^N A^k(i,j) \le d \cdot r_A^k \qquad \text{(cf. [28, Proposition 4.2.1]).}$$

Hence

$$\|\lambda_A^k(1)\| = \max_i \sum_{j=1}^N A^k(i,j) \le d \cdot r_A^k.$$

Lemma 3.7. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. If $(\mathcal{A}, \rho, \Sigma)$ satisfies (FP), then $(\mathcal{A}, \rho, \Sigma)$ is power-bounded.*

Proof. As $(\mathcal{A}, \rho, \Sigma)$ is irreducible and satisfies (FP), there exists a strictly positive fixed element x_0 of $\mathcal{A}$ under T_ρ. Since Ω is compact, one finds positive constants c_1, c_2 such that $0 < c_1 < x_0(\omega) < c_2$ for all $\omega \in \Omega$. It follows that

$$c_1 T_\rho^n(1) = T_\rho^n(c_1 1) \le T_\rho^n(x_0) = x_0 \le c_2, \qquad n \in \mathbf{N}.$$

Thus we have $\|T_\rho^n\| = \|T_\rho^n(1)\| \le \frac{c_2}{c_1}$ for $n \in \mathbf{N}$. □

We define the mean operator $M_n : \mathcal{A} \longrightarrow \mathcal{A}$ for $n \in \mathbf{N}$ by setting

$$M_n(a) = \frac{a + T_\rho(a) + T_\rho^2(a) + \cdots + T_\rho^{n-1}(a)}{n}, \qquad a \in \mathcal{A}. \tag{3.1}$$

Definition. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *mean ergodic* if for $a \in \mathcal{A}$ the limit $\lim_{n\to\infty} M_n(a)$ exists in $\mathcal{A}$ under norm-topology. For a mean ergodic $(\mathcal{A}, \rho, \Sigma)$, the limit $\lim_{n\to\infty} M_n(1)$ exists in $\mathcal{A}$ under norm-topology, which we denote by $x_\rho \in \mathcal{A}$

Lemma 3.8. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. For a mean ergodic $(\mathcal{A}, \rho, \Sigma)$, we have for $a \in \mathcal{A}$,*

$$\lim_{n\to\infty} M_n(a) = \lim_{n\to\infty} M_n(T_\rho(a)) = \lim_{n\to\infty} T_\rho(M_n(a)). \tag{3.2}$$

In particular x_ρ is a nonzero positive element which satisfies $x_\rho = T_\rho(x_\rho)$ and $\tau(x_\rho) = 1$ for an invariant state $\tau \in \mathcal{E}_{r_\rho}(\rho)$.

Proof. For $a \in \mathcal{A}$, the equality $T_\rho(M_n(a)) = M_n(T_\rho(a))$ is clear. As

$$(n+1)M_{n+1}(a) - nM_n(a) = T_\rho^n(a),$$

one has

$$\frac{1}{n} T_\rho^n(a) = M_{n+1}(a) - M_n(a) + \frac{1}{n} M_{n+1}(a)$$

so that $\lim_{n\to\infty}\frac{1}{n}T_\rho^n(a)=0$. By the equality

$$T_\rho(M_n(a))-M_n(a)=\frac{1}{n}(T_\rho^n(a)-a)$$

we have

$$\lim_{n\to\infty}(T_\rho(M_n(a))-M_n(a))=\lim_{n\to\infty}\frac{1}{n}(T_\rho^n(a)-a)=0.$$

Take a faithful invariant state τ on $\mathcal{A}$, we have

$$\tau(x_\rho)=\lim_{n\to\infty}\tau(M_n(1))=\tau(1)=1.$$

□

Proposition 3.9. *Assume that $(\mathcal{A},\rho,\Sigma)$ is irreducible. If $(\mathcal{A},\rho,\Sigma)$ is mean ergodic, there exists a faithful invariant state τ on $\mathcal{A}$ such taht*

$$\lim_{n\to\infty}M_n(a)=\tau(a)x_\rho,\qquad a\in\mathcal{A}. \tag{3.3}$$

Proof. For $a\in\mathcal{A}$, the limit $\Phi(a)=\lim_{n\to\infty}M_n(a)$ is fixed by T_ρ so that it is a scalar multiple of x_ρ by Lemma 3.5 (iv). One may put

$$\Phi(a)=\tau(a)x_\rho\quad\text{ for some }\tau(a)\in\mathbf{C}.$$

It is easy to see that $\tau:\mathcal{A}\longrightarrow\mathbf{C}$ is a state. As $\Phi(T_\rho(a))=\Phi(a)$, one sees $\tau(T_\rho(a))=\tau(a)$ for $a\in\mathcal{A}$. Hence τ is an invariant state on $\mathcal{A}$. Now $(\mathcal{A},\rho,\Sigma)$ is irreducible, the invariant state is faithful. □

Hence the following corollary is clear.

Corollary 3.10. *Assume that $(\mathcal{A},\rho,\Sigma)$ is irreducible. Then the following two assertions are equivalent:*

(i) *$(\mathcal{A},\rho,\Sigma)$ is mean ergodic.*
(ii) *There exist an invariant state τ on $\mathcal{A}$ and a positive element $x_0\in\mathcal{A}$ with $\tau(x_0)=1$ such that $\lim_{n\to\infty}M_n(a)=\tau(a)x_0$ for $a\in\mathcal{A}$.*

In this case x_0 is given by $\lim_{n\to\infty}M_n(1)(=x_\rho)$, and τ is faithful.

Theorem 3.11. *Assume that $(\mathcal{A},\rho,\Sigma)$ is irreducible. Then the following two assertions are equivalent:*

(i) *$(\mathcal{A},\rho,\Sigma)$ is mean ergodic.*
(ii) *$(\mathcal{A},\rho,\Sigma)$ is uniquely ergodic and satisfies (FP).*

Proof. (i) $\Rightarrow$ (ii): Suppose that $(\mathcal{A},\rho,\Sigma)$ is mean ergodic. Put $\Phi(a)=\lim_{n\to\infty}M_n(a)$ for $a\in\mathcal{A}$. The element $x_\rho=\Phi(1)$ is a nonzero fixed element of $\mathcal{A}$ under T_ρ. By the previous corollary, there exists an invariant state τ on $\mathcal{A}$ satisfying $\Phi(a)=\tau(a)x_\rho$ for $a\in\mathcal{A}$. For any invariant state ψ on $\mathcal{A}$, we have $\psi\circ M_n(a)=\psi(a)$ for $a\in\mathcal{A}$. Hence $\psi(\Phi(a))=\psi(a)$ so that $\psi(a)=\psi(\tau(a)x_\rho)=\tau(a)\psi(x_\rho)$. Since $\psi(x_\rho)=1$, we obtain $\psi(a)=\tau(a)$. Therefore $\psi=\tau$ so that $(\mathcal{A},\rho,\Sigma)$ is uniquely ergodic.
(ii) $\Rightarrow$ (i): Suppose that $(\mathcal{A},\rho,\Sigma)$ is uniquely ergodic and satisfies (FP). By Lemma 3.7, $(\mathcal{A},\rho,\Sigma)$ is power-bounded. Hence the sequence $\{\frac{1}{n}\sum_{k=0}^{n-1}T_\rho^k\}_{n\in\mathbf{N}}$

is uniformly bounded. This means that $T_\rho : \mathcal{A} \longrightarrow \mathcal{A}$ is Cesàro bounded (cf. [22, p.72]). As $\lim_{n\to\infty} \frac{T_\rho^{n-1}(a)}{n} = 0$ for $a \in \mathcal{A}$, the operator $T_\rho : \mathcal{A} \to \mathcal{A}$ satisfies the assumption of [22, p.74 Theorem 1.4]. To prove mean ergodicity, it suffices to show that $F = \{x \in \mathcal{A} \mid T_\rho x = x\}$ separates $F^* = \{\varphi \in \mathcal{A}^* \mid \varphi \circ T_\rho = \varphi\}$. By Lemma 3.6, one knows that $F^* = \mathbf{C}\tau$, where τ is a unique faithful invariant state on $\mathcal{A}$. Hence if $\varphi = c\tau \in F^*$ is nonzero, then $c \neq 0$ and $\varphi(x_\rho) = c\tau(x_\rho) = c \neq 0$. This implies that F separates F^*. Thus by [22, p.74 Theorem 1.4], $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic. $\square$

Remark. In [22, p.179], it is shown that a mean ergodic irreducible "Markov operator "is uniquely ergodic. In our situation, the operator T_ρ does not necessarily satisfy $T_\rho(1) = 1$. Hence the operator T_ρ is not necessarily a Markov operator.

We summarize results obtained in this section as in the following way:

Theorem 3.12. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. Then the following implications hold:*

$$\begin{array}{c} \textit{(ME)} \Longleftrightarrow \textit{(UE)} + \textit{(FP)} \Longrightarrow \textit{(FP)} \Longrightarrow \textit{(PB)} \\ \Downarrow \\ \dim \mathcal{E}_{r_\rho}(\rho) = 1 \Longrightarrow \textit{(UE)}, \end{array}$$

where (ME) means mean ergodic, (UE) means uniquely ergodic, and (PB) means power-bounded.

If in particular $(\mathcal{A}, \rho, \Sigma)$ is irreducible and mean ergodic, the following Perron-Frobenius type theorem holds.

Theorem 3.13. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and mean ergodic.*

(i) *There exists a unique pair of a faithful state τ on $\mathcal{A}$ and a strictly positive element x_ρ in $\mathcal{A}$ satisfying the conditions:*

$$\tau \circ \lambda_\rho = r_\rho \tau, \qquad \lambda_\rho(x_\rho) = r_\rho x_\rho \quad \text{and} \quad \tau(x_\rho) = 1,$$

where r_ρ is the spectral radius of the positive operator λ_ρ on $\mathcal{A}$.

(ii) *If there exists a continuous linear functional ψ on $\mathcal{A}$ satisfying*

$$\psi \circ \lambda_\rho = r_\rho \psi,$$

then $\psi = c\tau$ for some complex number $c \in \mathbf{C}$.

(iii) *If there exists a state φ on $\mathcal{A}$ and a complex number $\beta \in \mathbf{C}$ with $\beta \neq 0$ satisfying*

$$\varphi \circ \lambda_\rho = \beta\varphi,$$

then $\varphi = \tau$ and $\beta = r_\rho$.

(iv) *For any $a \in \mathcal{A}$, the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \frac{\lambda_\rho^k(a)}{r_\rho^k}$ exists in $\mathcal{A}$ in the norm topology such that*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \frac{\lambda_\rho^k(a)}{r_\rho^k} = \tau(a)x_\rho.$$

Proof. Under the assumption that $(\mathcal{A}, \rho, \Sigma)$ is irreducible, mean ergodicity is equivalent to unique ergodicity with (FP). (i) and (iv) follows from Corollary 3.10 and unique ergodicity. (ii) follows from Lemma 3.6 (ii). (iii) follows from Lemma 3.6 (i) and unique ergodicity. $\square$

4. Extension of eigenvectors to $\mathcal{F}_\rho$

In this section, we will study extendability of an eigenvector in $\mathcal{E}_\beta(\rho)$ to the subalgebra $\mathcal{F}_\rho$. We fix a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ satisfying condition (I) henceforth.

Lemma 4.1. *Fix a nonnegative integer $k \in \mathbf{Z}_+$. For any element $x \in \mathcal{F}_\rho^k$ there uniquely exists $x_{\mu,\nu}$ in $\mathcal{A}$ for each $\mu, \nu \in B_k(\Lambda)$ such that*

$$x = \sum_{\mu,\nu \in B_k(\Lambda)} S_\mu x_{\mu,\nu} S_\nu^* \qquad \text{and} \qquad x_{\mu,\nu} = \rho_\mu(1) x_{\mu,\nu} \rho_\nu(1). \tag{4.1}$$

If in particular x belongs to $\mathcal{D}_\rho^k$, there uniquely exists x_μ in $\mathcal{A}$ for each $\mu \in B_k(\Lambda)$ such that

$$x = \sum_{\mu \in B_k(\Lambda)} S_\mu x_\mu S_\mu^* \qquad \text{and} \qquad x_\mu = \rho_\mu(1) x_\mu \rho_\mu(1). \tag{4.2}$$

Proof. For an element x in $\mathcal{F}_\rho^k$ and $\mu, \nu \in B_k(\Lambda)$, put $x_{\mu,\nu} = S_\mu^* x S_\nu$ that belongs to $\mathcal{A}$ and satisfies the equalities (4.1). $\square$

We set

$$\begin{aligned} {\mathcal{D}_\rho}^{\mathrm{alg}} &= \text{the algebraic linear span of } S_\mu a S_\mu^* \text{ for } \mu \in B_*(\Lambda), a \in \mathcal{A}, \text{ and} \\ {\mathcal{F}_\rho}^{\mathrm{alg}} &= \text{the algebraic linear span of } S_\mu a S_\nu^* \text{ for } \mu,\nu \in B_*(\Lambda), |\mu| = |\nu|, a \in \mathcal{A}. \end{aligned}$$

Hence ${\mathcal{D}_\rho}^{\mathrm{alg}} = \cup_{k=0}^\infty \mathcal{D}_\rho^k$ and ${\mathcal{F}_\rho}^{\mathrm{alg}} = \cup_{k=0}^\infty \mathcal{F}_\rho^k$. They are dense $*$-subalgebras of $\mathcal{D}_\rho$ and $\mathcal{F}_\rho$ respectively.

Lemma 4.2. *For $\beta \in \mathbf{C}$ with $|\beta| > 1$ and $\varphi \in \mathcal{E}_\beta(\rho)$ on $\mathcal{A}$, put*

$$\tilde{\varphi}(S_\mu a S_\mu^*) = \frac{1}{\beta^{|\mu|}} \varphi(a \rho_\mu(1)), \qquad a \in \mathcal{A}, \mu \in B_*(\Lambda). \tag{4.3}$$

Then $\tilde{\varphi}$ is a well-defined (not necessarily continuous) linear functional on ${\mathcal{D}_\rho}^{alg}$, that is an extension of φ.

Proof. By the expansion (4.2) for an element $x \in \mathcal{D}_\rho^k$, the following definition of $\varphi_k(x)$ yields a linear functional φ_k on $\mathcal{D}_\rho^k$

$$\varphi_k(x) = \sum_{\mu \in B_k(\Lambda)} \frac{1}{\beta^k} \varphi(x_\mu). \tag{4.4}$$

We will show that $\varphi_k = \varphi_{k+1}$ on $\mathcal{D}_\rho^k$. As $S_\mu x_\mu S_\mu^* = \sum_{\alpha \in \Sigma} S_{\mu\alpha} \rho_\alpha(x_\mu) S_{\mu\alpha}^*$ and $\rho_{\mu\alpha}(1)\rho_\alpha(x_\mu)\rho_{\mu\alpha}(1) = S_\alpha^* \rho_\mu(1) x_\mu \rho_\mu(1) S_\alpha = \rho_\alpha(x_\mu)$, the following expression of x in $\mathcal{D}_\rho^{k+1}$

$$x = \sum_{\mu \in B_k(\Lambda), \alpha \in \Sigma} S_{\mu\alpha} \rho_\alpha(x_\mu) S_{\mu\alpha}^*$$

is the unique expression of (4.2). Hence we obtain

$$\varphi_{k+1}(x) = \sum_{\mu \in B_k(\Lambda), \alpha \in \Sigma} \frac{1}{\beta^{k+1}} \varphi(\rho_\alpha(x_\mu)) = \frac{1}{\beta^k} \sum_{\mu \in B_k(\Lambda)} \varphi(x_\mu) = \varphi_k(x).$$

The family $\{\varphi_k\}_{k \in \mathbf{Z}_+}$ of linear functionals on the subalgebras $\{\mathcal{D}_\rho^k\}_{k \in \mathbf{Z}_+}$ yields a linear functional on the algebra ${\mathcal{D}_\rho}^{\text{alg}}$. We denote it by $\tilde{\varphi}$. As the expansion $a = \sum_{\alpha \in \Sigma} S_\alpha \rho_\alpha(a) S_\alpha^*$ for $a \in \mathcal{A}$ is the unique expansion of a in (4.2) as an element of $\mathcal{D}_\rho^1$, we have $\tilde{\varphi}(a) = \frac{1}{\beta} \sum_{\alpha \in \Sigma} \varphi(\rho_\alpha(a)) = \varphi(a)$ so that $\tilde{\varphi} = \varphi$ on $\mathcal{A}$. ☐

We will extend λ_ρ on $\mathcal{A}$ to $\mathcal{F}_\rho$ such as

$$\lambda_\rho(x) = \sum_{\alpha \in \Sigma} S_\alpha^* x S_\alpha \qquad \text{for } x \in \mathcal{F}_\rho.$$

Lemma 4.3. *Let ψ be a linear functional on ${\mathcal{F}_\rho}^{alg}$ such that its restriction to $\mathcal{A}$ is continuous. Then the following three conditions are equivalent:*

(i) *ψ is tracial and $\psi \circ \lambda_\rho(x) = \beta\psi(x)$ for $x \in {\mathcal{F}_\rho}^{alg}$.*

(ii) *$\psi(S_\mu x S_\nu^*) = \delta_{\mu,\nu} \frac{1}{\beta^{|\mu|}} \psi(x S_\mu^* S_\mu)$ for $x \in {\mathcal{F}_\rho}^{alg}, \mu, \nu \in B_*(\Lambda)$ with $|\mu| = |\nu|$.*

(iii) *There exists $\varphi \in \mathcal{E}_\beta(\rho)$ such that*

$$\psi(S_\mu a S_\nu^*) = \delta_{\mu,\nu} \frac{1}{\beta^{|\mu|}} \varphi(a \rho_\mu(1)) \textit{ for } a \in \mathcal{A}, \mu, \nu \in B_*(\Lambda) \textit{with } |\mu| = |\nu|.$$

Proof. (i) ⇒ (ii): The equation (i) implies that for $k \in \mathbf{N}$,

$$\psi(x) = \frac{1}{\beta^k} \sum_{\gamma \in B_k(\Lambda)} \psi(S_\gamma^* x S_\gamma), \quad x \in {\mathcal{F}_\rho}^{\text{alg}}.$$

It then follows that for $\mu, \nu \in B_k(\Lambda)$

$$\psi(S_\mu x S_\nu^*) = \frac{1}{\beta^k} \sum_{\gamma \in B_k(\Lambda)} \psi(S_\gamma^* S_\mu x S_\nu^* S_\gamma) = \delta_{\mu,\nu} \frac{1}{\beta^k} \psi(x S_\mu^* S_\mu).$$

(ii) $\Rightarrow$ (iii): Define a linear functional φ on $\mathcal{A}$ by the restriction of ψ to the subalgebra $\mathcal{A}$. By the equation (ii) for $a \in \mathcal{A}$ and hence $S_\alpha^* a S_\alpha \in \mathcal{A}$, we see

$$\psi(S_\alpha S_\alpha^* a) = \psi(S_\alpha S_\alpha^* a S_\alpha S_\alpha^*) = \frac{1}{\beta}\psi(S_\alpha^* a S_\alpha S_\alpha^* S_\alpha) = \frac{1}{\beta}\psi(S_\alpha^* a S_\alpha)$$

so that $\varphi \in \mathcal{E}_\beta(\rho)$. The equation (iii) is clear.

(iii) $\Rightarrow$ (i): We will see that ψ is tracial. Let $x, y \in \mathcal{F}_\rho^k$ be expanded as in (4.1) so that $x = \sum_{\mu,\nu \in B_k(\Lambda)} S_\mu x_{\mu,\nu} S_\nu^*,\ y = \sum_{\mu,\nu \in B_k(\Lambda)} S_\mu y_{\mu,\nu} S_\nu^*$. We have

$$xy = \sum_{\mu,\nu,\gamma \in B_k(\Lambda)} S_\mu x_{\mu,\nu} \rho_\nu(1) y_{\nu,\gamma} S_\gamma^* = \sum_{\mu,\gamma \in B_k(\Lambda)} S_\mu (\sum_{\nu \in B_k(\Lambda)} x_{\mu,\nu} y_{\nu,\gamma}) S_\gamma^*$$

and $\sum_{\nu \in B_k(\Lambda)} x_{\mu,\nu} y_{\nu,\gamma} = \rho_\mu(1)(\sum_{\nu \in B_k(\Lambda)} x_{\mu,\nu} y_{\nu,\gamma})\rho_\gamma(1)$, similarly

$$yx = \sum_{\eta,\nu \in B_k(\Lambda)} S_\eta (\sum_{\gamma \in B_k(\Lambda)} y_{\eta,\gamma} x_{\gamma,\nu}) S_\nu^*$$

and $\sum_{\gamma \in B_k(\Lambda)} y_{\eta,\gamma} x_{\gamma,\nu} = \rho_\eta(1)(\sum_{\nu \in B_k(\Lambda)} y_{\eta,\gamma} x_{\gamma,\nu})\rho_\nu(1)$. It follows that

$$\psi(xy) = \sum_{\mu,\nu \in B_k(\Lambda)} \frac{1}{\beta^k}\varphi(x_{\mu,\nu} y_{\nu,\mu}) = \sum_{\gamma,\eta \in B_k(\Lambda)} \frac{1}{\beta^k}\varphi(y_{\eta,\gamma} x_{\gamma,\eta}) = \psi(yx).$$

Hence ψ is tracial on $\mathcal{F}_\rho^k$.

We will finally show that the equality in (i) holds. For $S_\mu a S_\nu^* \in \mathcal{F}_\rho^k$ with $a \in \mathcal{A}, \mu = (\mu_1, \dots, \mu_k), \nu = (\nu_1, \dots, \nu_k) \in B_k(\Lambda)$, put $\mu_{[2,k]} = (\mu_2, \dots, \mu_k), \nu_{[2,k]} = (\nu_2, \dots, \nu_k) \in B_{k-1}(\Lambda)$. One has

$$\begin{aligned}
&\sum_{\alpha \in \Sigma} \psi(S_\alpha^*(S_\mu a S_\nu^*) S_\alpha) \\
=&\delta_{\mu_1,\nu_1}\psi(\rho_{\mu_1}(1) S_{\mu_{[2,k]}} a S_{\nu_{[2,k]}}^* \rho_{\nu_1}(1)) \\
=&\delta_{\mu_1,\nu_1}\psi(S_{\mu_{[2,k]}} S_{\mu_{[2,k]}}^* \rho_{\mu_1}(1) S_{\mu_{[2,k]}} a S_{\nu_{[2,k]}}^* \rho_{\nu_1}(1) S_{\nu_{[2,k]}} S_{\nu_{[2,k]}}^*) \\
=&\delta_{\mu_1,\nu_1}\psi(S_{\mu_{[2,k]}} \rho_\mu(1) a \rho_\nu(1) S_{\nu_{[2,k]}}^*) \\
=&\delta_{\mu_1,\nu_1}\delta_{\mu_{[2,k]},\nu_{[2,k]}} \frac{1}{\beta^{k-1}}\varphi(\rho_\mu(1) a \rho_\nu(1) \rho_{\nu_{[2,k]}}(1)) \\
=&\delta_{\mu,\nu}\frac{1}{\beta^{k-1}}\varphi(\rho_\mu(1) a \rho_\nu(1)) \\
=&\beta\psi(S_\mu a S_\nu^*).
\end{aligned}$$

□

Let $E_\mathcal{D} : \mathcal{F}_\rho \longrightarrow \mathcal{D}_\rho$ denote the expectation satisfying

$$E_\mathcal{D}(S_\mu a S_\nu^*) = \delta_{\mu,\nu} S_\mu a S_\mu^*, \qquad a \in \mathcal{A}, \quad \mu, \nu \in B_*(\Lambda), |\mu| = |\nu|.$$

Once we have an extension $\tilde{\varphi}$ to $\mathcal{D}_\rho$ of $\varphi \in \mathcal{E}_\beta(\rho)$, $\tilde{\varphi}$ has a further extension to $\mathcal{F}_\rho$ by $\tilde{\varphi} \circ E_\mathcal{D}$. The extension $\tilde{\varphi} \circ E_\mathcal{D}$ on $\mathcal{F}_\rho$ is continuous if $\tilde{\varphi}$ is so on $\mathcal{D}_\rho$. It satisfies

$$\tilde{\varphi} \circ E_\mathcal{D}(S_\mu a S_\nu^*) = \delta_{\mu,\nu}\frac{1}{\beta^{|\mu|}}\varphi(a\rho_\mu(1)) \tag{4.5}$$

for $a \in \mathcal{A}, \mu, \nu \in B_*(\Lambda)$ with $|\mu| = |\nu|$. Hence the extension of a continuous linear functional on $\mathcal{D}_\rho$ to $\mathcal{F}_\rho$ is automatic. We have only to study extension of a linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ on $\mathcal{A}$ to $\mathcal{D}_\rho$. The condition (iii) of Lemma 4.3 is equivalent to $\psi = \tilde{\varphi} \circ E_{\mathcal{D}}$ where $\tilde{\varphi}$ is a linear functional on ${\mathcal{D}_\rho}^{\mathrm{alg}}$ obtained from $\varphi \in \mathcal{E}_\beta(\rho)$ as in Lemma 4.2, and so thst ψ is continuous if and only if $\tilde{\varphi}$ is continuous. We call the extension $\tilde{\varphi}$ on ${\mathcal{D}_\rho}^{\mathrm{alg}}$ of $\varphi \in \mathcal{E}_\beta(\rho)$ the *canonical extension* of φ.

Lemma 4.4. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and power-bounded. For $\beta \in \mathbf{C}$ with $|\beta| = r_\rho > 1$, a (not necessarily positive) continuous linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ on $\mathcal{A}$ extends to a continuous linear functional $\tilde{\varphi}$ on $\mathcal{D}_\rho$ satisfying* (4.3).

Proof. As $(\mathcal{A}, \rho, \Sigma)$ is irreducible, we may take a faithful invariant state τ on $\mathcal{A}$, which we will fix. By the hypothesis that $(\mathcal{A}, \rho, \Sigma)$ is power-bounded, there exists a positive number M such that $\frac{\|\lambda_\rho^k(1)\|}{r_\rho^k} < M$ for all $k \in \mathbf{N}$. By [43, Theorem 4.2], there exists a partial isometry $v \in \mathcal{A}^{**}$ and a positive linear functional $\psi \in \mathcal{A}^*$ such that

$$\varphi(a) = \psi(av), \qquad a \in \mathcal{A}.$$

For $x = \sum_{\mu \in B_k(\Lambda)} S_\mu x_\mu S_\mu^* \in \mathcal{D}_\rho^k$ as in (4.2). Define a linear functional φ_k on $\mathcal{D}_\rho^k$ by (4.4). As in Lemma 4.2, $\varphi_{k+1}|_{\mathcal{D}_\rho^k} = \varphi_k$ and hence $\{\varphi_k\}_{k \in \mathbf{N}}$ defines a linear functional on ${\mathcal{D}_\rho}^{\mathrm{alg}}$. It then follows that

$$|\varphi(x_\mu)| = |\psi(\rho_\mu(1) x_\mu \rho_\mu(1) v)| \le \psi(\rho_\mu(1))^{\frac{1}{2}} \|x_\mu^* x_\mu\|^{\frac{1}{2}} \psi(v^* \rho_\mu(1) v)^{\frac{1}{2}}.$$

Since $\rho_\mu(1)$ commutes with v and

$$\|x_\mu\| = \|S_\mu x_\mu S_\mu^*\| \le \max_{\nu \in B_k(\Lambda)} \|S_\nu x_\nu S_\nu^*\| = \|x\|, \tag{4.6}$$

we have

$$|\varphi(x_\mu)| \le \|x\| \psi(\rho_\mu(1))$$

and hence

$$|\varphi_k(x)| \le \frac{1}{|\beta|^k} \sum_{\mu \in B_k(\Lambda)} |\varphi(x_\mu)| \le \frac{1}{|\beta|^k} \|x\| \psi(\lambda_\rho^k(1)) = \frac{\|\lambda_\rho^k(1)\|}{r_\rho^k} \psi(1) \|x\|.$$

Therefore we have

$$|\varphi_k(x)| \le M \psi(1) \|x\|, \qquad x \in \mathcal{D}_\rho^k$$

and hence $\{\varphi_k\}_{k \in \mathbf{N}}$ extends to a continuous linear functional on the closure $\mathcal{D}_\rho$ of ${\mathcal{D}_\rho}^{\mathrm{alg}}$. □

If in particular a linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ is positive on $\mathcal{A}$, it always extends to a continuous linear functionl on $\mathcal{D}_\rho$ as in the following way:

Lemma 4.5. *Let $\beta \in \mathbf{C}$ be $|\beta| > 1$. If $\varphi \in \mathcal{E}_\beta(\rho)$ is a positive linear functional on $\mathcal{A}$, then β becomes a positive real number and the canonical extension $\tilde{\varphi}$ to $\mathcal{D}_\rho$ is continuous on $\mathcal{D}_\rho$.*

Proof. One may assume that $\varphi \neq 0$ and $\varphi(1) = 1$. We have $\beta = \beta\varphi(1) = \varphi(\lambda_\rho(1)) \geq 1$. For $k \in \mathbf{N}$, define a linear functional φ_k on $\mathcal{D}_\rho^k$ by (4.4). Since for $x = \sum_{\mu \in B_k(\Lambda)} S_\mu x_\mu S_\mu^* \in \mathcal{D}_\rho^k$ we have by (4.6),

$$|\varphi(\rho_\mu(1)x_\mu\rho_\mu(1))| \leq \varphi(\rho_\mu(1))^{\frac{1}{2}}\varphi(\rho_\mu(1)x_\mu^* x_\mu\rho_\mu(1))^{\frac{1}{2}} \leq \|x\|\varphi(\rho_\mu(1)),$$

it follows that

$$|\varphi_k(x)| \leq \frac{1}{|\beta|^k}\sum_{\mu \in B_k(\Lambda)}|\varphi(\rho_\mu(1)x_\mu\rho_\mu(1))| \leq \frac{1}{|\beta|^k}\|x\|\varphi(\lambda_\rho^k(1)) = \|x\|.$$

Therefore $\{\varphi_k\}_{k \in \mathbf{N}}$ extends to a state on $\mathcal{D}_\rho$. □

We are now assuming that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. By Lemma 3.4, there exists a faithful invariant state $\tau \in \mathcal{E}_{r_\rho}(\rho)$ on $\mathcal{A}$. By the previous lemma, the canonical extension $\tilde{\tau}$ is continuous on $\mathcal{D}_\rho$ which satisfies

$$\tilde{\tau}(S_\mu a S_\mu^*) = \frac{1}{r_\rho^{|\mu|}}\tau(a\rho_\mu(1)), \qquad a \in \mathcal{A}, \mu \in B_*(\Lambda). \tag{4.7}$$

Lemma 4.6. *For a faithful invariant state $\tau \in \mathcal{E}_{r_\rho}(\rho)$ on $\mathcal{A}$, the canonical extension $\tilde{\tau}$ is faithful on $\mathcal{D}_\rho$.*

Proof. Suppose that $\tilde{\tau}$ is not faithful on $\mathcal{D}_\rho$. Put

$$I_{\tilde{\tau}} = \{x \in \mathcal{D}_\rho \mid \tilde{\tau}(x^*x) = 0\}.$$

Since $\tilde{\tau}$ is tracial on $\mathcal{D}_\rho$, $I_{\tilde{\tau}}$ is a nonzero ideal of $\mathcal{D}_\rho$. By Lemma 4.3, the equality $\tilde{\tau} \circ \lambda_\rho = r_\rho\tilde{\tau}$ holds on $\mathcal{D}_\rho$ so that $I_{\tilde{\tau}}$ is λ_ρ-invariant. The sequence $\mathcal{D}_\rho^k, k \in \mathbf{N}$ of algebras is increasing such that $\cup_{k \in \mathbf{N}}\mathcal{D}_\rho^k$ is dense in $\mathcal{D}_\rho$. We may find $k \in \mathbf{N}$ such that $I_{\tilde{\tau}} \cap \mathcal{D}_\rho^k \neq 0$. It is easy to see that $\lambda_\rho^k(\mathcal{D}_\rho^k) \subset \mathcal{A}$ so that there exists a nonzero positive element $x \in I_{\tilde{\tau}} \cap \mathcal{D}_\rho^k$ such that $\lambda_\rho^k(x) \in I_{\tilde{\tau}} \cap \mathcal{A}$. Hence $I_{\tilde{\tau}} \cap \mathcal{A}$ is a nonzero λ_ρ-invariant ideal of $\mathcal{A}$. By the hypothesis that $(\mathcal{A}, \rho, \Sigma)$ is irreducible, we have a contradiction. □

For a faithful invariant state τ on $\mathcal{A}$, we will write the canonical extension $\tilde{\tau}$ of τ to $\mathcal{D}_\rho$ as still τ. Define a unital endomorphism $\phi_\rho : \mathcal{D}_\rho \longrightarrow \mathcal{D}_\rho$ by setting

$$\phi_\rho(y) = \sum_{\alpha \in \Sigma} S_\alpha y S_\alpha^*, \qquad y \in \mathcal{D}_\rho. \tag{4.8}$$

It induces a unital endomorphism on the enveloping von Neumann algebra ${\mathcal{D}_\rho}^{**}$ of $\mathcal{D}_\rho$, which we still denote by ϕ_ρ. The restriction of the positive map λ_ρ on $\mathcal{F}_\rho$ to $\mathcal{D}_\rho$ similarly induces a positive map on ${\mathcal{D}_\rho}^{**}$. We then need the following lemma for further discussions.

LEMMA 4.7. *The equality*

$$\lambda_\rho(x\phi_\rho(y)) = \lambda_\rho(x)y, \qquad x, y \in {\mathcal{D}_\rho}^{**} \tag{4.9}$$

holds.

Proof. Since $\mathcal{D}_\rho$ is dense in ${\mathcal{D}_\rho}^{**}$ under $\sigma({\mathcal{D}_\rho}^{**}, {\mathcal{D}_\rho}^{*})$-topology, it suffices to show the equality (4.9) for $x, y \in \mathcal{D}_\rho$. One has

$$\begin{aligned}\lambda_\rho(x\phi_\rho(y)) &= \sum_{\alpha,\gamma\in\Sigma} S_\alpha^* x S_\gamma y S_\gamma^* S_\alpha \\ &= \sum_{\alpha\in\Sigma} S_\alpha^* x S_\alpha y S_\alpha^* S_\alpha = \sum_{\alpha\in\Sigma} S_\alpha^* x S_\alpha y = \lambda_\rho(x)y.\end{aligned}$$

□

Recall that for a continuous linear functional ψ on a C^*-algebra $\mathcal{B}$ there exist a partial isometry $v \in \mathcal{B}^{**}$ and a positive linear functional $|\psi| \in \mathcal{B}^*$ in a unique way such that

$$v^*v = s(|\psi|), \qquad \psi(x) = |\psi|(xv) \quad \text{for } x \in \mathcal{B}, \tag{4.10}$$

where $s(|\psi|)$ denotes the support projection of $|\psi|$ (cf. [43, Theorem 4.2]). The decomposition (4.10) is called the polar decomposition of ψ. The linear functional $\psi : x \longrightarrow |\psi|(xv)$ is denoted by $v|\psi|$.

LEMMA 4.8. *Let $\beta = re^{i\theta} \in \mathbf{C}$ be $r, \theta \in \mathbf{R}$ with $r > 1$. For a (not necessarily positive) linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ on $\mathcal{A}$, let $\tilde{\varphi}$ be the extension on ${\mathcal{D}_\rho}^{alg}$ satisfying* (4.3). *Suppose that the linear functional $\tilde{\varphi}$ extends to a continuous linear functional on $\mathcal{D}_\rho$. Denote by $\tilde{\varphi} = v|\tilde{\varphi}|$ its polar decomposition for a partial isometry $v \in {\mathcal{D}_\rho}^{**}$ and a positive linear functional $|\tilde{\varphi}|$ on $\mathcal{D}_\rho$ such that $v^*v = s(|\tilde{\varphi}|)$. Then we have*

$$\phi_\rho(v) = e^{i\theta}v, \qquad |\tilde{\varphi}|(\lambda_\rho(x)) = r|\tilde{\varphi}|(x) \quad \textit{for } x \in \mathcal{D}_\rho.$$

Hence the restriction of $|\tilde{\varphi}|$ to $\mathcal{A}$ belongs to $\mathcal{E}_r(\rho)$ and $|\tilde{\varphi}|$ satisfies

$$|\tilde{\varphi}|(S_\mu a S_\mu^*) = \frac{1}{r^{|\mu|}}|\tilde{\varphi}|(a\rho_\mu(1)), \qquad a \in \mathcal{A}, \mu \in B_*(\Lambda).$$

Proof. Put a positive linear functional ψ on $\mathcal{D}_\rho$ and a partial isometry u in ${\mathcal{D}_\rho}^{**}$ by setting

$$\psi(x) = \frac{1}{r}|\tilde{\varphi}|(\lambda_\rho(x)) \quad \text{for } x \in \mathcal{D}_\rho \quad \text{and} \quad u = e^{-i\theta}\phi_\rho(v).$$

As $\lambda_\rho(xu) = e^{-i\theta}\lambda_\rho(x)v$ for $x \in \mathcal{D}_\rho$ by Lemma 4.7. It follows that for $x \in \mathcal{D}_\rho$

$$(u\psi)(x) = \frac{1}{r}|\tilde{\varphi}|(\lambda_\rho(xu)) = \frac{1}{\beta}|\tilde{\varphi}|(\lambda_\rho(x)v) = \tilde{\varphi}(x).$$

Hence we have

$$\tilde{\varphi} = u\psi \qquad \text{on } \mathcal{D}_\rho.$$

We will next show that $s(\psi) = u^*u$. For $y \in \mathcal{D}_\rho$, we have by Lemma 4.7

$$\psi(yu^*u) = \frac{1}{r}|\tilde{\varphi}|(\lambda_\rho(yu^*u)) = \frac{1}{r}|\tilde{\varphi}|(\lambda_\rho(y\phi_\rho(v^*v))) = \frac{1}{r}|\tilde{\varphi}|(\lambda_\rho(y)v^*v) = \psi(y).$$

Hence we have $u^*u \geq s(\psi)$. On the other hand, suppose that a projection $p \in {\mathcal{D}_\rho}^{**}$ satisfies

$$\psi(yp) = \psi(y) \qquad \text{for } y \in \mathcal{D}_\rho.$$

We then have $|\tilde{\varphi}|(\lambda_\rho(y(1-p))) = 0$ for all $y \in \mathcal{D}_\rho$. For $y = S_\alpha S_\alpha^*, \alpha \in \Sigma$, one has $|\tilde{\varphi}|(S_\alpha^*(1-p)S_\alpha)) = 0$. As $S_\alpha^*(1-p)S_\alpha$ is a projection in $\mathcal{D}_\rho$, one obtains that $S_\alpha^*(1-p)S_\alpha \leq 1 - v^*v$ so that $1 - p \leq 1 - \phi_\rho(v^*v)$. This implies that $u^*u \leq p$. Therefore we have $u^*u \leq s(\psi)$ and hence

$$u^*u = s(\psi).$$

By the uniqueness of the polar decomposition, we conclude that

$$v = u \qquad \text{and} \qquad |\tilde{\varphi}| = \psi \quad \text{on } \mathcal{D}_\rho$$

so that

$$\phi_\rho(v) = e^{i\theta}v, \qquad |\tilde{\varphi}|(\lambda_\rho(x)) = r|\tilde{\varphi}|(x) \quad \text{for} \quad x \in \mathcal{D}_\rho.$$

□

Therefore we have

Theorem 4.9. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and power-bounded. For $\beta \in \mathbf{C}$ with $|\beta| > 1$, a (not necessarily positive) linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ on $\mathcal{A}$ extends to $\mathcal{D}_\rho$ as a continuous linear functional $\tilde{\varphi}$ satisfying*

$$\tilde{\varphi}(S_\mu a S_\mu^*) = \frac{1}{\beta^{|\mu|}}\varphi(a\rho_\mu(1)), \qquad a \in \mathcal{A}, \mu \in B_*(\Lambda)$$

if $|\beta| = r_\rho$. If in particular, $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic, the converse implication holds.

Proof. The first part of the assertions is direct from Lemma 4.4. Under the condition that $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic, assume that the canonical extension $\tilde{\varphi}$ is continuous on $\mathcal{D}_\rho$. The preceding lemma says that the positive linear functional $|\tilde{\varphi}|$ belongs to $\mathcal{E}_{|\beta|}(\rho)$. Since the mean ergodicity implies (FP), by Lemma 3.6 (i) we see that $|\beta| = r_\rho$. □

Let us now assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and satisfies $\dim \mathcal{E}_{r_\rho}(\rho) = 1$, and hence it is uniquely ergodic. Take a unique invariant state τ on $\mathcal{A}$ and denote still by τ its canonical extension on $\mathcal{D}_\rho$. Denote by $p_\tau \in {\mathcal{D}_\rho}^{**}$ its support projection.

Lemma 4.10. *Let $w \in {\mathcal{D}_\rho}^{**}$ be a partial isometry satisfying*

$$w^*w = p_\tau \qquad \textit{and} \qquad \phi_\rho(w) = w. \tag{4.11}$$

Then w is a scalar multiple of the projection p_τ.

Proof. Put $w\tau(x) = \tau(xw)$ for $x \in \mathcal{D}_\rho$ and hence $w\tau \in {\mathcal{D}_\rho}^*$. Since $\lambda_\rho(x)w = \lambda_\rho(x\phi_\rho(w)) = \lambda_\rho(xw)$ by Lemma 4.7, it follows that for $x \in \mathcal{D}_\rho$

$$w\tau(\lambda_\rho(x)) = \tau(\lambda_\rho(xw)) = r_\rho\tau(xw) = r_\rho w\tau(x).$$

In particular, we have $w\tau \in \mathcal{E}_{r_\rho}(\rho)$. As $\dim \mathcal{E}_{r_\rho}(\rho) = 1$ by hypothesis, $w\tau$ is a scalar multiple of τ. Hence there exists $c \in \mathbf{C}$ such that $\tau(xw) = c\tau(x)$ for $x \in \mathcal{A}$. Since $w\tau$ is the canonical extension of $\tau(\cdot\, w) = w\tau$ on $\mathcal{A}$ to $\mathcal{D}_\rho$ and the canonical extension is unique, one has $\tau(xw) = c\tau(x)$ for $x \in \mathcal{D}_\rho$ so that

$$\tau(xw) = \tau(xcp_\tau) \qquad \text{for } x \in \mathcal{D}_\rho. \tag{4.12}$$

As $c = c\tau(1) = \tau(w)$, one has

$$1 = \tau(p_\tau) = \tau(w^*w) = c\tau(w^*) = c\overline{\tau(w)} = c\bar{c}$$

so that

$$(cp_\tau)^*(cp_\tau) = p_\tau = w^*w.$$

By the uniqueness of the polar decomposition, we have by (4.12) $w = cp_\tau$. $\square$

Proposition 4.11. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and satisfies $\dim \mathcal{E}_{r_\rho}(\rho) = 1$. Then $\dim \mathcal{E}_\beta(\rho) \le 1$ for $\beta \in \mathbf{C}$ with $|\beta| = r_\rho > 1$.*

Proof. Let $|\beta| = r_\rho > 1$. Take an arbitrary linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ with $\varphi \ne 0$. Its canonical extension $\tilde{\varphi}$ to $\mathcal{D}_\rho$ is continuous. Denote by $\tilde{\varphi} = v_{\tilde{\varphi}}|\tilde{\varphi}|$ the polar decomposition in ${\mathcal{D}_\rho}^*$ where $v_{\tilde{\varphi}}$ is a partial isometry in ${\mathcal{D}_\rho}^{**}$. By Lemma 4.7, the restriction of $|\tilde{\varphi}|$ to $\mathcal{A}$ is a positive linear functional belonging to $\mathcal{E}_{r_\rho}(\rho)$. Since $(\mathcal{A}, \rho, \Sigma)$ is uniquely ergodic, by putting $c_{\tilde{\varphi}} = |\tilde{\varphi}|(1)$ one has $|\tilde{\varphi}| = c_{\tilde{\varphi}}\tau$ as a positive linear functional on $\mathcal{A}$. The canonical extension to $\mathcal{D}_\rho$ which satisfies (4.3) is unique and determined by its behavior on $\mathcal{A}$. Hence the equalty $|\tilde{\varphi}| = c_{\tilde{\varphi}}\tau$ holds as a positive linear functional on $\mathcal{D}_\rho$ so that we have $\mathrm{supp}(|\tilde{\varphi}|) = \mathrm{supp}(\tau)$ and hence $v_{\tilde{\varphi}}^* v_{\tilde{\varphi}} = p_\tau$. For another linear functional $\psi \in \mathcal{E}_\beta(\rho)$ with $\psi \ne 0$, we have similar decompositions

$$\tilde{\psi} = v_{\tilde{\psi}}|\tilde{\psi}|, \qquad |\tilde{\psi}| = c_{\tilde{\psi}}\tau, \qquad v_{\tilde{\psi}}^* v_{\tilde{\psi}} = p_\tau.$$

Put a partial isometry $w = v_{\tilde{\varphi}}^* v_{\tilde{\psi}} \in {\mathcal{D}_\rho}^{**}$ so that $w^*w = p_\tau$. By Lemma 4.8, one has $\phi_\rho(w) = w$. Lemma 4.10 implies $w = cp_\tau$ for some $c \in \mathbf{C}$ with $|c| = 1$ so that $v_{\tilde{\psi}} = cv_{\tilde{\varphi}}$. Therefore we have

$$\tilde{\psi} = v_{\tilde{\psi}}|\tilde{\psi}| = cv_{\tilde{\varphi}}c_{\tilde{\psi}}\tau = c\frac{c_{\tilde{\psi}}}{c_{\tilde{\varphi}}}\tilde{\varphi}$$

on $\mathcal{D}_\rho$. In particular we have $\psi = c\frac{c_{\tilde{\psi}}}{c_{\tilde{\varphi}}}\varphi$ on $\mathcal{A}$ so that $\dim \mathcal{E}_\beta(\rho) \le 1$. $\square$

Corollary 4.12. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and mean ergodic. Then for $\beta \in \mathbf{C}$ with $|\beta| > 1$, we have $\dim \mathcal{E}_\beta(\rho) \le 1$ if $|\beta| = r_\rho$, otherwise $\mathcal{E}_\beta(\rho) = \{0\}$.*

Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and mean ergodic. Hence $(\mathcal{A}, \rho, \Sigma)$ is uniquely ergodic with a unique faithful invariant state $\tau \in \mathcal{E}_{r_\rho}(\rho)$. Denote by $p_\tau \in {\mathcal{D}_\rho}^{**}$ the support projection of the canonical extension of τ to $\mathcal{D}_\rho$, where the extension is still denoted by τ. For $\beta = re^{i\theta} \in \mathbf{C}$ with $r = r_\rho > 1$, we set

$$P_\beta(\mathcal{D}_\rho, \tau) = \{v \in {\mathcal{D}_\rho}^{**} \mid \phi_\rho(v) = e^{i\theta} v,\ v^* v = p_\tau\}.$$

Denote by $\mathbf{R}_+$ the set of all nonnegative real numbers. For $\varphi \in \mathcal{E}_\beta(\rho)$ denote by $\tilde{\varphi}$ its canonical extension to $\mathcal{D}_\rho$. As $|\beta| = r_\rho$, $\tilde{\varphi}$ is continuous and has a unique polar decomposition $\tilde{\varphi} = v_{\tilde{\varphi}} |\tilde{\varphi}|$ for some $v_{\tilde{\varphi}} \in {\mathcal{D}_\rho}^{**}$ and positive linear functional $|\tilde{\varphi}| \in {\mathcal{D}_\rho}^*$. By Lemma 4.8, we know the structure of the eigenspace $\mathcal{E}_\beta(\rho)$ as in the following way:

Proposition 4.13. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and mean ergodic. There exists a bijective correspondence between the eigenspace $\mathcal{E}_\beta(\rho)$ and the product set $P_\beta(\mathcal{D}_\rho, \tau) \times \mathbf{R}_+$ through the correspondences*

$$\begin{aligned} \varphi \in & \mathcal{E}_\beta(\rho) \longrightarrow (v_{\tilde{\varphi}}, |\tilde{\varphi}|(1)) \in P_\beta(\mathcal{D}_\rho, \tau) \times \mathbf{R}_+, \\ c\tau(\cdot\ v) \in & \mathcal{E}_\beta(\rho) \longleftarrow (v, c) \in P_\beta(\mathcal{D}_\rho, \tau) \times \mathbf{R}_+. \end{aligned}$$

5. Extension to $\mathcal{O}_\rho$ and KMS condition

In [9], Enomoto-Fujii-Watatani have proved that KMS states for gauge action on the Cuntz-Krieger algebra $\mathcal{O}_A$ exist if and only if its inverse temperature is $\log r_A$, where r_A is the Perron-Frobenius eigenvalue for the irreducible matrix A. They have showed that the KMS states bijectively correspond to the normalized positive eigenvectors of A for the eigenvalue r_A.

In this section, we will study KMS conditions for linear functionals without assuming its positivity at inverse temperature taking complex numbers. The extended notation is needed to study eigenvector spaces for C^*-symbolic dynamical systems.

Following after [3], KMS states for one-parameter group action α on a C^*-algebra $\mathcal{B}$ is defined as follows: For a positive real number $\gamma \in \mathbf{R}$, a state ψ on $\mathcal{B}$ is a KMS state at inverse temperature γ if ψ satisfies

$$\psi(y\alpha_{i\gamma}(x)) = \psi(xy), \qquad x \in \mathcal{B}^a, y \in \mathcal{B} \tag{5.1}$$

where $\mathcal{B}^a$ is the set of analytic elements of the action $\alpha : \mathbf{R} \longrightarrow \mathrm{Aut}(\mathcal{B})$ (cf.[3]). The equation (5.1) for ψ is called the KMS condition with respect to the action α.

In what follows, we restrict our interest to periodic actions so as to extend KMS condition to (not necessarily positive) linear functionals at inverse temperature taking complex numbers. We assume that an action α of $\mathbf{R}$ has its period 2π so that α is regarded as an action of one-dimensional torus group $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. Let $\mathcal{B}$ be a C^*-algebra and $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ a continuous action of $\mathbf{T}$ to the automorphism group $\mathrm{Aut}(\mathcal{B})$. We write a complex number $\beta \in \mathbf{C}$ as $\beta = re^{i\theta}$ where $r, \theta \in \mathbf{R}$ with $r > 1$.

DEFINITION. A continuous linear functional $\varphi \in \mathcal{B}^*$ on $\mathcal{B}$ is said to satisfy *KMS condition* at Logβ if φ satisfies the following condition

$$\varphi(y\alpha_{i\log r}(x)) = \varphi(\alpha_\theta(x)y), \qquad x \in \mathcal{B}^a, y \in \mathcal{B}. \tag{5.2}$$

REMARK.

(i) As $\alpha_\theta(x) = \alpha_{\theta+2\pi}(x)$, the right hand side $\varphi(\alpha_\theta(x)y)$ of (5.2) does not depend on the choice of $\theta \in \mathbf{R}$ as long as $\beta = re^{i\theta}$.

(ii) The above KMS condition (5.2) is equivalent to the following condition:

$$\varphi(y\alpha_{\zeta+i\log r}(x)) = \varphi(\alpha_{\zeta+\theta}(x)y), \qquad x \in \mathcal{B}^a, y \in \mathcal{B}, \quad \zeta \in \mathbf{C} \tag{5.3}$$

(iii) In case of $\theta = 0$, the above definition of KMS condition coincides with the original definition of KMS condition for states.

(iv) The above equality (5.2) can be written formally as

$$\varphi(y\alpha_{i\mathrm{Log}\beta}(x)) = \varphi(xy), \qquad x \in \mathcal{B}^a, y \in \mathcal{B}, \tag{5.4}$$

if we denote Log$\beta = \log r + i\theta$.

We will present some examples of linear functionals satisfying the extended KMS conditions.

EXAMPLES.

(i) Let $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ be an action of $\mathbf{T}$ to a C^*-algebra $\mathcal{B}$ such that there exists a projection $H \in \mathcal{B}$ satisfying $\alpha_t(a) = e^{itH}ae^{-itH}, a \in \mathcal{B}, t \in \mathbf{T}$. Assume that there exists an α-invariant tracial state tr on $\mathcal{B}$. Put

$$\varphi(x) = \frac{tr(e^{-\mathrm{Log}\beta H}x)}{tr(e^{-\mathrm{Log}\beta H})}, \qquad x \in \mathcal{B},$$

where Log$\beta = \log r + i\theta$. Then φ satisfies KMS condition at Logβ.

(ii) Let $\mathcal{B} = \otimes_{k=1}^\infty M_2$ be the UHF-algebra of type 2^∞ and $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ an action of $\mathbf{T}$ to $\mathcal{B}$ defined by

$$\alpha_t = \otimes_{k=1}^\infty Ad \begin{bmatrix} 1 & 0 \\ 0 & e^{it} \end{bmatrix}, \qquad t \in \mathbf{T}.$$

Put

$$\begin{aligned} \mathcal{B}_n &= \otimes_{k=1}^n M_2 = M_2 \otimes \cdots \otimes M_2, \\ u_t^n &= \otimes_{k=1}^n \begin{bmatrix} 1 & 0 \\ 0 & e^{it} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{it} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & 0 \\ 0 & e^{it} \end{bmatrix} \in \mathcal{B}_n, \\ \alpha_t^n &= Ad(u_t^n) \in \mathrm{Aut}(\mathcal{B}_n), \quad t \in \mathbf{T}. \end{aligned}$$

Let $\beta = re^{i\theta} \in \mathbf{C}$ be $r > 1$. Put

$$H = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in M_2, \qquad h_n = \otimes_{k=1}^n \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\beta} \end{bmatrix} \in \mathcal{B}_n,$$

and hence $h_n = \otimes_{k=1}^n e^{-\mathrm{Log}\beta H}$, $\alpha_t^n = \otimes_{k=1}^n Ad(e^{itH}), t \in \mathbf{T}$. It is straightforward to see that

$$tr(e^{-\mathrm{Log}\beta H} b\alpha_{i\log r}(a)) = tr(e^{-\mathrm{Log}\beta H}\alpha_\theta(a)b), \qquad a, b \in M_2.$$

Put

$$\varphi_n(x) = \otimes_{k=1}^n tr(xh_n) \qquad \text{for } x \in \mathcal{B}_n$$

so that we have

$$\varphi_n(y\alpha_{i\log r}(x)) = \varphi_n(\alpha_\theta(x)y), \qquad x, y \in \mathcal{B}_n.$$

As $\|h_n\| = 1$, φ_n extends to a continuous linear functional on $\mathcal{B}$, which we denote by φ. Then φ satisfies KMS condition at $\mathrm{Log}\beta$:

$$\varphi(y\alpha_{i\log r}(x)) = \varphi_n(\alpha_\theta(x)y), \qquad x \in \mathcal{B}^a, y \in \mathcal{B}.$$

We see the following two propositions whose proofs are similar to the case of usual KMS states.

Proposition 5.1 (cf. [39, 8.12.3]). *Let $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ be a continuous action of $\mathbf{T}$ to the automorphism group $\mathrm{Aut}(\mathcal{B})$ of a C^*-algebra $\mathcal{B}$ and β a complex number with $\beta = re^{i\theta}$, $r > 1$. The following conditions for a continuous linear functional φ on $\mathcal{B}$ are equivalent:*

(i) *φ satisfies the KMS condition at $\mathrm{Log}\beta$.*
(ii) *φ satisfies the equality (5.2) for just a dense set of elements in $\mathcal{B}^a$.*
(iii) *For all $x, y \in \mathcal{B}$, there is a bounded continuous function f on the strip*

$$\Omega_{\log r} = \{\zeta \in \mathbf{C} \mid 0 \le \mathrm{Im}\zeta \le \log r\}$$

such that f is holomorphic in the interior of $\Omega_{\log r}$ and

$$f(t) = \varphi(y\alpha_t(x)), \qquad f(t + i\log r) = \varphi(\alpha_{t+\theta}(x)y), \qquad t \in \mathbf{R}.$$

Proposition 5.2 (cf. [39, 8.12.4]). *Let $\mathcal{B}$ be a C^*-algebra and $\alpha : \mathbf{T} \longrightarrow \mathrm{Aut}(\mathcal{B})$ be a continuous action of $\mathbf{T}$ to the automorphism group $\mathrm{Aut}(\mathcal{B})$. Let φ be a continuous linear functional on $\mathcal{B}$. If φ satisfies KMS condition at $\mathrm{Log}\beta$ for some complex number β with $\beta = re^{i\theta}$ with $r > 1$, then φ is α-invariant, that is,*

$$\varphi \circ \alpha_t = \varphi, \qquad t \in \mathbf{T}.$$

We henceforth go back to our previous situations. Let $(\mathcal{A}, \rho, \Sigma)$ be a C^*-symbolic dynamical system. Recall that the positive operator λ_ρ on $\mathcal{A}$ extends to $\mathcal{F}_\rho$ by setting $\lambda_\rho(x) = \sum_{\alpha\in\Sigma} S_\alpha^* x S_\alpha, x \in \mathcal{F}_\rho$. For $\beta \in \mathbf{C}$ with $\beta \ne 0$, we set

$$\mathcal{E}_\beta^{\mathcal{D}}(\rho) = \{\varphi \in {\mathcal{D}_\rho}^* \mid \varphi(\lambda_\rho(x)) = \beta\varphi(x), x \in \mathcal{D}_\rho\}, \tag{5.5}$$

$$\mathcal{E}_\beta^{\mathcal{F}}(\rho) = \{\phi \in {\mathcal{F}_\rho}^* \mid \phi(\lambda_\rho(x)) = \beta\phi(x), x \in \mathcal{F}_\rho,\ \phi \text{ is tracial on } \mathcal{F}_\rho\}. \tag{5.6}$$

It is possible that both $\mathcal{E}_\beta^{\mathcal{D}}(\rho)$ and $\mathcal{E}_\beta^{\mathcal{F}}(\rho)$ are $\{0\}$. Recall that $E_{\mathcal{D}} : \mathcal{F}_\rho \longrightarrow \mathcal{D}_\rho$ is the canonical expectation satisfying by $E_{\mathcal{D}}(S_\mu a S_\nu^*) = \delta_{\mu,\nu} S_\mu a S_\nu^*$ for $a \in \mathcal{A}$ with $\mu, \nu \in B_*(\Lambda)$, $|\mu| = |\nu|$. By composing it to a given linear functional $\varphi \in \mathcal{E}_\beta^{\mathcal{D}}(\rho)$ on $\mathcal{D}_\rho$, φ extends to $\mathcal{F}_\rho$.

LEMMA 5.3. *Let $\beta \in \mathbf{C}$ with $|\beta| > 1$. A (not necessarily positive) continuous linear functional $\varphi \in \mathcal{E}_\beta^{\mathcal{D}}(\rho)$ on $\mathcal{D}_\rho$ uniquely extends to $\mathcal{F}_\rho$ as a tracial continuous linear functional $\phi = \varphi \circ E_{\mathcal{D}}$ such that*

$$\phi(S_\mu x S_\nu^*) = \delta_{\mu,\nu} \frac{1}{\beta^{|\mu|}} \phi(x S_\mu^* S_\nu), \qquad x \in \mathcal{F}_\rho, \mu, \nu \in B_*(\Lambda) \text{ with } |\mu| = |\nu|. \tag{5.7}$$

Hence the sets $\mathcal{E}_\beta^{\mathcal{D}}(\rho)$ and $\mathcal{E}_\beta^{\mathcal{F}}(\rho)$ bijectively correspond to each other.

Proof. For $\varphi \in \mathcal{E}_\beta^{\mathcal{D}}(\rho)$, as in the proof of Lemma 4.3 (i) $\Rightarrow$ (ii), the equality

$$\varphi(S_\mu a S_\mu^*) = \frac{1}{\beta^{|\mu|}} \varphi(a \rho_\mu(1)), \qquad a \in \mathcal{A}, \mu \in B_*(\Lambda)$$

holds so that

$$\phi(S_\mu a S_\nu^*) = \delta_{\mu,\nu} \frac{1}{\beta^{|\mu|}} \varphi(a \rho_\mu(1)), \qquad a \in \mathcal{A}, \ \mu, \nu \in B_*(\Lambda) \text{ with } |\mu| = |\nu|.$$

By Lemma 4.3 (iii) $\Rightarrow$ (i), ϕ belongs to $\mathcal{E}_\beta^{\mathcal{F}}(\rho)$. □

Recall that $E_\rho : \mathcal{O}_\rho \longrightarrow {\mathcal{O}_\rho}^{\hat{\rho}} = \mathcal{F}_\rho$ denotes the conditional expectation defined by (2.3).

PROPOSITION 5.4. *For any tracial continuous linear functional $\phi \in \mathcal{E}_\beta^{\mathcal{F}}(\rho)$, the composition $\psi = \phi \circ E_\rho$ is a continuous linear functional on $\mathcal{O}_\rho$ which satisfies KMS condition at $\mathrm{Log}\beta$ for gauge action $\hat{\rho}$ of $\mathbf{T}$.*

Proof. Let $\mathcal{P}_\rho$ be the dense $*$-subalgebra of $\mathcal{O}_\rho$ generated algebraically by $S_\alpha, \alpha \in \Sigma$ and $a \in \mathcal{A}$. It is clear that for each element $x \in \mathcal{P}_\rho$ the function $t \in \mathbf{T} = \mathbf{R}/2\pi\mathbf{R} \to \hat{\rho}_t(x) \in \mathcal{O}_\rho$ extends to an entire analytic function on $\mathbf{C}$. Put $\psi = \phi \circ E_\rho$. We will show that the equality (5.2) holds for ψ. Elements $x, y \in \mathcal{P}_\rho$ can be expanded as finite linear combinations

$$x = \sum x_{-\nu} S_\nu^* + x_0 + \sum S_\mu x_\mu, \quad y = \sum y_{-\nu} S_\nu^* + y_0 + \sum S_\mu y_\mu \tag{5.8}$$

for some $x_{-\nu}, x_0, x_\mu, y_{-\nu}, y_0, y_\mu \in {\mathcal{F}_\rho}^{\mathrm{alg}}$. As ψ is a tracial linear functional on $\mathcal{F}_\rho$, it suffices to check the equality (5.2) for the following two cases

$$(1) \quad x = S_\nu x_\nu, \quad y = y_{-\nu} S_\nu^*, \qquad (2) \quad x = x_{-\mu} S_\mu^*, \quad y = S_\mu y_\mu.$$

Case (1):

$$\begin{aligned}
\psi(y \hat{\rho}_{i \log r}(x)) &= \psi(y_{-\nu} S_\nu^* e^{-|\nu| \log r} S_\nu x_\nu) \\
&= \frac{1}{\beta^{|\nu|}} \psi(e^{i|\nu|\theta} x_\nu y_{-\nu} S_\nu^* S_\nu) \\
&= \psi(e^{i|\nu|\theta} S_\nu x_\nu y_{-\nu} S_\nu^*) \\
&= \psi(\hat{\rho}_\theta(x) y).
\end{aligned}$$

Case (2):

$$\begin{aligned}\psi(y\hat{\rho}_{i\log r}(x)) &= \psi(S_\mu y_\mu e^{|\mu|\log r}x_{-\mu}S_\mu^*)\\ &= \frac{r^{|\mu|}}{\beta^{|\mu|}}\psi(y_\mu x_{-\mu}S_\mu^* S_\mu)\\ &= \psi(e^{-i|\mu|\theta}x_{-\mu}S_\mu^* S_\mu y_\mu)\\ &= \psi(\hat{\rho}_\theta(x)y).\end{aligned}$$

This completes the proof. □

Conversely we have

Lemma 5.5. *If a continuous linear functional ψ on $\mathcal{O}_\rho$ satisfies KMS condition at* Logβ *for some $\beta \in \mathbf{C}$ with $|\beta| > 1$, then the restriction $\phi = \psi|_{\mathcal{F}_\rho}$ to $\mathcal{F}_\rho$ belongs to $\mathcal{E}_\beta^{\mathcal{F}}(\rho)$ and satisfies the equality $\psi = \phi \circ E_\rho$.*

Proof. Let $\beta = re^{i\theta}$ with $r > 1$. For any $x \in \mathcal{F}_\rho, \mu \in B_*(\Lambda)$, we see

$$\psi(S_\mu x) = \frac{1}{\beta^{|\mu|}}\psi(xS_\mu) = \frac{1}{\beta^{|\mu|}}\psi(S_\mu\hat{\rho}_{i\log r}(\alpha_{-\theta}(x))) = \frac{1}{\beta^{|\mu|}}\psi(S_\mu x)$$

so that $\psi(S_\mu x) = 0$ because $|\beta| > 1$. We similarly have $\psi(xS_\mu^*) = 0$. Since any element of $\mathcal{P}_\rho$ can be expanded as in (5.8), we get $\psi(y) = \phi \circ E_\rho(y)$ for $y \in \mathcal{P}_\rho$. We will next show that ϕ belongs to $\mathcal{E}_\beta^{\mathcal{F}}(\rho)$. For $x, y \in \mathcal{F}_\rho$, one sees $\hat{\rho}_{i\log r}(x) = \hat{\rho}_{-\theta}(x) = x$ so that $\psi(yx) = \psi(xy)$. Hence ψ gives rise to a tracial linear functional ϕ on $\mathcal{F}_\rho$. By KMS condition, we get for any $x \in \mathcal{F}_\rho, \mu \in B_*(\Lambda)$,

$$\psi(S_\mu \cdot xS_\mu^*) = \psi(xS_\mu^*\hat{\rho}_{i\log r}(\hat{\rho}_{-\theta}(S_\mu)) = \frac{1}{\beta^{|\mu|}}\psi(xS_\mu^* S_\mu).$$

Thus by Lemma 4.3, we know $\phi \in \mathcal{E}_\beta^{\mathcal{F}}(\rho)$. □

We set for $\beta \in \mathbf{C}$ with $|\beta| > 1$,

$$\begin{aligned}&KMS_\beta(\mathcal{O}_\rho)\\ =&\{\psi \in {\mathcal{O}_\rho}^* \mid \psi \text{ satisfies KMS condition at Log}\beta \text{ for gauge action}\}\end{aligned}$$

and

$$Sp(\rho) = \{\beta \in \mathbf{C} \mid \varphi \circ \lambda_\rho = \beta\varphi \text{ for some } \varphi \in \mathcal{A}^* \text{ with } \varphi \neq 0\}.$$

By Proposition 5.4 and Lemma 5.5, we have

Proposition 5.6. *Let $(\mathcal{A}, \rho, \Sigma)$ be an irreducible C^*-symbolic dynamical system. Assume that $(\mathcal{A}, \rho, \Sigma)$ is power-bounded. Let $\beta \in \mathbf{C}$ be a complex number with $|\beta| > 1$. If $|\beta| = r_\rho$ and $\beta \in Sp(\rho)$, we have $KMS_\beta(\mathcal{O}_\rho) \neq \{0\}$. If in particular, $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic, $KMS_\beta(\mathcal{O}_\rho) \neq \{0\}$ if and only if $|\beta| = r_\rho$ and $\beta \in Sp(\rho)$.*

Proof. Under the assumption that $(\mathcal{A}, \rho, \Sigma)$ is power-bounded, any continuous linear functional $\varphi \in \mathcal{E}_\beta(\rho)$ on $\mathcal{A}$ can uniquely extend to a continuous linear functional $\tilde{\varphi}$ on $\mathcal{D}_\rho$, that belongs to $\mathcal{E}_\beta^{\mathcal{D}}(\rho)$ if $|\beta| = r_\rho$. By Proposition 5.4,

$\tilde{\varphi} \circ E_{\mathcal{D}} \in \mathcal{E}^{\mathcal{F}}_{\beta}(\rho)$ has an extension on $\mathcal{O}_\rho$ as a continuous linear functional that satisfies KMS condition at Logβ.
Conversely, the restriction of a continuous linear functional $KMS_\beta(\mathcal{O}_\rho)$ to the subalgebra $\mathcal{A}$ yields a nonzero element of $\mathcal{E}_\beta(\rho)$ which has continuous extension to $\mathcal{D}_\rho$. If in particular, $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic, $|\beta|$ must be r_ρ by Theorem 4.9. $\square$

Therefore we conclude

Theorem 5.7. *Let $(\mathcal{A}, \rho, \Sigma)$ be an irreducible C^*-symbolic dynamical system. Let $\beta \in \mathbf{C}$ be a complex number with $|\beta| = r_\rho > 1$.*

(i) *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is power-bounded. Then there exist linear isomorphisms among the four spaces $\mathcal{E}_\beta(\rho)$, $\mathcal{E}^{\mathcal{D}}_{\beta}(\rho)$, $\mathcal{E}^{\mathcal{F}}_{\beta}(\rho)$ and $KMS_\beta(\mathcal{O}_\rho)$ through the correspondences $\varphi \in \mathcal{E}_\beta(\rho)$, $\tilde{\varphi} \in \mathcal{E}^{\mathcal{D}}_{\beta}(\rho)$, $\tilde{\varphi} \circ E_{\mathcal{D}} \in \mathcal{E}^{\mathcal{F}}_{\beta}(\rho)$, $\tilde{\varphi} \circ E_{\mathcal{D}} \circ E_\rho \in KMS_\beta(\mathcal{O}_\rho)$ respectively. In particular, there exists a bijective correspondence between the set $\mathcal{E}_\beta(\rho)$ of eigenvectors of λ^*_ρ for eigenvalue β consisting of continuous linear functionals on $\mathcal{A}$ and the set $KMS_\beta(\mathcal{O}_\rho)$ of continuous linear functionals on $\mathcal{O}_\rho$ satisfying KMS condition at* Logβ.

(ii) *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is mean ergodic. Then the dimension $\dim KMS_\beta(\mathcal{O}_\rho)$ of the space of continuous linear functionals on $\mathcal{O}_\rho$ satisfying KMS condition at* Logβ *is one if there exists a nonzero eigenvector of λ^*_ρ on $\mathcal{A}^*$ for the eigenvalue β. In particular there uniquely exists a faithful KMS state on $\mathcal{O}_\rho$ at* $\log r_\rho$.

The following corollary is a generalization of [9, Theorem 6].

Corollary 5.8. *Suppose that A is an irreducible matrix with entries in $\{0, 1\}$ with its period p_A. Let β be a complex number with $|\beta| > 1$.*

(i) *There exists a nonzero continuous linear functional on the Cuntz-Krieger algebra $\mathcal{O}_A$ satisfying KMS condition for gauge action at* Logβ *if and only if β is a p_A-th root of the Perron-Frobenius eigenvalue r_A of A.*

(ii) *The space of admitted continuous linear functionals on $\mathcal{O}_A$ satisfying KMS condition for gauge action at* Logβ *is of one-dimensional.*

(iii) *If in particular $\beta = r_A$, the space of admitted continuous linear functionals on $\mathcal{O}_A$ satisfying KMS condition for gauge action at $\log r_A$ is the scalar multiples of a unique KMS state.*

6. KMS states and invariant measures

In this section, we will study a relationship between KMS states on $\mathcal{O}_\rho$ and invariant measures on $\mathcal{D}_\rho$ under ϕ_ρ. In what follows we assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and fix a faithful invariant state τ on $\mathcal{A}$.

We denote by $\|a\|_2$ the L^2-norm $\tau(a^*a)^{\frac{1}{2}}$ for $a \in \mathcal{A}$, and by $\mathcal{H}_\tau$ the completion of $\mathcal{A}$ by the norm $\|\cdot\|_2$. By the inequalities for $n \in \mathbf{N}$, $a \in \mathcal{A}$

$$\tau(\lambda_\rho^n(a)^*\lambda_\rho^n(a)) \leq \|\lambda_\rho^n\|\tau(\lambda_\rho^n(a^*a)) = \|\lambda_\rho^n\| r_\rho^n \tau(a^*a) \leq \|\lambda_\rho^n\|^2\|a\|_2^2, \tag{6.1}$$

the operators $T_\rho^n, n \in \mathbf{N}$ induce bounded linear operators on $\mathcal{H}_\tau$. The induced operators on $\mathcal{H}_\tau$, which we also denote by $T_\rho^n, n \in \mathbf{N}$, are uniformly bounded in the operator norm on $\mathcal{H}_\tau$, if $(\mathcal{A}, \rho, \Sigma)$ is power-bonded. We provide the following lemma, which shows power-boundedness of $(\mathcal{A}, \rho, \Sigma)$ induces an ordinary mean ergodicity on $\mathcal{H}_\tau$, is a direct consequence from [22, p.73,Theorem 1.2]. We give a proof for the sake of completeness.

Lemma 6.1. *Suppose that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and power-bounded. Then*

$$\lim_{n\to\infty} \frac{1 + T_\rho + T_\rho^2 + \cdots + T_\rho^{n-1}}{n}$$

converges to an idempotent P_ρ on $\mathcal{H}_\tau$ under strong operator topology in $B(\mathcal{H}_\tau)$. The subspace $P_\rho\mathcal{H}_\tau$ consists of the vectors of $\mathcal{H}_\tau$ fixed under T_ρ.

Proof. The mean operators $M_n, n \in \mathbf{N}$ on $\mathcal{A}$ defined by (3.1) naturally act on $\mathcal{H}_\tau$. Since $(\mathcal{A}, \rho, \Sigma)$ is power-bounded, there exists a positive number $c > 0$ such that $\|T_\rho^n\| < c$ for all $n \in \mathbf{N}$. As $\|M_n\| < 1 + c, n \in \mathbf{N}$, the sequence $M_n v \in \mathcal{H}_\tau, n \in \mathbf{N}$ for a vector $v \in \mathcal{H}_\tau$ has a cluster point v_0 under the weak topology of $\mathcal{H}_\tau$. The identites

$$(I - T_\rho)M_n = M_n(I - T_\rho) = \frac{1}{n}(I - T_\rho^n)$$

imply the inequalites

$$\|(I - T_\rho)M_n\| = \|M_n(I - T_\rho)\| = \frac{1}{n}\|I - T_\rho^n\| < \frac{1}{n}(1 + c). \tag{6.2}$$

Hence we have $T_\rho v_0 = v_0$. Put

$$Q_n = \frac{1}{n}\{(I + T_\rho) + (I + T_\rho + T_\rho^2) + \cdots + (I + T_\rho + \cdots + T_\rho^{n-2})\}.$$

Then we have $v - M_n v = (I - T_\rho)Q_n v, n \in \mathbf{N}$. Hence $v - v_0$ belongs to the weak closure $\mathcal{K}_\tau$ of the subspace $(I - T_\rho)\mathcal{H}_\tau$. The weak closure $\mathcal{K}_\tau$ is also the norm closure of the subspace $(I - T_\rho)\mathcal{H}_\tau$. For $w \in \mathcal{K}_\tau$, take $w_j \in (I - T_\rho)\mathcal{H}_\tau$ such that $\|w - w_j\|_2 \to 0$ and $w_j = (I - T_\rho)x_j$ for some $x_j \in \mathcal{H}_\tau$. We then have by (6.2)

$$\begin{aligned}\|M_n w\|_2 &\leq \|M_n\|\|w - w_j\|_2 + \|M_n(I - T_\rho)x_j\|_2 \\ &\leq (1 + c)\|w - w_j\|_2 + \frac{1}{n}(1 + c)\|x_j\|_2\end{aligned}$$

so that $\lim_{n\to\infty} \|M_n w\|_2 = 0$. Since $M_n v - v_0 = M_n(v - v_0)$ and $v - v_0 \in \mathcal{K}_\tau$, one has

$$\lim_{n\to\infty} \|M_n v - v_0\|_2 = 0.$$

Put $P_\rho v = v_0$. The inequality

$$\|M_n v - T_\rho M_n v\|_2 = \|(I - T_\rho)M_n v\|_2 < \frac{1}{n}(1+c)\|v\|_2$$

implies that $P_\rho = T_\rho P_\rho$ that is equal to $P_\rho T_\rho$. Therefore $P_\rho = M_n P_\rho = P_\rho M_n$ and hence $P_\rho = P_\rho^2$. □

Remark. Under the same assumption above, one may prove that the limit

$$\lim_{r \downarrow r_\rho} (r - r_\rho) R(r)$$

for the resolvent $R(r) = (r - \lambda_\rho)^{-1}$ with $r > r_\rho$ converges to the idempotent P_ρ on $\mathcal{H}_\tau$ under strong operator topology in $B(\mathcal{H}_\tau)$. Hence the equality

$$\lim_{r \downarrow r_\rho} (r - r_\rho) R(r) = \lim_{n \to \infty} \frac{1 + T_\rho + T_\rho^2 + \cdots + T_\rho^{n-1}}{n} \tag{6.3}$$

holds. We will give a proof of the equality (6.3). It is enough to consider the limit $\lim_{n\to\infty} \frac{1}{n} R(r_\rho + \frac{1}{n})$ instead of $\lim_{r\downarrow r_\rho} (r - r_\rho) R(r)$. As in the above proof, there exists $c > 0$ such that $\|T_\rho^k(a)\|_2 \le c\|a\|_2$ for $a \in \mathcal{A}, k \in \mathbf{N}$. Put $R_n = \frac{1}{n} R(r_\rho + \frac{1}{n})$. Since for $y \in \mathcal{A}$

$$R(r_\rho + \frac{1}{n})y = \sum_{k=0}^{\infty} \frac{\lambda_\rho^k(y)}{(r_\rho + \frac{1}{n})^{k+1}}$$

one has

$$\|R(r_\rho + \frac{1}{n})y\|_2 \le \sum_{k=0}^{\infty} \|T_\rho^k(y)\| \frac{r_\rho^k}{(r_\rho + \frac{1}{n})^{k+1}} \le nc\|y\|_2$$

and hence $\|R_n\| \le c$ for $n \in \mathbf{N}$. The identites

$$(I - T_\rho)R_n = R_n(I - T_\rho) = \frac{1}{n}\frac{1}{r_\rho}(R_n - I)$$

hold so that we have

$$\|(I - T_\rho)R_n\| = \|R_n(I - T_\rho)\| \le \frac{1}{n}\frac{1}{r_\rho}(1+c).$$

A similar argument to the proof of Lemma 6.1 works so that for $u \in \mathcal{H}_\tau$ by taking a cluster point u_0 of the sequence $R_n u, n \in \mathbb{N}$ under the weak topology of $\mathcal{H}_\tau$ we have

$$\lim_{n\to\infty} \|R_n u - u_0\|_2 = 0.$$

Put $\widehat{P}_\rho u = u_0$. The inequality $\|R_n u - T_\rho R_n u\|_2 \le \frac{1}{n}\frac{1}{r_\rho}(1+c)\|u\|_2$ implies that $\widehat{P}_\rho = T_\rho \widehat{P}_\rho$ that is equal to $\widehat{P}_\rho T_\rho$. Hence $\widehat{P}_\rho = R_n \widehat{P}_\rho$ and $\widehat{P}_\rho = \widehat{P}_\rho^2$. The equality $\widehat{P}_\rho = T_\rho \widehat{P}_\rho$ implies $\widehat{P}_\rho = M_n \widehat{P}_\rho$ for all $n \in \mathbf{N}$ so that $\widehat{P}_\rho = P_\rho \widehat{P}_\rho$. Similarly the equalities $P_\rho = T_\rho P_\rho$ and $R_n = \sum_{k=0}^{\infty} T_\rho^k \frac{r_\rho^k}{(r_\rho + \frac{1}{n})^{k+1}}$ imply $P_\rho = R_n P_\rho$ for all $n \in \mathbf{N}$ so that $P_\rho = \widehat{P}_\rho P_\rho$. As $P_\rho \widehat{P}_\rho = \widehat{P}_\rho P_\rho$, one has $P_\rho = \widehat{P}_\rho$.

We denote by $\|a\|_1$ the L^1-norm $\tau(|a|)$ of $a \in \mathcal{A}$, and by $L^1(\mathcal{A},\tau)$ the completion of $\mathcal{A}$ by the norm $\|\cdot\|_1$. The positive operators $\lambda_\rho, T_\rho : \mathcal{A} \longrightarrow \mathcal{A}$ and the state $\tau : \mathcal{A} \longrightarrow \mathbf{C}$ extend to $L^1(\mathcal{A},\tau)$ in natural way, that are also denoted by λ_ρ, T_ρ and τ respectively.

Lemma 6.2. *Suppose that $(\mathcal{A},\rho,\Sigma)$ is uniquely ergodic and power-bounded. Then for $a \in \mathcal{A}$ the limit $\lim_{n\to\infty} M_n(a)$ converges in $L^1(\mathcal{A},\tau)$ under $\|\cdot\|_1$-topology. In particular $\lim_{n\to\infty} M_n(1) = x_\rho$ exists in $L^1(\mathcal{A},\tau)$ and satisfies the equalities*

$$\tau(x_\rho) = 1 \quad \text{and} \quad \lim_{n\to\infty} M_n(a) = \tau(a)x_\rho \qquad \text{for } a \in \mathcal{A}. \tag{6.4}$$

Proof. Since $(\mathcal{A},\rho,\Sigma)$ is irreducible and power-bounded, $\lim_{n\to\infty} M_n(a)$ for $a \in \mathcal{A}$ converges in $\mathcal{H}_\tau = L^2(\mathcal{A},\tau)$ under $\|\cdot\|_2$-norm by the previous lemma. By the inequality

$$\|M_n(a) - M_m(a)\|_1 \le \|M_n(a) - M_m(a)\|_2, \qquad a \in \mathcal{A}$$

the limit $\lim_{n\to\infty} M_n(a)$ exists in $L^1(\mathcal{A},\tau)$ under $\|\cdot\|_1$-norm. We denote it by $\Phi_1(a)$. Hence $x_\rho = \Phi_1(1)$. We will show that $\tau(f(\Phi_1(a) - \tau(a)x_\rho)) = 0$ for $f \in \mathcal{A}$. It suffices to show that $\tau(b\Phi_1(a)b^*) = \tau(a)\tau(bx_\rho b^*)$ for $b \in \mathcal{A}$. One may assume that $a \ge 0$. The inequality $a \le \|a\|1$ and hence $M_n(a) \le \|a\|M_n(1)$ implies $b\Phi_1(a)b^* \le \|a\|bx_\rho b^*$ so that we have $0 \le \tau(b\Phi_1(a)b^*) \le \|a\|\tau(bx_\rho b^*)$. Hence $\tau(bx_\rho b^*) = 0$ implies $\tau(b\Phi_1(a)b^*) = 0$. We may assume that $\tau(bx_\rho b^*) \ne 0$. Put $\omega(a) = \frac{\tau(b\Phi_1(a)b^*)}{\tau(bx_\rho b^*)}, a \in \mathcal{A}$. As $\Phi_1 \circ T_\rho(a) = \Phi_1(a)$, one sees that ω is an invariant state on $\mathcal{A}$. Hence we have $\omega = \tau$ by the unique ergodicity of $(\mathcal{A},\rho,\Sigma)$. Therefore we have $\tau(b\Phi_1(a)b^*) = \tau(a)\tau(bx_\rho b^*)$ for $b \in \mathcal{A}$.
The equality $\tau(x_\rho) = 1$ is clear. □

Lemma 6.3. *Keep the above assumptions and notations. The limit $\lim_{n\to\infty} M_n(f)$ for $f \in L^1(\mathcal{A},\tau)$ converges in $L^1(\mathcal{A},\tau)$ under $\|\cdot\|_1$-topology and satisfies the equality*

$$\lim_{n\to\infty} M_n(f) = \tau(f)x_\rho \qquad \text{for } f \in L^1(\mathcal{A},\tau).$$

Proof. Since for $f \in L^1(\mathcal{A},\tau)$ the inequality $|\lambda_\rho(f)| \le \lambda_\rho(|f|)$ holds, one has $|T_\rho(f)| \le T_\rho(|f|)$ and hence $\|M_n(f)\|_1 \le \|f\|_1$. Take $a_k \in \mathcal{A}$ such as $\|f - a_k\|_1 \to 0$ as $k \to \infty$. It then follows that

$$\begin{aligned}
&\|M_n(f) - \tau(f)x_\rho\|_1 \\
\le&\|M_n(f) - M_n(a_k)\|_1 + \|M_n(a_k) - \tau(a_k)x_\rho\|_1 + \|\tau(a_k)x_\rho - \tau(f)x_\rho\|_1 \\
\le&\|f - a_k\|_1 + \|M_n(a_k) - \tau(a_k)x_\rho\|_1 + |\tau(a_k) - \tau(f)|\|x_\rho\|_1,
\end{aligned}$$

and hence $\lim_{n\to\infty} \|M_n(f) - \tau(f)x_\rho\|_1 = 0$ by the preceding lemma. □

Proposition 6.4. *Keep the above assumptions and notations. If $f \in L^1(\mathcal{A},\tau)$ satisfies $T_\rho(f) = f$ and $\tau(f) = 1$, Then $f = x_\rho$. Namely the space of the fixed elements in $L^1(\mathcal{A},\tau)$ under T_ρ is one-dimensional.*

Proof. By the preceding lemma, we have for $f \in L^1(\mathcal{A}, \tau)$ $\lim_{n\to\infty} M_n(f) = \tau(f)x_\rho$ in $\|\cdot\|_1$-topology. By the condition $T_\rho(f) = f$, we have $M_n(f) = f$ with $\tau(f) = 1$ and hence $f = x_\rho$. ☐

Let us define the space $L^1(\mathcal{D}_\rho, \tau)$ in a similar way to $L^1(\mathcal{A}, \tau)$. The operators $\lambda_\rho, T_\rho : \mathcal{D}_\rho \longrightarrow \mathcal{D}_\rho$ and the state $\tau : \mathcal{D}_\rho \longrightarrow \mathbf{C}$ naturally act on $L^1(\mathcal{D}_\rho, \tau)$. The inclusion relation $\mathcal{A} \subset \mathcal{D}_\rho$ induces the inclusion relation $L^1(\mathcal{A}, \tau) \subset L^1(\mathcal{D}_\rho, \tau)$.

Lemma 6.5. *Keep the above assumptions and notations. Let x be an element of $L^1(\mathcal{D}_\rho, \tau)$ such that $T_\rho(x) = x$. Then x belongs to $L^1(\mathcal{A}, \tau)$.*

Proof. Take $x_n \in {\mathcal{D}_\rho}^{\mathrm{alg}}$ such that $\|x_n - x\|_1 \to 0$ as $n \to \infty$. As $|\lambda_\rho(y)| \le \lambda_\rho(|y|), y \in \mathcal{D}_\rho$, it then follows that

$$\|\lambda_\rho(x_n) - \lambda_\rho(x)\|_1 = \tau(|\lambda_\rho(x_n - x)|) \le \tau(\lambda_\rho(|x_n - x|) = r_\rho \|x_n - x\|_1$$

so that $\|T_\rho(x_n) - T_\rho(x)\|_1 \le \|x_n - x\|_1$. The element x is fixed by T_ρ so that

$$\|T_\rho^k(x_n) - x\|_1 \le \|x_n - x\|_1, \qquad n \in \mathbf{N}, \quad k \in \mathbf{N}.$$

Since $x_n \in {\mathcal{D}_\rho}^{\mathrm{alg}}$, there exists $k_n \in \mathbf{N}$ such that $T_\rho^{k_n}(x_n) \in \mathcal{A}$. Hence x belongs to $L^1(\mathcal{A}, \tau)$. ☐

Definition. A state μ on $\mathcal{D}_\rho$ is called a ϕ_ρ-*invariant measure* if it satisfies

$$\mu(y) = \mu(\phi_\rho(y)), \qquad y \in \mathcal{D}_\rho.$$

If the probability measure for a state μ on $\mathcal{D}_\rho$ is absolutely continuous with respect to the probability measure for the state τ on $\mathcal{D}_\rho$, we write it as $\mu \ll \tau$.

Proposition 6.6. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and uniquely ergodic. For a fixed positive element $x \in L^1(\mathcal{A}, \tau)$ by T_ρ satisfying $\tau(x) = 1$, the state μ_x on $\mathcal{D}_\rho$ defined by*

$$\mu_x(y) = \tau(yx), \qquad y \in \mathcal{D}_\rho$$

is a ϕ_ρ-invariant measure on $\mathcal{D}_\rho$ such that $\mu \ll \tau$. Conversely, for any ϕ_ρ-invariant measure μ on $\mathcal{D}_\rho$ such that $\mu \ll \tau$, there exists a fixed positive element $x_\mu \in L^1(\mathcal{A}, \tau)$ by T_ρ satisfying $\tau(x_\mu) = 1$ such that

$$\mu(y) = \tau(yx_\mu), \qquad y \in \mathcal{D}_\rho.$$

Proof. Let $x \in L^1(\mathcal{A}, \tau)$ be a fixed positive element by T_ρ satisfying $\tau(x) = 1$. As $\lambda_\rho(x) = r_\rho x$, it follows that from Lemma 4.7

$$\mu_x(\phi_\rho(y)) = \frac{1}{r_\rho}\tau(\lambda_\rho(\phi_\rho(y)x)) = \frac{1}{r_\rho}\tau(y\lambda_\rho(x)) = \mu_x(y), \qquad y \in \mathcal{D}_\rho$$

so that the state μ_x is a ϕ_ρ-invariant measure on $\mathcal{D}_\rho$ such that $\mu_x \ll \tau$. Conversely for a ϕ_ρ-invariant measure μ on $\mathcal{D}_\rho$ such that $\mu \ll \tau$, there exists a Radon-Nikodym derivative $x_\mu \in L^1(\mathcal{D}_\rho, \tau)$ such that $x_\mu \ge 0, \tau(x_\mu) = 1$ and

$$\mu(y) = \tau(yx_\mu), \qquad y \in \mathcal{D}_\rho.$$

By the equality $\tau(\phi_\rho(y)x_\mu) = \tau(yT_\rho(x_\mu)), y \in \mathcal{D}_\rho$, one sees that $\tau(yx_\mu) = \tau(yT_\rho(x_\mu))$, $y \in \mathcal{D}_\rho$ so that $T_\rho(x_\mu) = x_\mu$, $\tau - a.e.$ Hence x_μ is regarded as an element of $L^1(\mathcal{A}, \tau)$ by the preceding lemma. This completes the proof. $\square$

Especially the measure μ_ρ defined by $\mu_\rho(y) = \tau(yx_\rho),\ y \in \mathcal{D}_\rho$ is a ϕ_ρ-invariant measure on $\mathcal{D}_\rho$ such that $\mu_\rho \ll \tau$.
Therefore we have

Theorem 6.7. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible, uniquely ergodic and power-bounded. Then a ϕ_ρ-invariant measure on $\mathcal{D}_\rho$ absolutely continuous with respect to τ is unique and is of the form*

$$\mu_\rho(y) = \tau(yx_\rho), \qquad y \in \mathcal{D}_\rho. \tag{6.5}$$

The measure μ_ρ is faithful, and ergodic in the sense that the formula

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu_\rho(\phi_\rho^k(y)x) = \mu_\rho(y)\mu_\rho(x), \qquad x, y \in \mathcal{D}_\rho$$

holds.

Proof. Let μ be a ϕ_ρ-invariant measure on $\mathcal{D}_\rho$. By the preceding proposition there exists a fixed positive element $x_\mu \in L^1(\mathcal{A}, \tau)$ under T_ρ satisfying $\tau(x_\mu) = 1$ such that

$$\mu(y) = \tau(yx_\mu), \qquad y \in \mathcal{D}_\rho.$$

By Proposition 6.4 we have $x_\mu = x_\rho$. For $x, y \in \mathcal{D}_\rho$, the equality

$$\lambda_\rho^k(\phi_\rho^k(y)xx_\rho) = y\lambda_\rho^k(xx_\rho)$$

holds by Lemma 4.7 so that

$$\begin{aligned}
\frac{1}{n}\sum_{k=0}^{n-1} \mu_\rho(\phi_\rho^k(y)x) &= \frac{1}{n}\sum_{k=0}^{n-1} \tau(\phi_\rho^k(y)xx_\rho) \\
&= \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{r_\rho^k}\tau(\lambda_\rho^k(\phi_\rho^k(y)xx_\rho)) \\
&= \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{r_\rho^k}\tau(y\lambda_\rho^k(xx_\rho)) \\
&= \tau(yM_n(xx_\rho)).
\end{aligned}$$

Since

$$\|\cdot\|_1 - \lim_{n\to\infty} M_n(xx_\rho) = \tau(xx_\rho)x_\rho = \mu_\rho(x)x_\rho,$$

we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu_\rho(\phi_\rho^k(y)x) = \tau(y\mu_\rho(x)x_\rho)) = \mu_\rho(y)\mu_\rho(x).$$

$\square$

Corollary 6.8. *Assume that $(\mathcal{A}, \rho, \Sigma)$ is irreducible and mean ergodic.*

(i) *The unique ϕ_ρ-invariant probability measure absolutely continuous with respect to τ is obtained by $\mu_\rho(y) = \tau(yx_\rho), y \in \mathcal{D}_\rho$, where τ is the restriction of the unique KMS state on $\mathcal{O}_\rho$ and x_ρ is a positive element of $\mathcal{A}$ defined by the limit of the mean* $\lim_{n\to\infty} \frac{1}{n}(1 + T_\rho(1) + \cdots + T_\rho^{n-1}(1))$.

(ii) *The state μ_ρ is equivalent to the state τ as a measure on $\mathcal{D}_\rho$.*

Proof. (i) Under the assumption that $(\mathcal{A}, \rho, \Sigma)$ is irreducible. Mean ergodicity implies unique ergodicity and (FP), which implies power-boundedness. Therefore the assertion is immediate.
(ii) By the mean ergodicity, the fixed element x_ρ belongs to $\mathcal{A}$ and is strictly positive by Lemma 3.5 (ii). Hence we have $\tau(y) = \mu_\rho(yx_\rho^{-1}), y \in \mathcal{D}_\rho$ so that $\tau \ll \mu_\rho$. □

7. EXAMPLES

We will present examples of continuous linear functionals satisfying KMS conditions on some C^*-symbolic dynamical systems.

1. FINITE DIRECTED GRAPHS

Let $A = [A(i,j)]_{i,j=1,\dots,N}$ be an $N \times N$ matrix with entries in nonnegative integers. Denote by $G_A = (V_A, E_A)$ the associated finite directed graph with vertex set $V = \{v_1, \dots, v_N\}$ and edge set E_A. Let $\mathcal{O}_{A^{[2]}}$ be the Cuntz-Krieger algebra such that the generating partial isometries $S_e, e \in E_A$ indexed by the edges in G_A satisfy

$$\sum_{f\in E_A} S_f S_f^* = 1, \qquad S_e^* S_e = \sum_{f \in E_A} A^{[2]}(e,f) S_f S_f^*, \qquad e \in E_A,$$

where $A^{[2]}(e,f)$ is defined to be one if the edge f follows the edge e, otherwise zero. Put $\mathcal{A}_{G_A}$ the C^*-subalgebra of $\mathcal{O}_{A^{[2]}}$ generated by the projections $S_e^* S_e, e \in E_A$. Denote by ρ_e^A for $e \in E_A$ the endomorphism $\mathcal{A}_{G_A}$ defined by $\rho_e^A(a) = S_e^* a S_e, a \in \mathcal{A}_{G_A}$. Consider the C^*-symbolic dynamical system $(\mathcal{A}_{G_A}, \rho^A, E_A)$. Its associated C^*-algebra $\mathcal{O}_{\rho^A}$ is nothing but the Cuntz-Krieger algebra $\mathcal{O}_{A^{[2]}}$. The finite directed graphs G_A is naturally considered to be a finite labeled graph by regarding an edge itself as its label. Hence this example will be contained in the following examples.

2. FINITE LABELED GRAPHS

Let $\mathcal{G} = (G, \lambda)$ be a left-resolving finite labeled graph over Σ with underlying finite directed graph $G = (V, E)$ and labeling map $\lambda : E \to \Sigma$. Suppose that the graph G is irreducible. Let $\{v_1, \dots, v_N\}$ be the vertex set V. As in Section 2, we have a C^*-symbolic dynamical system $(\mathcal{A}_\mathcal{G}, \rho^\mathcal{G}, \Sigma)$ such that $\mathcal{A}_\mathcal{G} = \mathbf{C}E_1 \oplus \cdots \oplus \mathbf{C}E_N$ and $\rho_\alpha^\mathcal{G}(E_i) = \sum_{j=1}^N A^\mathcal{G}(i,\alpha,j)E_j$ for $i = 1, \dots, N, \alpha \in \Sigma$, where the $N \times N$-matrix $[A^\mathcal{G}(i,\alpha,j)]_{i,j=1,\dots,N}$ for $\alpha \in \Sigma$ is defined by (2.1). Put $A_\mathcal{G}(i,j) = \sum_{\alpha\in\Sigma} A^\mathcal{G}(i,\alpha,j)$ for $i,j = 1, \dots, N$. Then the matrix $A_\mathcal{G} = [A_\mathcal{G}(i,j)]_{i,j=1}^N$ is irreducible. Let $r_\mathcal{G}$ denote the Perron-Frobenius eigenvalue of the matrix $A_\mathcal{G}$. It is easy to see that $r_\mathcal{G}$ is equal to the spectral radius $r_{\rho^\mathcal{G}}$ of the positive operator $\lambda_{\rho^{\mathcal{G}}}(x) = \sum_{\alpha\in\Sigma}\rho^{\mathcal{G}}_\alpha(x), x \in \mathcal{A}_{\mathcal{G}}$. As

$$\lambda_{\rho^{\mathcal{G}}}(E_i) = \sum_{j=1}^{N} A_{\mathcal{G}}(i,j)E_j, \qquad i = 1,\dots,N,$$

by identifying $x = \sum_{i=1}^N x_i E_i \in \mathcal{A}_{\mathcal{G}}$ with the vector $[x_i]_{i=1}^N \in \mathbf{C}^N$, one may regard the operator $\lambda_{\rho^{\mathcal{G}}}$ as the transposed matrix $A^t_{\mathcal{G}}$ of $A_{\mathcal{G}}$. For a complex number $\beta \in \mathbf{C}$ with $|\beta| > 1$, let $\varphi \in \mathcal{A}^*_{\mathcal{G}}$ be a continuous linear functional belonging to $\mathcal{E}_\beta(\rho^{\mathcal{G}})$. The equality $\varphi \circ \lambda_{\rho^{\mathcal{G}}}(E_i) = \beta\varphi(E_i)$ implies

$$\sum_{j=1}^{N} A_{\mathcal{G}}(i,j)\varphi(E_j) = \beta\varphi(E_i), \qquad i = 1,\dots,N$$

so that the vector $[\varphi(E_j)]_{j=1}^N$ is an eigenvector of $A_{\mathcal{G}}$ for eigenvalue β. Conversely an eigenvector $[u_i]_{i=1}^N \in \mathbf{C}$ of the matrix $A_{\mathcal{G}}$ for an eigenvalue β gives rise to a continuous linear functional φ on $\mathcal{A}_{\mathcal{G}}$ by setting $\varphi(E_i) = u_i, i = 1,\dots,N$ so that $\varphi \in \mathcal{E}_\beta(\rho^{\mathcal{G}})$. Hence the space $\mathcal{E}_\beta(\rho^{\mathcal{G}})$ is identified with the eigenvector space of the matrix $A_{\mathcal{G}}$ for eigenvalue β. Especially a faithful invariant state τ on $\mathcal{A}_{\mathcal{G}}$ is the positive normalized eigenvector of $A_{\mathcal{G}}$ for eigenvalue $r_{\mathcal{G}}$. Similarly an element $x = \sum_{j=1}^N x_j E_j \in \mathcal{A}_{\mathcal{G}}$ is fixed by $T_{\rho^{\mathcal{G}}}$ if and only if the vector $[x_j]_{j=1}^N$ is an eigenvector of $A^t_{\mathcal{G}}$ for the eigenvalue $r_{\mathcal{G}}$. The ordinary Perron-Frobenius theorem for nonnegative matrices asserts that $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$ is mean ergodic if $A_{\mathcal{G}}$ is irreducible. The following proposition comes from the ordinary Perron-Frobenius theorem for irreducible nonnegative matrices, which is a special case of Theorem 3.13, and Corollary 6.8.

Proposition 7.1. *Suppose that the adjacency matrix $A_{\mathcal{G}} = [A_{\mathcal{G}}(i,j)]_{i,j=1}^N$ is irreducible. Let $[\tau_i]_{i=1}^N$ and $[x_i]_{i=1}^N$ be right and left Perron eigenvector of $A_{\mathcal{G}}$ respectively, that is,*

$$A_{\mathcal{G}}[\tau_i]_{i=1}^N = r_{\mathcal{G}}[\tau_i]_{i=1}^N, \qquad A^t_{\mathcal{G}}[x_i]_{i=1}^N = r_{\mathcal{G}}[x_i]_{i=1}^N,$$

such that $\sum_{i=1}^N \tau_i = 1$ and $\sum_{i=1}^N \tau_i x_i = 1$. Put $x_{\rho^{\mathcal{G}}} = \sum_{i=1}^N x_i E_i \in \mathcal{A}_{\mathcal{G}}$ and $\tau(a) = \sum_{i=1}^N \tau_i a_i$ for $a = \sum_{i=1}^N a_i E_i \in \mathcal{A}_{\mathcal{G}}$. Then τ is a unique faithful invariant state on $\mathcal{A}_{\mathcal{G}}$ such that the following equalities hold:

$$\lim_{n\to\infty} M_n(a) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} T^k_{\rho^{\mathcal{G}}}(a) = \tau(a)x_{\rho^{\mathcal{G}}}.$$

Furthermore the measure $\mu_{\rho^{\mathcal{G}}}$ on $\mathcal{D}_{\rho^{\mathcal{G}}}$ defined $\mu_{\rho^{\mathcal{G}}}(y) = \tau(yx_{\rho^{\mathcal{G}}})$ for $y \in \mathcal{D}_{\rho^{\mathcal{G}}}$ is a unique $\phi_{\rho^{\mathcal{G}}}$-invariant measure equivalent to the measure τ on $\mathcal{D}_{\rho^{\mathcal{G}}}$.

Remark. Let $X_{\mathcal{G}}$ be the right one-sided sofic shift presented by $\mathcal{G}$. The commutative C^*-algebra $C(X_{\mathcal{G}})$ on $X_{\mathcal{G}}$ is naturally regarded as a C^*-subalgebra of $\mathcal{D}_{\rho^{\mathcal{G}}}$ through the correspondence

$$\chi_\nu \in C(X_{\mathcal{G}}) \longrightarrow S_\nu S_\nu^* \in \mathcal{D}_{\rho^{\mathcal{G}}}, \qquad \nu \in B_k(\Lambda_{\mathcal{G}})$$

where χ_ν is the characteristic function for the cylinder

$$U_\nu = \{(x_i)_{i\in\mathbf{N}} \in X_{\mathcal{G}} \mid x_1 = \nu_1, \dots, \nu_k = x_k\}.$$

The restriction of the $\phi_{\rho^{\mathcal{G}}}$-invariant measure $\mu_{\rho^{\mathcal{G}}}$ on $\mathcal{D}_{\rho^{\mathcal{G}}}$ to the subalgebra $C(X_{\mathcal{G}})$ is nothing but a shift-invariant measure on $X_{\mathcal{G}}$ (cf. [21]).
We will next find continuous linear functionals on $\mathcal{O}_{\rho^{\mathcal{G}}}$ satisfying KMS conditions in concrete way. Now suppose that the irreducible matrix $A_{\mathcal{G}}$ has its period $p_{\mathcal{G}}$ and put

$$N_{\mathcal{G}}(i,j) = \{n \in \mathbf{Z}_+ \mid A_{\mathcal{G}}^n(i,j) > 0\}.$$

It is well-known that for $n, m \in N_{\mathcal{G}}(i,j)$ one has $n \equiv m \pmod{p_{\mathcal{G}}}$. Then for an eigenvalue $\beta \in \mathbf{C}$ of $A_{\mathcal{G}}$ with $|\beta| = r_{\mathcal{G}}$, $\frac{\beta}{r_{\mathcal{G}}}$ is a $p_{\mathcal{G}}$-th root of unity. We fix a vertex v_1 and for $k \in \{1, 2, \dots, N\}$ take $n_k \in N_{\mathcal{G}}(1,k)$. We set

$$u_k = (\frac{\beta}{r_{\mathcal{G}}})^{n_k} \tau(E_k).$$

Then u_k does not depend on the choice of n_k as long as $n_k \in N_{\mathcal{G}}(1,k)$.

LEMMA 7.2. $\sum_{j=1}^N A_{\mathcal{G}}(i,j) u_j = \beta u_i, \quad i = 1, \dots, N.$

Proof. If $A_{\mathcal{G}}(i,j) \neq 0$, one sees $n_i + 1 \in N(1,j)$ so that

$$A_{\mathcal{G}}(i,j) u_j = \frac{\beta}{r_{\mathcal{G}}} (\frac{\beta}{r_{\mathcal{G}}})^{n_i} A_{\mathcal{G}}(i,j) \tau(E_j) = \frac{\beta}{r_{\mathcal{G}}} \frac{u_i}{\tau(E_i)} A_{\mathcal{G}}(i,j) \tau(E_j).$$

It follows that

$$\sum_{j=1}^N A_{\mathcal{G}}(i,j) u_j = \frac{\beta}{r_{\mathcal{G}}} \frac{u_i}{\tau(E_i)} \sum_{j=1}^N A_{\mathcal{G}}(i,j) \tau(E_j) = \frac{\beta}{r_{\mathcal{G}}} \frac{u_i}{\tau(E_i)} r_{\mathcal{G}} \tau(E_i) = \beta u_i.$$

□

Hence $u = [u_k]_{k=1}^N$ yields a nonzero eigenvector of $A_{\mathcal{G}}$. Define a nonzero continuous linear functional φ on $\mathcal{A}_{\mathcal{G}}$ by setting

$$\varphi(E_k) = u_k, \qquad k = 1, \dots, N$$

so that the equality $\varphi \circ \lambda_{\mathcal{G}} = \beta \varphi$ on $\mathcal{A}_{\mathcal{G}}$ holds. Put $v_\varphi = \sum_{i=1}^N \frac{u_i}{\tau(E_i)} E_i \in \mathcal{A}_{\mathcal{G}}$. It is easy to see that v_φ is a partial isometry such that $\varphi(E_j) = \tau(E_j v_\varphi), j = 1, \dots, N$ so that

$$\varphi(x) = \tau(x v_\varphi), \qquad x \in \mathcal{A}_{\mathcal{G}}$$

holds. Therefore we have the following proposition.

PROPOSITION 7.3. *Let $\mathcal{G} = (G, \lambda)$ be a left-resolving finite labeled graph with underlying finite directed graph $G = (V, E)$ and labeling map $\lambda : E \to \Sigma$. Denote by $\{v_1, \dots, v_N\}$ the vertex set V. Assume that G is irreducible. Consider the N-dimensional commutative C^*-algebra $\mathcal{A}_{\mathcal{G}} = \mathbf{C}E_1 \oplus \cdots \oplus \mathbf{C}E_N$ where each minimal projection E_i corresponds to the vertex v_i for $i = 1, \dots, N$. Define an*

$N\times N$- nonnegative matrix $A_{\mathcal{G}} = [A_{\mathcal{G}}(i,j)]_{i,j=1}^{N}$ by $A_{\mathcal{G}}(i,j) = \sum_{\alpha\in\Sigma} A^{\mathcal{G}}(i,\alpha,j)$ where for $\alpha\in\Sigma$ and $i,j=1,\dots,N$

$$A^{\mathcal{G}}(i,\alpha,j) = \begin{cases} 1 & \text{if there exists an edge } e \text{ from } v_i \text{ to } v_j \text{ with } \lambda(e)=\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{O}_{A_{\mathcal{G}}}$ be the associated Cuntz-Krieger algebra and τ be the unique KMS state on $\mathcal{O}_{A_{\mathcal{G}}}$ for gauge action. Let $\beta\in\mathbf{C}$ be an eigenvalue of $A_{\mathcal{G}}$ such that $|\beta| = r_{\mathcal{G}}$ the Perron-Frobenius eigenvalue of the matrix $A_{\mathcal{G}}$. Then a continuous linear functional on $\mathcal{O}_{A_{\mathcal{G}}}$ satisfying KMS condition at $\mathrm{Log}\beta$ is a scalar multiple of $\varphi\in\mathcal{O}_{A_{\mathcal{G}}}^{*}$ giving by for $k=1,\dots,N$

$$\varphi(E_k) = (\frac{\beta}{r_{\mathcal{G}}})^{n_k}\tau(E_k) \qquad \text{where } n_k \text{ satisfies } A_{\mathcal{G}}^{n_k}(1,k)\neq 0.$$

Consider a finite labeled graph $\mathcal{G}$ whose adjacency matrix A is

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

As

$$A^3 = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

the period of the matrix is 3. The characteristic polynomial of A is $\det(t-A) = t^2(t^3-4)$ so that $Sp(A) = \{\sqrt[3]{4}, \sqrt[3]{4}e^{\frac{2\pi}{3}i}, \sqrt[3]{4}e^{\frac{4\pi}{3}i}, 0\}$ and $r_A = \sqrt[3]{4}$. Hence $\beta\in Sp(A)$ satisfying $|\beta| = \sqrt[3]{4}$ are

$$\sqrt[3]{4}, \quad \sqrt[3]{4}e^{\frac{2\pi}{3}i}, \quad \sqrt[3]{4}e^{\frac{4\pi}{3}i}.$$

Therefore the Cuntz-Krieger algebra $\mathcal{O}_A$ has three continuous linear functionals satisfying KMS conditions for gauge action at inverse temperatures

$$\frac{1}{3}\log 4, \quad \frac{1}{3}\log 4 + \frac{2\pi}{3}i, \quad \frac{1}{3}\log 4 + \frac{4\pi}{3}i$$

respectively.

3. Dyck shifts

We consider the Dyck shift D_N for a fixed natural number $N>1$ with alphabet $\Sigma = \Sigma^- \cup \Sigma^+$ where $\Sigma^- = \{\alpha_1,\dots,\alpha_N\}, \Sigma^+ = \{\beta_1,\dots,\beta_N\}$. The symbols α_i, β_i correspond to the brackets $(_i,)_i$ respectively. The Dyck inverse monoid has the relations

$$\alpha_i\beta_j = \begin{cases} \mathbf{1} & \text{if } i=j, \\ 0 & \text{otherwise} \end{cases} \tag{7.1}$$

for $i,j = 1,\dots,N$ (cf. [23],[26]). A word $\omega_1\cdots\omega_n$ of Σ is admissible for D_N precisely if $\prod_{m=1}^n \omega_m \neq 0$. For a word $\omega = \omega_1\cdots\omega_n$ of Σ, we denote by $\tilde{\omega}$ its reduced form. Namely $\tilde{\omega}$ is a word of $\Sigma \cup \{0, \mathbf{1}\}$ obtained after the operations (7.1). Hence a word ω of Σ is forbidden for D_N if and only if $\tilde{\omega} = 0$.
In [26], an irreducible λ-graph system presenting D_N called the Cantor horizon λ-graph system has been introduced. It is a minimal irreducible component of the canonical λ-graph system $\mathfrak{L}^{C(D_N)}$ and written as $\mathfrak{L}^{Ch(D_N)}$. Let us describe the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_N)}$ of D_N. Let Σ_N be the full N-shift $\{1,\dots,N\}^{\mathbf{Z}}$. We denote by $B_l(D_N)$ and $B_l(\Sigma_N)$ the set of admissible words of length l of D_N and that of Σ_N respectively. The vertices V_l of $\mathfrak{L}^{Ch(D_N)}$ at level l are given by the words of length l consisting of the symbols of Σ^+. That is,

$$V_l = \{(\beta_{\mu_1}\cdots\beta_{\mu_l}) \in B_l(D_N) \mid \mu_1\cdots\mu_l \in B_l(\Sigma_N)\}.$$

Hence the cardinal number of V_l is N^l. The mapping $\iota(=\iota_{l,l+1}) : V_{l+1} \to V_l$ deletes the rightmost symbol of a word in $B_l(\Sigma_N)$ such as

$$\iota((\beta_{\mu_1}\cdots\beta_{\mu_{l+1}})) = (\beta_{\mu_1}\cdots\beta_{\mu_l}), \qquad (\beta_{\mu_1}\cdots\beta_{\mu_{l+1}}) \in V_{l+1}.$$

There exists an edge labeled α_j from $(\beta_{\mu_1}\cdots\beta_{\mu_l}) \in V_l$ to $(\beta_{\mu_0}\beta_{\mu_1}\cdots\beta_{\mu_l}) \in V_{l+1}$ precisely if $\mu_0 = j$, and there exists an edge labeled β_j from $(\beta_j\beta_{\mu_1}\cdots\beta_{\mu_{l-1}}) \in V_l$ to $(\beta_{\mu_1}\cdots\beta_{\mu_{l+1}}) \in V_{l+1}$. The resulting labeled Bratteli diagram with ι-map becomes a λ-graph system over Σ, denoted by $\mathfrak{L}^{Ch(D_N)}$, that presents the Dyck shift D_N ([26]). It gives rise to a purely infinite simple C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ ([32]) such that

$$K_0(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong \mathbf{Z}/N\mathbf{Z} \oplus C(\mathfrak{K}, \mathbf{Z}), \qquad K_1(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong 0.$$

Let us denote by $(\mathcal{A}^{D_N}, \rho^{D_N}, \Sigma)$ the C^*-symbolic dynamical system associated to the λ-graph system $\mathfrak{L}^{Ch(D_N)}$ as in Section 2. Since the vertex set V_l is indexed by the set $B_l(\Sigma_N)$ of words, the family of projections denoted by $E_{\mu_1\dots\mu_l}$ for $\mu_1\cdots\mu_l \in B_l(\Sigma_N)$ in the C^*-algebra $\mathcal{A}^{D_N}$ forms the minimal projectins of $\mathcal{A}_l = C(V_l)$ such that

$$\sum_{\mu_1\cdots\mu_l \in B_l(\Sigma_N)} E_{\mu_1\dots\mu_l} = 1, \qquad E_{\mu_1\dots\mu_l} = \sum_{\mu_{l+1}=1}^{N} E_{\mu_1\dots\mu_{l+1}}.$$

As the algebra $\mathcal{A}_l$ is embedded into $\mathcal{A}_{l+1}$, the C^*-algebra $\mathcal{A}^{D_N}$ is a commutative AF-algebra generated by the subalgebras $\mathcal{A}_l, l \in \mathbf{N}$. The endomorphisms $\rho^{D_N}_\gamma : \mathcal{A}^{D_N} \longrightarrow \mathcal{A}^{D_N}$ for $\gamma \in \Sigma$ are defined by

$$\rho^{D_N}_{\alpha_j}(E_{\mu_1\dots\mu_l}) = E_{j\mu_1\dots\mu_l}, \qquad \rho^{D_N}_{\beta_j}(E_{j\mu_1\dots\mu_{l-1}}) = \sum_{\mu_l,\mu_{l+1}=1}^{N} E_{\mu_1\dots\mu_{l+1}}$$

for $\mu_1 \dots \mu_l \in B_l(\Sigma_N)$ and $j = 1, \dots, N$. It then follows that

$$
\begin{aligned}
\lambda_{\rho^{D_N}}(1) &= \sum_{j=1}^{N} \rho^{D_N}_{\alpha_j}(1) + \sum_{j=1}^{N} \rho^{D_N}_{\beta_j}(1) \\
&= \sum_{j=1}^{N} \sum_{\mu_1 \cdots \mu_l \in B_l(\Sigma_N)} E_{j\mu_1 \dots \mu_l} + \sum_{j=1}^{N} \sum_{\mu_2 \cdots \mu_l \in B_{l-1}(\Sigma_N)} \sum_{\mu_{l+1}, \mu_{l+2}=1}^{N} E_{\mu_2 \dots \mu_{l+2}} \\
&= 1 + N
\end{aligned}
$$

so that we have $\|\lambda_{\rho^{D_N}}\| = \|\lambda_{\rho^{D_N}}(1)\| = 1 + N$. Hence we obtain

$$
r_{\rho^{D_N}} = 1 + N, \qquad T_{\rho^{D_N}}(1) = 1.
$$

This implies that 1 is a fixed element by $T_{\rho^{D_N}}$ and hence $(\mathcal{A}^{D_N}, \rho^{D_N}, \Sigma)$ satisfies (FP). As in [32], $(\mathcal{A}^{D_N}, \rho^{D_N}, \Sigma)$ is irreducible and uniquely ergodic, so that it is mean ergodic. One then sees that there exists a KMS state at inverse temperature $\log \beta$ if and only if $\beta = 1 + N$. The admitted KMS state is unique ([32, Theorem 1.2]).

4. β-shifts

Let $\beta > 1$ be an arbitrary real number. Take a natural number N with $N - 1 < \beta \le N$. Put $\Sigma = \{0, 1, ..., N - 1\}$. For a nonnegative real number t, we denote by $[t]$ the integer part of t. Let $f_\beta : [0,1] \to [0,1]$ be the mapping defined by

$$
f_\beta(x) = \beta x - [\beta x], \qquad x \in [0,1]
$$

that is called the β-transformation ([38], [42]). The β-expansion of $x \in [0,1]$ is a sequence $\{d_i(x,\beta)\}_{i \in \mathbf{N}}$ of integers of Σ determined by

$$
d_i(x,\beta) = [\beta f_\beta^{i-1}(x)], \qquad i \in \mathbf{N}.
$$

By this sequence, we can write x as

$$
x = \sum_{i=1}^{\infty} \frac{d_i(x,\beta)}{\beta^i}.
$$

We endow the infinite product $\Sigma^{\mathbf{N}}$ with the product topology and the lexicographical order. Put $\zeta_\beta = \sup_{x \in [0,1)} (d_i(x,\beta))_{i \in \Sigma^{\mathbf{N}}}$. We define the shift-invariant compact subset X_β of $\Sigma^{\mathbf{N}}$ by

$$
X_\beta = \{\omega \in \Sigma^{\mathbf{N}} | \sigma^i(\omega) \le \zeta_\beta, i = 0, 1, 2, ...\},
$$

where σ denotes the shift $\sigma((\omega_i)_{i \in \mathbf{N}}) = (\omega_{i+1})_{i \in \mathbf{N}}$. The one-sided subshift (X_β, σ) is called the right one-sided β-shift (cf. [38], [42]). Its (two-sided) subshift

$$
\Lambda_\beta = \{(\omega_i)_{i \in \mathbf{Z}} \in \Sigma^{\mathbf{Z}} \mid (\omega_{i-k})_{i \in \mathbf{N}} \in X_\beta, k = 0, 1, 2, \dots\}
$$

is called the β-shift. In [17], the C^*-algbera $\mathcal{O}_\beta$ associated with the β-shift has been introduced and studied. It is simple and purely infinite for every $\beta > 1$ and generated by $N - 1$ isometries $S_0, S_1, \dots, S_{N-2}$ and one partial isometry S_{N-1} having certain operator relations (see [17]). The family $\mathcal{O}_\beta,\ 1 < \beta \in \mathbf{R}$

interpolates the Cuntz algebras $\mathcal{O}_n$, $1 < n \in \mathbf{N}$. Denote by $\mathcal{A}_\beta$ the C^*-subalgebra of $\mathcal{O}_\beta$ genertaed by the family of the projections $S_\mu^* S_\mu, \mu \in B_*(\Lambda_\beta)$. The algebra is commutative and is of infinite dimensional unless Λ_β is sofic, where Λ_β is sofic if and only if the sequence $(d_i(1,\beta))_{i\in\mathbf{N}}$ is ultimately periodic. Define a family $\{\rho_j^\beta\}_{j=0,1,\dots,N-1}$ of endomorphisms on $\mathcal{A}_\beta$ by

$$\rho_j^\beta(x) = S_j^* x S_j, \qquad x \in \mathcal{A}_\beta,\ j = 0,1,\dots,N-1$$

so that we have a C^*-symbolic dynamical system $(\mathcal{A}_\beta, \rho^\beta, \Sigma)$. It is direct to see that the C^*-algebra $\mathcal{O}_{\rho^\beta}$ is canonically isomorphic to the C^*-algebra $\mathcal{O}_\beta$. We set the positive operator λ_β on $\mathcal{A}_\beta$ by

$$\lambda_\beta(x) = \sum_{j=0}^{N-1} \rho_j^\beta(x), \qquad x \in \mathcal{A}_\beta.$$

LEMMA 7.4. *The spectral radius r_β of the positive operator λ_β on $\mathcal{A}_\beta$ is β.*

Proof. Denote by θ_k the cardinal number of the admissible words $B_k(\Lambda_\beta)$ of length k. Then we have

$$\|\lambda_\beta^k\| = \|\lambda_\beta^k(1)\| \le \sum_{\mu\in B_k(\Lambda_\beta)} \|S_\mu^* S_\mu\| = \theta_k.$$

As in [44, p. 179], $\lim_{k\to\infty} \frac{\theta_k}{\beta^k}$ converges to a positive real number so that there exists a positive constant $M > 0$ such that $\frac{\|\lambda_\beta^k\|}{\beta^k} < M$ for all $k \in \mathbf{N}$. Hence $\lim_{k\to\infty} \|\lambda_\beta^k\|^{\frac{1}{k}} \le \beta$ so that $r_\beta \le \beta$. As in [17], there exists a state τ on $\mathcal{A}_\beta$ satisfying $\tau \circ \lambda_\beta = \beta\tau$. This implies $\beta \in Sp(\lambda_\beta)$ so that $r_\beta = \beta$. □

PROPOSITION 7.5. *$(\mathcal{A}_\beta, \rho^\beta, \Sigma)$ is irreducible, uniquely ergodic and power-bounded.*

Proof. It has been proved in [17] that there is no nontrivial ideal of $\mathcal{A}_\beta$ invariant under λ_β and there exists a unique state τ on $\mathcal{A}_\beta$ satisfying $\tau\circ\lambda_\beta = r_\beta\tau$. Hence $(\mathcal{A}_\beta, \rho^\beta, \Sigma)$ is irreducible, uniquely ergodic. As in the proof of the above lemma, there exists a positive constant $M > 0$ such that $\frac{\|\lambda_\beta^k\|}{r_\beta^k} < M$ for all $k \in \mathbf{N}$. This means that $(\mathcal{A}_\beta, \rho^\beta, \Sigma)$ is power-bounded. □

By the above proposition, one knows that $(\mathcal{A}_\beta, \rho^\beta, \Sigma)$ satisfies the hypothesis of Theorem 6.7 so that there uniquely exists a ϕ_{ρ^β}-invariant measure on $\mathcal{D}_{\rho^\beta}$ absolutely continuous with respect to the restriction of the unique KMS-state τ to $\mathcal{D}_{\rho^\beta}$. We note that $C(X_\beta)$ is a C^*-subalgebra of $\mathcal{D}_{\rho^\beta}$ and the restriction of ϕ_{ρ^β} to $C(X_\beta)$ comes from the shift transformation σ. As in [17], the restriction of the KMS-state τ to $\mathcal{D}_{\rho^\beta}$ corresponds to the Lebesgue measure on $[0,1]$ in translating the β-shift to the β-transformation. Hence the uniqueness of the ϕ_{ρ^β}-invariant measure on $\mathcal{D}_{\rho^\beta}$ absolutely continuous with respect to τ exactly corresponds to the uniqueness of the invariant measure on $[0,1]$ under the β-transformation absolutely continuous with respect to the Lebesgue measure studied in [14], [38] and [42]. In fact, the density function h_β appeared in [14], [38] and [42] of the invariant measure for the β-transformation with respect to the Lebesgue measure is the element x_{ρ^β} realized as the mean $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \frac{\lambda_\beta^k(1)}{\beta^k}$ in Theorem 6.7.

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