# On the Maximal Unramified Quotients of $p$-Adic Étale Cohomology Groups and Logarithmic Hodge–Witt Sheaves

Dedicated to Professor K. Kato on his 50th birthday

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Abstract. Let $O_K$ be a complete discrete valuation ring of mixed characteristic $(0,p)$ with perfect residue field. From the semi-stable conjecture $(C_{\mathrm{st}})$ and the theory of slopes, we obtain isomorphisms between the maximal unramified quotients of certain Tate twists of $p$-adic étale cohomology groups and the cohomology groups of logarithmic Hodge-Witt sheaves for a proper semi-stable scheme over $O_K$. The object of this paper is to show that these isomorphisms are compatible with the symbol maps to the $p$-adic vanishing cycles and the logarithmic Hodge-Witt sheaves, and that they are compatible with the integral structures under certain restrictions. We also treats an open case and a proof of $C_{\mathrm{st}}$ in such a case is given for that purpose. The results are used in the work of U. Jannsen and S. Saito in this volume.

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We will study a description of the maximal unramified quotients of certain $p$-adic étale cohomology groups in terms of logarithmic Hodge-Witt sheaves. Let $K$ be a complete discrete valuation field of mixed characteristic $(0,p)$ whose residue field is perfect, let $O_K$ be the ring of integers of $K$, and let $\overline{K}$ be an algebraic closure of $K$. We consider a proper semi-stable scheme $X$ over $O_K$, i.e. a regular scheme $X$ proper and flat over $O_K$ such that the special fiber $Y$ of $X$ is reduced and is a divisor with normal crossings on $X$. For such an $X$, we have a comparison theorem (the semi-stable conjecture by Fontaine-Jannsen) between the $p$-adic étale cohomology $H^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ and the logarithmic crystalline cohomology of $X$ with some additional structures (Theorem 3.2.2). Combining this with the description of the maximal slope part of the logarithmic crystalline cohomology in terms of logarithmic Hodge-Witt sheaves (see §2.3 for details), we obtain canonical isomorphisms (=(3.2.6), (3.2.7)):

$$(0.1)\qquad \begin{aligned} &H^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))_I \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^0_{\text{ét}}(\overline{Y}, W\omega^r_{\overline{Y}/\overline{s},\log}) \\ &H^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(d))_I \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^{r-d}_{\text{ét}}(\overline{Y}, W\omega^d_{\overline{Y}/\overline{s},\log}) \qquad \text{if } r \geq d \end{aligned}$$

by a simple argument. Here $d = \dim X_K$, $I$ denotes the inertia subgroup of $\mathrm{Gal}(\overline{K}/K)$ and the subscript $I$ denotes the cofixed part by $I$, i.e. the maximal unramified quotient.

The purpose of this paper is to answer the following two questions (partially for the second one) on these isomorphisms.

First, if we denote by $i$ and $j$ the closed immersion and the open immersion $Y \to X$ and $X_K := X \times_{\mathrm{Spec}(O_K)} \mathrm{Spec}(K) \to X$ respectively, then we have a unique surjective homomorphism of sheaves on the étale site $Y_{\text{ét}}$ of $Y$:

$$(0.2)\qquad i^* R^r j_* \mathbb{Z}/p^n\mathbb{Z}(r) \longrightarrow W_n\omega^r_{Y/s,\log}$$

compatible with the symbol maps (see §3.1 for details), from which we obtain homomorphisms from the LHS's of (0.1) to the RHS's of (0.1). *Do these homomorphisms coincide with* (0.1) *constructed from the semi-stable conjecture?* In [Sat], it is stated without proof that they coincide. We will give its precise proof. This second construction via (0.2) is necessary in the applications [Sat] and [J-Sai].

Secondly, the both sides of the isomorphisms (0.1) have natural integral structures coming from $H^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p(s))$ and $H^{r-s}_{\text{ét}}(\overline{Y}, W\omega^s_{\overline{Y}/\overline{s},\log})$ for $s = r, d$. *Do the two integral structures coincide?* We will prove that it is true in the case $r \leq p-2$ and the base field $K$ is absolutely unramified by using the comparison theorem of C. Breuil (Theorem 3.2.4) between the $p$-torsion étale cohomology and certain log crystalline cohomology.

By [Ts4], one can easily extend the theorem of C. Breuil to any fine and saturated smooth log scheme whose special fiber is reduced (including the case that the log structure on the generic fiber is non-trivial), and we will discuss on the second question under this more general setting. For the $\mathbb{Q}_p$ case, G. Yamashita [Y] recently proved the semi-stable conjecture in the open case (more precisely under the condition (3.1.2)) by the syntomic method. If we use his result, we can prove our result also in the open case. Considering the necessity in [J-Sai] of our result in the open case, we will give an alternative proof in §4 when the horizontal divisors at infinity do not have self-intersections by proving the compatibility of the comparison maps with the Gysin sequences.

This paper is organized as follows. In §1, we will give a description of the maximal unramified quotients of semi-stable $\mathbb{Q}_p$-representations and semi-stable

$\mathbb{Z}_p$-representations (in the sense of C. Breuil) in terms of the corresponding objects in $\underline{MF}_K(\varphi, N)$ and in $\underline{\mathcal{MF}}_{W,\mathrm{tor}}(\varphi, N)$ respectively. In §2, we study the relation between the maximal slope part of the log crystalline cohomology and the logarithmic Hodge-Witt sheaves taking care of their integral structures. In §3, we will state our main theorem, review the construction of the comparison map in the semi-stable conjecture and then prove the main theorem.

I dedicate this paper to Professor K. Kato, who guided me to the $p$-adic world, especially to the $p$-adic Hodge theory. This paper is based on his work at many points. I would like to thank Professors U. Jannsen and S. Saito for fruitful discussions on the subject of this paper.

Notation. Let $K$ be a complete discrete valuation field of mixed characteristic $(0, p)$ whose residue field $k$ is perfect, and let $O_K$ denote the ring of integers of $K$. Let $W$ be the ring of Witt vectors with coefficients in $k$, and let $K_0$ denote the field of fractions of $W$. We denote by $\sigma$ the Frobenius endomorphisms of $k$, $W$ and $K_0$. Let $\overline{K}$ be an algebraic closure of $K$, and let $\overline{k}$ be the residue field of $\overline{K}$, which is an algebraic closure of $k$. Let $G_K$ (resp. $G_k$) be the Galois group $\mathrm{Gal}(\overline{K}/K)$ (resp. $\mathrm{Gal}(\overline{k}/k)$), and let $I_K$ be the inertia group of $G_K$. We have $G_K/I_K \cong G_k$. We denote by $P_0$ the field of fractions of the ring of Witt vectors $W(\overline{k})$ with coefficients in $\overline{k}$.

## §1. The maximal unramified quotients of semi-stable representations.

In this section, we study the maximal unramified quotients (i.e. the coinvariant by the inertia group $I_K$) of semi-stable $p$-adic or $\mathbb{Z}_p$-representations of $G_K$.

### §1.1. Review on slopes.

Let $D$ be a finite dimensional $P_0$-vector space with a semi-linear automorphism $\varphi$. For a rational number $\alpha = sr^{-1}$ $(s, r \in \mathbb{Z}, r > 0, (s, r) = 1)$, we denote by $D_\alpha$ the $P_0$-subspace of $D$ generated by the $\mathrm{Frac}(W(\mathbb{F}_{p^r}))$-vector space $D^{\varphi^r = p^s}$. Then, the natural homomorphisms $D^{\varphi^r = p^s} \otimes_{\mathrm{Frac}(W(\mathbb{F}_{p^r}))} P_0 \to D_\alpha$ and $\oplus_{\alpha\in\mathbb{Q}} D_\alpha \to D$ are isomorphisms. We call $D_\alpha$ the subspace of $D$ whose slope is $\alpha$, and we say that $\alpha$ is a slope of $D$ if $D_\alpha \neq 0$.

Let $D$ be a finite dimensional $K_0$-vector space with a $\sigma$-semi-linear automorphism $\varphi$. Then the above slope decomposition of $(P_0 \otimes_{K_0} D, \varphi \otimes \varphi)$ descends to the decomposition $D = \oplus_{\alpha\in\mathbb{Q}} D_\alpha$ of $D$. We call $D_\alpha$ the subspace of $D$ whose slope is $\alpha$ and we say that $\alpha$ is a slope of $D$ if $D_\alpha \neq 0$. For a subset $I \subset \mathbb{Q}$, we denote the sum $\oplus_{\alpha\in I} D_\alpha$ by $D_I$.

### §1.2. Review on semi-stable, crystalline and unramified $p$-adic representations ([Fo1], [Fo2]A1, [Fo3], [Fo4]).

Let $\underline{MF}_K(\varphi, N)$ denote the category of finite dimensional $K_0$-vector spaces endowed with $\sigma$-semilinear automorphisms $\varphi$, $K_0$-linear endomorphisms $N$ satisfying $N\varphi = p\varphi N$, and exhaustive and separated descending filtrations $Fil^\cdot$ on $D_K := K \otimes_{K_0} D$ by $K$-subspaces. Let $\underline{MF}_K(\varphi)$ denote the full subcategory of $\underline{MF}_K(\varphi, N)$ consisting of the objects such that $N = 0$. We denote by

$\underline{\mathrm{M}}_{K_0}(\varphi)$ the category of a finite dimensional $K_0$-vector space endowed with a $\sigma$-semilinear automorphism $\varphi$ whose slope is 0. We regard an object $D$ of $\underline{\mathrm{M}}_{K_0}(\varphi)$ as an object of $\underline{MF}_K(\varphi)$ by giving the filtration $Fil^0 D_K = D_K$, $Fil^1 D_K = 0$. By a $p$-adic representation of $G_K$, we mean a finite dimensional $\mathbb{Q}_p$-vector space endowed with a continuous and linear action of $G_K$, and we denote by $\underline{\mathrm{Rep}}(G_K)$ the category of $p$-adic representations. We denote by $\underline{\mathrm{Rep}}_{\mathrm{st}}(G_K)$ (resp. $\underline{\mathrm{Rep}}_{\mathrm{crys}}(G_K)$, resp. $\underline{\mathrm{Rep}}_{\mathrm{ur}}(G_K)$) be the full subcategory of $\underline{\mathrm{Rep}}(G_K)$ consisting of semi-stable (resp. crystalline, resp. unramified) $p$-adic representations.
Choose and fix a uniformizer $\pi$ of $K$. Then, by the theory of Fontaine, we have the following commutative diagram of categories and functors:

$$\begin{array}{ccc}
\underline{\mathrm{Rep}}_{\mathrm{ur}}(G_K) & \xrightarrow{D_{\mathrm{ur}}} & \underline{\mathrm{M}}_{K_0}(\varphi) \\
\cap & & \cap \\
\underline{\mathrm{Rep}}_{\mathrm{crys}}(G_K) & \xrightarrow{D_{\mathrm{crys}}} & \underline{MF}_K(\varphi) \\
\cap & & \cap \\
\underline{\mathrm{Rep}}_{\mathrm{st}}(G_K) & \xrightarrow{D_{\mathrm{st}}} & \underline{MF}_K(\varphi, N)
\end{array}$$

The functors $D_{\mathrm{crys}}$ and $D_{\mathrm{st}}$ are fully faithful and the functor $D_{\mathrm{ur}}$ is an equivalence of categories; they are defined by

$$D_\bullet(V) = (B_\bullet \otimes_{\mathbb{Q}_p} V)^{G_K} \quad (\bullet = \mathrm{st}, \mathrm{crys}, \mathrm{ur}),$$

where $B_{\mathrm{st}}$ and $B_{\mathrm{crys}}$ are the rings of Fontaine and $B_{\mathrm{ur}} = P_0$. A semi-stable representation $V$ is crystalline if and only if $N = 0$ on $D_{\mathrm{st}}(V)$. Recall that the embedding $B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$ and hence the functor $D_{\mathrm{st}}$ depends on the choice of $\pi$. The quasi-inverse of $D_{\mathrm{ur}}$ is given by

$$V_{\mathrm{ur}}(D) = (P_0 \otimes_{K_0} D)^{\varphi \otimes \varphi = 1}$$

We say that an object $D$ of $\underline{MF}_K(\varphi, N)$ (resp. $\underline{MF}_K(\varphi)$) is *admissible* if there exists a semi-stable (resp. crystalline) representation $V$ such that $D_{\mathrm{st}}(V) \cong D$ (resp. $D_{\mathrm{crys}}(V) \cong D$). We denote by $\underline{MF}^{\mathrm{ad}}_K(\varphi, N)$ (resp. $\underline{MF}^{\mathrm{ad}}_K(\varphi)$) the full subcategory of $\underline{MF}_K(\varphi, N)$ (resp. $\underline{MF}_K(\varphi)$) consisting of admissible objects. Then the quasi-inverse $V_{\mathrm{st}} \colon \underline{MF}^{\mathrm{ad}}_K(\varphi, N) \to \underline{\mathrm{Rep}}_{\mathrm{st}}(G_K)$ (resp. $V_{\mathrm{crys}} \colon \underline{MF}^{\mathrm{ad}}_K(\varphi) \to \underline{\mathrm{Rep}}_{\mathrm{crys}}(G_K)$) of the functor $D_{\mathrm{st}}$ (resp. $D_{\mathrm{crys}}$) is given by

$$V_{\mathrm{st}}(D) = Fil^0(B_{\mathrm{dR}} \otimes_K D_K) \cap (B_{\mathrm{st}} \otimes_{K_0} D)^{\varphi \otimes \varphi = 1, N \otimes 1 + 1 \otimes N = 0}$$
$$(\text{resp. } V_{\mathrm{crys}}(D) = Fil^0(B_{\mathrm{dR}} \otimes_K D_K) \cap (B_{\mathrm{crys}} \otimes_{K_0} D)^{\varphi \otimes \varphi = 1})$$

For an object $D$ of $\underline{MF}_K(\varphi, N)$ and an integer $r$, we denote by $D(r)$ the object of $\underline{MF}_K(\varphi, N)$ whose underlying $K_0$-vector space and monodromy operator $N$ are the same as $D$ and whose Frobenius endomorphism (resp. filtration) is

defined by $\varphi_{D(r)} = p^{-r}\varphi_D$ (resp. $Fil^i D(r)_K = Fil^{i+r} D_K$). If $D$ is admissible, then $D(r)$ is also admissible and there is a canonical isomorphism $V_{\mathrm{st}}(D)(r) \cong V_{\mathrm{st}}(D(r))$ induced by $(B_{\mathrm{st}} \otimes_{K_0} D) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r) \cong B_{\mathrm{st}} \otimes_{K_0} D(r); (a \otimes d) \otimes t^r \mapsto at^r \otimes d$. Here $(r)$ in the left-hand sides means the usual Tate twist
For an object $D$ of $\underline{MF}_K(\varphi, N)$, we define the integers $t_H(D)$ and $t_N(D)$ by

$$t_H(D) = \sum_{i \in \mathbb{Z}} (\dim_K \mathrm{gr}^i_{Fil} D_K) \cdot i$$
$$t_N(D) = \sum_{\alpha \in \mathbb{Q}} (\dim_{K_0} D_\alpha) \cdot \alpha,$$

where $D_\alpha$ denotes the subspace of $D$ whose slope is $\alpha$.
We say that an object $D$ of $\underline{MF}_K(\varphi, N)$ is *weakly admissible* if $t_H(D) = t_N(D)$ and if, for any $K_0$-subspace $D'$ of $D$ stable under $\varphi$ and $N$, $t_H(D') \leq t_N(D')$. Here we endow $D'_K$ with the filtration $Fil^i D'_K = Fil^i D_K \cap D'_K$ $(i \in \mathbb{Z})$. Note that $t_H(D') \leq t_N(D')$ is equivalent to $t_H(D/D') \geq t_N(D/D')$ under the assumption $t_H(D) = t_N(D)$. The admissibility implies the weak admissibility. P. Colmez and J.-M. Fontaine proved that the converse is also true ([C-Fo]).

§1.3. The maximal unramified quotients of semi-stable $p$-adic representations.

Lemma 1.3.1. *Let $D$ be a weakly admissible object of $\underline{MF}_K(\varphi, N)$. If $Fil^r D_K = D_K$ and $Fil^{s+1} D_K = 0$ for integers $r \leq s$, then the slopes of $\varphi$ on $D$ are contained in $[r, s]$.*

*Proof.* We prove that the slopes of $\varphi$ on $D$ are not less than $r$. The proof of slopes$\leq s$ is similar and is left to the reader. (Consider the projection $D \to D_\alpha$ for the largest slope $\alpha$.) Let $\alpha$ be the smallest slope of $D$, and let $D_\alpha$ be the subspace of $D$ whose slope is $\alpha$. Then, $D_\alpha$ is stable under $\varphi$. By the formula $N\varphi = p\varphi N$ and the choice of $\alpha$, we see $N = 0$ on $D_\alpha$, especially $D_\alpha$ is stable under $N$. Hence $t_H(D_\alpha) \leq t_N(D_\alpha) = \alpha \cdot \dim_{K_0} D_\alpha$. Since $t_H(D_\alpha) \geq r \cdot \dim_{K_0} D_\alpha$ by the assumption on $D$, we obtain $\alpha \geq r$. □

Let $V$ be a semi-stable $p$-adic representation of $G_K$, and set $D = D_{\mathrm{st}}(V)$. Let $s$ be an integer such that $Fil^{s+1} D_K = 0$, and let $V_s$ be the quotient $V(s)_{I_K}(-s)$ of $V$. We will construct explicitly the corresponding admissible quotient of $D$. For $\alpha \in \mathbb{Q}$, let $D_\alpha$ denote the subspace of $D$ whose slope is $\alpha$. By Lemma 1.3.1, $D_\alpha = 0$ if $\alpha > s$ and hence $D = \oplus_{\alpha \in \mathbb{Q}, \alpha \leq s} D_\alpha$. We define the monodromy operator on $D_s$ by $N = 0$ and the filtration on $(D_s)_K$ by $Fil^s (D_s)_K = (D_s)_K$ and $Fil^{s+1} (D_s)_K = 0$. Then, we see that $D_s(s)$ is an object of $\underline{\mathrm{M}}_{K_0}(\varphi)$. Especially $D_s$ is admissible. Using the relation $N\varphi = p\varphi N$ on $D$ and $Fil^{s+1} D_K = 0$, we see that the projection $D \to D_s$ is a morphism in the category $\underline{MF}^{\mathrm{ad}}_{K_0}(\varphi, N)$. Especially it is strictly compatible with the filtrations ([Fo4]4.4.4. Proposition i)), that is, the image of $Fil^i D_K$ is $Fil^i (D_s)_K$.

PROPOSITION 1.3.2. *Under the notation and the assumption as above, the quotient $V_s$ of $V$ corresponds to the admissible quotient $D_s$ of $D$.*

*Proof.* Since $D_s(s)$ is contained in $\underline{\mathrm{M}}_{K_0}(\varphi)$, $V_{\mathrm{st}}(D_s)(s) \cong V_{\mathrm{st}}(D_s(s))$ is unramified. Hence the natural surjection $V \to V_{\mathrm{st}}(D_s)$ factors through $V_s$. On the other hand, since $V_s(s)$ is unramified, $D_{\mathrm{st}}(V_s)(s) \cong D_{\mathrm{st}}(V_s(s))$ is contained in $\underline{\mathrm{M}}_{K_0}(\varphi)$. Hence the unique slope of $D_{\mathrm{st}}(V_s)$ is $s$ and the natural projection $D \to D_{\mathrm{st}}(V_s)$ factors as $D \to D_s \to D_{\mathrm{st}}(V_s)$. $D_s \to D_{\mathrm{st}}(V_s)$ is strictly compatible with the filtrations because $D \to D_s$ and $D \to D_{\mathrm{st}}(V_s)$ are strictly compatible with the filtrations. (Compatibility with $\varphi$ and $N$ is trivial). □

COROLLARY 1.3.3. *Under the above notations and assumptions, there exists a canonical $G_k$-equivariant isomorphism:*

$$V(s)_{I_K} \cong (P_0 \otimes_{K_0} D)^{\varphi=p^s}.$$

*Proof.* By Proposition 1.3.2, we have canonical $G_k$-equivariant isomorphisms:

$$\begin{aligned} V(s)_{I_K} &= V_s(s) \cong V_{\mathrm{st}}(D_s)(s) \cong V_{\mathrm{st}}(D_s(s)) = V_{\mathrm{ur}}(D_s(s)) \\ &= (P_0 \otimes_{K_0} D_s)^{\varphi \otimes p^{-s}\varphi=1} = (P_0 \otimes_{K_0} D)^{\varphi\otimes\varphi=p^s}. \end{aligned}$$

□

§1.4. REVIEW ON SEMI-STABLE, CRYSTALLINE AND UNRAMIFIED $p$-TORSION REPRESENTATIONS ([Fo-L], [Fo2]A1, [Br2], [Br3]§3.2.1).

In this section, we assume $K = K_0$. Following [Br2], we denote by $S$ the $p$-adic completion of the PD-polynomial ring in one variable $W\langle u\rangle = W\langle u-p\rangle$, and by $f_0$ (resp. $f_p$) the $W$-algebra homomorphism $S \to W$ defined by $u^{[n]} \mapsto 0$ (resp. $p^{[n]} = p^n/n!$) for $n \geq 1$. We define the filtration $Fil^iS$ ($i \in \mathbb{Z}, i > 0$) to be the $p$-adic completion of the $i$-th divided power of the PD-ideal of $W\langle u-p\rangle$ generated by $u-p$. We set $Fil^iS = S$ for $i \in \mathbb{Z}$, $i \leq 0$. Let $\varphi_S\colon S \to S$ denote the lifting of Frobenius defined by $\sigma$ on $W$ and $u^{[n]} \mapsto (u^p)^{[n]}$. For an integer $i$ such that $0 \leq i \leq p-2$, we have $\varphi(Fil^iS) \subset p^iS$ and we denote by $\varphi_i\colon Fil^iS \to S$ the homomorphism $p^{-i}\cdot\varphi|_{Fil^iS}$. Finally let $N$ denote the $W$-linear derivation $N\colon S \to S$ defined by $N(u^{[n]}) = nu^{[n]}$ ($n \in \mathbb{N}$).

Let $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ be the category of Fontaine-Laffaille of level within $[0, p-2]$, let $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ be the category $\underline{\mathcal{M}}^{p-2}$ of Breuil, and let $\underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi)$ be the category of $W$-modules of finite length endowed with $\sigma$-semilinear automorphisms. These categories are abelian and artinian.

An object of $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ is a $W$-module $M$ of finite length endowed with a descending filtration $Fil^iM$ ($i \in \mathbb{Z}$) by $W$-submodules such that $Fil^0M = M$, $Fil^{p-1}M = 0$ and $\sigma$-semilinear homomorphisms $\varphi_i\colon Fil^iM \to M$ ($0 \leq i \leq p-2$) such that $\varphi_i|_{Fil^{i+1}M} = p\varphi_{i+1}$ ($0 \leq i \leq p-3$) and $M = \sum_{0\leq i\leq p-2}\varphi_i(Fil^iM)$. We can prove that $Fil^iM$ ($i \in \mathbb{Z}$) are direct summands of $M$. For an integer $0 \leq r \leq p-2$, we say that $M$ is of level within $[0, r]$ if $Fil^{r+1}M = 0$. The sequence $M_1 \to M_2 \to M_3$ in $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ is exact if and only if it is exact as a sequence of $W$-modules. Furthermore, for an exact sequence $M_1 \to M_2 \to M_3$ in $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$, the sequence $Fil^i M_1 \to Fil^i M_2 \to Fil^i M_3$ is exact for any $i \in \mathbb{Z}$.

An object of $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ is an $S$-module $\mathcal{M}$ isomorphic to a finite sum of $S$-modules of the form $S/p^n S$ $(n \geq 1)$ endowed with the following three structures: A submodule $Fil^{p-2}\mathcal{M}$ such that $Fil^{p-2}S \cdot \mathcal{M} \subset Fil^{p-2}\mathcal{M}$. A $\varphi_S$-semi-linear homomorphism $\varphi_{p-2} \colon Fil^{p-2}\mathcal{M} \to \mathcal{M}$ such that $(\varphi_1(u-p))^{p-2}\varphi_{p-2}(ax) = \varphi_{p-2}(a)\varphi_{p-2}((u-p)^{p-2}x)$ $(a \in Fil^{p-2}S,\ x \in \mathcal{M})$ and that $\mathcal{M}$ is generated by $\varphi_{p-2}(Fil^{p-2}\mathcal{M})$ as an $S$-module. A $W$-linear map $N \colon \mathcal{M} \to \mathcal{M}$ such that $N(ax) = N(a)x + aN(x)$ $(a \in S, x \in \mathcal{M})$, $(u-p)N(Fil^{p-2}\mathcal{M}) \subset Fil^{p-2}\mathcal{M}$ and $\varphi_1(u-p)N\varphi_{p-2}(x) = \varphi_{p-2}((u-p)N(x))$ $(x \in Fil^{p-2}\mathcal{M})$. Note that $\varphi_1(u-p) = (p-1)!u^{[p]} - 1$ is invertible in $S$. For an object $\mathcal{M}$ of $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$, we define the filtration $Fil^i\mathcal{M}$ $(0 \leq i \leq p-2)$ by $Fil^i\mathcal{M} := \{x \in \mathcal{M} | (u-p)^{p-2-i}x \in Fil^{p-2}\mathcal{M}\}$ and the Frobenius $\varphi_i : Fil^i\mathcal{M} \to \mathcal{M}$ $(0 \leq i \leq p-2)$ by the formula $\varphi_i(x) = \varphi_1(u-p)^{-(p-2-i)}\varphi_{p-2}((u-p)^{p-2-i}x)$. We have $Fil^0\mathcal{M} = \mathcal{M}$ and $\varphi_i|Fil^{i+1}\mathcal{M} = p\varphi_{i+1}$ for $0 \leq i \leq p-3$. For an object $\mathcal{M}$ of $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ and an integer $0 \leq r \leq p-2$, we say that $\mathcal{M}$ is of level within $[0,r]$ if $Fil^{p-2-r}S \cdot \mathcal{M} \supset Fil^{p-2}\mathcal{M}$. The sequence $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3$ in $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ is exact if and only if it is exact as $S$-modules. Furthermore, for an exact sequence $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3$ in $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ and an integer $0 \leq r \leq p-2$, if $\mathcal{M}_s$ $(s = 1,2,3)$ are of level within $[0,r]$, then the sequence $Fil^r\mathcal{M}_1 \to Fil^r\mathcal{M}_2 \to Fil^r\mathcal{M}_3$ is exact.

We regard an object $M$ of $\underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi)$ as an object of $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ by setting $Fil^0 M = M$, $Fil^1 M = 0$ and $\varphi_0 = \varphi$. We have a canonical fully faithful exact functor $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi) \to \underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ defined as follows ([Br2]2.4.1): To an object $M$ of $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$, we associate the following object $\mathcal{M}$. The underlying $S$-module is $S \otimes_W M$ and $Fil^{p-2}\mathcal{M} = \sum_{0 \leq i \leq p-2} Fil^{p-2-i}S \otimes_W Fil^i M$. The Frobenius $\varphi_{p-2} : Fil^{p-2}\mathcal{M} \to \mathcal{M}$ is defined by the formula: $\varphi_{p-2}(a \otimes x) = \varphi_{p-2-i}(a) \otimes \varphi_i(x)$ $(0 \leq i \leq p-2, a \in Fil^{p-2-i}S, x \in Fil^i M)$. The monodromy operator is defined by $N(a \otimes x) = N(a) \otimes x$ $(a \in S, x \in M)$. To prove that $\varphi_{p-2}$ is well-defined, we use the fact that $Fil^i M$ $(i \in \mathbb{Z})$ are direct summands of $M$. Note that, for an integer $0 \leq r \leq p-2$, $\mathcal{M}$ is of level within $[0,r]$ if and only if $M$ is of level within $[0,r]$.

Let $\underline{\mathrm{Rep}}_{\mathrm{tor}}(G_K)$ be the category of $\mathbb{Z}_p$-modules of finite length endowed with continuous actions of $G_K$, and let $\underline{\mathrm{Rep}}_{\mathrm{tor,ur}}(G_K)$ be the full subcategory of $\underline{\mathrm{Rep}}_{\mathrm{tor}}(G_K)$ consisting of the objects such that the actions of $G_K$ are trivial on $I_K$.

First we have an equivalence of categories:

$$T_{\mathrm{ur}} \colon \underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi) \to \underline{\mathrm{Rep}}_{\mathrm{tor,ur}}(G_K)$$

defined by $T_{\mathrm{ur}}(M) = (W(\overline{k}) \otimes_W M)^{\varphi\otimes\varphi=1}$. Its quasi-inverse is given by $T \mapsto (W(\overline{k}) \otimes_{\mathbb{Z}_p} T)^{G_K}$ (cf. for example, [Fo2]A1).

J.-M. Fontaine and G. Laffaille constructed a covariant fully faithful exact functor ([Fo-L]):

$$T_{\mathrm{crys}}\colon \underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi) \to \underline{\mathrm{Rep}}_{\mathrm{tor}}(G_K).$$

(Strictly speaking, they constructed a contravariant functor $\underline{U}_S$. We define $T_{\mathrm{crys}}$ to be its dual: $T_{\mathrm{crys}}(M) := \mathrm{Hom}(\underline{U}_S(M), \mathbb{Q}_p/\mathbb{Z}_p)$.) For an object $M$ of $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ and an integer $0 \le r \le p-2$ such that $M$ is of level within $[0,r]$, we have the following exact sequence functorial on $M$ (cf. [Ka1]II §3):

$$(1.4.1) \quad 0 \longrightarrow T_{\mathrm{crys}}(M)(r) \longrightarrow Fil^r(A_{\mathrm{crys}} \otimes_W M) \xrightarrow{1-\varphi_r} A_{\mathrm{crys}} \otimes_W M \longrightarrow 0.$$

Here $Fil^i(A_{\mathrm{crys}} \otimes_W M) = \sum_{0\le j\le i} Fil^{i-j}A_{\mathrm{crys}} \otimes_W Fil^j M$ $(0 \le i \le p-2)$ and $\varphi_i$ is defined by $\varphi_i(a \otimes x) = \varphi_{i-j}(a) \otimes \varphi_j(x)$ $(0 \le j \le i,\ a \in Fil^{i-j}A_{\mathrm{crys}},\ x \in Fil^j M)$.

C. Breuil constructed a covariant fully faithful exact functor ([Br2]):

$$T_{\mathrm{st}}\colon \underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N) \to \underline{\mathrm{Rep}}_{\mathrm{tor}}(G_K).$$

(Strictly speaking, he constructed a contravariant functor $V_{\mathrm{st}}$. We define $T_{\mathrm{st}}$ to be its dual: $T_{\mathrm{st}}(\mathcal{M}) := \mathrm{Hom}(V_{\mathrm{st}}(\mathcal{M}), \mathbb{Q}_p/\mathbb{Z}_p)$.) For an object $\mathcal{M}$ of $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ and an integer $0 \le r \le p-2$ such that $\mathcal{M}$ is of level within $[0,r]$, we have the following exact sequence functorial on $\mathcal{M}$ ([Br3]§3.2.1):

$$(1.4.2) \quad 0 \longrightarrow T_{\mathrm{st}}(\mathcal{M})(r) \longrightarrow (Fil^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}))^{N=0} \xrightarrow{1-\varphi_r} (\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M})^{N=0} \longrightarrow 0$$

Here we define $Fil^{\cdot}$, $\varphi_r$ and $N$ on $\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}$ as follows (see [Br1]§2 for the definition of $\widehat{A_{\mathrm{st}}}$): We define $Fil^i(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M})$ $(0 \le i \le p-2)$ to be the sum of the images of $Fil^{i-j}\widehat{A_{\mathrm{st}}} \otimes_S Fil^j\mathcal{M}$ $(0 \le j \le i)$. The homomorphism $\varphi_r\colon Fil^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}) \to \widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}$ is defined by $\varphi_r(a \otimes x) = \varphi_{r-i}(a) \otimes \varphi_i(x)$ $(0 \le i \le r, a \in Fil^{r-i}\widehat{A_{\mathrm{st}}}, x \in Fil^i\mathcal{M})$. (The well-definedness is non-trivial). The monodromy operator $N$ is defined by $N(a \otimes x) = N(a) \otimes x + a \otimes N(x)$ $(a \in \widehat{A_{\mathrm{st}}}, x \in \mathcal{M})$.

Now we have the following diagram of categories and functors commutative up to canonical isomorphisms:

$$(1.4.3) \quad \begin{array}{ccc} \underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi) & \xrightarrow{T_{\mathrm{ur}}} & \underline{\mathrm{Rep}}_{\mathrm{tor,ur}}(G_K) \\ \cap & & \cap \\ \underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi) & \xrightarrow{T_{\mathrm{crys}}} & \underline{\mathrm{Rep}}_{\mathrm{tor}}(G_K) \\ \downarrow & & \| \\ \underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N) & \xrightarrow{T_{\mathrm{st}}} & \underline{\mathrm{Rep}}_{\mathrm{tor}}(G_K). \end{array}$$

For an object $M$ of $\underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi)$, the isomorphism $T_{\mathrm{ur}}(M) \cong T_{\mathrm{crys}}(M)$ is induced by the natural homomorphism $(W(\overline{k}) \otimes_W M)(r) \to Fil^r(A_{\mathrm{crys}} \otimes_W$

$M) = (Fil^r A_{\mathrm{crys}}) \otimes_W M$ $(0 \leq r \leq p-2)$ defined by the natural inclusion $W(\overline{k})(r) \subset Fil^r A_{\mathrm{crys}}$. (The isomorphism is independent of the choice of $r$.) For an object $M$ of $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$, if we denote by $\mathcal{M}$ the corresponding object of $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$, then the natural homomorphisms $Fil^i(A_{\mathrm{crys}} \otimes_W M) \to Fil^i(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M})^{N=0}$ $(0 \leq i \leq p-2)$ are isomorphisms and we obtain the natural isomorphism $T_{\mathrm{crys}}(M) \cong T_{\mathrm{st}}(\mathcal{M})$ from the two exact sequences (1.4.1) and (1.4.2).

§1.5. Review of the comparison between $\mathbb{Q}_p$ and $p$-torsion theories ([Fo-L], [Br1], [Br2]).

We assume $K = K_0$ and keep the notation of §1.4. We review the relation between the functor $T_\bullet$ (§1.4) and the functor $V_\bullet$ (§1.2) ($\bullet \in \{\mathrm{ur}, \mathrm{crys}, \mathrm{st}\}$).

Let us begin with $\bullet = \mathrm{ur}$. Let $(M_n)_{n\geq 1}$ be a projective system of objects of $\underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi)$ such that the underlying $W$-module of $M_n$ is a free $W/p^nW$-module of finite rank and the morphism $M_{n+1}/p^nM_{n+1} \to M_n$ is an isomorphism. Associated to such a system, we define the object $D$ of $\underline{\mathrm{M}}_{K_0}(\varphi)$ to be $K_0 \otimes_W \varprojlim_n M_n$. Then, we have a canonical isomorphism defined in an obvious way:

$$V_{\mathrm{ur}}(D) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n T_{\mathrm{ur}}(M_n). \tag{1.5.1}$$

Next consider the case $\bullet = \mathrm{crys}$. Let $(M_n)_{n\geq 1}$ be a projective system of objects of $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ such that the underlying $W$-module of $M_n$ is a free $W/p^nW$-module of finite rank and the morphism $M_{n+1}/p^nM_{n+1} \to M_n$ is an isomorphism. We define the object $D$ of $\underline{MF}_K(\varphi)$ associated to this system as follows: The underling vector space is $K_0 \otimes_W (\varprojlim_n M_n)$, the filtration is defined by $K_0 \otimes_W (\varprojlim_n Fil^i M_n)$ and the Frobenius endomorphism is defined to be the projective limit of $\varphi_0$ of $M_n$. Then $D$ is admissible, and we have a canonical isomorphism ([Fo-L]§7, §8):

$$V_{\mathrm{crys}}(D) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\varprojlim_n T_{\mathrm{crys}}(M_n)). \tag{1.5.2}$$

The homomorphism from the RHS to the LHS is constructed as follows: By taking the projective limit of the exact sequence (1.4.1) for $M = M_n$ and $0 \leq r \leq p-2$ such that $M_n$ are of level within $[0,r]$ for all $n \geq 1$ and tensoring with $\mathbb{Q}_p$, we obtain an isomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\varprojlim_n T_{\mathrm{crys}}(M_n))(r) \cong Fil^r(B^+_{\mathrm{crys}} \otimes_{K_0} D)^{\varphi = p^r},$$

whose RHS is contained in

$$\begin{aligned} V_{\mathrm{crys}}(D)(r) &\cong Fil^r(B_{\mathrm{crys}} \otimes_{K_0} D)^{\varphi=p^r} \\ v \otimes t^{\otimes r} &\mapsto t^r v \quad (v \in V_{\mathrm{crys}}(D), t \in \mathbb{Q}_p(1)). \end{aligned}$$

Finally let us explain the case $\bullet = \mathrm{st}$. In this case, we need to introduce another category $\underline{\mathcal{MF}}_K(\varphi, N)$ defined by C. Breuil [Br1]§6 ($\underline{\mathcal{MF}}_{S\otimes_W K_0}(\Phi, \mathcal{N})$

in his notation), which is categorically equivalent to $\underline{MF}_K(\varphi,N)$. Set $S_{K_0} := K_0 \otimes_W S$, and let $f_0$ (resp. $f_p$) also denote the surjective homomorphism $S_{K_0} \to K_0$ induced by $f_0$ (resp. $f_p$) : $S \to W$. The filtration $Fil^{\cdot}$, the Frobenius endomorphism $\varphi$ and the monodromy operator $N$ on $S$ naturally induce those on $S_{K_0}$, which we also denote by $Fil^{\cdot}$, $\varphi$ and $N$. An object of $\underline{\mathcal{MF}}_K(\varphi,N)$ is a free $S_{K_0}$-module $\mathcal{D}$ of finite rank endowed with the following three structures: A descending filtration $Fil^i\mathcal{D}$ $(i \in \mathbb{Z})$ by $S_{K_0}$-submodules such that $Fil^iS_{K_0} \cdot Fil^j\mathcal{D} \subset Fil^{i+j}\mathcal{D}$ $(i,j \in \mathbb{Z})$, $Fil^i\mathcal{D} = \mathcal{D}$ $(i << 0)$ and $Fil^i\mathcal{D} \subset Fil^1S_{K_0} \cdot \mathcal{D}$ $(i >> 0)$. A $\varphi_{S_{K_0}}$-semilinear endomorphism $\mathcal{D} \to \mathcal{D}$ whose linearization $\mathcal{D} \otimes_{S_{K_0},\varphi} S_{K_0} \to \mathcal{D}$ is an isomorphism. A homomorphism $N \colon \mathcal{D} \to \mathcal{D}$ such that $N(ax) = N(a)x + aN(x)$ $(a \in S_{K_0}, x \in \mathcal{D})$, $N(Fil^i\mathcal{D}) \subset Fil^{i-1}\mathcal{D}$ $(i \in \mathbb{Z})$ and $N\varphi = p\varphi N$. We can construct a functor $\underline{MF}_K(\varphi,N) \to \underline{\mathcal{MF}}_K(\varphi,N)$ easily as follows: Let $D$ be an object of $\underline{MF}_K(\varphi,N)$. The corresponding object $\mathcal{D}$ is the $S_{K_0}$-module $S_{K_0} \otimes_{K_0} D$ with the Frobenius $\varphi \otimes \varphi$ and the monodromy operator defined by $N(a \otimes x) = N(a) \otimes x + a \otimes N(x)$. The filtration is defined inductively by the following formula, where $i_0$ is an integer such that $Fil^{i_0}D = D$.

$$Fil^i\mathcal{D} = \mathcal{D} \quad (i \leq i_0)$$
$$Fil^i\mathcal{D} = \{x \in Fil^{i-1}\mathcal{D} | f_p(x) \in Fil^iD, N(x) \in Fil^{i-1}\mathcal{D}\} \quad (i > i_0)$$

Here $f_p$ denotes the natural projection $\mathcal{D} \to \mathcal{D} \otimes_{S_{K_0},f_p} K_0$ and we identify $\mathcal{D} \otimes_{S_{K_0},f_p} K_0$ with $D$ by the natural isomorphism $(D \otimes_{S_{K_0}} S_{K_0}) \otimes_{S_{K_0},f_p} K_0 \cong D$. To construct the quasi-inverse of the above functor, we need the following proposition. (Compare with $B_{\mathrm{st}}^+ = \widehat{B_{\mathrm{st}}^+}^{N\text{-nilp}}$.)

Proposition 1.5.3. ([Br1]§6.2.1). *Let $\mathcal{D}$ be an object of $\underline{\mathcal{MF}}_K(\varphi,N)$. Then $D := \mathcal{D}^{N\text{-nilp}}$ is a finite dimensional vector space over $K_0$ and the natural homomorphism $D \otimes_{K_0} S_{K_0} \to \mathcal{D}$ is an isomorphism. Here $N$-nilp denotes the part where $N$ is nilpotent.*

With the notation of Proposition 1.5.3, we can verify easily that $D$ is stable under $N$ and $\varphi$ and we can define an exhaustive and separated filtration on $D$ by the image of $Fil^i\mathcal{D}$ under the homomorphism $\mathcal{D} \to \mathcal{D} \otimes_{S_{K_0},f_p} K_0 \cong D$. Thus we obtain an object $D$ of $\underline{MF}_K(\varphi,N)$. The functor associating $D$ to $\mathcal{D}$ is the quasi-inverse of the above functor.

Let $(\mathcal{M}_n)_{n \geq 1}$ be a projective system of objects of $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi,N)$ such that the underlying $S$-module of $\mathcal{M}_n$ is a free $S/p^nS$-module and the morphism $\mathcal{M}_{n+1}/p^n\mathcal{M}_{n+1} \to \mathcal{M}_n$ is an isomorphism for $n \geq 1$. Set $\mathcal{M} := \varprojlim_n \mathcal{M}_n$, $Fil^{p-2}\mathcal{M} := \varprojlim_n Fil^{p-2}\mathcal{M}_n$, and define the Frobenius $\varphi_{p-2} \colon Fil^{p-2}\mathcal{M} \to \mathcal{M}$ and the monodromy operator $N \colon \mathcal{M} \to \mathcal{M}$ by taking the projective limit of those on $\mathcal{M}_n$ $(n \in \mathbb{N})$. Then $\mathcal{M}$ is a free $S$-module of finite rank and the additional three structures satisfy the same conditions required in the definition of the category $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi,N)$. Furthermore $\mathcal{M}/Fil^{p-2}\mathcal{M}$ is $p$-torsion free. Indeed, we have the following injective morphism between two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M} & \xrightarrow{p} & \mathcal{M} & \longrightarrow & \mathcal{M}_1 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & Fil^{p-2}\mathcal{M} & \xrightarrow{p} & Fil^{p-2}\mathcal{M} & \longrightarrow & Fil^{p-2}\mathcal{M}_1 & \longrightarrow & 0. \end{array}$$

(With the terminology of [Br2]Définition 4.1.1.1, $\mathcal{M}$ with the three additional structures is a strongly divisible $S$-module.) For each integer $i$ such that $0 \leq i \leq p-2$, we define the $S$-submodule $Fil^i\mathcal{M}$ of $\mathcal{M}$ by $\{x \in \mathcal{M} | (u-p)^{p-2-i}x \in Fil^{p-2}\mathcal{M}\}$ and the Frobenius $\varphi_i\colon Fil^i\mathcal{M} \to \mathcal{M}$ by $\varphi_i(x) = \varphi_1(u-p)^{-(p-2-i)}\varphi_{p-2}((u-p)^{p-2-i}x)$. We have $Fil^0\mathcal{M} = \mathcal{M}$, $\varphi_i|Fil^{i+1}\mathcal{M} = p\varphi_{i+1}$ for $0 \leq i \leq p-3$ and $Fil^i\mathcal{M} = \varprojlim_n Fil^i\mathcal{M}_n$ $(0 \leq i \leq p-2)$. If $\mathcal{M}_n$ are of level within $[0,r]$ for an integer $0 \leq r \leq p-2$, then we see that $\mathcal{M}/Fil^r\mathcal{M}$ is $p$-torsion free. Now we define the object $\mathcal{D}$ of $\underline{\mathcal{MF}}_K(\varphi, N)$ associated to the projective system $(\mathcal{M}_n)_n$ as follows ([Br2]§4.1.1): The underlying $S_{K_0}$-module is $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{M}$. The filtration is defined by $Fil^i\mathcal{D} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} Fil^i\mathcal{M}$ $(0 \leq i \leq p-2)$ and $Fil^i\mathcal{D} = \sum_{0 \leq j \leq p-2} Fil^{i-j}S_{K_0} \cdot Fil^j\mathcal{D}$ $(i \geq p-1)$. The Frobenius and the monodromy operator on $\mathcal{D}$ are defined to be the endomorphisms induced by $\varphi_0$ and $N$ on $\mathcal{M}$. Finally let $D$ be the object of $\underline{MF}_K(\varphi, N)$ corresponding to $\mathcal{D}$. (If $\mathcal{M}_n$ are of level within $[0,r]$ for an integer $0 \leq r \leq p-2$, then so is $D$, that is, $Fil^0 D = D$ and $Fil^{r+1}D = 0$.) Then $D$ is admissible and there is a canonical isomorphism

$$V_{\mathrm{st}}(D) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n T_{\mathrm{st}}(\mathcal{M}_n) \tag{1.5.4}$$

functorial on $(\mathcal{M}_n)_n$ ([Br2]4.2).

This isomorphism is constructed as follows (cf. [Br3]4.3.2). First, since $\mathcal{M}$ is a free $S$-module of finite rank and $\mathcal{M}_n \cong \mathcal{M}/p^n\mathcal{M}$, we have

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\varprojlim_n (\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}_n)) \cong \widehat{B^+_{\mathrm{st}}} \otimes_{S_{K_0}} \mathcal{D}. \tag{1.5.5}$$

See [Br1]§2 for the definition of $\widehat{B^+_{\mathrm{st}}}$.

Lemma 1.5.6. *Let $r$ be an integer such that $0 \leq r \leq p-2$ and $\mathcal{M}_n$ $(n \geq 1)$ are of level within $[0,r]$. Then the above isomorphism induces an isomorphism:*

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\varprojlim_n (Fil^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}_n)^{N=0})) \cong Fil^r(\widehat{B^+_{\mathrm{st}}} \otimes_{S_{K_0}} \mathcal{D})^{N=0}.$$

*Here $Fil^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}_n)$ is the $\widehat{A_{\mathrm{st}}}$-submodule of $\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}_n$ defined after* (1.4.2), *and $Fil^r(\widehat{B^+_{\mathrm{st}}} \otimes_{S_{K_0}} \mathcal{D})$ is the sum of the images of $Fil^{r-j}\widehat{B^+_{\mathrm{st}}} \otimes_{S_{K_0}} Fil^j\mathcal{D}$ $(0 \leq j \leq r)$ in $\widehat{B^+_{\mathrm{st}}} \otimes_{S_{K_0}} \mathcal{D}$.*

This is stated in [Br3] Lemme 4.3.2.2 in the special case that $(\mathcal{M}_n)$ comes from crystalline cohomology with only an outline of a proof, and his proof seems to work for a general $(\mathcal{M}_n)$. To make our argument certain, I will give a proof, which seems a bit different from his.

*Proof.* We use the terminology and the notation in [Br3]3.2.1 freely. We define another filtration $\overline{Fil}^i\mathcal{M}$ $(0 \leq i \leq r)$ of $\mathcal{M}$ by $\overline{Fil}^i\mathcal{M} := Fil^i\mathcal{D} \cap \mathcal{M}$. We see easily that this filtration satisfies the three conditions of [Br3] Définition 3.2.1.1, and $\overline{Fil}^i\mathcal{M} \supset Fil^i\mathcal{M}$. Note that $\mathcal{M}/Fil^r\mathcal{M}$ is $p$-torsion free. Define another filtration $\overline{Fil}^i\mathcal{M}_n$ $(0 \leq i \leq r)$ of $\mathcal{M}_n$ to be the images of $\overline{Fil}^i\mathcal{M}$. Then this filtration is admissible in the sense of [Br3] Définition 3.2.1.1. Define $\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}_n)$ using $\overline{Fil}^{\cdot}\mathcal{M}_n$ instead of $Fil^{\cdot}\mathcal{M}_n$, and $\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M})$ similarly using $\overline{Fil}^{\cdot}\mathcal{M}$. Then, by [Br3] Proposition 3.2.1.4, we have

$$(\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}_n))^{N=0} = (Fil^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}_n))^{N=0}.$$

Hence it suffices to prove that (1.5.5) induces an isomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\varprojlim_n \overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}_n)) \cong Fil^r(\widehat{B^+_{\rm st}} \otimes_{S_{K_0}} \mathcal{D}).$$

To prove this isomorphism, we choose a basis $e_\lambda$ $(\lambda \in \Lambda)$ of $\mathcal{D}$ over $S_{K_0}$ and integers $0 \leq r_\lambda \leq r$ such that

$$Fil^i\mathcal{D} = \oplus_\lambda Fil^{i-r_\lambda} S_{K_0} e_\lambda \quad (0 \leq i \leq r)$$

([Br1] A). Let $\mathcal{M}'$ be the free $S$-module generated by $e_\lambda$. By multiplying $p^{-m}$ for some $m > 0$ if necessary, we may assume that there exists an integer $\nu \geq 0$ such that $p^\nu\mathcal{M}' \subset \mathcal{M} \subset \mathcal{M}'$. We define the filtration $\overline{Fil}^i\mathcal{M}'$ $(0 \leq i \leq r)$ and $\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}')$ in the same way as $\mathcal{M}$. Then, for $0 \leq i \leq r$, we have

$$\overline{Fil}^i\mathcal{M}' = \oplus_\lambda Fil^{i-r_\lambda} S \cdot e_\lambda$$

and hence

$$\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}') = \oplus_\lambda Fil^{r-r_\lambda}\widehat{A_{\rm st}} \cdot e_\lambda.$$

Especially $\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}')$ and $(\widehat{A_{\rm st}} \otimes_S \mathcal{M}')/(\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}'))$ are $p$-adically complete and separated, and $p$-torsion free. On the other hand, we have $p^\nu\overline{Fil}^i\mathcal{M}' \subset \overline{Fil}^i\mathcal{M} \subset \overline{Fil}^i\mathcal{M}'$ and hence

$$p^\nu\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}') \subset \overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}) \subset \overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}').$$

Since $\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}_n)$ is the image of $\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M})$ and $(\widehat{A_{\rm st}} \otimes_S \mathcal{M})/p^n = \widehat{A_{\rm st}} \otimes_S \mathcal{M}_n$, we have the following commutative diagram:

$$\begin{array}{ccccc}
\overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}')/p^n & \xrightarrow{\text{“}p^\nu\text{”}} & \overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}_n) & \longrightarrow & \overline{Fil}^r(\widehat{A_{\rm st}} \otimes_S \mathcal{M}')/p^n \\
\cap & & \cap & & \cap \\
(\widehat{A_{\rm st}} \otimes_S \mathcal{M}')/p^n & \xrightarrow{\text{“}p^\nu\text{”}} & \widehat{A_{\rm st}} \otimes_S \mathcal{M}_n & \longrightarrow & (\widehat{A_{\rm st}} \otimes_S \mathcal{M}')/p^n.
\end{array}$$

By taking the projective limit with respect to $n$, we obtain the following commutative diagram:

$$\begin{array}{ccccc} \overline{Fil}^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}') & \xrightarrow{p^\nu} & \varprojlim_n(\overline{Fil}^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}_n)) & \longrightarrow & \overline{Fil}^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}') \\ \cap & & \cap & & \cap \\ \widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}' & \xrightarrow{p^\nu} & \widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M} & \longrightarrow & \widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}'. \end{array}$$

By tensoring with $\mathbb{Q}_p$, we obtain the claim. □

Let $r$ be an integer as in Lemma 1.5.6. Then by taking the projective limit of the exact sequence (1.4.2) for $\mathcal{M}_n$ and $r$ and using the isomorphism (1.5.5) and Lemma 1.5.6, we obtain an isomorphism:

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n (T_{\mathrm{st}}(\mathcal{M}_n)(r)) \cong Fil^r(\widehat{B^+_{\mathrm{st}} \otimes_{S_{K_0}} \mathcal{D}})^{N=0,\varphi=p^r},$$

where $Fil^r(\widehat{B^+_{\mathrm{st}} \otimes_{S_{K_0}} \mathcal{D}})$ is as in Lemma 1.5.6. Recall that $D$ denotes the object of $\underline{MF}_K(\varphi, N)$ corresponding to $\mathcal{D}$. By definition, $D = \mathcal{D}^{N\text{-Nilp}}$, the canonical homomorphism $D \otimes_{K_0} S_{K_0} \to \mathcal{D}$ is an isomorphism, $\varphi$ and $N$ on $D$ are induced from those on $\mathcal{D}$, and the filtration $Fil^i D$ is the image of $Fil^i\mathcal{D}$ by $f_p\colon \mathcal{D} \to \mathcal{D} \otimes_{S_{K_0}, f_p} K_0 \cong D$. (Recall that we assume $K = K_0$.) On the other hand, $B^+_{\mathrm{st}} = \widehat{B^+_{\mathrm{st}}}^{N\text{-Nilp}}$ ([Ka3]Theorem (3.7)) and we have a canonical homomorphism $\widehat{B^+_{\mathrm{st}}} \to B^+_{\mathrm{dR}}$ ([Br1]§7, see also [Ts2]§4.6) compatible with the filtrations and $f_p\colon S_{K_0} \to K_0$ such that the composite with $B_{\mathrm{st}} \subset \widehat{B^+_{\mathrm{st}}}$ is the inclusion $B^+_{\mathrm{st}} \subset B^+_{\mathrm{dR}}$ (associated to $p$). Hence we have $(\widehat{B^+_{\mathrm{st}} \otimes_{S_{K_0}} \mathcal{D}})^{N\text{-Nilp}} = (\widehat{B^+_{\mathrm{st}}} \otimes_{K_0} D)^{N\text{-Nilp}} = B^+_{\mathrm{st}} \otimes_{K_0} D$ and the image of $Fil^i(\widehat{B^+_{\mathrm{st}} \otimes_{S_{K_0}} \mathcal{D}}) \cap (B^+_{\mathrm{st}} \otimes_{K_0} D)$ by the homomorphism $B^+_{\mathrm{st}} \otimes_{K_0} D \to B^+_{\mathrm{dR}} \otimes_{K_0} D$ is contained in $Fil^i(B^+_{\mathrm{dR}} \otimes_{K_0} D)$. Thus we obtain an injective homomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\varprojlim_n T_{\mathrm{st}}(\mathcal{M}_n)(r)) \hookrightarrow (B^+_{\mathrm{st}} \otimes_{K_0} D)^{N=0,\varphi=p^r} \cap Fil^r(B^+_{\mathrm{dR}} \otimes_{K_0} D)$$

and the RHS is contained in

$$V_{\mathrm{st}}(D)(r) \cong (B_{\mathrm{st}} \otimes_{K_0} D)^{N=0,\varphi=p^r} \cap Fil^r(B_{\mathrm{dR}} \otimes_{K_0} D).$$

§1.6. Unramified quotients of semi-stable $\mathbb{Z}_p$-representations.
We assume $K = K_0$ and keep the notation of §1.4 and §1.5.

Let $(\mathcal{M}_n)_{n\geq 1}$ be a projective system of objects of $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ such that the underlying $S$-module $\mathcal{M}_n$ is a free $S/p^nS$-module of finite rank and the morphism $\mathcal{M}_{n+1}/p^n\mathcal{M}_{n+1} \to \mathcal{M}_n$ in $\underline{\mathcal{MF}}_{W,[0,p-2],\mathrm{tor}}(\varphi, N)$ is an isomorphism for every integer $n \geq 1$. Let $\mathcal{D}$ and $D$ be the objects of $\underline{\mathcal{MF}}_K(\varphi, N)$ and $\underline{MF}_K(\varphi, N)$ associated to $(\mathcal{M}_n)_{n\geq 1}$. If we denote by $\mathcal{M}$ the projective limit of $\mathcal{M}_n$ with respect to $n$ as $S$-modules, then $\mathcal{D} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{M}$, $D = \mathcal{D}^{N\text{-Nilp}}$ and

$D \otimes_{K_0} S_{K_0} \cong \mathcal{D}$. Set $T_n := T_{\mathrm{st}}(\mathcal{M}_n)$, $T := \varprojlim_n T_n$, and $V := V_{\mathrm{st}}(D)$. Recall that $D$ is admissible. We have a canonical isomorphism $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} T \cong V$ (1.5.4) and we will regard $T$ as a lattice of $V$ by this isomorphism in the following. Let $r$ be an integer such that $0 \leq r \leq p-2$ and let $V'$ be a quotient of the representation $V$ such that $V'(r)$ is unramified. Since $V'(r)$ is unramified and hence crystalline, $V'$ is also crystalline, and if we denote by $D'$ the corresponding quotient of $D$ in the category $\underline{MF}_K(\varphi, N)$, then $Fil^r D' = D'$, $Fil^{r+1} D' = 0$, and the slope of the Frobenius $\varphi$ is $r$. Furthermore, we have $G_K$-equivariant isomorphisms

(1.6.1)
$$V' \cong V_{\mathrm{crys}}(D') \cong V_{\mathrm{crys}}(D'(r))(-r) \cong V_{\mathrm{ur}}(D'(r))(-r) = ((P_0 \otimes_{K_0} D')^{\varphi = p^r})(-r).$$

Let $T'$ be the image of $T$ in $V'$ and let $M'$ be the image of $\mathcal{M}$ under the composite

$$\mathcal{D} \to \mathcal{D} \otimes_{S_{K_0}, f_0} K_0 \cong D \to D'.$$

Theorem 1.6.2. *Let the notation and the assumption be as above. Then the composite of the isomorphisms* (1.6.1) *induces an isomorphism*

$$T' \cong ((W(\overline{k}) \otimes_W M')^{\varphi \otimes \varphi = p^r})(-r).$$

*Proof.* Set $T'_n := T'/p^n T'$. Since $T'_n(r)$ is unramified, $T'_n$ is crystalline. If we denote by $\mathcal{M}'_n$ the corresponding quotient of $\mathcal{M}_n$, the projective system $(\mathcal{M}'_n)_{n \geq 0}$ comes from the projective system $(M'_n)_{n \geq 1}$ in $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ such that, for every integer $n \geq 1$, $Fil^r M'_n = M'_n$ and $Fil^{r+1} M'_n = 0$, the underlying $W_n$-module of $M'_n$ is free of rank $\mathrm{rank}_{\mathbb{Z}/p^n\mathbb{Z}} T'_n = \dim_{\mathbb{Q}_p} V'$, and the morphism $M'_{n+1}/p^n M'_{n+1} \to M'_n$ in $\underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ is an isomorphism. We define $M'_n(r)$ to be the object of $\underline{\mathrm{M}}_{W,\mathrm{tor}}(\varphi) \subset \underline{MF}_{W,[0,p-2],\mathrm{tor}}(\varphi)$ whose underlying $W_n$-module is the same as $M'_n$ and whose Frobenius $\varphi$ is $\varphi_r$. Then, we have isomorphisms:

$$\begin{aligned} T'_n &\cong T_{\mathrm{st}}(\mathcal{M}'_n) \cong T_{\mathrm{crys}}(M'_n) \cong T_{\mathrm{crys}}(M'_n(r))(-r) \\ &\cong T_{\mathrm{ur}}(M'_n(r))(-r) = ((W(\overline{k}) \otimes_W M'_n)^{\varphi \otimes \varphi_r = 1})(-r). \end{aligned}$$

The third isomorphism follows from the exact sequence (1.4.1). By taking the projective limit, we obtain an isomorphism

$$T' \cong (W(\overline{k}) \otimes_W (\varprojlim_n M'_n))^{\varphi \otimes \varphi_r = 1})(-r).$$

Let $\mathcal{M}'$ be the projective limit of $\mathcal{M}'_n$ as $S$-modules. By taking the projective limit of the surjective homomorphisms $\mathcal{M}_n \to \mathcal{M}'_n$ of $S/p^n S$-modules, we obtain a surjective homomorphisms $\mathcal{M} \to \mathcal{M}'$ of $S$-modules such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{M}' \\ \cap\downarrow & & \cap\downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{D}', \end{array}$$

where $\mathcal{D}'$ denotes the quotient of $\mathcal{D}$ corresponding to the quotient $D'$ of $D$. Note that $\mathcal{D}'$ and $D'$ are the objects of $\underline{\mathcal{MF}}_{K_0}(\varphi, N)$ and $\underline{MF}_{K_0}(\varphi, N)$ associated to $(\mathcal{M}'_n)_n$. On the other hand, since $\mathcal{M}'_n \otimes_{S,f_0} W \cong M'_n$, we have $\mathcal{M}' \otimes_{S,f_0} W \cong \varprojlim_n M'_n$. Hence there exists an injective homomorphism $\varprojlim_n M'_n \to D'$ which makes the following diagram commutative:

$$\begin{array}{ccccc} \mathcal{M}' & \longrightarrow & \mathcal{M}' \otimes_{S_{K_0},f_0} W & \cong & \varprojlim_n M'_n \\ \downarrow & & & & \downarrow \\ \mathcal{D}' & \longrightarrow & \mathcal{D}' \otimes_{S_{K_0},f_0} K_0 & \cong & D'. \end{array}$$

By the definition of $M'$, the image of $\varprojlim_n M'_n$ in $D'$ is $M'$, and $\varphi_r$ on $\varprojlim_n M'_n$ is induced by $p^{-r}\varphi$ on $D'$.
Thus we obtain an isomorphism:

$$T' \cong ((W(\overline{k}) \otimes_W M')^{\varphi \otimes \varphi = p^r})(-r).$$

Now it remains to prove that this is compatible with (1.6.1), which is straightforward. $\square$

## §2. The maximal slope of log crystalline cohomology.

Set $s := \mathrm{Spec}(k)$ and $\overline{s} := \mathrm{Spec}(\overline{k})$. Let $L$ be any fine log structure on $s$, and let $\overline{L}$ denote its inverse image on $\overline{s}$. We consider a fine log scheme $(Y, M_Y)$ smooth and of Cartier type over $(s, L)$ such that $Y$ is proper over $s$. Let $H^q((Y, M_Y)/(W, W(L)))$ be the crystalline cohomology $\varprojlim_n H^q_{\mathrm{crys}}((Y, M_Y)/(W_n, W_n(L)))$ defined by Hyodo and Kato in [H-Ka] (3.2), which is a finitely generated $W$-module endowed with a $\sigma$-semi-linear endomorphism $\varphi$ called the Frobenius. The Frobenius $\varphi$ becomes bijective after $\otimes_W K_0$. In this section, we will construct a canonical decomposition $M_1 \oplus M_2$ of $H^q((Y, M_Y)/(W, W(L)))$ stable under $\varphi$ such that $K_0 \otimes_W M_1$ (resp. $K_0 \otimes_W M_2$) is the direct factor of slope $q$ (resp. slopes$< q$), and a canonical isomorphism

$$W(\overline{k}) \otimes_W M_1 \cong W(\overline{k}) \otimes_{\mathbb{Z}_p} H^0_{\mathrm{\acute{e}t}}(\overline{Y}, W\omega^q_{\overline{Y}/\overline{s},\log}).$$

We will also construct a canonical decomposition $M'_1 \oplus M'_2$ stable under $\varphi$ such that $K_0 \otimes_W M'_1$ (resp. $K_0 \otimes_W M'_2$) is the direct factor of slope $d$ (resp. slopes$< d$), and a canonical isomorphism

$$W(\overline{k}) \otimes_W M'_1 \cong W(\overline{k}) \otimes_{\mathbb{Z}_p} H^{q-d}_{\mathrm{\acute{e}t}}(\overline{Y}, W\omega^d_{\overline{Y}/\overline{s},\log}).$$

Here $(\overline{Y}, M_{\overline{Y}}) := (Y, M_Y) \times_{(s,L)} (\overline{s}, \overline{L})$. See §2.2 for the definition of the RHS's.

### §2.1. De Rham-Witt complex.

First, we will review the de Rham-Witt complex (with log poles) $W_\bullet\omega^\bullet_{Y/s}$ associated to $(Y, M_Y)/(s, L)$. We don't assume that $Y$ is proper over $s$ in §2.1. Noting the crystalline description of the de Rham-Witt complex given in [I-R] III (1.5) for a usual smooth scheme over $s$, we define the sheaf of $W_n$-modules $W_n\omega^i_{Y/s}$ on $Y_{\mathrm{\acute{e}t}}$ to be

$$\sigma^n_* R^i u_{(Y,M_Y)/(W_n,W_n(L))*}\mathcal{O}_{(Y,M_Y)/(W_n,W_n(L))}$$

for integers $i \geq 0$ and $n > 0$ ([H-Ka] (4.1)). Here $u_{(Y,M_Y)/(W_n,W_n(L))}$ denotes the canonical morphism of topoi: $((Y,M_Y)/(W_n,W_n(L)))^{\sim}_{\text{crys}} \to Y^{\sim}_{\text{ét}}$. These sheaves are endowed with the canonical projections $\pi\colon W_{n+1}\omega^i_{Y/s} \to W_n\omega^i_{Y/s}$ ([H-Ka] (4.2)) and the differentials $d\colon W_n\omega^i_{Y/s} \to W_n\omega^{i+1}_{Y/s}$ ([H-Ka] (4.1)) such that $\pi d = d\pi$ and $((W_n\omega^\bullet_{Y/s}, d)_{n\geq 1}, \pi)_n$ becomes a projective system of graded differential algebras over $W$. The projections $\pi$ are surjective ([H-Ka] Theorem (4.4)). Furthermore, we have the operators $F\colon W_{n+1}\omega^i_{Y/s} \to W_n\omega^i_{Y/s}$ and $V\colon W_n\omega^i_{Y/s} \to W_{n+1}\omega^i_{Y/s}$ ([H-Ka] (4.1)) compatible with the projections $\pi$. As in the classical case (cf. [I1] I Introduction), one checks easily that we have

(1) $FV = VF = p$, $FdV = d$.

(2) $Fx.Fy = F(xy)$ $(x \in W_n\omega^i_{Y/s}, y \in W_n\omega^j_{Y/s})$,
$xVy = V(Fx.y)$ $(x \in W_{n+1}\omega^i_{Y/s}, y \in W_n\omega^j_{Y/s})$.

(3) $V(xdy) = Vx.dVy$ $(x \in W_n\omega^i_{Y/s}, y \in W_n\omega^j_{Y/s})$.

(The property (3) follows from (1) and (2): $V(xdy) = V(x.FdVy) = Vx.dVy$.)

We have $W$-algebra isomorphisms $\tau\colon W_n(\mathcal{O}_Y) \xrightarrow{\sim} W_n\omega^0_{Y/s}$ (see [H-Ka] (4.9) for the construction of $\tau$ and the proof of the isomorphism) compatible with $\pi$, $F$ and $V$, where the projections $\pi$ and the operators $F$ and $V$ on $W_n(\mathcal{O}_Y)$ are defined in the usual manner. With the notation in [H-Ka] (4.9), $C^\bullet_{Y/W_n}$ is a complex of quasi-coherent $W_n(\mathcal{O}_Y)$-modules and hence $W_n\omega^i_{Y/s} = \mathcal{H}^i(C^\bullet_{Y/W_n})$ are quasi-coherent $W_n(\mathcal{O}_Y)$-modules.

In the special case $n = 1$, we have an isomorphism $W_1\omega^i_{Y/s} \cong \mathcal{H}^i(\omega^\bullet_{Y/s}) \underset{C^{-1}}{\xleftarrow{\sim}} \omega^i_{Y/s}$ ([Ka2] (6.4), (4.12)), which is $\mathcal{O}_Y$-linear and compatible with the differentials. Recall that we regard $\mathcal{H}^i(\omega^\bullet_{Y/s})$ as an $\mathcal{O}_Y$-module by the action via $\mathcal{O}_Y \to \mathcal{O}_Y; x \mapsto x^p$. We will identify $(W_1\omega^\bullet_{Y/s}, d)$ with $(\omega^\bullet_{Y/s}, d)$ in the following by this isomorphism.

With the notation of [H-Ka] Definition (4.3) and Theorem (4.4) $B_{n+1}\omega^i_{Y/s} \oplus Z_n\omega^{i-1}_{Y/s}$ (resp. $B_1\omega^i_{Y/s}$) are coherent subsheaves of the coherent $\mathcal{O}_Y$-module $F^{n+1}_*(\omega^i_{Y/s} \oplus \omega^{i-1}_{Y/s})$ (resp. $F_*\omega^i_{Y/s}$), the homomorphism $(C^n, dC^n)\colon B_{n+1}\omega^i_{Y/s} \oplus Z_n\omega^{i-1}_{Y/s} \to B_1\omega^i_{Y/s}$ is $\mathcal{O}_Y$-linear and the isomorphism ([H-Ka] Theorem (4.4)):

(2.1.1)
$$(V^n, dV^n)\colon F^{n+1}_*(\omega^i_{Y/s} \oplus \omega^{i-1}_{Y/s})/\mathrm{Ker}(C^n, dC^n) \xrightarrow{\sim} \mathrm{Ker}(\pi\colon W_{n+1}\omega^i_{Y/s} \to W_n\omega^i_{Y/s})$$

is $\mathcal{O}_Y$-linear, where we regard the right-hand side as an $\mathcal{O}_Y$-module via $F\colon \mathcal{O}_Y = W_{n+1}\mathcal{O}_Y/VW_{n+1}\mathcal{O}_Y \to W_{n+1}\mathcal{O}_Y/pW_{n+1}\mathcal{O}_Y$. Especially this implies, by induction on $n$, that $W_n\omega^i_{Y/s}$ is a coherent $W_n(\mathcal{O}_Y)$-module.

We set $W\omega^i_{Y/s} := \varprojlim_n W_n\omega^i_{Y/s}$ and denote also by $d$, $F$ and $V$ the projective limits of $d$, $F$ and $V$ for $W_\bullet\omega^\bullet_{Y/s}$. Then, by the same argument as in the proof of [I1] II Proposition 2.1 (a) using the above coherence, we see that, if $Y$ is proper over $k$, then $H^j(Y, W_n\omega^i_{Y/s})$ and $H^j(Y, W_n\omega^\bullet_{Y/s})$ are finitely generated

$W_n$-modules and the canonical homomorphisms

$$H^j(Y, W\omega^i_{Y/s}) \to \varprojlim_n H^j(Y, W_n\omega^i_{Y/s}) \tag{2.1.2}$$

$$H^j(Y, W\omega^\bullet_{Y/s}) \to \varprojlim_n H^j(Y, W_n\omega^\bullet_{Y/s}) \tag{2.1.3}$$

are isomorphisms.

Theorem 2.1.4. ([H-Ka] Theorem (4.19), cf. [I1] II Théorème 1.4). *There exists a canonical isomorphism in $D^+(Y_{\text{ét}}, W_n)$:*

$$Ru_{(Y,M_Y)/(W_n,W_n(L))*}\mathcal{O}_{(Y,M_Y)/(W_n,W_n(L))} \cong W_n\omega^\bullet_{Y/s}$$

*functorial on $(Y, M_Y)$ and compatible with the products, the Frobenius and the transition maps. Here the Frobenius on the RHS is defined by $p^iF$ in degree $i$.*

Recall that $p\colon W_n\omega^i_{Y/s} \to W_n\omega^i_{Y/s}$ factors through the canonical projection $\pi\colon W_n\omega^i_{Y/s} \to W_{n-1}\omega^i_{Y/s}$ and that the induced homomorphism $\underline{p}\colon W_{n-1}\omega^i_{Y/s} \to W_n\omega^i_{Y/s}$ is injective ([H-Ka] Corollary (4.5) (1), cf. [I1] I Proposition 3.4). Hence $p^iF\colon W_n\omega^i_{Y/s} \to W_n\omega^i_{Y/s}$ is well-defined for $i \geq 1$. For $i = 0$, it is defined by the usual $F$ on $W_n\mathcal{O}_Y$.

*Remark.* Strictly speaking, the compatibility with the products is not mentioned in [H-Ka] Theorem (4.19), but one can verify it simply by looking at the construction of the map carefully as follows: We use the notation of the proof in [H-Ka]. We have a PD-homomorphism $\mathcal{O}_{D^\cdot_n} \to W_n(\mathcal{O}_{Y^\cdot})$ and $\omega^1_{Z^\cdot_n/(W_n,W_n(L))}|_{Y^\cdot} \to \omega^1_{W_n(Y^\cdot)/(W_n,W_n(L))}$, which induce a morphism of graded algebras $C_n = \mathcal{O}_{D^\cdot_n} \otimes \omega^\bullet_{Z^\cdot_n/(W_n,W_n(L))} \to \omega^\bullet_{W_n(Y^\cdot)/(W_n,W_n(L))}$. By taking the quotient $\omega^\bullet_{W_n(Y^\cdot)/(W_n,W_n(L),[\,])}$ of the target, we obtain a morphism $C_n \to \omega^\bullet_{W_n(Y^\cdot)/(W_n,W_n(L),[\,])}$ of differential graded algebras. The homomorphism $W_n(\mathcal{O}_{Y^\cdot}) \to W_n\omega^0_{Y^\cdot/s}$ is extended uniquely to a $W_n(\mathcal{O}_{Y^\cdot})$-linear morphism of differential graded algebras: $\omega^\bullet_{W_n(Y^\cdot)/(W_n,W_n(L))} \to W_n\omega^\bullet_{Y^\cdot}$ compatible with $d\log$'s from $W_n(M^\cdot) = M \oplus \operatorname{Ker}(W_n(\mathcal{O}_{Y^\cdot})^* \to \mathcal{O}^*_{Y^\cdot})$. It factors through the quotient $\omega^\bullet_{W_n(Y^\cdot)/(W_n,W_n(L),[\,])}$. Thus we see that the morphism $C_n \to W_n\omega^\bullet_{Y^\cdot/s}$ constructed in the proof of [H-Ka] Theorem (4.19) is a morphism of differential graded algebras.

From the definition, we immediately obtain the following exact sequences (cf. [I1] Remarques 3.21.1):

$$\begin{gathered}
W_{2n+k}\omega^i_{Y/s} \xrightarrow{F^{n+k}} W_n\omega^i_{Y/s} \xrightarrow{dV^k} W_{n+k}\omega^{i+1}_{Y/s},\\
W_k\omega^{i-1}_{Y/s} \xrightarrow{F^kdV^n} W_n\omega^i_{Y/s} \xrightarrow{V^k} W_{n+k}\omega^i_{Y/s},\\
W_{2n+k}\omega^i_{Y/s} \xrightarrow{F^n} W_{n+k}\omega^i_{Y/s} \xrightarrow{F^kd} W_n\omega^{i+1}_{Y/s},\\
W_k\omega^i_{Y/s} \xrightarrow{V^n} W_{n+k}\omega^i_{Y/s} \xrightarrow{F^k} W_n\omega^i_{Y/s}.
\end{gathered} \tag{2.1.5}$$

For example, the first exact sequence is obtained by considering the following morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^{\bullet}_{n+k} & \xrightarrow[\underline{p}^{n+k}]{} & C^{\bullet}_{2n+2k} & \longrightarrow & C^{\bullet}_{n+k} & \longrightarrow & 0 \\ & & \| & & \uparrow \underline{p}^k & & \uparrow \underline{p}^k & & \\ 0 & \longrightarrow & C^{\bullet}_{n+k} & \xrightarrow[\underline{p}^{n}]{} & C^{\bullet}_{2n+k} & \longrightarrow & C^{\bullet}_{n} & \longrightarrow & 0, \end{array}$$

where $C^{\bullet}_n$ is the same complex as in [H-Ka] (4.1).

Finally we review a generalization of the Cartier isomorphism to the de Rham-Witt complex (cf. [I-R] III Proposition (1.4)) and the operator $V'$ (cf. [I-R] III (1.3.2)), which will be used in the proof of our main result in §2.

By the exact sequences (2.1.5), $F^n\colon W_{2n}\omega^i_{Y/s} \to W_n\omega^i_{Y/s}$ induces an isomorphism $F^n\colon W_{2n}\omega^i_{Y/s}/V^nW_n\omega^i_{Y/s} \xrightarrow{\sim} ZW_n\omega^i_{Y/s}$. We define the operator $V'$ on $ZW_{\bullet}\omega^i_{Y/s}$ by the following commutative diagram:

$$(2.1.6)\qquad \begin{array}{ccc} W_{2n+2}\omega^i_{Y/s}/V^{n+1}W_{n+1}\omega^i_{Y/s} & \xrightarrow{\pi^2} & W_{2n}\omega^i_{Y/s}/V^nW_n\omega^i_{Y/s} \\ \wr\downarrow F^{n+1} & & \wr\downarrow F^n \\ ZW_{n+1}\omega^i_{Y/s} & \xrightarrow{V'} & ZW_n\omega^i_{Y/s}. \end{array}$$

We see easily that this operator satisfies the relations:

$$(2.1.7)\qquad V'\pi = \pi V',\ FV' = V'F = \pi^2,\ V'd = dV\pi^2.$$

The last relation implies $V'(BW_{n+1}\omega^i_{Y/s}) \subset BW_n\omega^i_{Y/s}$ and hence $V'$ induces a morphism $\mathcal{H}^i(W_{n+1}\omega^{\bullet}_{Y/s}) \to \mathcal{H}^i(W_n\omega^{\bullet}_{Y/s})$, which we will also denote by $V'$.

Using the property $F^ndV^n = d$, we also see that $F^n$ induces an isomorphism $W_{2n}\omega^i_{Y/s}/(V^nW_n\omega^i_{Y/s}+dV^nW_n\omega^{i-1}_{Y/s}) \xrightarrow{\sim} \mathcal{H}^i(W_n\omega^{\bullet}_{Y/s})$. On the other hand, from [H-Ka] Theorem (4.4), we can easily derive

$$(2.1.8)\qquad \mathrm{Ker}(\pi^m\colon W_{n+m}\omega^i_{Y/s} \to W_n\omega^i_{Y/s}) = V^nW_m\omega^i_{Y/s} + dV^nW_m\omega^{i-1}_{Y/s}$$

by induction on $m$ (cf. [I1] I Proposition 3.2). Hence $F^n$ induces an isomorphism:

$$(2.1.9)\qquad C^{-n}\colon W_n\omega^i_{Y/s} \xrightarrow{\sim} \mathcal{H}^i(W_n\omega^{\bullet}_{Y/s}),$$

which we can regard as a generalization of the Cartier isomorphism. Indeed, if $n = 1$, this coincides with the Cartier isomorphism by the identification $W_1\omega^i_{Y/s} = \omega^i_{Y/s}$. (To prove this coincidence, we need $d\log$ which will be explained in the next subsection.)

We need the following lemma in the proof of Lemma 3.4.4.

Lemma 2.1.10. *The composite of the following homomorphisms is the identity.*

$$W_n\omega^r_{Y/s} \xrightarrow[C^{-n}]{\sim} \mathcal{H}^r(W_n\omega^\bullet_{Y/s}) \overset{2.1.4}{\cong} R^r u_{(Y,M_Y)/(W_n,W_n(L))*}\mathcal{O} = W_n\omega^r_{Y/s}.$$

*Proof.* Denote by $\alpha$ the homomorphism in question. Then $\alpha$ is compatible with the products. Since $W_n\omega^r_{Y/s}$ $(r \geq 2)$ is generated by $x_1.x_2\ldots x_r$ $(x_i \in W_n\omega^1_{Y/s})$ as a sheaf of modules ([H-Ka] Proposition (4.6)), it suffices to prove the lemma in the case $r = 0,\ 1$. We use the notation in the proof of [H-Ka] Theorem (4.19). The lemma for $r = 0$ follows from the fact that the composite of $W_n\mathcal{O}_{Y^\bullet} \xrightarrow{\tau} \mathcal{O}_{D^\bullet} \to W_n\mathcal{O}_{Y^\bullet}$ coincides with $F^n$, where $\tau(x_0,\ldots,x_{n-1}) = \sum_{i=0}^{n-1} p^i\tilde{x_i}^{p^{n-i}}$. By [H-Ka] Proposition (4.6), $W_n\omega^1_{Y/s}$ is generated by $dx$ $(x \in W_n\mathcal{O}_Y)$ and $d\log(a)$ $(a \in M_Y^{\mathrm{gp}})$ as a $W_n\mathcal{O}_Y$-module. Using the quasi-isomorphism $W_{2n}\omega^\bullet_{Y/s}/\underline{p}^n W_n\omega^\bullet_{Y/s} \to W_n\omega^\bullet_{Y/s}$ ([H-Ka] Corollary (4.5)), we see that $\alpha$ commutes with the differentials. Hence $\alpha(dx) = d(\alpha(x)) = dx$. For $a \in M_Y^{\mathrm{gp}}$, we have $C^{-n}(d\log(a)) =$ the class of $d\log(a)$. On the other hand, by the construction of the quasi-isomorphism of [H-Ka] Theorem (4.19), the following diagram is commutative:

$$\begin{array}{ccc} M^{\mathrm{gp}}_{D^\bullet} & \longrightarrow & W_n(M_{Y^\bullet})^{\mathrm{gp}} \\ {\scriptstyle d\log}\downarrow & & {\scriptstyle d\log}\downarrow \\ C^1_n & \longrightarrow & W_n\omega^1_{Y^\bullet/s}. \end{array}$$

Choose a lifting $\tilde{a} \in M^{\mathrm{gp}}_{D^\bullet}$ of $a$ and let $au$, $u \in \mathrm{Ker}(W_n\mathcal{O}^*_{Y^\bullet} \to \mathcal{O}^*_{Y^\bullet})$ be its image in $W_n(M^\bullet)^{\mathrm{gp}}$. Then the image of $d\log(\tilde{a}) \in C^1_n$ in $W_n\omega^1_{Y^\bullet/s}$ is $d\log(a) + d(\log(u))$, which is congruent to $d\log(a)$ modulo $dW_n(\mathcal{O}_{Y^\bullet})$. $\square$

## §2.2. Logarithmic Hodge-Witt sheaves.

We will review the logarithmic Hodge-Witt sheaves $W_\bullet\omega^\bullet_{Y/s,\,\log}$ associated to $(Y, M_Y)/(s, L)$ (See [I1] for usual smooth schemes and [H], [L] for log smooth schemes). We still don't assume that $Y$ is proper over $s$.

As in [H-Ka] (4.9), we have natural homomorphisms

$$d\log\colon M_Y^{\mathrm{gp}} \to W_n\omega^1_Y \quad (n \geq 1),$$

which satisfy $d\log(g^{-1}(L^{\mathrm{gp}})) = 0$, $[b]d\log(a) = d([b])$ for $a \in M_Y$, its image $b$ in $\mathcal{O}_Y$ and $[b] = (b, 0, 0, \cdots) \in W_n(\mathcal{O}_Y)$, $\pi d\log = d\log$, $Fd\log = d\log$, and $d(d\log(M_Y^{\mathrm{gp}})) = 0$. For $n = 1$, $d\log$ coincides with the usual $d\log\colon M_Y^{\mathrm{gp}} \to \omega^1_{Y/s}$. We define the *logarithmic Hodge-Witt sheaves* $W_n\omega^i_{Y/s,\,\log}$ to be the subsheaves of abelian groups of $W_n\omega^i_{Y/s}$ generated by local sections of the form $d\log(a_1) \wedge \cdots \wedge d\log(a_i)$ $(a_1, \ldots, a_i \in M_Y^{\mathrm{gp}})$.

Theorem 2.2.1. ([I1] 0 Théorème 2.4.2, [Ts1] Theorem (6.1.1)). *The following sequence is exact for any integer $i \geq 0$:*

$$0 \longrightarrow \omega^i_{Y/s,\,\log} \longrightarrow Z\omega^i_{Y/s} \xrightarrow{1-C} \omega^i_{Y/s} \longrightarrow 0.$$

Now, by the same argument as the proof of [I1] I (3.26) and [I1] I (5.7.2) plus some additional calculation, we can derive the following theorem from Theorem 2.2.1, the exact sequences (2.1.5) and the isomorphism (2.1.1) (cf. [L] 1.5.2).

Theorem 2.2.2. (cf. [I1] I (3.26), (5.7.2)). *For any integers $n \geq 1$ and $i \geq 0$, the following sequence is exact:*

$$0 \longrightarrow W_n\omega^i_{Y/s,\,\log} + V^{n-1}W_1\omega^i_{Y/s} \longrightarrow W_n\omega^i_{Y/s} \xrightarrow{\pi-F} W_{n-1}\omega^i_{Y/s} \longrightarrow 0.$$

Note that we easily obtain

$$\begin{aligned} &\mathrm{Ker}(V^n\colon \omega^i_{Y/s} \to W_{n+1}\omega^i_{Y/s}) = B_n\omega^i_{Y/s}, \\ &\mathrm{Ker}(V^n\colon \omega^i_{Y/s} \to W_{n+1}\omega^i_{Y/s}/dV^n\omega^{i-1}_{Y/s}) = B_{n+1}\omega^i_{Y/s} \end{aligned} \tag{2.2.3}$$

(cf. [I1] I (3.8)) from the isomorphism (2.1.1) ([L] Proposition 1.2.7).

Corollary 2.2.4. (cf. [H] (2.6), [L] (1.5.4)). *The following sequence is exact for any integers $i \geq 0$, $n, m \geq 1$:*

$$0 \longrightarrow W_n\omega^i_{Y/s,\,\log} \xrightarrow{\underline{p}^m} W_{n+m}\omega^i_{Y/s,\,\log} \longrightarrow W_m\omega^i_{Y/s,\,\log} \longrightarrow 0.$$

Corollary 2.2.5. (cf. [I-R] IV §3). *The homomorphism $W_n\omega^i_{Y/s,\,\log} \to \mathcal{H}^i(W_n\omega^\bullet_{Y/s})$ is injective and the following sequence is exact:*

$$0 \longrightarrow K^i_n \longrightarrow \mathcal{H}^i(W_n\omega^\bullet_{Y/s}) \xrightarrow{V'-\pi} \mathcal{H}^i(W_{n-1}\omega^\bullet_{Y/s}) \longrightarrow 0,$$

*where $K^i_n$ denotes the image of $W_n\omega^i_{Y/s,\,\log} + p^{n-1}F(W_{n+1}\omega^i_{Y/s})$ in $\mathcal{H}^i(W_n\omega^\bullet_{Y/s})$.*

*Proof.* This immediately follows from Theorem 2.2.2 using the following commutative diagram:

$$\begin{array}{ccc} W_n\omega^i_{Y/s} & \xrightarrow{\pi-F} & W_{n-1}\omega^i_{Y/s} \\ \wr\downarrow C^{-n} & & \wr\downarrow C^{-(n-1)} \\ \mathcal{H}^i(W_n\omega^\bullet_{Y/s}) & \xrightarrow{V'-\pi} & \mathcal{H}^i(W_{n-1}\omega^\bullet_{Y/s}). \end{array}$$

□

Corollary 2.2.6. (cf. [I-R] IV §3). *The homomorphism* $V'-\pi\colon ZW_n\omega^i_{Y/s}\to ZW_{n-1}\omega^i_{Y/s}$ *is surjective and, if we denote its kernel by* $L^i_n$*, then* $W_n\omega^i_{Y/s,\,\log}\subset L^i_n$ *and* $L^i_n/W_n\omega^i_{Y/s,\,\log}$ *is killed by* $\pi^3$.

*Proof.* The surjectivity follows from Theorem 2.2.2 using the commutative diagram:

$$\begin{array}{ccc} W_{2n}\omega^i_{Y/s} & \xrightarrow{\pi^2-\pi F} & W_{2n-2}\omega^i_{Y/s} \\ {\scriptstyle F^n}\downarrow & & \downarrow{\scriptstyle F^{n-1}} \\ ZW_n\omega^i_{Y/s} & \xrightarrow{V'-\pi} & ZW_{n-1}\omega^i_{Y/s}. \end{array}$$

The assertion on the kernel follows from Theorem 2.2.2 and Lemma 2.2.7 below by considering the commutative diagram:

$$\begin{array}{ccc} ZW_{n+1}\omega^i_{Y/s} & \xrightarrow{\pi-F} & ZW_n\omega^i_{Y/s} \\ \| & & \downarrow{\scriptstyle V'} \\ ZW_{n+1}\omega^i_{Y/s} & \xrightarrow{\pi V'-\pi^2} & ZW_{n-1}\omega^i_{Y/s}. \end{array}$$

□

Lemma 2.2.7. *The homomorphism* $V'\colon ZW_{n+1}\omega^i_{Y/s}\to ZW_n\omega^i_{Y/s}$ *is surjective and its kernel is killed by* $\pi^2$.

*Proof.* In the diagram (2.1.6), the upper horizontal map is a surjection with its kernel $(V^nW_{n+2}\omega^i_{Y/s}+dV^{2n}W_2\omega^{i-1}_{Y/s})/V^{n+1}W_{n+1}\omega^i_{Y/s}$ (2.1.8). Hence $V'$ is surjective and its kernel is $p^nFW_{n+2}\omega^i_{Y/s}+dV^{n-1}W_2\omega^{i-1}_{Y/s}$. □

§2.3. The maximal slope.

In §2.3, we always assume that $Y$ is proper over $s$. For $i\in\mathbb{N}$, we define the projective system of morphisms of complexes (cf. [I-R] III (1.7)):

$$V'_{\le i}\colon\{\tau_{\le i}W_n\omega^\bullet_{Y/s}\to\tau_{\le i}W_{n-1}\omega^\bullet_{Y/s}\}_{n\ge1}$$

by the morphism $p^{i-j-1}\pi^2V$ in degree $j\le i-1$ and $V'\colon ZW_n\omega^i_{Y/s}\to ZW_{n-1}\omega^i_{Y/s}$ in degree $i$. If $j\le i$ or $i=d:=\dim Y$, then the natural homomorphism $H^j(Y,\tau_{\le i}W_n\omega^\bullet_{Y/s})\to H^j(Y,W_n\omega^\bullet_{Y/s})$ is an isomorphism and hence $V'_{\le i}$ induces an endomorphism on $H^j(Y,W\omega^\bullet_{Y/s})$, which we will also denote by $V'_{\le i}$. We need the following lemma, which is well-known for a $\sigma$-semilinear endomorphism and is proven in the same way as in the $\sigma$-semilinear case.

Lemma 2.3.1. *Let* $M$ *be a finitely generated* $W$*-module and let* $V$ *be a* $\sigma^{-1}$*-semilinear endomorphism of* $M$.

(1) *There exists a unique decomposition* $M_{\rm bij}\oplus M_{\rm nil}$ *of* $M$ *stable under* $V$ *such that* $V$ *is bijective on* $M_{\rm bij}$ *and* $p$*-adically nilpotent on* $M_{\rm nil}$.

(2) *If $k$ is algebraically closed, $V-1$ is surjective on $M$ and bijective on $M_{\rm nil}$. Furthermore, the natural homomorphism:*

$$W\otimes_{\mathbb{Z}_p} M^{V=1} = W\otimes_{\mathbb{Z}_p}(M_{\rm bij})^{V=1}\to M_{\rm bij}$$

*is bijective.*

Recall that we have canonical isomorphisms:

$$H^j(Y, W_n\omega^\bullet_{Y/s}) \cong H^j_{\rm crys}((Y,M_Y)/(W_n, W_n(L)))$$
$$H^j(Y, W\omega^\bullet_{Y}) \cong H^j_{\rm crys}((Y,M_Y)/(W, W(L)))$$

by Theorem 2.1.4 and the right hand sides are finitely generated modules over $W_n$ and over $W$ respectively. By applying Lemma 2.3.1 to $H^j(Y, W\omega^\bullet_{Y/s})$ and $V'_{\le i}$ for $i\ge j$ or $i=d$, we obtain a decomposition

$$H^j(Y,W\omega^\bullet_{Y/s}) = H^j(Y,W\omega^\bullet_{Y/s})_{V'_{\le i}-{\rm bij}}\oplus H^j(Y,W\omega^\bullet_{Y/s})_{V'_{\le i}-{\rm nil}} \tag{2.3.2}$$

and a natural isomorphism

$$W\otimes_{\mathbb{Z}_p} H^j(Y,W\omega^\bullet_{Y/s})^{V'_{\le i}=1}\xrightarrow{\sim} H^j(Y,W\omega^\bullet_{Y/s})_{V'_{\le i}-{\rm bij}} \tag{2.3.3}$$

if $k$ is algebraically closed.

LEMMA 2.3.4. *For any integers $i$ and $j$ such that $i\ge j$ or $i=d$, we have*

$$K_0\otimes_W H^j(Y,W\omega^\bullet_{Y/s})_{V'_{\le i}-{\rm bij}} = (K_0\otimes_W H^j(Y,W\omega^\bullet_{Y/s}))_{[i]},$$
$$K_0\otimes_W H^j(Y,W\omega^\bullet_{Y/s})_{V'_{\le i}-{\rm nil}} = (K_0\otimes_W H^j(Y,W\omega^\bullet_{Y/s}))_{[0,i[}.$$

(*See* §1.1 *for the definition of $D_I$ ($I\subset\mathbb{Q}$) for an $F$-isocrystal $D$ over $k$.*)

*Proof.* Let $\mathcal{F}$ denote the morphism $\tau_{\le i}W_n\omega^\bullet_{Y/s}\to\tau_{\le i}W_{n-1}\omega^\bullet_{Y/s}$ whose degree $q$-part is $p^qF$, which induces the Frobenius endomorphism $\varphi$ on $H^j(Y,W\omega^\bullet_{Y/s})$. Then $\mathcal{F}V'_{\le i} = V'_{\le i}\mathcal{F} = p^i\pi^2\colon \tau_{\le i}W_n\omega^\bullet_{Y/s}\to\tau_{\le i}W_{n-2}\omega^\bullet_{Y/s}$. Hence, we have $\varphi V'_{\le i} = V'_{\le i}\varphi = p^i$ on $H^j(Y,W\omega^\bullet_{Y/s})$, which implies the lemma. □

We set $H^j(Y, W\omega^i_{Y/s,\,\log}) := \varprojlim_n H^j(Y, W_n\omega^i_{Y/s,\,\log})$.

PROPOSITION 2.3.5. *Assume that $k$ is algebraically closed. Then, for any integers $i$ and $j$, we have*

$$H^0(Y,W\omega^i_{Y/s,\,\log})\xrightarrow{\sim} H^i(Y,W\omega^\bullet_{Y/s})^{V'_{\le i}=1},$$
$$H^j(Y,W\omega^d_{Y/s,\,\log})\xrightarrow{\sim} H^{j+d}(Y,W\omega^\bullet_{Y/s})^{V'_{\le d}=1}.$$

*Proof.* First note that $V\pi$ is a nilpotent endomorphism of $W_n\omega^i_{Y/s}$. By Corollary 2.2.6, the morphism of complexes

$$V'_{\leq i} - \pi\colon \tau_{\leq i}W_n\omega^\bullet_{Y/s} \longrightarrow \tau_{\leq i}W_{n-1}\omega^\bullet_{Y/s}$$

is surjective, and if we denote its kernel by $K^\bullet_{i,n}$, $K^j_{i,n} = \mathrm{Ker}(\pi\colon W_n\omega^j_{Y/s} \to W_{n-1}\omega^j_{Y/s})$ if $j \leq i-1$, $W_n\omega^i_{Y/s,\,\log} \subset K^i_{i,n}$ and $K^i_{i,n}/W_n\omega^i_{Y/s,\,\log}$ is annihilated by $\pi^3$. Hence we have a long exact sequence

$$\cdots \to H^j(Y, K^\bullet_{n,i}) \longrightarrow H^j(Y, \tau_{\leq i}W_n\omega^\bullet_{Y/s}) \overset{V'_{\leq i}-\pi}{\longrightarrow} H^j(Y, \tau_{\leq i}W_{n-1}\omega^\bullet_{Y/s}) \longrightarrow \cdots$$

and the natural homomorphism

$$H^{j-i}(Y, W\omega^i_{Y/s,\,\log}) \longrightarrow \varprojlim_n H^j(Y, K^\bullet_{n,i})$$

is bijective. By Lemma 2.3.1 (2), if $j \leq i$ or $i = d$, the endomorphism $V'_{\leq i} - 1$ on $H^j(Y, W\omega^\bullet_{Y/s})$ is surjective. On the other hand, we have the following morphism of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^j(Y/W)/p^n & \longrightarrow & H^j(Y/W_n) & \longrightarrow & H^{j+1}(Y/W)_{p^n} & \longrightarrow & 0 \\
 & & \Big\uparrow \text{proj} & & \Big\uparrow \text{proj} & & \Big\uparrow p & & \\
0 & \longrightarrow & H^j(Y/W)/p^{n+1} & \longrightarrow & H^j(Y/W_{n+1}) & \longrightarrow & H^{j+1}(Y/W)_{p^{n+1}} & \longrightarrow & 0,
\end{array}$$

where we abbreviate $(Y, M_Y)/(W, W(L))$ or $(W_n, W_n(L))$ to $Y/W$ or $W_n$. Hence, if the torsion part of $H^{j+1}((Y, M_Y)/(W, W(L)))$ is killed by $p^{\nu_{j+1}}$, then the cokernel of the homomorphism

$$V'_{\leq i} - \pi\colon H^j(Y, W_n\omega^\bullet_{Y/s}) \to H^j(Y, W_{n-1}\omega^\bullet_{Y/s})$$

is killed by $\pi^{\nu_{j+1}}$ if $j \leq i$ or $i = d$. By taking $\varprojlim_n$ of the above long exact sequences, we obtain

$$\varprojlim_n H^i(Y, K^\bullet_{n,i}) \xrightarrow{\sim} H^i(Y, W\omega^\bullet_{Y/s})^{V'_{\leq i}=1}$$
$$\varprojlim_n H^j(Y, K^\bullet_{n,d}) \xrightarrow{\sim} H^j(Y, W\omega^\bullet_{Y/s})^{V'_{\leq d}=1}.$$

□

We also need the following lemma (in (3.4.5)):

Lemma 2.3.6. *Assume that $k$ is algebraically closed. Then, for any integers $i \geq 0$ and $j \geq 0$, we have an isomorphism*

$$H^j(Y, W\omega^i_{Y/s,\,\log}) \xrightarrow{\sim} \varprojlim_n H^j(Y, \mathcal{H}^i(W_n\omega^\bullet_{Y/s}))^{V'=1}.$$

*Proof.* Set $M_n^j := H^j(Y, \mathcal{H}^i(W_n\omega^\bullet_{Y/s}))$, $M^j := \varprojlim_n M_n^j$ and let $M_n^{j\prime}$ be the image of $M^j$ in $M_n^j$. By (2.1.9), $M_n^j$ are finitely generated $W_n$-modules and hence $\{(V'-\pi)(M_{n+1}^j)\}_{n\geq 1}$ satisfies the Mittag-Leffler condition. On the other hand, by Lemma 2.3.7 below, $(V'-\pi)(M_{n+1}^{j\prime}) = M_n^{j\prime}$, which implies

$$\varprojlim_n((V'-\pi)(M_{n+1}^j)) \supset \varprojlim_n((V'-\pi)(M_{n+1}^{j\prime})) = \varprojlim_n M_n^{j\prime} = \varprojlim_n M_n^j.$$

Hence $\varprojlim_n(M_n^j/(V'-\pi)(M_{n+1}^j)) = 0$. Since $\{M_n^{j-1}/(V'-\pi)(M_{n+1}^{j-1})\}_{n\geq 1}$ satisfies the Mittag-Leffler condition, Corollary 2.2.5 implies $\varprojlim H^j(Y, K_n^i) \cong (M^j)^{V'=1}$. Since $\pi(p^{n-1}FW_{n+1}\omega^i_{Y/s}) = 0$, the LHS is isomorphic to $H^j(Y, W\omega^i_{Y/s,\,\log})$. □

Lemma 2.3.7. *Let $M_1$ and $M_2$ be $W$-modules of finite length, let $\pi\colon M_1 \to M_2$ be a surjective $W$-linear homomorphism and let $V'\colon M_1 \to M_2$ be a $\sigma^{-1}$-linear homomorphism. If $k$ is algebraically closed, then $V'-\pi\colon M_1 \to M_2$ is surjective.*

*Proof.* Using the short exact sequences $0 \to pM_i \to M_i \to M_i/pM_i \to 0$ $(i = 1, 2)$, we are easily reduced to the case $pM_i = 0$ $(i = 1, 2)$. In this case, $\pi$ has a $W$-linear section $s\colon M_2 \to M_1$ and $V' \circ s - \pi \circ s = V' \circ s - 1$ is surjective by Lemma 2.3.1 (2). □

## §3. The maximal unramified quotient of $p$-adic étale cohomology.

### §3.1. Statement of the main theorem.

Let $(S, N)$ be the scheme $\mathrm{Spec}(O_K)$ endowed with the canonical log structure (i.e. the log structure defined by its closed point). Let $f\colon (X, M) \to (S, N)$ be a smooth fs(=fine and saturated) log scheme and let $g\colon (Y, M_Y) \to (s, L)$ be the reduction of $f$ modulo the maximal ideal of $O_K$. We assume that $X$ is proper over $S$ and that $f$ is universally saturated, which is equivalent to saying that $g$ is of Cartier type, or also to saying that $Y$ is reduced ([Ts3]). Let $X_{\mathrm{triv}}$ denote the locus where the log structure $M$ is trivial, which is open and contained in the generic fiber of $X$. Let $(\overline{s}, \overline{L})$ be the scheme $\mathrm{Spec}(\overline{k})$ endowed with the inverse image of $L$, and set $(\overline{Y}, M_{\overline{Y}}) := (Y, M_Y) \times_{(s,L)} (\overline{s}, \overline{L})$. Set $(X_{\mathrm{triv}})_{\overline{K}} := X_{\mathrm{triv}} \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$. We will describe the maximal unramified quotients

$$H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}}, \mathbb{Q}_p(r))_{I_K}, \quad H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}}, \mathbb{Q}_p(d))_{I_K} \quad (r \geq d)$$

of $p$-adic étale cohomology groups and the images of $H^r_{\text{ét}}((X_{\text{triv}})_{\overline{K}}, \mathbb{Z}_p(r'))$ ($r' = r$ or $d$) in them in terms of the logarithmic Hodge-Witt sheaves of $(\overline{Y}, M_{\overline{Y}})/(\overline{s}, \overline{L})$ (Theorem 3.1.11).

In the rest of §3.1, we choose and fix an integer $r \geq 0$, and assume that $(X, M)/(S, N)$ and $r$ satisfy one of the following conditions:

(3.1.1) $r \leq p - 2$.

(3.1.2) Étale locally on $X$, there exists an étale morphism over $S$:

$$X \to \operatorname{Spec}((O_K[T_1, \ldots, T_u]/(T_1 \cdots T_u - \pi^e))[U_1, \ldots, U_s, V_1, \ldots, V_t])$$

for some integers $u \geq 1$, $s, t \geq 0$ and $e \geq 1$ such that $e \mid [K : K_0]$ and $X_{\text{triv}} = X - (Y \cup D)$, where $D$ is the inverse image of $\{U_1 \cdots U_s = 0\}$.

Let $i$ and $j$ denote the immersions $Y \to X$ and $X_{\text{triv}} \to X$ respectively. By [Ka4]Theorem (11.6), we have

$$M = \mathcal{O}_X \cap j_* \mathcal{O}^*_{X_{\text{triv}}} \quad \text{and} \quad M^{\text{gp}} = j_* \mathcal{O}^*_{X_{\text{triv}}} \tag{3.1.3}$$

and, hence, from the Kummer sequence on $(X_{\text{triv}})_{\text{ét}}$, we obtain a symbol map:

$$(i^* M^{\text{gp}})^{\otimes r} \longrightarrow i^* R^r j_* \mathbb{Z}/p^n\mathbb{Z}(r). \tag{3.1.4}$$

Theorem 3.1.5. ([Bl-Ka]Theorem (1.4), [H](1.6.1), [Ts2], [Ts4]). *The homomorphism* (3.1.4) *is surjective.*

*Proof.* In the case (3.1.2) with $s = 0$, this is [H](1.6.1) (=[Bl-Ka]Theorem (1.4) in the good reduction case). The case (3.1.2) for a general $s$ is reduced to the case $s = 0$ as in the proof of [Ts2] Lemma 3.4.7. (The proof of Lemma 3.4.7 (1) works without the assumption $\mu_{p^n} \subset K$). In the case (3.1.1), we are easily reduced to the case $n = 1$, and the theorem follows from [Ts3]Theorem 5.1 and Proposition A15 with $r = q$. □

We have a surjective homomorphism (§2.2):

$$(M_Y^{\text{gp}})^{\otimes r} \longrightarrow W_n\omega^r_{Y/s,\,\log}; a_1 \otimes \cdots \otimes a_r \mapsto d\log(a_1) \wedge \cdots \wedge d\log(a_r). \tag{3.1.6}$$

Proposition 3.1.7. (cf. [Bl-Ka]Theorem (1.4) (i), [H](1.6.2)). *There exists a unique surjective homomorphism:*

$$i^* R^r j_* \mathbb{Z}/p^n\mathbb{Z}(r) \longrightarrow W_n\omega^r_{Y/s,\,\log} \tag{3.1.8}$$

*such that the following diagram commutes:*

$$\begin{array}{ccc} (i^* M^{\text{gp}})^{\otimes r} & \longrightarrow & i^* R^r j_* \mathbb{Z}/p^n\mathbb{Z}(r) \\ \downarrow & & \downarrow \\ (M_Y^{\text{gp}})^{\otimes r} & \longrightarrow & W_n\omega^r_{Y/s,\,\log}. \end{array}$$

*Proof.* We have the required map $\nu^*(i^*R^rj_*\mathbb{Z}/p^n\mathbb{Z}(r)) \to \nu^*(W_n\omega^r_{Y/s,\,\log})$ for each generic point $\nu\colon \eta \to Y$ of $Y$ ([Bl-Ka] (6.6)). Note that $Y$ is reduced. Since the homomorphism $\omega^r_{Y/s} \to \oplus_\nu \nu_*\nu^*\omega^r_{Y/s}$ is injective, the homomorphism $W_n\omega^r_{Y/s,\,\log} \to \oplus_\nu\nu_*\nu^*W_n\omega^r_{Y/s,\,\log}$ is also injective by Corollary 2.2.4. Now the proposition follows from Theorem 3.1.5 (cf. [Bl-Ka] (6.6)). □

The condition (3.1.1) or (3.1.2) still holds for the base change of $(X,M)/(S,N)$ by any finite extension of $K$ contained in $\overline{K}$. Hence, by taking the inductive limit, we obtain:

$$\overline{i}^*R^r\overline{j}_*\mathbb{Z}/p^n\mathbb{Z}(r) \longrightarrow W_n\omega^r_{\overline{Y}/\overline{s},\log} \tag{3.1.9}$$

Here $\overline{X} = X \times_{\mathrm{Spec}(O_K)} \mathrm{Spec}(O_{\overline{K}})$, and $\overline{i}$ and $\overline{j}$ denote the morphisms $\overline{Y} \to \overline{X}$ and $(X_{\mathrm{triv}})_{\overline{K}} \to \overline{X}$ respectively. Note that, for any fs log structure $L'$ on $s$ and a morphism $(s,L') \to (s,L)$, $W_n\omega^\bullet$ and $W_n\omega^\bullet_{\log}$ associated to $(Y,M_Y)/(s,L)$ and $(Y,M_Y)\times_{(s,L)}(s,L')/(s,L')$ coincide by the base change theorem [H-Ka]Proposition (2.23) (or [Ts2] Proposition 4.3.1).
Let $d = \dim X_K$. Then, for any affine scheme $U$ étale over $(X_{\mathrm{triv}})_{\overline{K}}$, we have $H^i(U,\mathbb{Z}/p^n\mathbb{Z}(d)) = 0$ $(i > d)$. Hence we have $\overline{i}^*R^i\overline{j}_*\mathbb{Z}/p^n\mathbb{Z}(d) = 0$ $(i > d)$. By the proper base change theorem, we obtain from (3.1.9) homomorphisms:

$$\begin{aligned} &H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}},\mathbb{Z}/p^n\mathbb{Z}(r)) \longrightarrow H^0_{\text{ét}}(\overline{Y},W_n\omega^r_{\overline{Y}/\overline{s},\log}),\\ &H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}},\mathbb{Z}/p^n\mathbb{Z}(d)) \longrightarrow H^{r-d}_{\text{ét}}(\overline{Y},W_n\omega^d_{\overline{Y}/\overline{s},\log})\end{aligned} \tag{3.1.10}$$

THEOREM 3.1.11. (1) (cf. [Sat] Lemma 3.3). *Assume $K = K_0$ in the case* (3.1.1) *and $s = 0$ in the case* (3.1.2). *Then the homomorphisms* (3.1.10) *induce isomorphisms:*
(3.1.12)

$$\begin{aligned} &H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}},\mathbb{Q}_p(r))_{I_K} \xrightarrow{\sim} \mathbb{Q}_p\otimes_{\mathbb{Z}_p} H^0_{\text{ét}}(\overline{Y},W\omega^r_{\overline{Y}/\overline{s}\,,\log})\\ &H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}},\mathbb{Q}_p(d))_{I_K} \xrightarrow{\sim} \mathbb{Q}_p\otimes_{\mathbb{Z}_p} H^{r-d}_{\text{ét}}(\overline{Y},W\omega^d_{\overline{Y}/\overline{s}\,,\log}) \qquad \textit{if } r \geq d\end{aligned}$$

(2) *In the case* (3.1.1), *if $K = K_0$,* (3.1.10) *induce isomorphisms:*
(3.1.13)

$$\begin{aligned} &H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}},\mathbb{Z}_p(r))_{I_K}/tor \xrightarrow{\sim} H^0_{\text{ét}}(\overline{Y},W\omega^r_{\overline{Y}/\overline{s},\,\log})\\ &H^r_{\text{ét}}((X_{\mathrm{triv}})_{\overline{K}},\mathbb{Z}_p(d))_{I_K}/tor \xrightarrow{\sim} H^{r-d}_{\text{ét}}(\overline{Y},W\omega^d_{\overline{Y}/\overline{s},\,\log})/tor \qquad \textit{if } r \geq d\end{aligned}$$

*Remark 3.1.14.* For (1) in the case (3.1.2), if we use [Y] (resp. Theorem 4.1.2), the proof of Theorem 3.1.11 in 3.2–3.4 works without the assumption $s = 0$ (resp. under the assumption (4.1.1)). See Remark 3.2.3.

§3.2. REVIEW OF THE SEMI-STABLE CONJECTURE.
We will review comparison theorems between $p$-adic étale cohomology and crystalline cohomology.

Let $f\colon (X,M) \to (S,N)$ and $g\colon (Y,M_Y) \to (s,L)$ be the same as in §3.1. Then the crystalline cohomology $D^q := K_0 \otimes_W \varprojlim_n H^q((Y,M_Y)/(W_n, W_n(L))$ ([H-Ka] (3.2)) is a finite dimensional $K_0$-vector space endowed with a $\sigma$-semi-linear automorphism $\varphi$ and a linear endomorphism $N$ ([H-Ka] (3.4), (3.5), (3.6)) satisfying the relation $N\varphi = p\varphi N$. We choose and fix a uniformizer $\pi$ of $K$. Then there exists a canonical isomorphism ([H-Ka] Theorem (5.1)):

$$\rho_\pi\colon K \otimes_{K_0} D^q \cong H^q(X_K, \Omega^\bullet_{X_K}(\log M_K)). \tag{3.2.1}$$

Using the Hodge filtration on the RHS, the crystalline cohomology $D^q$ becomes an object of $\underline{MF}_K(\varphi, N)$ (§1.2). Set $V^q := H^q_{\text{ét}}((X_{\text{triv}})_{\overline{K}}, \mathbb{Q}_p)$, which is a finite dimensional $\mathbb{Q}_p$-vector space endowed with a continuous and linear action of $G_K$.

Theorem 3.2.2. (The semi-stable conjecture by Fontaine-Jannsen:[Ka3], [Ts2]). *Assume that $(X,M)/(S,N)$ satisfies the condition (3.1.2) with $s=0$. Then, for any integer $q \geq 0$, $V^q$ is a semi-stable $p$-adic representation and there exists a canonical isomorphism $D_{\text{st}}(V^q) \cong D^q$ in $\underline{MF}_K(\varphi, N)$. Here we define $D_{\text{st}}$ using the same uniformizer $\pi$ as (3.2.1).*

*Remark 3.2.3.* G. Faltings ([Fa]) proved the theorem without the assumption on $(X,M)/(S,N)$. However his construction of the comparison map is different from that in [Ka3], [Ts2] via syntomic cohomology, and we will use the latter construction in the proof of Theorem 3.1.11. Recently, G. Yamashita [Y] proved that the comparison map via syntomic cohomology is an isomorphism for any $(X,M)$ satisfying (3.1.2). We give an alternative proof in §4 when the horizontal divisors at infinity do not have self-intersections.

To prove Theorem 3.1.11 (2), we need the following refinement by C. Breuil for the integral $p$-adic étale cohomology $T^q := H^q_{\text{ét}}((X_{\text{triv}})_{\overline{K}}, \mathbb{Z}_p)/\text{tor}$. We assume $K = K_0$ and $q \leq p-2$. Let $(E_n, M_{E_n})$ be the scheme $\text{Spec}(W_n\langle u\rangle) = \text{Spec}(W_n\langle u-p\rangle)$ endowed with the log structure associated to $\mathbb{N} \to W_n\langle u\rangle; 1 \mapsto u$. We have a closed immersion $i_p\colon (S_n, N_n) \hookrightarrow (E_n, M_{E_n})$ defined by $u \mapsto p$. Then the crystalline cohomology $\mathcal{M}^q_n := H^q((X_n, M_n)/(E_n, M_{E_n})) \cong H^q((X_1, M_1)/(E_n, M_{E_n}))$ is naturally regarded as an object of $\underline{\mathcal{MF}}_{W,[0,q],\text{tor}}(\varphi, N)$ ([Br3] Théorème 2.3.2.1). Set $\mathcal{M}^q := (\varprojlim_n \mathcal{M}^q_n)/\text{tor}$ and $\mathcal{M}'^q_n := \mathcal{M}^q/p^n\mathcal{M}^q$. Then $\{\mathcal{M}'^q_n\}_n$ becomes a projective system of objects of $\underline{\mathcal{MF}}_{W,[0,p-2],\text{tor}}(\varphi, N)$ satisfying the condition in the beginning of §1.6 ([Br3] 4.1).

Theorem 3.2.4. ([Br3] Théorème 3.2.4.7, §4.2, [Ts4]). *Assume $K = K_0$ and let $q$ be any integer such that $0 \leq q \leq p-2$. Then there exist canonical $G_K$-equivariant isomorphisms:*

$$T_{\text{st}}(\mathcal{M}^q_n) \cong H^q_{\text{ét}}((X_{\text{triv}})_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z}), \quad T_{\text{st}}(\mathcal{M}'^q_n) \cong T^q/p^nT^q$$

The object of $\underline{MF}_K(\varphi, N)$ associated to the projective system $\{\mathcal{M}'^q_n\}_{n\geq 0}$ (§1.6) is canonically isomorphic to $D^q$ ([Br3] Proposition 4.3.2.3 and the remark after Corollaire 4.3.2.4). Hence, Theorem 3.2.4 implies (§1.5):

Theorem 3.2.5. ([Br3]Corollaire 4.3.2.4 and the following remark). *If $K = K_0$, for any integer $0 \leq q \leq p-2$, the p-adic étale cohomology $V^q$ is a semi-stable p-adic representation and there exists a canonical isomorphism $D_{\mathrm{st}}(V^q) \cong D^q$ in $\underline{MF}_K(\varphi, N)$*

By Corollary 1.3.3, Lemma 2.3.4 and Proposition 2.3.5, we obtain the following isomorphisms form Theorem 3.2.2 (resp. 3.2.5) for an integer $r \geq 0$ such that the condition (3.1.2) with $s = 0$ is satisfied (resp. the condition (3.1.1) is satisfied and $K = K_0$):

$$V^r(r)_{I_K} \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^0_{\text{ét}}(\overline{Y}, W\omega^r_{\overline{Y}/\overline{s},\log}) \tag{3.2.6}$$

$$V^r(d)_{I_K} \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^{r-d}_{\text{ét}}(\overline{Y}, W\omega^d_{\overline{Y}/\overline{s},\log}) \qquad \text{if } r \geq d. \tag{3.2.7}$$

Here $d = \dim(X_K)$. See the proof of Lemma 2.3.4 for the relation between $\varphi$ and $V'_{\leq i}$. In the case (3.1.2), if we use the result in [Y] (resp. Theorem 4.1.2), we obtain the isomorphisms without the assumption $s = 0$ (resp. under the condition (4.1.1)).

In the case (3.1.1) and $K = K_0$, the image of $\mathcal{M}^r$ under the projection $\mathcal{M}^r \to D^r$ given by $u \mapsto 0$ coincides with the image of $H^r((Y, M)/(W, W(L)))(\cong H^r(Y, W\omega^\bullet_{Y/s}))$ ([Br3] Proposition 4.3.1.3). Hence, by Corollary 1.3.3, Theorem 1.6.2, (2.3.2), Lemma 2.3.4, Proposition 2.3.5 and Theorem 3.2.4, we see that the isomorphisms (3.2.6) and (3.2.7) induce isomorphisms:

$$T^r(r)_{I_K}/\mathrm{tor} \cong H^0_{\text{ét}}(\overline{Y}, W\omega^r_{\overline{Y}/\overline{s},\log}) \tag{3.2.8}$$

$$T^r(d)_{I_K}/\mathrm{tor} \cong H^{r-d}_{\text{ét}}(\overline{Y}, W\omega^d_{\overline{Y}/\overline{s},\log})/\mathrm{tor} \quad \text{if } r \geq d. \tag{3.2.9}$$

To prove Theorem 3.1.11, it remains to prove that (3.2.6) and (3.2.7) are induced by (3.1.10).

§3.3. Review of the construction of the comparison map.

We will review the construction of the comparison map in Theorems 3.2.2 and 3.2.5. First recall that to give an isomorphism $D_{\mathrm{st}}(V^q) \cong D^q$ is equivalent to give a $B_{\mathrm{st}}$-linear isomorphism:

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^q \xrightarrow{\sim} B_{\mathrm{st}} \otimes_{K_0} D^q \tag{3.3.1}$$

compatible with $\varphi$, $N$, the actions of $G_K$ and $Fil^\cdot$ after $B_{\mathrm{dR}} \otimes_{B_{\mathrm{st}}}$. Recall also that we fixed a uniformizer $\pi$ of $K$ (in the case $K = K_0$, we choose $p$ as $\pi$) in order to define the functor $D_{\mathrm{st}}$ (or equivalently the embedding $B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$) and to define the filtration on $K \otimes_{K_0} D^q$.

Let $(E_n, M_{E_n})$ be the PD-envelope of the exact closed immersion $(S_n, N_n) \hookrightarrow (\mathrm{Spec}(W_n[u]), \mathcal{L}(u))$ defined by $u \mapsto \pi$ compatible with the canonical PD-structure on $pW_n$. Here $\mathcal{L}(u)$ denotes the log structure defined by $\mathbb{N} \to W[u]; 1 \mapsto u$. Since $(E_n, M_{E_n})$ is isomorphic to the PD-envelope of $(S_1, N_1) \hookrightarrow (\mathrm{Spec}(W_n[u]), \mathcal{L}(u))$, the lifting of Frobenius of $(\mathrm{Spec}(W_n[u]), \mathcal{L}(u))$ defined by

$\sigma\colon W_n \to W_n$ and $u \mapsto u^p$ induces that of $(E_n, M_{E_n})$, which is compatible with the PD-structure $\overline{\delta}$ on the ideal $\overline{J}_{E_n}$ of $\mathcal{O}_{E_n}$ defining $S_1$. We denote by $i_{E_n,\pi}$ the canonical exact closed immersion $(S_n, N_n) \hookrightarrow (E_n, M_{E_n})$. We have $\Gamma(E_n, \mathcal{O}_{E_n}) = W[u, \frac{u^{en}}{n!}\ (n \in \mathbb{N})]/p^n$ ($e = [K : K_0]$) and $(E_n, M_{E_n})$ coincides with the log scheme appearing before Theorem 3.2.4 when $K = K_0$ and $\pi = p$. Let $W_n(L)$ be the "Teichmüller lifting" ([H-Ka] Definition (3.1)) of the log structure $L$ on $s$ to $\mathrm{Spec}(W_n)$, which already appeared in the definition of $D^q$. Then, we have a closed immersion $i_{E_n,0}\colon (\mathrm{Spec}(W_n), W_n(L)) \hookrightarrow (E_n, M_{E_n})$ defined by $u \mapsto 0$, which is compatible with the lifting of Frobenius.

First we review the crystalline interpretation of $(B^+_{\mathrm{st}} \otimes_{K_0} D)^{N=0}$. We set $R_{E,\mathbb{Q}_p} := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n \Gamma(E_n, \mathcal{O}_{E_n})$. We define the crystalline cohomology $\mathcal{D}^q$ to be

$$\begin{aligned}&\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^q_{\mathrm{crys}}((X_n, M_n)/(E_n, M_{E_n}, \overline{J}_{E_n}, \overline{\delta}))\\ \cong&\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^q_{\mathrm{crys}}((X_1, M_1)/(E_n, M_{E_n}, \overline{J}_{E_n}, \overline{\delta})),\end{aligned}$$

which is an $R_{E,\mathbb{Q}_p}$-module endowed with $\varphi$ and $N$ satisfying $N\varphi = p\varphi N$ ([Ts2] 4.3). The projection $\mathrm{pr}_0\colon \mathcal{D}^q \to D^q$ induced by the exact closed immersions $\{i_{E_n,0}\}$, which is compatible with $\varphi$, $N$, has a unique $K_0$-linear section $s\colon D^q \to \mathcal{D}^q$ compatible with $\varphi$ and $N$, and it induces an isomorphism $R_{E,\mathbb{Q}_p} \otimes_{K_0} D^q \xrightarrow{\sim} \mathcal{D}^q$ ([H-Ka] Lemma (5.2), [Ts2] Propositions 4.4.6, 4.4.9).

We define $H^q_{\mathrm{crys}}((\overline{X}_n, \overline{M}_n)/(E_n, M_{E_n}))$ to be the inductive limit of $H^q_{\mathrm{crys}}((X'_n, M'_n)/(E_n, M_{E_n}))$, where $(X', M')$ ranges over the base changes of $(X, M)$ by all finite extensions $K'$ of $K$ contained in $\overline{K}$. We define $\overline{\mathcal{D}}^q$ to be $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^q_{\mathrm{crys}}((\overline{X}_n, \overline{M}_n)/(E_n, M_{E_n}))$, which is also endowed with $\varphi$ and $N$ ([Ts2] 4.3). In the special case $(X, M) = (S, N)$, we have $\mathcal{D}^q = 0$ ($q > 0$) ([Ka3] Proposition (3.1), [Ts2] Lemma 1.6.7) and $\widehat{B^+_{\mathrm{st}}} = \mathcal{D}^0$ by definition ([Br1]§2). Furthermore, there exists a canonical isomorphism (the crystalline interpretation of $B^+_{\mathrm{st}}$) $\iota\colon B^+_{\mathrm{st}} \cong \widehat{B^+_{\mathrm{st}}}^{N\text{-nil}}$ compatible with the actions of $G_K$, $\varphi$ and $N$ ([Ka3] Theorem (3.7)). Here $N$-nil denotes the part where $N$ is nilpotent. Let us return to a general $(X, M)$. Then we have a Künneth isomorphism $\widehat{B^+_{\mathrm{st}}} \otimes_{R_{E,\mathbb{Q}_p}} \mathcal{D}^q \xrightarrow{\sim} \overline{\mathcal{D}}^q$ ([Ka3]§4, [Ts2]Proposition 4.5.4). By taking $N = 0$, we obtain the following crystalline interpretation of $(B^+_{\mathrm{st}} \otimes_{K_0} D^q)^{N=0}$ ([Ka3]§4, [Ts2]§4.5).

$$(3.3.2)\qquad (B^+_{\mathrm{st}} \otimes_{K_0} D^q)^{N=0} \xrightarrow[\iota\otimes s]{\sim} (\widehat{B^+_{\mathrm{st}}} \otimes_{R_{E,\mathbb{Q}_p}} \mathcal{D}^q)^{N=0} \xrightarrow{\sim} (\overline{\mathcal{D}}^q)^{N=0}.$$

To compare $\overline{\mathcal{D}}^q$ with $V^q$, we use syntomic cohomology. The syntomic complex $\mathcal{S}^{\sim}_n(r)_{(X,M)}$ ([Ts2]§2.1) is an object of $D^+(Y_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ such that there exists a canonical distinguished triangle:

$$(3.3.3)\qquad \to \mathcal{S}^{\sim}_n(r)_{(X,M)} \to Ru_{(X_n,M_n)/W_n *} J^{[r]}_{(X_n,M_n)/W_n} \xrightarrow{p^r-\varphi} Ru_{(X_n,M_n)/W_n *}\mathcal{O}_{(X_n,M_n)/W_n},$$

where $u_{(X_n,M_n)/W_n}$ denotes the canonical morphism of topoi $((X_n,M_n)/W_n)^{\sim}_{\mathrm{crys}} \to (X_n)^{\sim}_{\text{ét}}$. We define the syntomic cohomology $H^q(\overline{Y},\mathcal{S}_{\mathbb{Q}_p}(r)_{(\overline{X},\overline{M})})$ to be $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\varprojlim_n(\varinjlim_{K'} H^q_{\text{ét}}(Y',\mathcal{S}^{\sim}_n(r)_{(X',M')}))$, where $K'$ ranges over all finite extensions of $K$ contained in $\overline{K}$ and $(X',M')$ denotes the base change of $(X,M)$ by $O_K\to O_{K'}$. From the above distinguished triangle (3.3.3), we obtain a natural homomorphism

$$(3.3.4)\quad H^q(\overline{Y},\mathcal{S}_{\mathbb{Q}_p}(r)_{(\overline{X},\overline{M})})\to (\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\varprojlim_n H^q_{\mathrm{crys}}((\overline{X}_n,\overline{M}_n)/W_n))^{\varphi=p^r}\to(\overline{\mathcal{D}}^q)^{N=0,\varphi=p^r}.$$

On the other hand, if we set $\mathbb{Z}/p^n\mathbb{Z}(r)'=(\frac{1}{p^a a!}\mathbb{Z}_p(r))/p^n$ $(r=a(p-1)+b, a\geq 0, 0\leq b\leq p-2)$, we have a canonical map

$$(3.3.5)\qquad \mathcal{S}^{\sim}_n(r)_{(X,M)}\longrightarrow i^*Rj_*\mathbb{Z}/p^n\mathbb{Z}(r)'$$

in $D^+(Y_{\text{ét}},\mathbb{Z}/p^n\mathbb{Z})$ ([Ts2] §3.1), which induces a homomorphism

$$(3.3.6)\qquad H^q(\overline{Y},\mathcal{S}_{\mathbb{Q}_p}(r)_{(\overline{X},\overline{M})})\longrightarrow V^q(r)$$

by taking $H^q_{\text{ét}}(Y,-)$ and the inductive limit with respect to all finite base changes of $(X,M)$. We do not identify $\mathbb{Z}/p^n\mathbb{Z}(r)'$ with $\mathbb{Z}/p^n\mathbb{Z}(r)$ by the multiplication by $p^a a!$ because (3.3.5) is compatible with the products and the Tate twist if we use the natural maps $\mathbb{Z}/p^n\mathbb{Z}(r)'\otimes\mathbb{Z}/p^n\mathbb{Z}(s)'\to\mathbb{Z}/p^n\mathbb{Z}(r+s)'$ and $\mathbb{Z}/p^n\mathbb{Z}(r)'(1)\to\mathbb{Z}/p^n\mathbb{Z}(r+1)'$.

Theorem 3.3.7. ([Ku], [Ka3]Theorem (5.4), [Ts2]Theorem 3.4.4, [Ts4]Theorem 5.1). *Assume $(X,M)/(S,N)$ and $r$ satisfy one of the conditions* (3.1.1) *and* (3.1.2). *Then the homomorphism* (3.3.6) *is an isomorphism if $q\leq r$ or $r=d=\dim X_K$.*

For the case $r=d$, note $\overline{i}^*R^i\overline{j}_*\mathbb{Z}/p^n\mathbb{Z}(d)=0$ $(i>d)$, $\mathcal{H}^i(\mathcal{S}^{\sim}_n(d)_{(\overline{X},\overline{M})})=0$ $(i>d+1)$ ([Ts2] (2.3.3) and Lemma 2.3.4), and $p^N\mathcal{H}^{d+1}(\mathcal{S}^{\sim}_n(d)_{(\overline{X},\overline{M})})=0$ for some $N>0$ independent of $n$ ([Ts2] Lemma 2.3.19: $\mathcal{H}^{d+1}(C_n(d))=0$ and the proof of [Ts2] Theorem 2.3.2).
Thus, under the condition (3.1.1) or (3.1.2), we obtain a homomorphism

$$(3.3.8)\quad V^q(r)\underset{p^{-r}\cdot(3.3.6)}{\overset{\sim}{\longleftarrow}} H^q(\overline{Y},\mathcal{S}_{\mathbb{Q}_p}(r)_{(\overline{X},\overline{M})})\underset{(3.3.4)}{\longrightarrow}(\overline{\mathcal{D}}^q)^{N=0,\varphi=p^r}\underset{(3.3.2)}{\overset{\sim}{\longleftarrow}}(B^+_{\mathrm{st}}\otimes_{K_0}D^q)^{N=0,\varphi=p^r}.$$

if $q\leq r$ or $r=d$. By using $\mathbb{Q}_p(-r)\subset B_{\mathrm{st}}$, we obtain a homomorphism

$$(3.3.9)\qquad B_{\mathrm{st}}\otimes_{\mathbb{Q}_p}V^q\longrightarrow B_{\mathrm{st}}\otimes_{K_0}D^q,$$

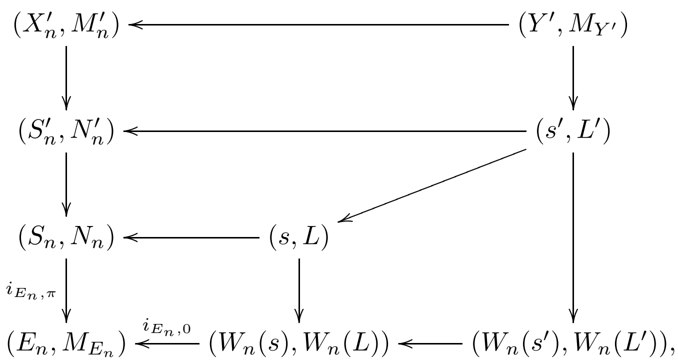
which is independent of $r$ ([Ts2] Corollary 4.8.8 and Lemma 4.9.1). If $K=K_0$ in the case (3.1.1) and $s=0$ in the case (3.1.2), then (3.3.9) gives the required isomorphism $D_{\mathrm{st}}(V^q)\cong D^q$. In the case (3.1.2), the result of [Y] (resp. Theorem 4.1.2) says that (3.3.9) still gives the isomorphism without the condition $s=0$ (resp. under (4.1.1)).

§3.4. Proof of the main theorem.

Let us prove the main theorem: Theorem 3.1.11. Let $(X, M)/(S, N)$ be as in the beginning of §3.1. We fix an integer $r \geq 0$ and assume that $(X, M)/(S, N)$ and $r$ satisfy (3.1.1) and $K = K_0$, or (3.1.2) with $s = 0$. (In the case (3.1.2), we can remove (resp. replace) the condition $s = 0$ (resp. by (4.1.1)) if we use [Y] (resp. Theorem 4.1.2).) The remaining problem is only to prove that the isomorphisms (3.2.6) and (3.2.7) induced by the comparison isomorphism $D_{\mathrm{st}}(V^r) \cong D^r$ (Theorems 3.2.2 and 3.2.5) coincide with the homomorphisms induced by (3.1.10).

First we define canonical projections $\overline{f}_0 \colon \widehat{B^+_{\mathrm{st}}} \to P_0$ and $\overline{\mathrm{pr}}_0 \colon \overline{\mathcal{D}}^q \to P_0 \otimes_{K_0} D^q$ compatible with $\varphi$. Let $K'$ be any finite extension of $K$ contained in $\overline{K}$, let $(S', N')$ be the scheme $\mathrm{Spec}(O_{K'})$ endowed with the canonical log structure, and set $(X', M') := (X, M) \times_{(S,N)} (S', N')$. Then we have a commutative diagram:

(3.4.1)



which induces a homomorphism

$$H^q((X'_n, M'_n)/(E_n, M_{E_n})) \longrightarrow H^q((Y', M_{Y'})/(W_n(s'), W_n(L')))$$
$$\xleftarrow{\sim} W_n(k') \otimes_{W_n} H^q((Y, M_Y)/(W_n, W_n(L))).$$

By taking the inductive limit with respect to $K'$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n$, we obtain

$$\overline{\mathrm{pr}}_0 \colon \overline{\mathcal{D}}^q \longrightarrow P_0 \otimes_W D^q.$$

This is compatible with $\varphi$ and the action of $G_K$, but not with $N$. In the special case $(X, M) = (S, N)$ and $q = 0$, this becomes a ring homomorphism

$$\overline{f}_0 \colon \widehat{B^+_{\mathrm{st}}} \longrightarrow P_0$$

and the above homomorphism $\overline{\mathrm{pr}}_0$ is compatible with $\overline{f}_0$. By definition, $\overline{\mathrm{pr}}_0$ and $\overline{f}_0$ are compatible with $\mathrm{pr}_0 \colon \mathcal{D}^q \to D^q$ and $f_0 \colon R_{E,\mathbb{Q}_p} \to K_0$ induced by $\{i_{E_n,0}\}$.

LEMMA 3.4.2. *Let $V$ be a semi-stable $p$-adic representation of $G_K$ and set $D := D_{\mathrm{st}}(V)$. Let $s$ be a positive integer such that $Fil^{s+1}D_K = 0$ and let $D'$ be the quotient of $D$ corresponding to the quotient $V(s)_{I_K}(-s)$ of $V$ (§1.3). Then the following diagram is commutative:*

$$\begin{array}{ccc} Fil^s(B_{\mathrm{st}}^+ \otimes_{K_0} D)^{N=0,\varphi=p^s} & \xrightarrow[\overline{f}_0\otimes 1]{} & (P_0 \otimes_{K_0} D)^{\varphi=p^s} \\ \downarrow & & \downarrow \wr \\ Fil^s(B_{\mathrm{st}}^+ \otimes_{K_0} D')^{N=0,\varphi=p^s} & \xleftarrow{\sim} & (P_0 \otimes_{K_0} D')^{\varphi=p^s}. \end{array}$$

*Recall $D' = D_{[s]}$, $Fil^s D'_K = D'_K$, $Fil^{s+1}D'_K = 0$ and $N = 0$ on $D'$.*

*Proof.* We are reduced to the case $D = D'$, in which case the claim is trivial because $\overline{f}_0$ is a $P_0$-algebra homomorphism. □

Note that, with the notation of Lemma 3.4.2, the isomorphism in Corollary 1.3.3 is characterized by the following commutative diagram:
(3.4.3)

$$\begin{array}{ccccc} V(s) & & = & & Fil^s(B_{\mathrm{st}} \otimes_{K_0} D)^{N=0,\varphi=p^s} \\ \downarrow & & & & \downarrow \\ V(s)_{I_K} & \underset{\text{Cor. 1.3.3}}{\cong} & (P_0 \otimes_{K_0} D')^{\varphi=p^s} & \xrightarrow{\sim} & Fil^s(B_{\mathrm{st}} \otimes_{K_0} D')^{N=0,\varphi=p^s}. \end{array}$$

LEMMA 3.4.4. *Let $K'$, $(S', N')$ and $(X', M')$ be as in the definition of $\overline{pr}_0$ above. Then the composites of the following two sequences of homomorphisms coincide:*

$$\mathcal{H}^r(\mathcal{S}_n^{\sim}(r)_{(X',M')}) \xrightarrow{p^r\cdot(\mathrm{A})} R^r u_{(X'_n,M'_n)/W_n *}\mathcal{O} \xrightarrow{(\mathrm{B})} R^r u_{(X'_n,M'_n)/(E_n,M_{E_n})*}\mathcal{O}$$
$$\xrightarrow{(\mathrm{C})} R^r u_{(Y',M_{Y'})/(W_n(s'),W_n(L'))*}\mathcal{O} \overset{(\mathrm{D})}{\cong} \mathcal{H}^r(W_n\omega^\bullet_{Y'/s'}) \xrightarrow{p^m} \mathcal{H}^r(W_n\omega^\bullet_{Y'/s'}),$$
$$\mathcal{H}^r(\mathcal{S}_n^{\sim}(r)_{(X',M')}) \xrightarrow{(\mathrm{E})} i'^* R^r j'_* \mathbb{Z}/p^n\mathbb{Z}(r)' \xrightarrow{\text{“}p^m\text{”}} i'^* R^r j'_* \mathbb{Z}/p^n\mathbb{Z}(r)$$
$$\xrightarrow{(\mathrm{F})} W_n\omega^r_{Y'/s',\,\log} \longrightarrow \mathcal{H}^r(W_n\omega^\bullet_{Y'/s'}).$$

*Here the integer $m$ is defined by $p^a a! = p^m\cdot$unit in $\mathbb{Z}_p$, $r = a(p-1)+b$, $a, b \in \mathbb{Z}$, $0 \le b \le p-2$. The homomorphisms (A), (B), (C), (D), (E) and (F) are induced by the distinguished triangle* (3.3.3)*, the morphism $(E_n, M_{E_n}) \to$ Spec$(W_n)$, the diagram* (3.4.1)*, Theorem* 2.1.4*,* (3.3.5) *and* (3.1.8) *respectively.*

*Proof.* The question is étale local on $X'$. Étale locally on $X'$, by choosing a closed immersion of $(X', M')$ into a smooth fine log scheme $(Z, M_Z)$ over $W$ with liftings of Frobenius $\{F_{Z_n}\}$ of $\{(Z_n, M_{Z_n})\}$ satisfying the condition [Ts2] (2.1.1), we can define a complex $\mathcal{S}'_n(r)$ ([Ts2]§2.1) and a natural homomorphism $\mathcal{S}_n^{\sim}(r) \to \mathcal{S}'_n(r)$ ([Ts2](2.1.2)). By the definition of the morphisms [Ts2](2.1.2) and [Ts2](3.1.1) (=(3.3.5)), the homomorphisms $p^r\cdot$ (A) and (E) canonically factor through $\mathcal{H}^r(\mathcal{S}'_n(r))$. Hence we may replace $\mathcal{H}^r(\mathcal{S}_n^{\sim}(r))$ with $\mathcal{H}^r(\mathcal{S}'_n(r))$.

We denote by $\{a_1,\dots,a_r\}_{\text{ét}}$ (resp. $\{a_1,\dots,a_r\}_{\text{syn}}$) the image of $a_1\otimes\cdots\otimes a_r$ ($a_i\in M'^{\text{gp}}$) under the symbol map (3.1.4) (resp. $(i'^*M'^{\text{gp}})^{\otimes r}\to\mathcal{H}^r(\mathcal{S}'_n(r))$ ([Ts2](2.2.1)). Then the local section $\{a_1,\dots,a_r\}_{\text{syn}}$ is sent to $\{a_1,\dots,a_r\}_{\text{ét}}$ by (E) ([Ts2] Proposition 3.2.4 (2)) and hence to $p^m d\log(\overline{a_1})\wedge\cdots\wedge d\log(\overline{a_r})$ by the composite of the second sequence. On the other hand, by the explicit description of the syntomic symbol map [Ts2] Lemma 2.4.6, which is still valid for $\mathcal{S}_n(r)'$, and Lemma 2.1.10, the image of $\{a_1,\dots,a_r\}_{\text{syn}}$ under the composite of the first sequence is also $p^m d\log(\overline{a_1})\wedge\cdots\wedge d\log(\overline{a_r})$. By [Ts2] Proposition 2.4.1 (1), we see that the symbol map $(i'^*M'^{\text{gp}})^{\otimes r}\to\mathcal{H}^r(\mathcal{S}'_n(r))$ is surjective by induction on $n$. □

By taking the inductive limit with respect to $K'$ and $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\varprojlim_n$, we obtain the following commutative diagram from Lemma 3.4.4:

(3.4.5)

$$\begin{array}{ccccc}
V^r(s) & \xleftarrow[p^{-s}\cdot(3.3.6)]{\sim} & H^r(\overline{Y},\mathcal{S}_{\mathbb{Q}_p}(s)_{(\overline{X},\overline{M})}) & \longrightarrow & (\overline{\mathcal{D}}^r)^{N=0,\varphi=p^s} \\
\downarrow & & & & \downarrow{\scriptstyle \overline{\text{pr}}_0} \\
H^{r-s}(\overline{Y},W\omega^s_{\overline{Y}/\overline{s},\,\log})_{\mathbb{Q}_p} & \xrightarrow{\sim} & (\varprojlim_n H^{r-s}(\overline{Y},\mathcal{H}^s(W_n\omega^\bullet_{\overline{Y}/\overline{s}})))^{\varphi=p^s}_{\mathbb{Q}_p} & \longleftarrow & (P_0\otimes_{K_0}D)^{\varphi=p^s}.
\end{array}$$

where $s=r$ or $s=d=\dim X_K$ and we assume $r\ge d$ in the latter case. The left vertical (resp. lower right) homomorphism is induced by (3.1.10) (resp. $\tau_{\le s}W_n\omega^\bullet_{Y/s}\to\mathcal{H}^s(W_n\omega^\bullet_{Y/s})$). Note $W_n\omega^\bullet_{Y/s}=\tau_{\le d}W_n\omega^\bullet_{Y/s}$ in the case $s=d$. The lower left one is an isomorphism by Lemma 2.3.6. See (2.1.7) for the relation between $\varphi$ and $V'$. To prove the commutativity in the case $s=d$, we need the remark after Theorem 3.3.7.

On the other hand, we have a commutative diagram:

(3.4.6)

$$\begin{array}{ccccc}
(\overline{\mathcal{D}}^r)^{N=0,\varphi=p^s} & \xleftarrow{\sim} & (\widehat{B^+_{\text{st}}\otimes_{R_{E,\mathbb{Q}_p}}}\mathcal{D}^r)^{N=0,\varphi=p^s} & \xleftarrow[\iota\otimes s]{\sim} & (B^+_{\text{st}}\otimes_{K_0}D^r)^{N=0,\varphi=p^s} \\
& {\scriptstyle\overline{\text{pr}}_0}\searrow & \downarrow{\scriptstyle \overline{f}_0\otimes\text{pr}_0} & \swarrow{\scriptstyle \overline{f}_0\otimes 1} & \\
& & (P_0\otimes_{K_0}D^r)^{\varphi=p^s} & &
\end{array}$$

Note that the first line is (3.3.2). Combining the above two commutative diagrams, we obtain a commutative diagram:

$$\begin{array}{ccc}
V^r(s) & \xrightarrow[(3.3.8)]{\sim} & Fil^r(B^+_{\text{st}}\otimes_{K_0}D^r)^{N=0,\varphi=p^s} \\
\downarrow & & \downarrow{\scriptstyle \overline{f}_0\otimes 1} \\
\mathbb{Q}_p\otimes_{\mathbb{Z}_p}H^{r-s}(\overline{Y},W\omega^s_{\overline{Y}/\overline{s},\,\log}) & \xrightarrow[\text{Prop. 2.3.5}]{\sim} & (P_0\otimes_{K_0}D^r)^{\varphi=p^s}.
\end{array}\tag{3.4.7}$$

Note that the composite of the lower horizontal map of (3.4.7) with the lower right one of (3.4.5) coincides with the lower left one of (3.4.5). By Lemma 3.4.2, (3.4.3) and (3.4.7), we see that the isomorphisms (3.2.6) and (3.2.7) induced by the comparison isomorphism $D_{\text{st}}(V^r)\cong D^r$ are induced by (3.1.10).

## §4. The semi-stable conjecture in the open case.

### §4.1. Statement of the theorem.

Recently, G. Yamashita [Y] gave a proof of the semi-stable conjecture via syntomic cohomology for $(X,M)/(S,N)$ satisfying (3.1.2) in the open case, i.e. without assuming $s=0$. (Moreover he proved it also for cohomologies with partial compact supports.) In this section, we will give an alternative proof in the special case that the "horizontal divisors at infinity do not have self-intersections" i.e. when $(X,M)/(S,N)$ satisfies the following condition:

(4.1.1) There exist a finite number of divisors $D_i$ $(i\in I)$ on $X$ such that $X_{\mathrm{triv}}=X-(Y\cup(\cup_{i\in I}D_i))$ and an étale covering $\{X_\lambda\to X\}_{\lambda\in\Lambda}$ satisfying the following condition. For each $\lambda\in\Lambda$, there exists an étale morphism over $S$:

$$X\to\mathrm{Spec}((O_K[T_1,\dots,T_u]/(T_1\cdots T_u-\pi^e))[U_i\ (i\in I_\lambda),V_1,\dots,V_t])$$

for some integers $u\geq 1$, $t\geq 0$ and $e\geq 1$ such that $e|[K:K_0]$ and $D_i\times_X X_\lambda$ is the inverse image of $\{U_i=0\}$ for each $i\in I_\lambda$. Here $I_\lambda$ denotes the set of all $i\in I$ such that $D_i\times_X X_\lambda\neq\emptyset$.

Theorem 4.1.2. *Assume that $(X,M)/(S,N)$ satisfies the condition* (4.1.1)*. Then, for any integer $q\geq 0$, the homomorphism* (3.3.9) *is an isomorphism preserving the filtrations after taking $B_{\mathrm{dR}}\otimes_{B_{\mathrm{st}}}$. Hence $V^q$ is a semi-stable $p$-adic representation and* (3.3.9) *induces an isomorphism: $D_{\mathrm{st}}(V^q)\cong D^q$ in $\underline{MF}_K(\varphi,N)$.*

We will prove this theorem by removing the divisors $D_i$ at infinity one by one and using the Gysin exact sequences.

### §4.2 Gysin sequence for crystalline cohomology.

Associated to an effective Cartier divisor $\mathcal{X}'$ on a scheme $\mathcal{X}$, one can construct a log structure on $\mathcal{X}'$ as follows: If $\mathcal{X}'$ is defined by a global section $f\in\Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})$, then $f$ is a non-zero divisor and unique up to multiplication by units. Hence the fine log structure on $\mathcal{X}$ associated to the pre-log structure $\mathbb{N}_{\mathcal{X}}\to\mathcal{O}_{\mathcal{X}};1\mapsto f$ is independent on the choice of $f$ up to canonical isomorphisms. In the general case, one obtains a fine log structure by gluing the above log structures étale locally. For a fine log scheme $(\mathcal{X},\mathcal{M})$, we define the fine log structure associated to a Cartier divisor $\mathcal{X}'\subset\mathcal{X}$ to be the co-product of $\mathcal{M}$ and the log structure constructed above.

We say that a morphism of fine log scheme $f\colon(\mathcal{X},\mathcal{M})\to(\mathcal{Y},\mathcal{N})$ is *syntomic* ([Ka3] (2.5)) if it is integral, the underlying morphism of schemes $\mathcal{X}\to\mathcal{Y}$ is flat and locally of finite presentation, and étale locally on $\mathcal{X}$, there exists a $(\mathcal{Y},\mathcal{N})$-exact closed immersion of $(\mathcal{X},\mathcal{M})$ into a smooth fine log scheme $(\mathcal{Z},\mathcal{L})$ over $(\mathcal{Y},\mathcal{N})$ such that the underlying closed immersion of schemes is transversally regular relative to $\mathcal{Y}$ ([EGA IV] Définition (19.2.2). See also Proposition (19.2.4)). If $f$ is syntomic, for any $(\mathcal{Y},\mathcal{N})$-exact closed immersion of $(\mathcal{X},\mathcal{M})$ into a smooth fine log scheme $(\mathcal{Z},\mathcal{L})$ over $(\mathcal{Y},\mathcal{N})$, the underlying closed immersion is transversally regular relative to $\mathcal{S}$. Syntomic morphisms are stable under base changes and compositions.

Proposition 4.2.1. *Let $(\mathcal{S},\mathcal{N})$ be a fine log scheme endowed with a PD-ideal $(I,\gamma)$ such that $p$ is nilpotent on $\mathcal{O}_{\mathcal{S}}$, let $\mathcal{S}_0$ be the closed subscheme of $\mathcal{S}$ defined by a sub PD-ideal of $I$, and let $\mathcal{N}_0$ be the inverse image of $\mathcal{N}$. Let $(\mathcal{X}_0,\mathcal{M}_0)$ be a syntomic fine log scheme over $(\mathcal{S}_0,\mathcal{N}_0)$, let $\mathcal{X}'_0\subset\mathcal{X}_0$ be a Cartier divisor flat over $\mathcal{S}_0$, and let $\mathcal{M}'_0$ be the inverse image of $\mathcal{M}_0$ on $\mathcal{X}'_0$.*
*(1) Étale locally on $\mathcal{X}_0$, there exist an $(\mathcal{S},\mathcal{N})$-closed immersion of $(\mathcal{X}_0,\mathcal{M}_0)$ into a smooth fine log scheme $(\mathcal{Y},\mathcal{L})$ over $(\mathcal{S},\mathcal{N})$ and a Cartier divisor $\mathcal{Y}'\subset\mathcal{Y}$ such that $\mathcal{X}'_0$ is the inverse image of $\mathcal{Y}'$ and $\mathcal{Y}'$ endowed with the inverse image $\mathcal{L}'$ of $\mathcal{L}$ is smooth over $(\mathcal{S},\mathcal{N})$.*
*(2) Suppose that we are given $i\colon(\mathcal{X}_0,\mathcal{M}_0)\hookrightarrow(\mathcal{Y},\mathcal{L})$ and $\mathcal{Y}'$ as in (1) globally. Let $(\mathcal{D},\mathcal{M}_{\mathcal{D}})$ (resp. $(\mathcal{D}',\mathcal{M}_{\mathcal{D}'})$) be the PD-envelope of $(\mathcal{X}_0,\mathcal{M}_0)$ in $(\mathcal{Y},\mathcal{L})$ (resp. $(\mathcal{X}'_0,\mathcal{M}'_0)$ in $(\mathcal{Y}',\mathcal{L}')$) compatible with the PD-structure $(I,\gamma)$. Let $J_{\mathcal{D}}$ (resp. $J_{\mathcal{D}'}$) be the PD-ideal of $\mathcal{O}_{\mathcal{D}}$ (resp. $\mathcal{O}_{\mathcal{D}'}$). If $\mathcal{Y}'$ is defined by a global section $f\in\Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$, then $f$ is a non-zero divisor on $\mathcal{O}_{\mathcal{D}}$ and we have isomorphisms $\mathcal{O}_{\mathcal{D}}/f\mathcal{O}_{\mathcal{D}}\cong\mathcal{O}_{\mathcal{D}'}$ and $J_{\mathcal{D}}^{[r]}/fJ_{\mathcal{D}}^{[r]}\cong J_{\mathcal{D}'}^{[r]}$ $(r\geq 1)$.*
*(3) Under the notation of (2), let $\mathcal{L}^\circ$ (resp. $\mathcal{M}_0^\circ$) be the log structure on $\mathcal{Y}$ (resp. $\mathcal{X}_0$) defined by the log structure $\mathcal{L}$ (resp. $\mathcal{M}_0$) and the Cartier divisor $\mathcal{Y}'$ (resp. $\mathcal{X}'_0$). Then the PD-envelope of $(\mathcal{X}_0,\mathcal{M}_0^\circ)$ in $(\mathcal{Y},\mathcal{L}^\circ)$ compatible with the PD-structure $(I,\gamma)$ has the same underlying scheme as $(\mathcal{D},\mathcal{M}_{\mathcal{D}})$. Furthermore $(\mathcal{Y},\mathcal{L}^\circ)$ is smooth over $(\mathcal{S},\mathcal{N})$, and, for each integer $r\geq 0$, we have a canonical exact sequence:*

$$\begin{aligned}0\to J_{\mathcal{D}}^{[r-\bullet]}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^\bullet_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{L}/\mathcal{N}))\to J_{\mathcal{D}}^{[r-\bullet]}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^\bullet_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{L}^\circ/\mathcal{N}))\\ \overset{(*)}{\to}(J_{\mathcal{D}'}^{[r-1-\bullet]}\otimes_{\mathcal{O}_{\mathcal{Y}'}}\Omega^\bullet_{\mathcal{Y}'/\mathcal{S}}(\log(\mathcal{L}'/\mathcal{N})))[-1]\to 0\end{aligned}$$

*such that $(*)$ sends $\omega_1+d\log(g)\wedge\omega_2$ to $\overline{\omega_2}$ for $\omega_1\in J_{\mathcal{D}}^{[r-q]}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^q_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{L}/\mathcal{N}))$, $\omega_2\in J_{\mathcal{D}}^{[r-q]}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^{q-1}_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{L}/\mathcal{N}))$ and a local equation $g=0$ defining $\mathcal{Y}'$ in $\mathcal{Y}$, where $\overline{\omega_2}$ denotes the image of $\omega_2$ in $J_{\mathcal{D}'}^{[r-q]}\otimes_{\mathcal{O}_{\mathcal{Y}'}}\Omega^{q-1}_{\mathcal{Y}'/\mathcal{S}}(\log(\mathcal{L}'/\mathcal{N}))$.*

*Proof.* (1) Étale locally on $\mathcal{X}_0$, there exists an $(\mathcal{S},\mathcal{N})$-closed immersion $i$ of $(\mathcal{X}_0,\mathcal{M}_0)$ into a smooth fine log scheme $(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})$ over $(\mathcal{S},\mathcal{N})$, and $\mathcal{X}'_0$ is defined by the global equation $f=0$ for some $f\in\Gamma(\mathcal{X}_0,\mathcal{O}_{\mathcal{X}_0})$. Set $(\mathcal{Y},\mathcal{L}):=(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})\times_{\mathrm{Spec}(\mathbb{Z})}\mathrm{Spec}(\mathbb{Z}[T])$ an let $\mathcal{Y}'$ be the closed subscheme of $\mathcal{Y}$ defined by $T=0$. Then the closed immersion $(\mathcal{X}_0,\mathcal{M}_0)\hookrightarrow(\mathcal{Y},\mathcal{L})$ defined by $i$ and $T\mapsto f$ satisfies the required condition.
(2) Since the question is étale local on $\mathcal{X}_0$, we may assume that we have a factorization $(\mathcal{X}_0,\mathcal{M}_0)\overset{j}{\hookrightarrow}(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})\overset{\alpha}{\to}(\mathcal{Y},\mathcal{L})$ such that $j$ is an exact closed immersion and $\alpha$ is étale, and that $\mathcal{Y}'$ is defined by the global equation $g=0$ for some $g\in\Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$. Since $(\mathcal{X}_0,\mathcal{M}_0)$ is integral over $(\mathcal{S}_0,\mathcal{N}_0)$, we may also assume that $(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})$ is integral over $(\mathcal{S},\mathcal{N})$. Let $\mathcal{Z}'\subset\mathcal{Z}$ be the pull-back of $\mathcal{Y}'\subset\mathcal{Y}$ and let $h$ be the inverse image of $g$ in $\Gamma(\mathcal{Z},\mathcal{O}_{\mathcal{Z}})$. Since $(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})$ and $\mathcal{Z}'$ endowed with the pull-back of $\mathcal{M}_{\mathcal{Z}}$ are smooth and integral over $(\mathcal{S},\mathcal{N})$, the morphism $(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})\to(\mathcal{S},\mathcal{N})\times_{\mathrm{Spec}(\mathbb{Z})}\mathrm{Spec}(\mathbb{Z}[T])$ defined by $T\mapsto g$ is smooth

and integral on a neighbourhood of $\mathcal{Z}'$. Especially the underlying morphism of schemes is flat on the neighbourhood. Hence $h$ is a non-zero divisor. Thus, by the construction of PD-envelopes ([Ka2] (5.6)), we may replace $(\mathcal{Y},\mathcal{L})$ with $(\mathcal{Z},\mathcal{M}_{\mathcal{Z}})$ and assume that $i$ is an exact closed immersion and $(\mathcal{Y},\mathcal{L})$ is integral over $(\mathcal{S},\mathcal{N})$. Since $(\mathcal{X}_0,\mathcal{M}_0)$ is syntomic over $(\mathcal{S}_0,\mathcal{N}_0)$, the closed immersion $\mathcal{X}_0\hookrightarrow\mathcal{Y}_0:=\mathcal{Y}\times_{\mathcal{S}}\mathcal{S}_0$ is transversally regular relative to $\mathcal{S}_0$ and hence we may assume that there exists a sequence $g_1,\dots,g_d\in\Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$ whose image in $\mathcal{O}_{\mathcal{Y}_0}$ is transversally $\mathcal{O}_{\mathcal{Y}_0}$-regular relative to $\mathcal{S}_0$, and $\mathcal{X}_0\subset\mathcal{Y}_0$ is defined by the ideal $\sum_{1\leq i\leq d}g_i\cdot\mathcal{O}_{\mathcal{Y}_0}$. Since $\mathcal{S}_0\hookrightarrow\mathcal{S}$ is a nilimmersion and $\mathcal{Y}$ is flat over $\mathcal{S}$, we see that the sequence $g_1,\dots,g_d$ is transversally $\mathcal{O}_{\mathcal{Y}}$-regular relative to $\mathcal{S}$. (Since $\mathcal{Y}$ is locally of finite presentation over $\mathcal{S}$, we are reduced to the case $\mathcal{S}$ is noetherian and then to the case $\mathcal{S}_0$ is defined by an ideal $J$ of $\mathcal{O}_{\mathcal{S}}$ such that $J^2=0$). Let $\mathcal{X}$ be the closed subscheme of $\mathcal{Y}$ defined by the ideal $\sum_{1\leq i\leq d}g_i\cdot\mathcal{O}_{\mathcal{Y}}$ and set $\mathcal{X}':=\mathcal{X}\times_{\mathcal{Y}}\mathcal{Y}'(\subset\mathcal{Y}')$. Since the image of the sequence $g_1,\cdots,g_d,g$ in $\mathcal{O}_{\mathcal{Y}_0}$ is transversally $\mathcal{O}_{\mathcal{Y}_0}$-regular relative to $\mathcal{S}_0$ and $\mathcal{S}_0\hookrightarrow\mathcal{S}$ is a nilimmersion, the sequence $g_1,\dots,g_d,g$ is transversally $\mathcal{O}_{\mathcal{Y}}$-regular relative to $\mathcal{S}$ and hence the image of $g_1,\dots,g_d$ in $\mathcal{O}_{\mathcal{Y}'}$ is transversally $\mathcal{O}_{\mathcal{Y}'}$-regular relative to $\mathcal{S}$. Hence the morphism $\mathcal{Y}\to\mathcal{S}[T_1,\dots,T_d]$ (resp. $\mathcal{Y}'\to\mathcal{S}[T_1,\dots,T_d]$) defined by $T_i\mapsto g_i$ (resp. $T_i\mapsto$ the image of $g_i$ in $\mathcal{O}_{\mathcal{Y}'}$) is flat on a neighbourhood of $\mathcal{X}$ (resp. $\mathcal{X}'$). (Since $\mathcal{Y}$ and $\mathcal{Y}'$ are locally of finite presentation over $\mathcal{S}$, we are easily reduced to the case $\mathcal{S}$ is noetherian, where we can use [EGA IV] Chap. 0 Proposition (15.1.21).) Furthermore, since $\mathcal{X}_0=\mathcal{X}\times_{\mathcal{S}}\mathcal{S}_0$ (resp. $\mathcal{X}'_0=\mathcal{X}'\times_{\mathcal{S}}\mathcal{S}_0$), $\mathcal{D}$ (resp. $\mathcal{D}'$) is isomorphic to the PD-envelope of $\mathcal{X}$ in $\mathcal{Y}$ (resp. $\mathcal{X}'$ in $\mathcal{Y}'$). Hence, by [Be-O] 3.2.1, we have $\mathcal{D}\cong\mathcal{Y}\times_{\mathcal{S}[T_1,\dots,T_d]}\mathcal{S}<T_1,\dots,T_d>$, $\mathcal{D}'\cong\mathcal{Y}'\times_{\mathcal{S}[T_1,\dots,T_d]}\mathcal{S}<T_1,\dots,T_d>$, which implies the claim.

(3) As in (2), we may assume that $i$ is an exact closed immersion, and $(\mathcal{Y},\mathcal{L})$ is integral over $(\mathcal{S},\mathcal{N})$, and $\mathcal{Y}'\subset\mathcal{Y}$ is defined by the global equation $g=0$ for some $g\in\Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$. Then $(\mathcal{X},\mathcal{M}^{\circ})\hookrightarrow(\mathcal{Y},\mathcal{L}^{\circ})$ is an exact closed immersion and we obtain the first claim. For the second claim, we may replace $\mathcal{Y}$ with a neighbourhood of $\mathcal{Y}'$. Hence, we may assume $(\mathcal{Y},\mathcal{L})\to(\mathcal{S},\mathcal{N})\times_{\mathrm{Spec}(\mathbb{Z})}\mathrm{Spec}(\mathbb{Z}[T])$ defined by $T\mapsto g$ is smooth, and there exists a chart $P\to\Gamma(\mathcal{S},\mathcal{N})$, $Q\to\Gamma(\mathcal{Y},\mathcal{L})$, $P\to Q$ of $(\mathcal{Y},\mathcal{L})\to(\mathcal{S},\mathcal{N})$ such that $(\mathcal{Y},\mathcal{L})\to(\mathcal{S},\mathcal{N})\times_{\mathrm{Spec}(\mathbb{Z}[P])}\mathrm{Spec}(\mathbb{Z}[Q][T])$ is étale and the kernel and the torsion part of the cokernel of $P^{\mathrm{gp}}\to Q^{\mathrm{gp}}$ have orders invertible on $\mathcal{S}$. Then $Q\oplus\mathbb{N}\to\Gamma(\mathcal{Y},\mathcal{L}^{\circ});(0,1)\mapsto g$ becomes a chart of $\mathcal{L}^{\circ}$. Hence $(\mathcal{Y},\mathcal{L}^{\circ})$ is smooth over $(\mathcal{S},\mathcal{N})$, and we have

$$\begin{aligned}\Omega_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{L}^{\circ}))&\cong\mathcal{O}_{\mathcal{Y}}\otimes_{\mathbb{Z}}P^{\mathrm{gp}}\oplus\mathcal{O}_{\mathcal{Y}}\cdot d\log(g),\\ \Omega_{\mathcal{Y}/\mathcal{S}}(\log(\mathcal{L}))&\cong\mathcal{O}_{\mathcal{Y}}\otimes_{\mathbb{Z}}P^{\mathrm{gp}}\oplus\mathcal{O}_{\mathcal{Y}}\cdot dg,\\ \Omega_{\mathcal{Y}'/\mathcal{S}'}(\log(\mathcal{L}'))&\cong\mathcal{O}_{\mathcal{Y}'}\otimes_{\mathbb{Z}}P^{\mathrm{gp}}.\end{aligned}$$

Now the claim follows from (2). □

Proposition 4.2.2. *Let $(\mathcal{S},\mathcal{N},I,\gamma)$, $(\mathcal{S}_0,\mathcal{N}_0)$, $(\mathcal{X}_0,\mathcal{M}_0)$ and $\mathcal{X}'_0$ be the same as in Proposition* 4.2.1. *Assume that $\mathcal{X}_0$ is quasi-compact and separated. Then, there exist an étale hypercovering $\mathcal{X}_0^{\bullet}$ of $\mathcal{X}_0$, a simplicial smooth and integral* *fine log scheme $(\mathcal{Y}^\bullet, \mathcal{L}^\bullet)$ over $(\mathcal{S}, \mathcal{N})$ with a Cartier divisor $\mathcal{Y}'^\nu \subset \mathcal{Y}^\nu$ for each $\nu \geq 0$ and an $(\mathcal{S}, \mathcal{N})$-closed immersion of $(\mathcal{X}_0^\bullet, \mathcal{M}|_{\mathcal{X}_0^\bullet})$ into $(\mathcal{Y}^\bullet, \mathcal{L}^\bullet)$ such that $\mathcal{X}_0^\nu$ $(\nu \geq 0)$ is affine, $\mathcal{Y}'^\nu$ $(\nu \geq 0)$ endowed with the inverse image of $\mathcal{L}^\nu$ is smooth over $(\mathcal{S}, \mathcal{N})$, $\mathcal{Y}'^0$ is defined by the global equation $g = 0$ for some $g \in \Gamma(\mathcal{Y}^0, \mathcal{O}_{\mathcal{Y}^0})$, for any non-decreasing map $s\colon \{0, 1, \dots, \nu\} \to \{0, 1, \dots, \mu\}$, $\mathcal{Y}'_\mu$ is the pull-back of $\mathcal{Y}'_\nu$ by the morphism $\mathcal{Y}_\mu \to \mathcal{Y}_\nu$ corresponding to $s$, and $\mathcal{X}_0'^\nu := \mathcal{X}_0' \times_{\mathcal{X}_0} \mathcal{X}_0^\nu$ $(\nu \geq 0)$ is the pull-back of $\mathcal{Y}'^\nu$ by the closed immersion $\mathcal{X}_0^\nu \hookrightarrow \mathcal{Y}^\nu$.*

*Proof.* We will write $\mathcal{X}$ for $\mathcal{X}_0$ to simplify the notation. Since $\mathcal{X}$ is quasi-compact, by Proposition 4.2.1 (1), there exist an étale covering $\mathcal{X}^0 \to \mathcal{X}$ with $\mathcal{X}^0$ affine, an $(\mathcal{S}, \mathcal{N})$-closed immersion of $(\mathcal{X}^0, \mathcal{M}|_{\mathcal{X}^0})$ into a fine log scheme $(\mathcal{Y}^0, \mathcal{L}^0)$ smooth and integral over $(\mathcal{S}, \mathcal{N})$, and a Cartier divisor $\mathcal{Y}'^0 \subset \mathcal{Y}^0$ defined by the global equation $g = 0$ for some $g \in \Gamma(\mathcal{Y}^0, \mathcal{O}_{\mathcal{Y}^0})$ such that $\mathcal{X}'^0 \subset \mathcal{X}^0$ is the pull-back of $\mathcal{Y}'^0$ by the closed immersion $\mathcal{X}^0 \hookrightarrow \mathcal{Y}^0$ and $\mathcal{Y}'^0$ endowed with the inverse image of $\mathcal{L}^0$ is smooth over $(\mathcal{S}, \mathcal{N})$. For each $\nu \geq 0$, we define $\mathcal{X}^\nu$ to be the fiber product of $\nu+1$ copies of $\mathcal{X}^0$ over $\mathcal{X}$, which is affine, and $\mathcal{X}'^\nu$ to be $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}^\nu$. We define $(\mathcal{Y}(\nu), \mathcal{L}(\nu))$ (resp. $\mathcal{Y}'(\nu)$) to be the fiber product of $\nu + 1$ copies of $(\mathcal{Y}^0, \mathcal{L}^0)$ (resp. $\mathcal{Y}'^0$) over $(\mathcal{S}, \mathcal{N})$ (resp. $\mathcal{S}$), and $\overline{\mathcal{Y}^\nu}$ to be the blowing-up of $\mathcal{Y}(\nu)$ along $\mathcal{Y}'(\nu)$. We define $\mathcal{Y}''(\nu)$ to be the sum of the pull-backs of $\mathcal{Y}'^0$ by the $\nu + 1$ projections $\mathcal{Y}(\nu) \to \mathcal{Y}$ and $\mathcal{Y}^\nu$ to be the complement of the strict transform of $\mathcal{Y}''(\nu) \subset \mathcal{Y}(\nu)$ on $\overline{\mathcal{Y}^\nu}$. We denote by $\overline{\mathcal{L}^\nu}$ and $\mathcal{L}^\nu$ the inverse images of $\mathcal{L}(\nu)$ on $\overline{\mathcal{Y}^\nu}$ and $\mathcal{Y}^\nu$ respectively. We define $\mathcal{Y}'^\nu$ to be $\mathcal{Y}'(\nu) \times_{\mathcal{Y}(\nu)} \mathcal{Y}^\nu$, which is a Cartier divisor on $\mathcal{Y}^\nu$. Let $\mathrm{pr}_i^\nu\colon \mathcal{Y}(\nu) \to \mathcal{Y}^1$ $(0 \leq i \leq \nu)$ be the projection to the $(i+1)$-th component and set $g_i^\nu := (\mathrm{pr}_i^\nu)^*(g)$. Then the morphism $(\mathcal{Y}(\nu), \mathcal{L}(\nu)) \to (\mathcal{S}, \mathcal{N})[T_0, \dots, T_\nu]$ defined by $T_i \mapsto g_i^\nu$ is smooth and integral on a neighbourhood of $\mathcal{Y}'(\nu)$ in $\mathcal{Y}(\nu)$, especially, the underlying morphism of scheme is flat on the neighbourhood. Hence $\overline{\mathcal{Y}^\nu}$ is the pull-back of the blowing-up of $\mathcal{S}[T_0, \dots, T_\nu]$ along $T_0 = T_1 = \cdots = T_\nu = 0$. If we choose an integer $i_0$ such that $0 \leq i_0 \leq \nu$, $\mathcal{Y}^\nu$ is the pull-back of $\mathcal{S}[T_{i_0}, U_i, U_i^{-1}(0 \leq i \leq \nu, i \neq i_0)] \to \mathcal{S}[T_0, \dots, T_\nu]$ where $T_i = T_{i_0} U_i$ $(i \neq i_0)$.This implies that $(\overline{\mathcal{Y}^\nu}, \overline{\mathcal{L}^\nu})$, $(\mathcal{Y}^\nu, \mathcal{L}^\nu)$ and $\mathcal{Y}'^\nu$ endowed with the inverse images of $\mathcal{L}^\nu$ are smooth and integral over $(\mathcal{S}, \mathcal{N})$. We also see that $\mathcal{Y}'^\nu \subset \mathcal{Y}^\nu$ is defined by the equation $g_{i_0}^\nu = 0$ and $g_{i_0}^\nu$ is a non-zero divisor on $\mathcal{Y}^\nu$. By the universality of blowing-up, the closed immersion $i(\nu)\colon (\mathcal{X}^\nu, \mathcal{M}|_{\mathcal{X}^\nu}) \hookrightarrow (\mathcal{Y}(\nu), \mathcal{L}(\nu))$ canonically factors through a closed immersion $\overline{i^\nu}\colon (\mathcal{X}^\nu, \mathcal{M}|_{\mathcal{X}^\nu}) \hookrightarrow (\overline{\mathcal{Y}^\nu}, \overline{\mathcal{L}^\nu})$. If we denote by $h_i^\nu$ the inverse image of $g_i^\nu$ in $\mathcal{O}_{\mathcal{X}^\nu}$, then, for each $i$, the closed subscheme $\mathcal{X}'^\nu$ of $\mathcal{X}^\nu$ is defined by $h_i^\nu = 0$. Hence $h_i^\nu = h_{i_0}^\nu \cdot u_i$ for some $u_i \in \mathcal{O}^*_{\mathcal{X}^\nu}$. This implies that $\overline{i^\nu}$ factors through $(\mathcal{Y}^\nu, \mathcal{L}^\nu)$, which we denote by $i^\nu$. Furthermore we see that $\mathcal{X}'^\nu$ is the pull-back of $\mathcal{Y}'^\nu$. Let $s\colon \{0, 1, \cdots, \nu\} \to \{0, 1, \dots, \mu\}$ be a non-decreasing map. By the universality of blowing-up, the composite of $(\mathcal{Y}^\mu, \mathcal{L}^\mu) \to (\mathcal{Y}(\mu), \mathcal{L}(\mu))$ with the morphism $(\mathcal{Y}(\mu), \mathcal{L}(\mu)) \to (\mathcal{Y}(\nu), \mathcal{L}(\nu))$ corresponding to $s$ uniquely factors through $(\overline{\mathcal{Y}^\nu}, \overline{\mathcal{L}^\nu})$. The inverse images of $g_i^\nu$ $(0 \leq i \leq \nu)$ in $\mathcal{O}_{\mathcal{Y}^\mu}$ are $g_{s(i)}^\mu$ and coincide up to the multiplication by units. Hence it further factors through $(\mathcal{Y}^\nu, \mathcal{L}^\nu)$ and $\mathcal{Y}'^\mu$ is the pull-back of $\mathcal{Y}'^\nu$. Thus

$\{(\mathcal{Y}^\nu,\mathcal{L}^\nu)\}_{\nu\geq 0}$ become a simplicial fine log scheme and $\{i^\nu\}$ are compatible with the simplicial structures. □

Corollary 4.2.3. *Let $(\mathcal{S},\mathcal{N},I,\gamma)$, $(\mathcal{S}_0,\mathcal{N}_0)$, $(\mathcal{X}_0,\mathcal{M}_0)$ and $(\mathcal{X}'_0,\mathcal{M}'_0)$ be as in Proposition* 4.2.1 *and let $\mathcal{M}_0^\circ$ be the fine log structure on $\mathcal{X}_0$ defined by $\mathcal{M}_0$ and the Cartier divisor $\mathcal{X}'_0\subset\mathcal{X}_0$. Assume that $\mathcal{X}_0$ is quasi-compact and separated. Let $u_{(\mathcal{X}_0,\mathcal{M}_0)/(\mathcal{S},\mathcal{N})}$ denote the morphism of topoi*

$$((\mathcal{X}_0,\mathcal{M}_0)/(\mathcal{S},\mathcal{N},I,\gamma))^\sim_{\mathrm{crys}}\longrightarrow(\mathcal{X}_0)^\sim_{\mathrm{\acute{e}t}},$$

*and define $u_{(\mathcal{X}_0,\mathcal{M}_0^\circ)/(\mathcal{S},\mathcal{N})}$ and $u_{(\mathcal{X}'_0,\mathcal{M}'_0)/(\mathcal{S},\mathcal{N})}$ similarly. Then we have a canonical distinguished triangle:*

$$Ru_{(\mathcal{X}_0,\mathcal{M}_0)/(\mathcal{S},\mathcal{N}),*}\mathcal{J}^{[r]}\to Ru_{(\mathcal{X}_0,\mathcal{M}_0^\circ)/(\mathcal{S},\mathcal{N}),*}\mathcal{J}^{[r]}$$
$$\to Ru_{(\mathcal{X}'_0,\mathcal{M}'_0)/(\mathcal{S},\mathcal{N}),*}\mathcal{J}^{[r-1]}[-1]\to$$

*for each integer $r$. Here $\mathcal{O}$ denotes the structure sheaf on the relevant crystalline site, $\mathcal{J}$ denotes the PD-ideal of $\mathcal{O}$ and, for an integer $r$, $\mathcal{J}^{[r]}$ denotes the $r$-th divided power of $\mathcal{J}$ if $r\geq 1$ and $\mathcal{O}$ if $r\leq 0$.*

*Proof.* Choose $(\mathcal{X}_0^\bullet,\mathcal{M}_0|_{\mathcal{X}_0^\bullet})\hookrightarrow(\mathcal{Y}^\bullet,\mathcal{L}^\bullet)$, $\mathcal{Y}'^\bullet\subset\mathcal{Y}^\bullet$ and $g$ as in Proposition 4.2.2 Then we can apply Proposition 4.2.1 (3) to $(\mathcal{X}_0^\nu,\mathcal{M}_0|_{\mathcal{X}_0^\nu})\hookrightarrow(\mathcal{Y}^\nu,\mathcal{L}^\nu)$, $\mathcal{X}_0'^\nu$ and $\mathcal{Y}'^\nu$ for each $\nu\geq 0$. Furthermore, for each non-decreasing map $s\colon\{0,1,\dots,\nu\}\to\{0,1,\dots,\mu\}$, since $\mathcal{Y}'^\mu$ is the pull-back of $\mathcal{Y}'^\nu$ by the morphism $f_s\colon\mathcal{Y}^\mu\to\mathcal{Y}^\nu$ corresponding to $s$, the short exact sequences are functorial with respect to $f_s$. Hence, by the cohomological descent ([Ka3](2.18)–(2.21)), we obtain the required distinguished triangles. If we are given another $(\tilde{\mathcal{X}}_0^\bullet,\mathcal{M}_0|_{\tilde{\mathcal{X}}_0^\bullet})\hookrightarrow(\tilde{\mathcal{Y}}^\bullet,\tilde{\mathcal{L}}^\bullet)$, $\tilde{\mathcal{Y}}'^\bullet\subset\tilde{\mathcal{Y}}^\bullet$ and $\tilde{g}$, we define $\overline{\mathcal{Z}^\nu}$ to be the blowing-up of $\mathcal{Y}^\nu\times_{\mathcal{S}}\tilde{\mathcal{Y}}^\nu$ along $\mathcal{Y}'^\nu\times_{\mathcal{S}}\tilde{\mathcal{Y}}'^\nu$ and let $\mathcal{Z}^\nu$ be the complement of the strict transform of $\mathcal{Y}'^\nu\times\tilde{\mathcal{Y}}^\nu\cup\mathcal{Y}^\nu\times\tilde{\mathcal{Y}}'^\nu$ on $\overline{\mathcal{Z}^\nu}$. Let $\mathcal{M}_{\mathcal{Z}^\nu}$ be the inverse image of the log structure of $(\mathcal{Y}^\nu,\mathcal{L}^\nu)\times_{(\mathcal{S},\mathcal{N})}(\tilde{\mathcal{Y}}^\nu,\tilde{\mathcal{L}}^\nu)$ to $\mathcal{Z}^\nu$, and let $\mathcal{Z}'^\nu\subset\mathcal{Z}^\nu$ be the pull-back of $\mathcal{Y}'^\nu\times\tilde{\mathcal{Y}}'^\nu$. Then, similarly as the proof of Proposition 4.2.2, using $g$ and $\tilde{g}$, we see that $\{(\mathcal{Z}^\nu,\mathcal{M}_{\mathcal{Z}^\nu})\}_{\nu\geq 0}$ naturally become a simplicial fine log scheme, there exists a closed immersion $(\mathcal{X}_0^\nu\times_{\mathcal{X}_0}\tilde{\mathcal{X}}_0^\nu,$ the inverse image of $\mathcal{M}_0)\hookrightarrow(\mathcal{Z}^\nu,\mathcal{M}_{\mathcal{Z}^\nu})$ inducing a morphism between simplicial fine log schemes, and this closed immersion with $\mathcal{Z}'^\nu\subset\mathcal{Z}^\nu$ satisfies the conditions in Proposition 4.2.2. Furthermore, we have natural morphisms $(\mathcal{Z}^\bullet,\mathcal{M}_{\mathcal{Z}^\bullet})\to(\mathcal{Y}^\bullet,\mathcal{L}^\bullet)$ and $(\mathcal{Z}^\bullet,\mathcal{M}_{\mathcal{Z}^\bullet})\to(\tilde{\mathcal{Y}}^\bullet,\tilde{\mathcal{L}}^\bullet)$ compatible with the closed immersions, and isomorphisms $\mathcal{Z}'^\nu\cong\mathcal{Y}'^\nu\times_{\mathcal{Y}^\nu}\mathcal{Z}^\nu\cong\tilde{\mathcal{Y}}'^\nu\times_{\tilde{\mathcal{Y}}^\nu}\mathcal{Z}^\nu$. Hence the distinguished triangle is independent of the choice of $\mathcal{X}_0^\bullet$ etc. □

The distinguished triangle in Corollary 4.2.3 is functorial with respect to $(\mathcal{X}_0,\mathcal{M}_0)\to(\mathcal{S}_0,\mathcal{N}_0)\hookrightarrow(\mathcal{S},\mathcal{N},I,\gamma)$ and $\mathcal{X}'_0$ as follows: Suppose that we are given another $(\tilde{\mathcal{X}}_0,\tilde{\mathcal{M}}_0)\to(\tilde{\mathcal{S}}_0,\tilde{\mathcal{N}}_0)\hookrightarrow(\tilde{\mathcal{S}},\tilde{\mathcal{N}},\tilde{I},\tilde{\gamma})$ and $\tilde{\mathcal{X}}'_0$, a morphism $\alpha\colon(\tilde{\mathcal{X}}_0,\tilde{\mathcal{M}}_0)\to(\mathcal{X}_0,\mathcal{M}_0)$ and a PD-morphism $\beta\colon(\tilde{\mathcal{S}},\tilde{\mathcal{N}})\to(\mathcal{S},\mathcal{N})$ inducing a morphism $\beta_0\colon(\tilde{\mathcal{S}}_0,\tilde{\mathcal{N}}_0)\to(\mathcal{S}_0,\mathcal{N}_0)$ in a compatible manner in the obvious sense.

We further assume that $\tilde{\mathcal{X}}'_0 = \mathcal{X}'_0 \times_{\mathcal{X}_0} \tilde{\mathcal{X}}_0$. Then the distinguished triangles in Corollary 4.2.3 for $(\mathcal{X}_0, \mathcal{M}_0)/(\mathcal{S}, \mathcal{N}, I, \gamma)$, $\mathcal{X}'_0$ and for $(\tilde{\mathcal{X}}_0, \tilde{\mathcal{M}}_0)/(\tilde{\mathcal{S}}, \tilde{\mathcal{N}}, \tilde{I}, \tilde{\gamma})$, $\tilde{\mathcal{X}}'_0$ are compatible with the morphisms between the each component induced by $\alpha$ and $\beta$: Choose $(\mathcal{X}_0^\bullet, \mathcal{M}_0^\bullet) \hookrightarrow (\mathcal{Y}^\bullet, \mathcal{L}^\bullet)$, $\mathcal{Y}'^\bullet \subset \mathcal{Y}^\bullet$ for $(\mathcal{X}_0, \mathcal{M}_0)/(\mathcal{S}, \mathcal{N})$, and $(\tilde{\mathcal{X}}_0^\bullet, \tilde{\mathcal{M}}_0^\bullet) \hookrightarrow (\tilde{\mathcal{Y}}^\bullet, \tilde{\mathcal{L}}^\bullet)$, $\tilde{\mathcal{Y}}'^\bullet \subset \tilde{\mathcal{Y}}^\bullet$ for $(\tilde{\mathcal{X}}_0, \tilde{\mathcal{M}}_0)/(\tilde{\mathcal{S}}, \tilde{\mathcal{N}})$ as in Proposition 4.2.2. Let $\overline{\mathcal{Z}^\nu}$ be the blowing-up of $\tilde{\mathcal{Y}}^\nu \times_{\mathcal{S}} \mathcal{Y}^\nu$ along $\tilde{\mathcal{Y}}'^\nu \times_{\mathcal{S}} \mathcal{Y}'^\nu$, let $\mathcal{Z}^\nu$ be the complement of the strict transform of $\tilde{\mathcal{Y}}'^\nu \times_{\mathcal{S}} \mathcal{Y}^\nu \cup \tilde{\mathcal{Y}}^\nu \times_{\mathcal{S}} \mathcal{Y}'^\nu$ on $\overline{\mathcal{Z}^\nu}$, and let $\mathcal{Z}'^\nu$ be the inverse image of $\tilde{\mathcal{Y}}'^\nu \times_{\mathcal{S}} \mathcal{Y}'^\nu$. Let $\mathcal{M}_{\mathcal{Z}^\nu}$ be the inverse image of the log structure on $(\tilde{\mathcal{Y}}^\nu, \tilde{\mathcal{L}}^\nu) \times_{(\mathcal{S}, \mathcal{N})} (\mathcal{Y}^\nu, \mathcal{L}^\nu)$ to $\mathcal{Z}^\nu$. Then similarly as in the proof of Corollary 4.2.3, we see that $\{(\mathcal{Z}^\nu, \mathcal{M}_{\mathcal{Z}^\nu})\}_{\nu \geq 0}$ naturally become a simplicial fine log scheme smooth and integral over $(\tilde{\mathcal{S}}, \tilde{\mathcal{N}})$, there exists a closed immersion of $(\tilde{\mathcal{X}}_0^\nu \times_{\mathcal{X}_0} \mathcal{X}_0^\nu$, the inverse image of $\tilde{\mathcal{M}}_0)$ into $(\mathcal{Z}^\nu, \mathcal{M}_{\mathcal{Z}^\nu})$ over $(\tilde{\mathcal{S}}, \tilde{\mathcal{N}})$ compatible with the simplicial structures, and this closed immersion with $\mathcal{Z}'^\nu \subset \mathcal{Z}^\nu$ satisfies the conditions in Proposition 4.2.2. Furthermore we have a natural morphism to the closed immersion $(\mathcal{X}_0^\bullet, \mathcal{M}_0^\bullet) \hookrightarrow (\mathcal{Y}^\bullet, \mathcal{L}^\bullet)$ (resp. $(\tilde{\mathcal{X}}_0^\bullet, \tilde{\mathcal{M}}_0^\bullet) \hookrightarrow (\tilde{\mathcal{Y}}^\bullet, \tilde{\mathcal{L}}^\bullet)$) such that $\mathcal{Z}'^\nu$ is the pull-back of $\mathcal{Y}'^\nu$ (resp. $\tilde{\mathcal{Y}}'^\nu$). This implies the required functoriality.

## §4.3. Gysin sequence for syntomic cohomology.

Let the notation and the assumption as in §4.1. Assume that $I$ is non-empty and choose one $i_0 \in I$. We will change the notation as follows: We write $M^\circ$ for $M$, and $M$ will denote the log structure defined by the union of the special fiber of $X$ and the divisors $D_i$ $(i \in I, i \neq i_0)$. We define $(X', M')$ to be $D_{i_0}$ endowed with the inverse image of $M$. Then $(X, M)$, $(X, M^\circ)$ and $(X', M')$ satisfies the condition (4.1.1). Note that $X'$ is a Cartier divisor on $X$ and $M^\circ$ is the co-product of $M$ and the log structure on $X$ defined by $X'$ (cf. §4.2).

We can construct Gysin sequence for syntomic cohomology as follows. We choose an affine étale covering $X^0 \to X$, a closed immersion of $(X^0, M|_{X^0})$ into a fine log scheme $(Z^0, M_{Z^0})$ smooth over $\mathrm{Spec}(W)$ endowed with a Cartier divisor $Z'^0$ defined by a global equation $g = 0$ and with a compatible system of liftings of Frobenius $\{F_{Z_n^0} : (Z_n^0, M_{Z_n^0}) \to (Z_n^0, M_{Z_n^0})\}_{n \geq 1}$ such that $X' \times_X X^0$ is the pull-back of $Z'^0$ and $F^*_{Z_n^0}(g) = g^p \cdot (1 + py)$ for some $y \in \mathcal{O}_{Z_n^0}$. Here the subscript $n$ denotes the reduction mod $p^n$. Such a covering and an embedding exist by a similar argument as the proof of Proposition 4.2.1 (1). Starting from this embedding, we can construct an étale hypercovering $X^\bullet \to X$, a closed immersion $(X^\bullet, M^\bullet) \hookrightarrow (Z^\bullet, M_{Z^\bullet})$ and a Cartier divisor $Z'^\bullet \subset Z^\bullet$ as in Proposition 4.2.2 endowed with a compatible system of liftings of Frobenius on $\{(Z_n^\bullet, M_{Z_n^\bullet})\}_{n \geq 1}$. By taking the PD-envelope of $(X_n^\bullet, M_n^\bullet)$ in $(Z_n^\bullet, M_{Z_n^\bullet})$ compatible with the canonical PD-structure on $pW_n$ and applying Proposition 4.2.1 (3), we obtain a short exact sequence on the étale site of the simplicial scheme $X_1^\bullet$ for each $r \geq 0$. By using the property $F^*_{Z_n^0}(g) = g^p \cdot (1 + py)$, we see that the short exact sequences are compatible with the Frobenius induced by $F_{Z_n^\bullet}$ and obtain a short exact sequence:

$$0 \to \mathcal{S}_n^{\sim}(r)_{(X^\bullet, M^\bullet), (Z^\bullet, M_{Z^\bullet})} \to \mathcal{S}_n^{\sim}(r)_{(X^\bullet, M^{\circ\bullet}), (Z^\bullet, M^\circ_{Z^\bullet})} \to \mathcal{S}_n^{\approx}(r-1)_{(X'^\bullet, M'^\bullet), (Z'^\bullet, M_{Z'^\bullet})}[-1] \to 0.$$

Here $\mathcal{S}_n^{\sim}(s)$ denotes the syntomic complex defined in [Ts2] §2.1, and $\mathcal{S}_n^{\approx}(s)$ denotes the complex obtained by replacing $p^s-\varphi$ with $p^{s+1}-p\varphi$ in the definition of $\mathcal{S}_n^{\sim}(s)$. By taking the derived direct image by the morphism of topoi $(X_1^{\bullet})^{\sim}_{\text{ét}}\to(X_1)^{\sim}_{\text{ét}}$. We obtain the required distinguished triangle:

$$\to\mathcal{S}_n^{\sim}(r)_{(X,M)}\to\mathcal{S}_n^{\sim}(r)_{(X,M^{\circ})}\to\mathcal{S}_n^{\approx}(r-1)_{(X',M')}[-1] \tag{4.3.1}$$

on $(X_1)_{\text{ét}}$. We can verify the independence and the functoriality similarly as in the crystalline case by taking care of liftings of Frobenius.

We define $V^q_{\text{syn},1}(r)$ to be the syntomic cohomology $H^q(\overline{Y},\mathcal{S}_{\mathbb{Q}_p}(r)_{(\overline{X},\overline{M})})$ (cf.§3.3) of $(\overline{X},\overline{M})$. We define $V^q_{\text{syn},2}(r)$ and $V^q_{\text{syn},3}(r)$ to be the syntomic cohomology of $(\overline{X},\overline{M^{\circ}})$ and $(\overline{X'},\overline{M'})$ respectively. Then, by taking $\mathbb{Q}\otimes\varprojlim_n\varinjlim_{K'}H^*((X_1\times_{\text{Spec}(O_K)}\text{Spec}(O_{K'}))_{\text{ét}},-)$ of the triangle (4.3.1) for the base changes of $(X_n,M_n)$, $(X_n,M_n^{\circ})$ and $(X'_n,M'_n)$ by $O_K\to O_{K'}$, we obtain a complex:

$$\cdots\to V^q_{\text{syn},1}(r)\to V^q_{\text{syn},2}(r)\to V^{q-1}_{\text{syn},3}(r-1)\to V^{q+1}_{\text{syn},1}(r)\to\cdots \tag{4.3.2}$$

§4.4 Compatibility of Gysin sequences 1.

We will prove the compatibility of Gysin sequences with the isomorphisms (3.2.1), (3.3.2) and the homomorphism (3.3.4). We follow the notation in §3.3. We denote by $D_1^q$ (resp. $\mathcal{D}_1^q$, resp. $\overline{\mathcal{D}}_1^q$) for the crystalline cohomology $D^q$ (resp. $\mathcal{D}^q$, resp. $\overline{\mathcal{D}}^q$) for $(X,M)/(S,N)$ defined in §3.2 (resp. §3.3, resp. §3.3). We denote by $D_2^q$, $\mathcal{D}_2^q$ and $\overline{\mathcal{D}}_2^q$ for the cohomologies of $(X,M^{\circ})/(S,N)$, and $D_3^q$, $\mathcal{D}_3^q$ and $\overline{\mathcal{D}}_3^q$ for the cohomologies of $(X',M')/(S,N)$. We denote by $D_3^q(-1)$, $\mathcal{D}_3^q(-1)$ and $\overline{\mathcal{D}}_3^q(-1)$ the same modules as $D_3^q$, $\mathcal{D}_3^q$ and $\overline{\mathcal{D}}_3^q$ whose Frobenius endomorphisms $\varphi$ are replaced with $p\varphi$. By taking $\mathbb{Q}\otimes\varprojlim_n$ of the Gysin sequences for the crystalline cohomologies over the bases $(W_n,W_n(L),pW_n,\gamma)$ and $(E_n,M_{E_n},\overline{J_{E_n}},\overline{\delta})$, we obtain an exact sequence:

$$\cdots\to D_1^q\to D_2^q\to D_3^{q-1}(-1)\to D_1^{q+1}\to\cdots \tag{4.4.1}$$

and a complex:

$$\cdots\to\mathcal{D}_1^q\to\mathcal{D}_2^q\to\mathcal{D}_3^{q-1}(-1)\to\mathcal{D}_1^{q+1}\to\cdots. \tag{4.4.2}$$

Lemma 4.4.3. *Let $\mathcal{D}_i$ $(i=1,2)$ be finitely generated free $R_{E,\mathbb{Q}_p}$-modules endowed with $\varphi_E$-semilinear endomorphisms $\varphi_{\mathcal{D}_i}$ whose linearizations $R_{E,\varphi}\otimes_{R_E}\mathcal{D}_i\to\mathcal{D}_i$ are isomorphisms. Let $D_i$ be the reduction of $\mathcal{D}_i$ with respect to $R_{E,\mathbb{Q}_p}\to K_0$ induced by $\{i_{E_n,0}\}$ (§3.3) and let $\varphi_{D_i}$ be the $\sigma$-semilinear automorphism of $D_i$ induced by $\varphi_{\mathcal{D}_i}$. Suppose that we are given an $R_{E,\mathbb{Q}_p}$-linear homomorphism $f\colon\mathcal{D}_1\to\mathcal{D}_2$ compatible with $\varphi_{\mathcal{D}_i}$ and $K_0$-linear sections $s_i\colon D_i\to\mathcal{D}_i$ of the canonical surjections $p_i\colon\mathcal{D}_i\to D_i$ compatible with $\varphi_{\mathcal{D}_i}$ and $\varphi_{D_i}$. Let $g\colon D_1\to D_2$ be the $K_0$-linear homomorphism induced by $f$. Then we have $f\circ s_1=s_2\circ g$.*

*Proof.* We apply [Ts2] Lemma 4.4.11. Let $I_n$ be as in loc. cit. For any $a \in D_1$, $f(s_1(\varphi_{D_1}^{-n}(a)))$ is a lifting of $g(\varphi_{D_1}^{-n}(a)) = \varphi_{D_2}^{-n}(g(a)) \in D_2$ in $\mathcal{D}_2$. Hence, by loc. cit., $s_2(g(a)) \equiv \varphi_{\mathcal{D}_2}^{n}(f(s_1(\varphi_{D_1}^{-n}(a)))) = f(s_1(a)) \mod I_n\mathcal{D}_2$. Since $\cap_n(I_n \otimes \mathbb{Q}) = 0$ and $\mathcal{D}_2$ is a free $R_{E,\mathbb{Q}_p}$-module, this implies $s_2(g(a)) = f(s_1(a))$. □

By functoriality, the projections $\mathcal{D}_i^q \to D_i^q$ (cf. §3.3) are compatible with (4.4.1) and (4.4.2). On the other hand, by a similar argument as the proof of the functoriality of the Gysin sequences, we also see that the sequences (4.4.1) and (4.4.2) are compatible with the Frobenius endomorphisms. Hence, from Lemma 4.4.3, we obtain the following:

Lemma 4.4.4. *The isomorphisms* $R_{E,\mathbb{Q}_p} \otimes_{K_0} D_i^q \cong \mathcal{D}_i^q$ $(i = 1, 2, 3)$ (§3.3) *are compatible with the sequences* (4.4.1) *and* (4.4.2).

Note that this and the exactness of (4.4.1) implies that (4.4.2) is also exact. We denote by $D_{\mathrm{dR},1}^q$ the de Rham cohomology $H^q(X_K, \Omega_{X_K}^{\bullet}(\log(M_K)))$ endowed with the Hodge filtration (cf. (3.2.1)), which is canonically isomorphic to the projective limit of the crystalline cohomology of $(X_n, M_n)$ over $(S_n, N_n, p\mathcal{O}_{S_n}, \gamma)$ with respect to $n$ tensored with $K$ over $O_K$. We denote by $D_{\mathrm{dR},2}^q$ and $D_{\mathrm{dR},3}^q$ the de Rham cohomology of $(X_K, M_K^{\circ})$ and $(X'_K, M_{K'})$ respectively. Recall that the Hodge spectral sequences for $D_{\mathrm{dR},i}^q$ degenerate (cf. [Ts2] Proposition 4.7.9). We denote by $D_{\mathrm{dR},3}^q(-1)$ the same $K$-vector space as $D_{\mathrm{dR},3}^q$ whose filtration is defined by $Fil^r(D_{\mathrm{dR},3}^q(-1)) = Fil^{r-1}D_{\mathrm{dR},3}^q$. Then, by taking $\mathbb{Q} \otimes \varprojlim_n$ of the Gysin sequence for the crystalline cohomology over the base $(S_n, N_n, p\mathcal{O}_{S_n}, \gamma)$, we obtain an exact sequence of filtered $K$-vector spaces:

$$(4.4.5) \qquad \cdots \to D_{\mathrm{dR},1}^q \to D_{\mathrm{dR},2}^q \to D_{\mathrm{dR},3}^{q-1}(-1) \to D_{\mathrm{dR},1}^{q+1} \to \cdots$$

By functoriality, the projections $\mathcal{D}_i^q \to D_{\mathrm{dR},i}^q$ induced by $\{i_{E_n,\pi}\}$ (§3.3) are compatible with the exact sequences (4.4.2) and (4.4.5). Hence by combining with Lemma 4.4.4, we obtain the following compatibility:

Lemma 4.4.6. *The isomorphisms* $\rho_\pi\colon K \otimes_{K_0} D_i^q \cong D_{\mathrm{dR},i}^q$ (3.2.1) *are compatible with the exact sequences* (4.4.1) *and* (4.4.5).

By taking $\mathbb{Q} \otimes \varprojlim_n \varinjlim_{K'}$ of the Gysin sequence for the base changes of $(X_n, M_n)$, $(X_n, M_n^{\circ})$ and $(X'_n, M'_n)$ by $(S', N') \to (S, N)$ over $(E_n, M_{E_n})$, we obtain a complex:

$$(4.4.7) \qquad \cdots \to \overline{\mathcal{D}}_1^q \to \overline{\mathcal{D}}_2^q \to \overline{\mathcal{D}}_3^{q-1}(-1) \to \overline{\mathcal{D}}_1^{q+1} \to \cdots .$$

Here $S' = \mathrm{Spec}(O_{K'})$ and $N'$ is the log structure defined by the closed point. By functoriality, the natural homomorphisms $\mathcal{D}_i^q \to \overline{\mathcal{D}}_i^q$ are compatible with (4.4.2) and (4.4.7). Hence, by Lemma 4.4.4, we obtain the following compatibility:

Lemma 4.4.8. *The isomorphisms* $\widehat{B_{\mathrm{st}}^+} \otimes_{K_0} D_i^q \cong \overline{\mathcal{D}}_i^q$ ([Ts2] *Proposition* 4.4.6) *are compatible with the sequence* (4.4.1) *and* (4.4.7).

Note that this lemma and the exactness of (4.4.1) imply the exactness of (4.4.7). By construction, it is clear that the distinguished triangle (4.3.1) is compatible with the distinguished triangle of Corollary 4.2.3 for $(X_n, M_n)$, $(X'_n, M'_n)$, ... over $(W_n, pW_n, \gamma)$. Hence, by the functoriality of the Gysin sequence for crystalline cohomology, we obtain:

LEMMA 4.4.9. *The natural homomorphisms* $V^q_{\mathrm{syn},i} \to \overline{\mathcal{D}}^q_i$ (3.3.4) *are compatible with the sequences* (4.3.2) *and* (4.4.7).

§4.5. COMPATIBILITY OF GYSIN SEQUENCES 2.

To prove Theorem 4.1.2, we also need to verify the compatibility of the Gysin sequence (4.3.2) of the syntomic cohomology with that of the étale cohomology. For simplicity, we omit the log structures from the notation of log schemes; we simply write $X$, $X^\circ$ and $X'$ for the log schemes $(X, M)$, $(X, M^\circ)$ and $(X', M')$ appearing in §4.4. As in [O], we denote by $\underline{X}$, $\underline{S}$,... the underlying schemes of log schemes $X$, $S$,... (We do not adopt the notation $\overset{\circ}{X}$, $\overset{\circ}{S}$,... in [Na] Notation (1.1.2) and [I2]1.2 to avoid the confusion with the notation $X^\circ$. )

Let $X_{\mathrm{triv}}$ (resp. $(X^\circ)_{\mathrm{triv}}$, resp. $X'_{\mathrm{triv}}$) be the maximal open subschemes of $X$ (resp. $X^\circ$, resp. $X'$) on which the log structure is trivial. We have $(X^\circ)_{\mathrm{triv}} = X_{\mathrm{triv}} \backslash X'_{\mathrm{triv}}$. We denote by $V^q_1$, $V^q_2$ and $V^q_3$ the $q$-th étale cohomology of $(X_{\mathrm{triv}})_{\overline{K}}$, $((X^\circ)_{\mathrm{triv}})_{\overline{K}}$ and $(X'_{\mathrm{triv}})_{\overline{K}}$ with coefficients $\mathbb{Q}_p$ respectively. Then we have the Gysin exact sequence:

$$\cdots \to V^q_1 \to V^q_2 \to V^{q-1}_3(-1) \to V^{q+1}_1 \to \cdots \qquad (4.5.1)$$

PROPOSITION 4.5.2. *For any integer* $r \geq 0$*, the homomorphisms* $V^q_{\mathrm{syn},i}(r) \to V^q_i(r)$ *defined by* $p^{-r} \cdot$ (3.3.6) *are compatible with the sequences* (4.5.1) *and* (4.3.2).

Let $i$ and $i'$ denote the closed immersions $\underline{Y} \to \underline{X}$ and $\underline{Y'} \to \underline{X'}$ and let $j$, $j^\circ$ and $j'$ denote the open immersions $X_{\mathrm{triv}} \to X$, $(X^\circ)_{\mathrm{triv}} \to X$ and $X'_{\mathrm{triv}} \to X'$ respectively. Proposition 4.5.2 follows from the following local version:

PROPOSITION 4.5.3. *For any integer* $r \geq 0$*, the following diagram is commutative:*

$$\begin{array}{ccccccc}
\longrightarrow & \mathcal{S}^{\sim}_n(r)_X & \longrightarrow & \mathcal{S}^{\sim}_n(r)_{X^\circ} & \longrightarrow & \mathcal{S}^{\approx}_n(r-1)_{X'}[-1] \\
& \downarrow{\scriptstyle (3.3.5)} & & \downarrow{\scriptstyle (3.3.5)} & & \downarrow \\
\longrightarrow & i^* Rj_* \mathbb{Z}/p^n\mathbb{Z}(r)' & \longrightarrow & i^* Rj^\circ_* \mathbb{Z}/p^n\mathbb{Z}(r)' & \longrightarrow & i'^* Rj'_* \mathbb{Z}/p\mathbb{Z}(r)'(-1)[-1].
\end{array}$$

*Here the right vertical homomorphism is the composite of*

(4.5.4)

$$\mathcal{S}^{\approx}_n(r-1)_{X'} \to \mathcal{S}^{\sim}_n(r-1)_{X'} \xrightarrow{(3.3.5)} i'^* Rj'_* \mathbb{Z}/p^n(r-1)' \to i'^* Rj'_* \mathbb{Z}/p^n(r)'(-1),$$

*where the first map is defined by the multiplication by* $p$ *on* $J^{[r-1-\bullet]}_{D'} \otimes \Omega^\bullet$ *and the identity map on* $\mathcal{O}_{D'} \otimes \Omega^\bullet$.

We will prove Proposition 4.5.3 in §4.8 after some preliminaries in §4.6 and §4.7. We will prove it along the following lines. By using the Gysin sequence (4.3.1) and explicit descriptions of $i^*Rj_*\mathbb{Z}/p^n\mathbb{Z}(r)'$ and $i^*Rj^\circ_*\mathbb{Z}/p^n\mathbb{Z}(r)'$ as complexes in terms of Godment resolutions, we construct a map

$$\alpha\colon \mathcal{S}_n^{\approx}(r-1)_{X'} \to i'^*Rj'_*\mathbb{Z}/p^n\mathbb{Z}(r)'(-1) \tag{4.5.5}$$

such that the diagram in Proposition 4.5.3 with (4.5.4) replaced by $\alpha$ is commutative. The main difficulty to compare $\alpha$ with (4.5.4) comes from the fact that the resolution $\overline{\mathcal{S}}_n(r)$ of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ relating $\mathbb{Z}/p^n\mathbb{Z}(r)'$ with $\mathcal{S}_n^{\sim}(r)$ (cf. [Ts2] §3.1) does not behave well with respect to the closed immersion $X' \hookrightarrow X$. We overcome this problem by replacing $X' \to X \leftarrow X^\circ$ with $X' \stackrel{\mathrm{id}}{\to} X' \leftarrow X'^\circ$, where $X'^\circ$ is the scheme $\underline{X'}$ endowed with the inverse image of $M_{X^\circ}$. Although $X'^\circ$ is not log smooth, we still have a Gysin sequence for $\mathcal{S}_n^{\sim}(r)$ (4.7.11) and we can construct a map

$$\beta\colon \mathcal{S}_n^{\approx}(r-1)_{X'} \to i'^*Rj'_*\mathbb{Z}/p^n\mathbb{Z}(r)'(-1) \tag{4.5.6}$$

in the same way as $\alpha$, which is easily seen to be equal to $\alpha$ above. (For the étale side, we need to use the Kummer étale sites of fs log schemes ([Na]).) For $X' \stackrel{\mathrm{id}}{\to} X' \leftarrow X'^\circ$, we also have a Gysin sequence for $\overline{\mathcal{S}}_n(r)$ (4.7.10), which allows us to compare the morphism $\beta$ with (4.5.4).

§4.6. Preliminaries on log fundamental groups.

We will summarize some basic facts on log fundamental groups ([I2]§4) which we use in the proof of the compatibility of Gysin sequences for syntomic and étale cohomologies. We leave the most of their proofs to the readers. We continue to omit the log structures in the notation of log schemes.

A *logarithmic point* $s$ is $\mathrm{Spec}(k)$ for a separably closed field $k$ with a saturated log structure such that the multiplication by $n$ is bijective on $M_s/k^*$ for any positive integer $n$ prime to the characteristic of $k$. A *log geometric point* of an fs log scheme $S$ is a morphism $s \to S$ for a log geometric point $s$ ([Na] Definition (2.5), [I2] Definition 4.1).

Lemma 4.6.1. *Let $T \to S$ be a Kummer étale morphism* ([Na] *Definition* (2.1.2)*,* [I2] 1.6) *and let $\tilde{s} \to S$ be a log geometric point. If the image of $\tilde{s}$ in $S$ is contained in the image of $T$ in $S$, then there exists a lifting $\tilde{s} \to T$.*

Using the fact that a Kummer étale closed immersion is an open immersion, we can prove the following lemma in the same way as [SGA1] 5.3, 5.4.

Lemma 4.6.2. *Let $T \to S$ be a Kummer étale separated morphism of fs log schemes, let $U$ be an fs log scheme over $S$ and let $\varphi\colon \tilde{u} \to U$ be a log geometric point of $U$. Then the map $Hom_S(U,T) \to Hom_S(\tilde{u},T); f \mapsto f \circ \varphi$ is injective.*

Let $S$ be a locally noetherian fs log scheme and let $\tilde{s} \to S$ be a log geometric point. Then the category $\mathrm{Kcov}(S)$ of Kummer étale covers of $S$ ([I2] 3.1) with the fiber functor $F_{\tilde{s}}\colon \mathrm{Kcov}(S) \to (\mathrm{Sets})$ satisfies the axioms (G1) to (G6) of [SGA1] V§4. We define the log fundamental group of $S$ at $\tilde{s}$ to be $\mathrm{Aut}(F_{\tilde{s}})$ ([I2]4.6). By Lemma 4.6.2, we see that the fiber functor $F_{\tilde{s}}(-)$ is canonically identified with $\mathrm{Hom}_S(\tilde{s}, -)$.

We consider an equi-characteristic connected normal scheme $X$ with an fs log structure such that there exists a global chart $\alpha\colon \mathbb{N} \to \Gamma(X, M_X)$ whose composite with $\Gamma(X, M_X) \to \Gamma(X, \mathcal{O}_X)$ sends $\mathbb{N}\backslash\{0\}$ to 0.

Let $x = \mathrm{Spec}(k)$ be the generic point of $X$ with the inverse image log structure. Let $k^{\mathrm{sep}}$ be a separable closure of $k$ and let $\overline{x}$ be $\mathrm{Spec}(k^{\mathrm{sep}})$ endowed with the inverse image of $M_x$. Choose a chart $\alpha$ as above and define a log geometric point $\tilde{x}$ of $X$ to be $\mathrm{Spec}(k^{\mathrm{sep}})$ with the log structure associated to $\widetilde{\mathbb{N}} \to k; a \neq 0 \mapsto 0$. Here $\widetilde{\mathbb{N}} = \cup_{n\in\mathbb{N}, n\in k^*} \frac{1}{n}\mathbb{N}$. We define the morphism $\tilde{x} \to \overline{x} \to x \to X$ by the natural inclusion $\mathbb{N} \to \widetilde{\mathbb{N}}$ and the chart $\alpha$. $\tilde{x}$ is independent of the choice of the chart $\alpha$ up to non-canonical isomorphisms over $\overline{x}$.

For $n \in \mathbb{N}$ invertible in $k$, let $t_n$ denote the image of $\frac{1}{n} \in \widetilde{\mathbb{N}}$ in $\Gamma(\tilde{x}, M_{\tilde{x}})$. Then the homomorphism $\mathrm{Aut}(\tilde{x}/\overline{x}) \to \hat{\mathbb{Z}}'(1) := \varprojlim_{n\in\mathbb{N}, n\in k^*} \mu_n(k^{\mathrm{sep}})$ defined by $\sigma \mapsto (\sigma(t_n)t_n^{-1})_{n\geq 1}$ is an isomorphism. Any automorphism of $\tilde{x}$ over $x$ induces an automorphism of $\overline{x}$ over $x$ and thus we obtain a surjective homomorphism $\mathrm{Aut}(\tilde{x}/x) \to \mathrm{Aut}(\overline{x}/x) = \mathrm{Gal}(k^{\mathrm{sep}}/k)^\circ$ with kernel $\mathrm{Aut}(\tilde{x}/\overline{x}) \cong \hat{\mathbb{Z}}'(1)$. Here $(-)^\circ$ denotes the the opposite group.

We define $k^{\mathrm{ur}}$ to be the union of all finite extensions $k'$ of $k$ contained in $k^{\mathrm{sep}}$ such that the normalizations of the underlying scheme of $X$ in $k'$ are unramified. We define $x^{\mathrm{ur}}$ and $\widetilde{x^{\mathrm{ur}}}$ similarly as $\overline{x}$ and $\tilde{x}$ using $k^{\mathrm{ur}}$ instead of $k^{\mathrm{sep}}$. We have a canonical surjective homomorphism $\mathrm{Aut}(\tilde{x}/x) \to \mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x)$ inducing an isomorphism $\hat{\mathbb{Z}}'(1) \cong \mathrm{Aut}(\tilde{x}/\overline{x}) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x^{\mathrm{ur}})$. We also have a natural surjection $\mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x) \to \mathrm{Aut}(x^{\mathrm{ur}}/x) \cong \mathrm{Gal}(k^{\mathrm{ur}}/k)^\circ$.

The fiber functor $F_{\tilde{x}}\colon \mathrm{Kcov}(X) \to (\mathrm{Sets})$ is explicitly pro-represented as follows. For each finite extension $k'$ of $k$ contained in $k^{\mathrm{ur}}$, let $X_{k'}$ be a strict étale cover of $X$ whose function field is $k'$, and for a positive integer $n$ invertible on $X$, we define $X_{k',n}$ to be the Kummer étale cover $X_{k'} \times_{\mathrm{Spec}(\mathbb{Z}[\mathbb{N}])} \mathrm{Spec}(\mathbb{Z}[\frac{1}{n}\mathbb{N}])$ of $X$. Then, the inclusions $k' \hookrightarrow k^{\mathrm{sep}}$ and $\frac{1}{n}\mathbb{N} \hookrightarrow \widetilde{\mathbb{N}}$ define a morphism $\tilde{x} \to X_{k',n}$. If $k'/k$ is Galois and $\mu_n(k^{\mathrm{sep}}) \subset k'$, then $X_{k',n}/X$ is Galois i.e. $\mathrm{Aut}(X_{k',n}/X)$ acts transitively on $F_{\tilde{x}}(X_{k',n}) = \mathrm{Hom}_X(\tilde{x}, X_{k',n})$. We assert that $F_{\tilde{x}}$ is pro-represented by $\{X_{k',n}\}$ i.e. $\varinjlim_{k',n} \mathrm{Hom}_X(X_{k',n}, Y) \to \mathrm{Hom}_X(\tilde{x}, Y)$ is an isomorphism for any $Y \in \mathrm{Kcov}(X)$. The injectivity follows from Lemma 4.6.2. For the surjectivity, by Lemma 4.6.1 and Lemma 4.6.3 below, we may replace $Y$ by $X_{k',n}$ with $k'/k$ Galois and $\mu_n(k^{\mathrm{sep}}) \subset k'$. In this case $\mathrm{Hom}_X(X_{k',n}, X_{k',n}) \to \mathrm{Hom}_X(\tilde{x}, X_{k',n})$ is surjective.

Lemma 4.6.3. *For any Kummer étale cover $Y \to X$, there exists $n \in \mathbb{N}$ invertible on $X$ and a strict étale cover $X' \to X$ such that $Y \times_X X'_n \to X'_n$ is trivial. Here $X'_n = X_n \times_{\mathrm{Spec}(\mathbb{Z}[\mathbb{N}])}$ Spec$(\mathbb{Z}[\frac{1}{n}\mathbb{N}])$.*

The automorphism group $\mathrm{Aut}(\tilde{x}/x)$ naturally acts on the fiber functor $F_{\tilde{x}}\colon \mathrm{Kcov}(X) \to (\mathrm{Sets})$ and we obtain a homomorphism $\mathrm{Aut}(\tilde{x}/x)^\circ \to \pi_1(X, \tilde{x})$.

Proposition 4.6.4. *The above homomorphism factors through an isomorphism* $Aut(\widetilde{x^{\mathrm{ur}}}/x)^\circ \xrightarrow{\cong} \pi_1(X,\tilde{x})$.

*Proof.* Since $F_{\tilde{x}}$ is pro-represented by $\{X_{k',n}\}$ and $\tilde{x}\to X_{k',n}$ factors through $\widetilde{x^{\mathrm{ur}}}$, we see that the action of $\mathrm{Aut}(\tilde{x}/x)$ on $F_{\tilde{x}}$ factors through $\mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x)$. For $m\in\mathbb{N}$ invertible on $X$, we denote by $t_m$ the image of $(\frac{1}{m},1)$ by the (non-canonical) isomorphism $\tilde{\mathbb{N}}\oplus(k^{\mathrm{ur}})^*\cong M_{\widetilde{x^{\mathrm{ur}}}}$. We have a bijection as sets $\mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x)\to\mathrm{Gal}(k^{\mathrm{ur}}/k)\times\hat{\mathbb{Z}}'(1)(k^{\mathrm{ur}})$ sending $\sigma$ to the pair of $\sigma^*\colon k^{\mathrm{ur}}\to k^{\mathrm{ur}}$ and $(\sigma^*(t_m)t_m^{-1})_m$. On the other hand, we have $\mathrm{Aut}(F_{\tilde{x}})^\circ\cong\varprojlim_{k',n}\mathrm{Aut}(X_{k',n})$, where $(k',n)$ ranges over all finite Galois extensions $k'$ of $k$ contained in $k^{\mathrm{ur}}$ and $n\in\mathbb{N}$ invertible on $X$ such that $\mu_n(k^{\mathrm{ur}})\subset k'$. For such $(k',n)$, we have a bijection $\mathrm{Aut}(X_{k',n})\to\mathrm{Gal}(k'/k)\times\mu_n(k')$ sending $\tau$ to the pair of $\tau^*\colon k'\to k'$ and $\tau^*(t'_n)(t'_n)^{-1}$, where $t'_n$ denotes the image of $\frac{1}{n}$ by the chart $\frac{1}{n}\mathbb{N}\to\Gamma(X_{k',n},M_{X_{k',n}})$. Hence $\mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x)^\circ\cong\pi_1(X,\tilde{x})$. □

Next we consider an equi-characteristic connected regular scheme $Z$ with the fs log structure associated to a regular divisor defined by the equation $t=0$ for some $t\in\Gamma(Z,\mathcal{O}_Z)$, and assume that $X$ is the divisor with the inverse image of $M_Z$. Set $Z_{\mathrm{triv}}=Z\backslash X$. Then the functor $\mathrm{Kcov}(Z)\to\mathrm{Etcov}(Z_{\mathrm{triv}}); W\mapsto W\times_Z Z_{\mathrm{triv}}$ induces an equivalence of categories from $\mathrm{Kcov}(Z)$ to the subcategory consisting of étale covers of $Z_{\mathrm{triv}}$ tamely ramified along $X$ ([I2] Theorem 7.6). Let $z=\mathrm{Spec}(K)$ be the generic point of $Z$, choose a separable closure $K^{\mathrm{sep}}$ of $K$ and set $\overline{z}:=\mathrm{Spec}(K^{\mathrm{sep}})$. Let $K^{\mathrm{ur}}$ be the union of all finite extensions $K'$ of $K$ contained in $K^{\mathrm{sep}}$ such that the normalizations of $Z_{\mathrm{triv}}$ in $K'$ are unramified and tamely ramified along $X$. Set $z^{\mathrm{ur}}:=\mathrm{Spec}(K^{\mathrm{ur}})$.

We will give a way to construct a path from $\overline{z}$ to $\tilde{x}$. For a finite extension $K'$ of $K$ contained in $K^{\mathrm{ur}}$, we denote by $Z_{K'}$ a Kummer étale cover of $Z$ whose function field is $K'$. Then we have a natural morphism $\overline{z}\to Z_{K'}$ and the fiber functor $F_{\overline{z}}\colon\mathrm{Kcov}(Z)\to(\mathrm{Sets})$ is pro-represented by $\{Z_{K'}\}$. By Lemma 4.5.1, $\varprojlim_{K'}\mathrm{Hom}_Z(\tilde{x},Z_{K'})$ is non-empty. An element $\varphi=\{\varphi_{K'}\}_{K'}$ of this set defines a path from $\overline{z}$ to $\tilde{x}$; it induces a map

$$\mathrm{Hom}_Z(\overline{z},W)\xleftarrow{\sim}\varinjlim_{K'\subset K^{\mathrm{ur}}}\mathrm{Hom}_Z(Z_{K'},W)\xrightarrow{\varphi\circ-}\mathrm{Hom}_Z(\tilde{x},W)$$

for $W\in\mathrm{Kcov}(Z)$. For any $\sigma\in\mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x)$, there exists a unique automorphism $\sigma_{K'}$ of $Z_{K'}$ such that $\sigma_{K'}\circ\varphi_{K'}=\varphi_{K'}\circ\sigma$ for each finite Galois extension $K'$ of $K$ contained in $K^{\mathrm{ur}}$, and $\{\sigma_{K'}\}$ defines an automorphism $\tau\in\mathrm{Aut}(z^{\mathrm{ur}}/z)$. The homomorphism $\mathrm{Aut}(\widetilde{x^{\mathrm{ur}}}/x)^\circ\cong\pi_1(X,\tilde{x})\to\pi_1(Z,\overline{z})\cong\mathrm{Aut}(z^{\mathrm{ur}}/z)^\circ$ induced by the above path sends $\sigma$ to $\tau$. For another $\varphi'$, there exists a unique $\tau\in\mathrm{Aut}(z^{\mathrm{ur}}/z)$ such that $\varphi'=\sigma\circ\varphi$.

## §4.7. The complexes $\overline{\mathcal{S}}_n(r)$ and $\mathcal{S}_n^{\sim}(r)$.

We keep the notation of §4.5. Working étale locally on $\underline{X}$, we assume that $\underline{X}$ is affine and we are given a $W$-closed immersion of $X$ into a fine log scheme $Z$ smooth over $W$ endowed with a Cartier divisor $\underline{Z'}\subset\underline{Z}$ defined by a global equation $g = 0$ ($g \in \Gamma(Z, \mathcal{O}_Z)$) and with a compatible system of liftings of Frobenius $\{F_{Z_n} \colon Z_n \to Z_n\}_{n\geq 1}$ such that $\underline{X'}$ is the pull-back of $\underline{Z'}$ and $\underline{Z'}$ endowed with the pull-back of $M_Z$ is smooth over $W$. We denote by $Z^\circ$ the scheme $\underline{Z}$ endowed with the log structure defined by $M_Z$ and the Cartier divisor $\underline{Z'}$ (cf. the beginning of §4.2), and by $Z'$ (resp. $Z'^\circ$) the scheme $\underline{Z'}$ with the the inverse image of $M_Z$ (resp. $M_{Z^\circ}$). Note that $Z'^\circ$ is not smooth over $W$. We further assume that there exists $t_1, \ldots, t_d \in \Gamma(Z, M_Z)$ and $t \in \Gamma(Z, \mathcal{O}_Z)$ such that $\underline{Z'}$ is defined by $t = 0$ in $\underline{Z}$, $\{d\log(t_i)(1 \leq i \leq d), dt\}$ (resp. $\{d\log(t_i)(1 \leq i \leq d), d\log(t)\}$, resp. $\{d\log(t_i)\}$) form a basis of $\Omega_{Z/W}$ (resp. $\Omega_{Z^\circ/W}$, resp. $\Omega_{Z'/W}$), and $F^*_{Z_n}(t_i) = t_i^p$, $F^*_{Z_n}(t) = t^p$ for each $n \geq 1$. Choose and fix such $t_i$ and $t$. We have closed immersions $X^\circ \hookrightarrow Z^\circ$, $X' \hookrightarrow Z'$ and $X'^\circ \hookrightarrow Z'^\circ$. Recall that $X'^\circ$ is $\underline{X'}$ with the inverse image of $M_{X^\circ}$.

Let $U = \mathrm{Spec}(A) \to X$ be a strict étale morphism and set $U^\circ := X^\circ \times_X U$, $U' := X' \times_X U$ and $U'^\circ := X'^\circ \times_X U$. By replacing $U$ with a suitable affine open covering, we assume that $U$, $U^\circ$ and $U'$ satisfy the condition [Ts2] (1.5.2). (See [Ts2] Lemma 1.3.3). We may further assume that $U'^\circ$ also satisfies the equivalent conditions in [Ts2] Lemma 1.3.2 and $\Gamma(U', M_{U'})/\Gamma(U', \mathcal{O}^*_{U'}) \to \Gamma(U', M_{U'}/\mathcal{O}^*_{U'})$ is an isomorphism. (See the proof of [Ts2] Lemma 1.3.3 for the latter.) Let $A'$ be the coordinate ring of $U'$. As in [Ts2] 1.4, let $A^h$ be the henselization of $A$ with respect to the ideal $pA$.

Let $U^h$ be $\mathrm{Spec}(A^h)$ with the inverse image of $M_U$ and set $U^{h\circ} := X^\circ \times_X U^h$, $U'^h := X' \times_X U^h$ and $U'^{h\circ} := X'^\circ \times_X U^h$. The coordinate ring of $\underline{U'^h} = \underline{U'^{h\circ}}$ is the henselization of $A'$ with respect to $pA'$, which we denote by $A'^h$. Let $U^h_{\mathrm{triv}}$, $(U^{h\circ})_{\mathrm{triv}}$ and $U'^h_{\mathrm{triv}}$ denote the maximal open subschemes of $U^h$, $U^{h\circ}$ and $U'^h$ respectively on which the log structures are trivial and let $A^h_{\mathrm{triv}}$, $(A^{h\circ})_{\mathrm{triv}}$ and $A'^h_{\mathrm{triv}}$ denote their coordinate rings. Finally we define $(X_{\mathrm{triv}})^\circ$ (resp. $(X'_{\mathrm{triv}})^\circ$) to be $X_{\mathrm{triv}}$ (resp. $X'_{\mathrm{triv}}$) endowed with the inverse image of $M_{X^\circ}$, and define $(U^h_{\mathrm{triv}})^\circ$ and $(U'^h_{\mathrm{triv}})^\circ$ similarly. Note that the log structure of $(X_{\mathrm{triv}})^\circ$ is the one defined by the divisor $X'_{\mathrm{triv}} \hookrightarrow X_{\mathrm{triv}}$.

Now we have the following commutative diagrams:

(4.7.1)
$$\begin{array}{ccccc} ((X^\circ)_{\mathrm{triv}})_{\text{ét}} & \to & ((X_{\mathrm{triv}})^\circ)_{\text{Két}} & \xrightarrow{\varepsilon} & (X_{\mathrm{triv}})_{\text{ét}} \\ & & \uparrow & & \uparrow \\ & & ((X'_{\mathrm{triv}})^\circ)_{\text{Két}} & \xrightarrow{\varepsilon'} & (X'_{\mathrm{triv}})_{\text{ét}} \end{array}$$

(4.7.2)
$$\begin{array}{ccccc} ((U^{h\circ})_{\mathrm{triv}})_{\text{ét}} & \to & ((U^h_{\mathrm{triv}})^\circ)_{\text{Két}} & \xrightarrow{\varepsilon_U} & (U^h_{\mathrm{triv}})_{\text{ét}} \\ & & \uparrow & & \uparrow \\ & & ((U'^h_{\mathrm{triv}})^\circ)_{\text{Két}} & \xrightarrow{\varepsilon'_U} & (U'^h_{\mathrm{triv}})_{\text{ét}} \end{array}$$

Here Két denotes the Kummer étale site ([Na]). Note ét = Két for schemes with trivial log structures. We have natural morphisms from (4.7.2) to (4.7.1). We will construct a resolution $\overline{\mathcal{S}}_n(r)$ of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ on each site in the diagram (4.7.2) in a compatible manner. For $(U^{h\circ})_{\mathrm{triv}}$, $(U^h_{\mathrm{triv}})^\circ$ and $U'^h_{\mathrm{triv}}$, we can directly apply [Ts2]§3.1, but for the other two, we need some modifications.

Let $\eta := \mathrm{Spec}(\mathcal{K})$ be the generic point of $U^h_{\mathrm{triv}}$. Choose an algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ and set $\overline{\eta} := \mathrm{Spec}(\overline{\mathcal{K}})$. We define $\mathcal{K}^{\mathrm{ur}}$ (resp. $\mathcal{K}^{\circ\mathrm{ur}}$) to be the union of all finite extensions $\mathcal{L}$ of $\mathcal{K}$ contained in $\overline{\mathcal{K}}$ such that the normalizations of $U^h_{\mathrm{triv}}$ (resp. $(U^{h\circ})_{\mathrm{triv}}$) in $\mathcal{L}$ are unramified. We define $\eta'$, $\mathcal{K}'$, $\overline{\mathcal{K}'}$, $\overline{\eta'}$ and $\mathcal{K}'^{\mathrm{ur}}$ similarly using $U'^h_{\mathrm{triv}}$. We set

$$\begin{aligned} G_U &:= \mathrm{Gal}(\mathcal{K}^{\mathrm{ur}}/\mathcal{K}) \cong \pi_1(U^h_{\mathrm{triv}}, \overline{\eta}),\\ G^\circ_U &:= \mathrm{Gal}(\mathcal{K}^{\circ\mathrm{ur}}/\mathcal{K}) \cong \pi_1((U^{h\circ})_{\mathrm{triv}}, \overline{\eta}) \cong \pi_1((U^h_{\mathrm{triv}})^\circ, \overline{\eta}),\\ G'_U &:= \mathrm{Gal}(\mathcal{K}'^{\mathrm{ur}}/\mathcal{K}') \cong \pi_1(U'^h_{\mathrm{triv}}, \overline{\eta'}). \end{aligned}$$

See [I2]Theorem 7.6 for the last isomorphism in the second line. We define $\eta'^\circ$ to be $\mathrm{Spec}(\mathcal{K}')$ with the inverse image of $M_{(U'^h_{\mathrm{triv}})^\circ}$ and define $\widetilde{\eta'^\circ}$ and $\widetilde{\eta'^{\circ\mathrm{ur}}}$ similarly as $\tilde{x}$ and $\widetilde{x^{\mathrm{ur}}}$ in §4.6 using $\overline{\mathcal{K}'}$ and $\mathcal{K}'^{\mathrm{ur}}$. We set

$$G'^\circ_U := \mathrm{Aut}(\widetilde{\eta'^{\circ\mathrm{ur}}}/\eta'^\circ)^\circ \cong \pi_1((U'^h_{\mathrm{triv}})^\circ, \widetilde{\eta'^\circ}).$$

See Proposition 4.6.4 for the second isomorphism.

For a finite extension $\mathcal{L}$ of $\mathcal{K}$ contained in $\mathcal{K}^{\circ\mathrm{ur}}$, denote by $V_{\mathcal{L}}$ a Kummer étale cover of $(U^h_{\mathrm{triv}})^\circ$ whose function field is $\mathcal{L}$. We choose and fix a compatible system $\{f_{\mathcal{L}} \colon \widetilde{\eta'^\circ} \to V_{\mathcal{L}}\}_{\mathcal{L}\subset\mathcal{K}^{\circ\mathrm{ur}}}$, which gives a path from $\widetilde{\eta'^\circ} \to (U'^h_{\mathrm{triv}})^\circ \to (U^h_{\mathrm{triv}})^\circ$ to $\overline{\eta} \to (U^h_{\mathrm{triv}})^\circ$ (§4.6). It also induces a compatible system $\{\underline{f_{\mathcal{L}}} \colon \overline{\eta'} \to \underline{V_{\mathcal{L}}}\}_{\mathcal{L}\subset\mathcal{K}^{\mathrm{ur}}}$, which gives a path from $\overline{\eta'}$ to $\overline{\eta}$. These paths induce homomorphisms $G'^\circ_U \to G^\circ_U$ and $G'_U \to G_U$ which are compatible with the natural homomorphisms $G^\circ_U \to G_U$ and $G'^\circ_U \to G'_U$.

We define $\overline{A^h}$ (resp. $\overline{A^h_{\mathrm{triv}}}$) to be the normalization of $A^h$ (resp. $A^h_{\mathrm{triv}}$) in $\mathcal{K}^{\mathrm{ur}}$. Similarly, we define $\overline{A^{h\circ}}$ and $\overline{(A^{h\circ})_{\mathrm{triv}}}$ (resp. $\overline{A'^h}$ and $\overline{A'^h_{\mathrm{triv}}}$) using $A^h$, $(A^{h\circ})_{\mathrm{triv}}$ and $\mathcal{K}^{\circ\mathrm{ur}}$ (resp. $A'^h$, $A'^h_{\mathrm{triv}}$ and $\mathcal{K}'^{\mathrm{ur}}$). By applying [Ts2]§1.4 and §1.5 to $U$, $U^\circ$, $U'$ and $\overline{A^h}$, $\overline{A^{h\circ}}$, $\overline{A'^h}$, we obtain a commutative diagram:

$$\begin{array}{ccccc} U'^h & \leftarrow & \overline{U'} & \hookrightarrow & \overline{D'}\\ \downarrow & & \downarrow & & \downarrow\\ U^h & \leftarrow & \overline{U} & \hookrightarrow & \overline{D}\\ \uparrow & & \uparrow & & \uparrow\\ U^{h\circ} & \leftarrow & \overline{U^\circ} & \hookrightarrow & \overline{D^\circ} \end{array}$$

compatible with the actions of $G^*_U$ on $\overline{U^*}$ and $\overline{D^*}$ and with the liftings of Frobenius on $\overline{D^*}$ $(* = \emptyset, \prime, \circ)$. The upper vertical maps are induced by the path from $\overline{\eta'}$ to $\overline{\eta}$ chosen above.

Since $Z^\circ$ and $Z'$ satisfy the condition [Ts2] (2.1.1), we can construct resolutions $\overline{\mathcal{S}}_n(r)_{U^\circ,Z^\circ}$ and $\overline{\mathcal{S}}_n(r)_{U',Z'}$ of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ on $((U^{h\circ})_{\mathrm{triv}})_{\mathrm{\acute{e}t}}$ (or $((U^h_{\mathrm{triv}})^\circ)_{\mathrm{K\acute{e}t}}$) and on $(U'^h_{\mathrm{triv}})_{\mathrm{\acute{e}t}}$ as in [Ts2] §3.1. Although $Z$ does not satisfy [Ts2] (2.1.1), the construction in [Ts2] §3.1 still works as follows.

Lemma 4.7.3. (1) *The absolute Frobenius of* $\overline{A^h}/p\overline{A^h}$ *is surjective.*

(2) *The homomorphism* $Fil^1 A_{\mathrm{crys}}(\overline{A^h}) \to \widehat{\overline{A^h}}$ *defined by* $x \mapsto p^{-1}\varphi(x)$ mod $Fil^1$ *is surjective.*

*Proof.* (1) For any $a \in \overline{A^h}$, there exists $u \in (\overline{A^h})^*$ such that $1 + p^{1/2}a = u^p$ and hence $p^{1/2}a \equiv (u-1)^p \mod p\overline{A^h}$. Set $v = (u-1)\cdot p^{-1/2p}$. Then $v^p \in \overline{A^h}$ and hence $v \in \overline{A^h}$. Thus we obtain $a \equiv v^p \mod p^{1/2}\overline{A^h}$. Set $b = (a - v^p)\cdot p^{-1/2}$. Then, by the same argument, there exists $w \in \overline{A^h}$ such that $b \equiv w^p$ mod $p^{1/2}\overline{A^h}$. Now we have $a \equiv (v + p^{1/2p}w)^p \mod p\overline{A^h}$. (2) By (1), the homomorphism $A_{\mathrm{crys}}(\overline{A^h}) \to \widehat{\overline{A^h}}$ is surjective (cf. [Ts2] Lemma A1.1). Hence the claim follows from the surjectivity of $1-p^{-1}\varphi\colon Fil^1 A_{\mathrm{crys}}(\overline{A^h}) \to A_{\mathrm{crys}}(\overline{A^h})$ (cf. [Ts2] Theorem A3.26 and Proposition A3.33). □

As in [Ts2] §3.1, let $\overline{E}_n$ be the PD-envelope of $\overline{U}_n \hookrightarrow \overline{D}_n \times_{W_n} Z_n$ compatible with the PD-structure on $J_{\overline{D}_n} + p\mathcal{O}_{\overline{D}_n}$. For each $t_i \in \Gamma(Z, M_Z)$, choose and fix a lifting $a_i \in \Gamma(\overline{D}, M_{\overline{D}})$ of the image of $t_i$ in $\Gamma(\overline{U}, M_{\overline{U}})$ such that $F^*_{\overline{D}}(a_i) = a_i^p$ ([Ts2] Lemma 3.1.5). Let $u_i \in \Gamma(\overline{E}_n, 1 + J_{\overline{E}_n})$ be the unique element such that $t_i = a_i \cdot u_i$ in $\Gamma(\overline{E}_n, M_{\overline{E}_n})$. For $t \in \Gamma(Z, \mathcal{O}_Z)$, we choose a lifting $a \in \Gamma(\overline{D}, \mathcal{O}_{\overline{D}}) = A_{\mathrm{crys}}(\overline{A^h})$ of the image of $t$ in $\Gamma(\overline{U}, \mathcal{O}_{\overline{U}}) = \widehat{\overline{A^h}}$. By Lemma 4.7.3 (2), we may assume that $a^p - \varphi(p) \in pFil^1 A_{\mathrm{crys}}(\overline{A^h})$. Similarly as [Ts2] Lemma 3.1.4, we see that there exists a PD-isomorphism over $\mathcal{O}_{\overline{D}_n}$:

$$\mathcal{O}_{\overline{D}_n}\langle V_1, \dots, V_d, V\rangle \xrightarrow{\sim} \mathcal{O}_{\overline{E}_n}; V_i \mapsto u_i - 1, V \mapsto t - a$$

We define $\widetilde{J^{[r]\prime}_{\overline{D}_n}}$ and $\widetilde{J^{[r]\prime}_{\overline{E}_n}}$ as in [Ts2] §3.1.

Lemma 4.7.4. (cf. [Ts2] Lemma 3.1.6). *For each* $r$*, we have*

$$\widetilde{J^{[r]\prime}_{\overline{E}_n}} = \bigoplus_{\underline{m}\in\mathbb{N}^{d+1}} \widetilde{J^{[r]\prime}_{\overline{D}_n}} \prod_{1\le i\le d} (u_i - 1)^{[m_i]} \cdot (t-a)^{[m]},$$

*where* $\underline{m} = (m_1, \dots, m_d, m)$*.*

*Proof.* Since we can apply the same argument as the proof of [Ts2] Lemma 3.1.6 to the ring $R_n := \mathcal{O}_{\overline{D}_n}\langle u_1 - 1, \dots, u_d - 1\rangle$, it suffices to show the following: For $x = \sum_{m\in\mathbb{N}} x_m (t-a)^{[m]} \in \mathcal{O}_{\overline{E}_n} = R_n\langle t - a\rangle$, we have $\varphi(x) \in p^r\mathcal{O}_{\overline{E}_n}$ if and only if $\varphi(x_m) \in p^{\max\{r-m,0\}}R_n$ for all $m \in \mathbb{N}$. The sufficiency follows from

$$\begin{aligned}\varphi(t-a) &= t^p - \varphi(a) = (t - a + a)^p - \varphi(a)\\ &= p\big\{(p-1)!(t-a)^{[p]} + \sum_{\nu=1}^{p-1} \frac{1}{p}\binom{p}{\nu}(t-a)^{p-\nu}a^\nu\big\} + a^p - \varphi(p) \in pJ_{\overline{E}_n}.\end{aligned}$$

Assume $x \neq 0$ and $\varphi(x) \in p^r R_n\langle t-a\rangle$, and let $M$ be the largest integer such that $x_M \neq 0$. Then the coefficient of $(t-a)^{[Mp]}$ in $\varphi(x)$ is $\varphi(x_M)p^M c$ for some $c \in \mathbb{Z}_p^* \cap \mathbb{Q}$. Hence $p^M\varphi(x_M) \in p^r R_n$, which implies $\varphi(x_M) \in p^{\max\{r-M,0\}}R_n$. By the sufficiency, we can subtract $x_M(t-a)^{[M]}$, repeat the argument and show $\varphi(x_m) \in p^{\max\{r-m,0\}}R_n$ for all $m$. □

By Lemma 4.7.4, we can construct a complex $J^{[r]\prime}_{\overline{E}_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^\bullet_{Z_n/W_n}$ and show that it gives a resolution of $J^{[r]\prime}_{\overline{D}_n}$ as in [Ts2] Lemma 3.1.7. (Since $d(t-a)^{[m]} = (t-a)^{[m-1]}dt$, it is enough to use $dt$ instead of $u_i d\log(t_i)$ for the indeterminate $t-a$. ) Thus we obtain a resolution $\overline{\mathcal{S}}_n(r)_{U,Z}$ of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ on $(U^h_{\mathrm{triv}})_{\text{ét}}$ by the same method as [Ts2] §3.1. We have natural maps from the pull-backs of $\overline{\mathcal{S}}_n(r)_{U,Z}$ to $\overline{\mathcal{S}}_n(r)_{U',Z'}$ and $\overline{\mathcal{S}}_n(r)_{U^\circ,Z^\circ}$. We will need the following lemma to construct the map $\alpha$ (4.5.5).

Lemma 4.7.5. *The natural map $\overline{\mathcal{S}}_n(r)_{U,Z} \to \varepsilon_{U*}\overline{\mathcal{S}}_n(r)_{U^\circ,Z^\circ}$ is injective.* (*See* (4.7.2) *for* $\varepsilon_U$.)

Let $\overline{E^\circ}_n$ be the PD-envelope of $\overline{U^\circ}_n \hookrightarrow \overline{D^\circ}_n \times Z^\circ_n$. Choose a lifting $a' \in \Gamma(\overline{D^\circ}, M_{\overline{D^\circ}})$ of the image of $t$ in $\Gamma(\overline{U^\circ}, M_{\overline{U^\circ}})$ such that $F^*_{\overline{D^\circ}}(a') = (a')^p$ ([Ts2] Lemma 3.1.5) and let $u \in \Gamma(\overline{E^\circ}_n, 1+J_{\overline{E^\circ}_n})$ be the unique element such that $a' \cdot u = t$ in $\Gamma(\overline{E^\circ}_n, M_{\overline{E^\circ}_n})$. We denote the image of $u_i \in \Gamma(\overline{E}_n, 1+J_{\overline{E}_n})$ (defined above) in $\Gamma(\overline{E^\circ}_n, 1+J_{\overline{E^\circ}_n})$ by the same letter $u_i$. Then we have an isomorphism:

$$\mathcal{O}_{\overline{E^\circ}_n} \cong \mathcal{O}_{\overline{D^\circ}_n}\langle u_1-1,\dots,u_d-1,u-1\rangle.$$

*Proof Lemma 4.7.5.* It suffices to prove that the natural map $J^{[r]\prime}_{\overline{E}_n} \to J^{[r]\prime}_{\overline{E^\circ}_n}$ and the multiplication by $t$ on $J^{[r]\prime}_{\overline{E^\circ}_n}$ are injective. We first prove the claim for $r=0$. Since the map $\mathcal{O}_{\overline{E}_n} \to \mathcal{O}_{\overline{E^\circ}_n}$ factors through $\mathcal{O}_{\overline{D^\circ}_n}\langle u_i-1, t-a\rangle \cong \mathcal{O}_{\overline{D^\circ}_n}\langle u_i-1, t-a'\rangle$ and $t=a'u$, it is enough to prove that $\mathcal{O}_{\overline{D}_n} \to \mathcal{O}_{\overline{D^\circ}_n}$ and the multiplication by $a'$ on $\mathcal{O}_{\overline{D^\circ}_n}$ are injective. We are easily reduced to the case $n=1$. Define $R_{\overline{A^h}}$ and $R_{\overline{A^{h\circ}}}$ as in [Ts2] §1.1 and let $z$ be a generator of the kernel of $W(R_{\overline{A^h}}) \to \widehat{\overline{A^h}}$ ([Ts2] Corollary A2.2). Then, by [Ts2] Lemma A2.11, we see that $\Gamma(\overline{D}_1, \mathcal{O}_{\overline{D}_1}) = A_{\mathrm{crys}}(\overline{A^h})/p$ (resp. $\Gamma(\overline{D^\circ}_1, \mathcal{O}_{\overline{D^\circ}_1}) = A_{\mathrm{crys}}(\overline{A^{h\circ}})/p$) is a free $R_{\overline{A^h}}/z^p$ (resp. $R_{\overline{A^{h\circ}}}/z^p$)-module with a base $\{z^{[pn]} | n \geq 0\}$. Especially, the filtration is separated and each graded quotient is isomorphic to $R_{\overline{A^h}}/z \cong \overline{A^h}/p$ (resp. $R_{\overline{A^{h\circ}}}/z \cong \overline{A^{h\circ}}/p$) (Lemma 4.7.3 (1) and [Ts2] Lemma A2.1). Hence the claim follows from the injectivity of the natural map $\overline{A^h}/p \to \overline{A^{h\circ}}/p$ and the multiplication by $t$ on $\overline{A^{h\circ}}/p$. Next we consider the case $r \geq 1$. Let $x \in \widetilde{J^{[r]\prime}_{\overline{E}_{n+r}}}$ and assume that its image in $\widetilde{J^{[r]\prime}_{\overline{E^\circ}_{n+r}}}$ is contained in $p^n\widetilde{J^{[r]\prime}_{\overline{E^\circ}_{n+r}}}$. Then it is contained in $p^n J^{[r]}_{\overline{E^\circ}_{n+r}}$ and hence $x \in p^n J^{[r]}_{\overline{E}_{n+r}}$ by the case $r=0$. Choose

$y \in J^{[r]}_{\overline{E}_{n+r}}$ such that $x = p^n y$ and let $y^\circ$ be the image of $y$ in $J^{[r]}_{\overline{E^\circ}_{n+r}}$. Then $p^n y^\circ \in p^n \widetilde{J^{[r]\prime}_{\overline{E^\circ}_{n+r}}}$, which implies $y^\circ \in \widetilde{J^{[r]\prime}_{\overline{E^\circ}_{n+r}}}$ i.e. $\varphi(y^\circ) \in p^r \mathcal{O}_{\overline{E^\circ}_{n+r}}$. By the case $r = 0$, this implies $\varphi(y) \in p^r \mathcal{O}_{\overline{E}_{n+r}}$ and hence $y \in \widetilde{J^{[r]\prime}_{\overline{E}_{n+r}}}$. We can prove the second assertion similarly. Note $\varphi(a') = (a')^p$. $\square$

Now it remains to construct a resolution $\overline{\mathcal{S}}_n(r)_{U'^\circ, Z'^\circ}$ of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ on $((U'_{\mathrm{triv}})^\circ)_{\mathrm{K\acute{e}t}}$. To do this we need to construct $\overline{U'}^\circ \hookrightarrow \overline{D'}^\circ$ for the non-smooth $U'^\circ$ modifying the construction in [Ts2] §1.4, §1.5.

The underlying scheme of $\overline{D'}^\circ$ is the same as $\overline{D'}$ i.e. $\mathrm{Spec}(A_{\mathrm{crys}}(\overline{A'^h}))$. For the log structure, we use the fiber product $Q'^\circ$ of the diagram of monoids:

$$\varprojlim(M_{\widetilde{\eta'^\circ\mathrm{ur}}} \overset{f}{\leftarrow} M_{\widetilde{\eta'^\circ\mathrm{ur}}} \overset{f}{\leftarrow} M_{\widetilde{\eta'^\circ\mathrm{ur}}} \overset{f}{\leftarrow} \cdots) \to M_{\widetilde{\eta'^\circ\mathrm{ur}}} \leftarrow \Gamma(U'^\circ, M_{U'^\circ}),$$

where $f(x) = x^p$ and the left map is the projection to the first component. Choose a chart $\widetilde{\mathbb{N}} \to M_{\widetilde{\eta'^\circ\mathrm{ur}}}$ compatible with the chart $\mathbb{N} \to \Gamma(U'^\circ, M_{U'^\circ})$ sending 1 to the image of $t$ (cf. the definition of $\widetilde{x^{\mathrm{ur}}}$ in §4.6). The chart induces an isomorphism $\widetilde{\mathbb{N}} \oplus (\mathcal{K}'^{\mathrm{ur}})^* \cong M_{\widetilde{\eta'^\circ\mathrm{ur}}}$. By $\Gamma(U', M_{U'}/\mathcal{O}^*_{U'}) \oplus \mathbb{N} \cong \Gamma(U'^\circ, M_{U'^\circ}/\mathcal{O}^*_{U'^\circ})$ and the assumption $\Gamma(U', M_{U'})/\Gamma(U', \mathcal{O}^*_{U'}) \cong \Gamma(U', M_{U'}/\mathcal{O}^*_{U'})$, we see $\Gamma(U', M_{U'})/\Gamma(U', \mathcal{O}^*_{U'}) \oplus \mathbb{N} \cong \Gamma(U'^\circ, M_{U'^\circ})/\Gamma(U'^\circ, \mathcal{O}^*_{U'^\circ})$ and hence $\Gamma(U'^\circ, M_{U'^\circ})$ is generated by the images of $t$ and $\Gamma(U', M_{U'})$. This implies that the image of $\Gamma(U'^\circ, M_{U'^\circ})$ in $M_{\widetilde{\eta'^\circ\mathrm{ur}}} \cong \widetilde{\mathbb{N}} \oplus (\mathcal{K}'^{\mathrm{ur}})^*$ is contained in $\mathbb{N} \oplus ((A'^h_{\mathrm{triv}})^* \cap A'^h)$. Hence $Q'^\circ$ coincides with the fiber product of

$$\varprojlim_f \big(\widetilde{\mathbb{N}} \oplus ((\overline{A'^h_{\mathrm{triv}}})^* \cap \overline{A'^h})\big) \to \widetilde{\mathbb{N}} \oplus ((\overline{A'^h_{\mathrm{triv}}})^* \cap \overline{A'^h}) \leftarrow \Gamma(U'^\circ, M_{U'^\circ})$$

and the morphism $Q'^\circ \to \varprojlim_f M_{\widetilde{\eta'^\circ\mathrm{ur}}} \to \varprojlim_f \mathcal{K}'^{\mathrm{ur}}$ factors through $\varprojlim_f \overline{A'^h}$. We define the log structure of $\overline{D'}^\circ$ to be the one associated to

$$Q'^\circ \to \varprojlim \overline{A^h} \to R_{\overline{A^h}} \overset{[\,]}{\to} W(R_{\overline{A'^h}}) \subset A_{\mathrm{crys}}(\overline{A'^h}).$$

Using the natural action of $G'^\circ_U$ on $Q'^\circ$ and the multiplication by $p$ on $Q'^\circ$, we can define the action of $G'^\circ_U$ and the lifting of Frobenius on $\overline{D'}^\circ$.

We define $\overline{U'}^\circ$ to be $\mathrm{Spec}(\widehat{\overline{A'^h}})$ with the log structure associated to $\Gamma(U'^\circ, M_{U'^\circ}) \to \widehat{\overline{A'^h}}$. We have a natural action of $G'^\circ_U$ (through $G'_U$) on $\overline{U'}^\circ$ and $G'^\circ_U$-equivariant morphism $\overline{U'}^\circ \to \overline{D'}^\circ$ induced by the surjection $A_{\mathrm{crys}}(\overline{A'^h}) \to \widehat{\overline{A'^h}}$ (cf. Lemma 4.7.3) and $Q'^\circ \to \Gamma(U'^\circ, M_{U'^\circ})$.

We assert that $\overline{U'}^\circ$ and $\overline{D'}^\circ$ are fs log schemes and the morphism $\overline{U'}^\circ \to \overline{D'}^\circ$ is an exact closed immersion. To prove this, we choose a chart $P \to \Gamma(U'^\circ, M_{U'^\circ})$ such that $P \to \Gamma(U'^\circ, M_{U'^\circ})/\Gamma(U'^\circ, \mathcal{O}^*_{U'^\circ})$ is an isomorphism ([Ts2] Lemma

1.3.2). Then the composite $P \to \Gamma(U'^{\circ}, M_{U'^{\circ}}) \to \widetilde{\mathbb{N}} \oplus ((\overline{A'^h_{\mathrm{triv}}})^* \cap \overline{A'^h})$ can be lifted to the projective limit $\varprojlim_f(\widetilde{\mathbb{N}} \oplus ((\overline{A'^h_{\mathrm{triv}}})^* \cap \overline{A'^h}))$. Hence $P \to \Gamma(U'^{\circ}, M_{U'^{\circ}})$ can be lifted to $P \to Q'^{\circ}$. On the other hand, since the image of $\Gamma(U'^{\circ}, \mathcal{O}^*_{U'^{\circ}})$ in $\widetilde{\mathbb{N}} \oplus (\overline{A'^h_{\mathrm{triv}}}^* \cap \overline{A'^h})$ is contained in $\overline{A'^h}^*$, the inverse image $G$ of $\{1\}$ under $Q'^{\circ} \to \Gamma(U'^{\circ}, M_{U'^{\circ}})/\Gamma(U'^{\circ}, \mathcal{O}^*_{U'^{\circ}})$ is a group and we have $Q'^{\circ}/G \cong \Gamma(U'^{\circ}, M_{U'^{\circ}})/\Gamma(U'^{\circ}, \mathcal{O}^*_{U'^{\circ}}) \cong P$. Hence, by [Ts2] Lemma 1.3.1, $P \to \Gamma(U'^{\circ}, M_{U'^{\circ}}) \to \Gamma(\overline{U'}^{\circ}, M_{\overline{U'}^{\circ}})$ and $P \to Q'^{\circ} \to \Gamma(\overline{D'}^{\circ}, M_{\overline{D'}^{\circ}})$ are charts, and $\overline{U'}^{\circ} \to \overline{D'}^{\circ}$ is an exact closed immersion.

Next we compare $\overline{U'}^{\circ} \to \overline{D'}^{\circ}$ with $\overline{U'} \to \overline{D'}$ and $\overline{U^{\circ}} \to \overline{D^{\circ}}$. The fiber product $Q'$ of $\varprojlim_f \overline{A'^h} \to \overline{A'^h} \leftarrow \Gamma(U', M_{U'})$ used in the definition of the log structure of $\overline{D'}$ is the same as the fiber product of the diagram with $\overline{A'^h}$ replaced by $\overline{A'^h_{\mathrm{triv}}}^* \cap \overline{A'^h}$. Hence, there exists a natural map $Q' \to Q'^{\circ}$ compatible with the actions of $G'_U$ and $G'^{\circ}_U$. Using this and the natural map $\Gamma(U', M_{U'}) \to \Gamma(U'^{\circ}, M_{U'^{\circ}})$, we obtain a commutative diagram:

$$\begin{array}{ccccc} U'^{h\circ} & \leftarrow & \overline{U'}^{\circ} & \hookrightarrow & \overline{D'}^{\circ} \\ \downarrow & & \downarrow & & \downarrow \\ U'^h & \leftarrow & \overline{U'} & \hookrightarrow & \overline{D'} \end{array} \tag{4.7.6}$$

compatible with the actions of $G'_U$, $G'^{\circ}_U$ and the liftings of Frobenius.

Similarly, the fiber product $Q^{\circ}$ of $\varprojlim_f \overline{A^{h\circ}} \to \overline{A^{h\circ}} \leftarrow \Gamma(U^{\circ}, M_{U^{\circ}})$ used in the construction of the log structure of $\overline{D^{\circ}}$ is the same as the the fiber product of the diagram with $\overline{A^{h\circ}}$ replaced with $\overline{(A^{h\circ})_{\mathrm{triv}}}^* \cap \overline{A^{h\circ}}$. On the other hand, we have $\overline{(A^{h\circ})_{\mathrm{triv}}}^* \cap \overline{A^{h\circ}} \subset \varinjlim_{\mathcal{L} \subset \mathcal{K}^{\mathrm{our}}} \Gamma(V_{\mathcal{L}}, M_{\mathcal{L}})$, where $V_{\mathcal{L}}$ is as in the construction of a path from $\widetilde{\eta'^{\circ}}$ to $\overline{\eta}$. The fixed system of morphisms $\{f_{\mathcal{L}} \colon \widetilde{\eta'^{\circ}} \to V_{\mathcal{L}}\}$ induces a morphism $Q^{\circ} \to Q'^{\circ}$ compatible with the actions of $G^{\circ}_U$ and $G'^{\circ}_U$. Using this and the natural map $\Gamma(U^{\circ}, M_{U^{\circ}}) \to \Gamma(U'^{\circ}, M_{U'^{\circ}})$, we obtain a commutative diagram:

$$\begin{array}{ccccc} U^{h\circ} & \leftarrow & \overline{U^{\circ}} & \hookrightarrow & \overline{D^{\circ}} \\ \uparrow & & \uparrow & & \uparrow \\ U'^{h\circ} & \leftarrow & \overline{U'}^{\circ} & \hookrightarrow & \overline{D'}^{\circ} \end{array} \tag{4.7.7}$$

compatible with the actions of $G'^{\circ}_U$, $G^{\circ}_U$ and the liftings of Frobenius.

Now we are ready to construct $\overline{\mathcal{S}}_n(r)_{U'^{\circ}, Z'^{\circ}}$. Let $\overline{E'^{\circ}}_n$ be the PD-envelope of $\overline{U'}^{\circ}_n \hookrightarrow \overline{D'}^{\circ}_n \times Z^{\circ}_n$, which is endowed with an action of $G'^{\circ}_U$ and a lifting of Frobenius by a natural way. The diagram (4.7.7) and $\mathrm{id} \colon Z^{\circ} \to Z^{\circ}$ induce a PD-morphism $\overline{E'^{\circ}}_n \to \overline{E^{\circ}}_n$ compatible with the actions of $G^{\circ}_U$ and $G'^{\circ}_U$ and with the liftings of Frobenius. If we denote by the images of $u_i, u \in \Gamma(\overline{E^{\circ}}_n, \mathcal{O}^*_{\overline{E^{\circ}}_n})$ in $\Gamma(\overline{E'^{\circ}}_n, \mathcal{O}^*_{\overline{E'^{\circ}}_n})$ by the same symbols, we have (cf. [Ts2] Lemma 3.1.4)

$$\mathcal{O}_{\overline{E'^{\circ}}_n} \cong \mathcal{O}_{\overline{D'}^{\circ}_n} \langle u_i - 1, u - 1 \rangle.$$

Hence, if we define $\widetilde{J^{[r]\prime}_{\overline{E'^{\circ}}_n}}$ similarly as in [Ts2] §3.1, [Ts2] Lemma 3.1.6 still holds. Similarly as in [Ts2] Lemma 3.1.7, we obtain a resolution $J^{[r]\prime}_{\overline{D'}^{\circ}_n} \to J^{[r-\bullet]\prime}_{\overline{E'^{\circ}}_n} \otimes \Omega^{\bullet}_{Z^{\circ}_n/W_n}$. Thus, as in [Ts2] §3.1, we obtain the required resolution $\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}}$ of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ on $((U'^{h}_{\mathrm{triv}})^{\circ})_{\mathrm{K\acute{e}t}}$ by taking the global section of the mapping fiber of

$$1-\varphi_r\colon J^{[r-\bullet]\prime}_{\overline{E'^{\circ}}_n} \otimes \Omega^{\bullet}_{Z^{\circ}_n/W_n} \to \mathcal{O}_{\overline{E'^{\circ}}_n} \otimes \Omega^{\bullet}_{Z^{\circ}_n/W_n}.$$

The morphism $\overline{E'^{\circ}}_n \to \overline{E^{\circ}}_n$ induces a map from the pull-back of $\overline{\mathcal{S}}_n(r)_{U^{\circ},Z^{\circ}}$ to $\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}}$. The comparison with $\overline{\mathcal{S}}_n(r)_{U',Z'}$ is non-trivial. Consider the morphism $\overline{D'}^{\circ} \times Z^{\circ} \to \mathrm{Spec}(\mathbb{Z}[\mathbb{N}\oplus\mathbb{N}])$ defined by sending $(1,0),(0,1) \in \mathbb{N}\oplus\mathbb{N}$ to the images of $a' \in \Gamma(\overline{D^{\circ}}, M_{\overline{D^{\circ}}})$ and $t \in \Gamma(Z^{\circ}, M_{Z^{\circ}})$ in $\Gamma(\overline{D'}^{\circ} \times Z^{\circ}, M)$. Define the log étale morphism $\mathrm{Spec}(\mathbb{Z}[\mathbb{N}\oplus\mathbb{Z}]) \to \mathrm{Spec}(\mathbb{Z}[\mathbb{N}\oplus\mathbb{N}])$ by $\mathbb{N}\oplus\mathbb{N} \to \mathbb{N}\oplus\mathbb{Z}; (m,n) \mapsto (m, m-n)$, and consider the following cartesian diagrams:

$$\begin{array}{ccccc}
(\overline{D'}^{\circ} \times Z'^{\circ})^{\sim} & \to & (\overline{D'}^{\circ} \times Z^{\circ})^{\sim} & \to & \mathrm{Spec}(\mathbb{Z}[\mathbb{N}\oplus\mathbb{Z}]) \\
\downarrow & & \downarrow & & \downarrow \\
\overline{D'}^{\circ} \times Z'^{\circ} & \to & \overline{D'}^{\circ} \times Z^{\circ} & \to & \mathrm{Spec}(\mathbb{Z}[\mathbb{N}\oplus\mathbb{N}]).
\end{array}$$

Then, the closed immersion $\overline{U'}^{\circ} \hookrightarrow \overline{D'}^{\circ} \times Z'^{\circ}$ naturally factors through $(\overline{D'}^{\circ} \times Z'^{\circ})^{\sim}$ because the images of $a'$ and $t$ coincide in $\Gamma(\overline{U'}^{\circ}, M)$. On the other hand, the vanishing of $a'$ in $\Gamma(\overline{D'}^{\circ} \times Z^{\circ}, \mathcal{O})$ implies that $(\overline{D'}^{\circ} \times Z'^{\circ})^{\sim} \to (\overline{D'}^{\circ} \times Z^{\circ})^{\sim}$ is an isomorphism. Hence $\overline{E'^{\circ}}_n$ is isomorphic to the PD-envelope of $\overline{U'}^{\circ}_n \hookrightarrow \overline{D'}^{\circ}_n \times Z'^{\circ}_n$, and the diagram (4.7.6) and $Z'^{\circ} \to Z'$ induce a PD-morphism $\overline{E'^{\circ}}_n \to \overline{E'}_n$. Here $\overline{E'}_n$ denotes the PD-envelope of $\overline{U'}_n \hookrightarrow \overline{D'}_n \times Z'_n$ used in the construction of $\overline{\mathcal{S}}_n(r)_{U',Z'}$. One can verify the compatibility with the actions of $G'_U$, $G'^{\circ}_U$, with the liftings of Frobenius and with the connections. Thus we obtain a canonical map $\overline{\mathcal{S}}_n(r)_{U',Z'} \to \varepsilon'_{U*}\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}}$.

LEMMA 4.7.8. *The map* $\overline{\mathcal{S}}_n(r)_{U',Z'} \to \varepsilon'_{U*}\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}}$ *is injective.*

*Proof.* This follows from $\mathcal{O}_{\overline{E'}_n} \cong \mathcal{O}_{\overline{D'}_n}\langle u_i - 1\rangle$ and $\mathcal{O}_{\overline{E'^{\circ}}_n} \cong \mathcal{O}_{\overline{D'}_n}\langle u_i - 1, u-1\rangle$. □

Next we discuss on Gysin sequence for $\overline{\mathcal{S}}_n(r)$ on $(U'_{\mathrm{triv}})_{\mathrm{\acute{e}t}}$. Recall that $Z'^{\circ}$ is not smooth over $W$.

LEMMA 4.7.9. (1) *The natural map* $\Omega_{Z^{\circ}/W} \otimes_{\mathcal{O}_{Z^{\circ}}} \mathcal{O}_{Z'^{\circ}} \to \Omega_{Z'^{\circ}/W}$ *is an isomorphism.*
(2) *Let* $\mathcal{F}$ *be an* $\mathcal{O}_{Z'}(=\mathcal{O}_{Z'^{\circ}})$*-module with an integrable connection* $\nabla : \mathcal{F} \to \mathcal{F}\otimes\Omega_{Z'/W}$. *Then the composite*

$$\mathcal{F} \to \mathcal{F}\otimes_{\mathcal{O}_{Z'}}\Omega_{Z'/W} \to \mathcal{F}\otimes_{\mathcal{O}_{Z'}}\Omega_{Z'^{\circ}/W} \cong \mathcal{F}\otimes_{\mathcal{O}_{Z}}\Omega_{Z^{\circ}/W}$$

*is an integrable connection on $\mathcal{F}$ as an $\mathcal{O}_{Z^\circ}$-module. Furthermore, the natural maps $\mathcal{F}\otimes\Omega^q_{Z'} \to \mathcal{F}\otimes\Omega^q_{Z'^\circ} \cong \mathcal{F}\otimes\Omega^q_{Z^\circ}$ induce a morphism between the de Rham complexes: $\mathcal{F}\otimes\Omega^\bullet_{Z'} \to \mathcal{F}\otimes\Omega^\bullet_{Z^\circ}$.*

*Proof.* Straightforward. □

By Lemma 4.7.9 above, we can replace $\Omega^q_{Z'}$ with $\Omega^q_{Z'^\circ}$ in $\overline{\mathcal{S}}_n(r)_{U',Z'}$, and obtain a complex on $(U'^h_{\text{triv}})_{\text{ét}}$, which we denote by $\overline{\mathcal{S}}_n(r)^\circ_{U',Z'}$. We can construct a short exact sequence:

$$(4.7.10)\qquad 0 \longrightarrow \overline{\mathcal{S}}_n(r)_{U',Z'} \longrightarrow \overline{\mathcal{S}}_n(r)^\circ_{U',Z'} \longrightarrow \overline{\mathcal{S}}_n(r-1)_{U',Z'}[-1] \longrightarrow 0$$

in an obvious way. On the other hand, the PD-morphism $\overline{E'^\circ}_n \to \overline{E'}_n$ induces a map $\overline{\mathcal{S}}_n(r)^\circ_{U',Z'} \to \varepsilon'_{U*}\overline{\mathcal{S}}_n(r)_{U'^\circ,Z'^\circ}$ compatible with the map in Lemma 4.7.8. Finally, we discuss on the complex $\mathcal{S}^\sim_n(r)$. As in [Ts2] §2.1, using the PD-envelopes of $X \hookrightarrow Z$, $X^\circ \hookrightarrow Z^\circ$ and $X' \hookrightarrow Z'$, we can define complexes $\mathcal{S}^\sim_n(r)_{X,Z}$, $\mathcal{S}^\sim_n(r)_{X^\circ,Z^\circ}$ and $\mathcal{S}^\sim_n(r)_{X',Z'}$ on $(\underline{X_1})_{\text{ét}}$, $(\underline{X^\circ_1})_{\text{ét}} = (\underline{X_1})_{\text{ét}}$ and $(\underline{X'_1})_{\text{ét}}$. We have natural maps from the first complex to the latter two. We also have natural maps from the sections over $\underline{U_1}$, $\underline{U^\circ_1} = \underline{U_1}$ and $\underline{U'_1}$ to the global sections of $\overline{\mathcal{S}}_n(r)_{U,Z}$, $\overline{\mathcal{S}}_n(r)_{U^\circ,Z^\circ}$ and $\overline{\mathcal{S}}_n(r)_{U',Z'}$ (cf. [Ts2] (3.1.9) and (2.1.2)). For $X'^\circ \hookrightarrow Z'^\circ$, we define the complex $\mathcal{S}^\sim_n(r)_{X'^\circ,Z'^\circ}$ on $(\underline{X'^\circ_1})_{\text{ét}} = (\underline{X'_1})_{\text{ét}}$ to be the one obtained by replacing $\Omega^q_{Z'}$ with $\Omega^q_{Z'^\circ}$ in $\mathcal{S}^\sim_n(r)_{X',Z'}$, using Lemma 4.7.9. We can construct a short exact sequence:

$$(4.7.11)\qquad 0 \longrightarrow \mathcal{S}^\sim_n(r)_{X',Z'} \longrightarrow \mathcal{S}^\sim_n(r)_{X'^\circ,Z'^\circ} \longrightarrow \mathcal{S}^\approx_n(r-1)_{X',Z'}[-1] \longrightarrow 0.$$

Here $\mathcal{S}^\approx_n(r-1)_{X',Z'}$ is the complex obtained from $\mathcal{S}^\sim_n(r-1)_{X',Z'}$ by replacing $p^{r-1}-\varphi$ with $p^r - p\varphi$. We have a natural map from $\Gamma(\underline{U'_1}, \mathcal{S}^\sim_n(r)_{X'^\circ,Z'^\circ})$ to the global section of $\overline{\mathcal{S}}_n(r)^\circ_{U',Z'}$, and (4.7.11) is compatible with (4.7.10). Noting that the PD-envelope of $U'^\circ \hookrightarrow Z'^\circ$ is isomorphic to the PD-envelope of $U' \hookrightarrow Z'$, one can also construct a natural map from the pull-back of $\mathcal{S}^\sim_n(r)_{X^\circ,Z^\circ}$ to $\mathcal{S}^\sim_n(r)_{X'^\circ,Z'^\circ}$ and (4.7.11) is compatible with the short exact sequence (cf. (4.3.1)):

$$(4.7.12)\qquad 0 \longrightarrow \mathcal{S}^\sim_n(r)_{X,Z} \longrightarrow \mathcal{S}^\sim_n(r)_{X^\circ,Z^\circ} \longrightarrow \mathcal{S}^\approx_n(r-1)_{X',Z'}[-1] \longrightarrow 0.$$

## §4.8. Proof of Proposition 4.5.3.

We keep the notation and the assumption in §4.7. We first construct the morphisms $\alpha$ (4.5.5) and $\beta$ (4.5.6).

Choose sufficiently large algebraically closed fields $\Omega$ of characteristic 0 and $\Omega'$ of characteristic $p$. Let $\mathcal{S}$ be the set of all isomorphic classes of fs monoids $P$ such that $P^* = \{1\}$. For each isomorphic class $c \in \mathcal{S}$, choose a representative $P_c$ of $c$ and define the log geometric point $\Omega_c$ to be Spec($\Omega$) with $M_{\Omega_c} = \Omega \oplus \cup_{n\in\mathbb{N},n\neq 0}\frac{1}{n}P_c$ and $\Omega'_c$ to be Spec($\Omega'$) with $M_{\Omega'_c} = \Omega' \oplus \cup_{n\in\mathbb{N},p\nmid n}\frac{1}{n}P_c$. In the following, we denote by $C^*$ the Godement resolution with respect to all log geometric points whose sources are $\Omega_c$ or $\Omega'_c$ for some $c \in \mathcal{S}$. Note that such log geometric points form a set.

To simplify the notation, we write $\Theta$ for the operation $C^* i_* i^* j_* C^*$ and $\Theta'$ for $C^* i'_* i'^* j'_* C^*$. Denote by $i_U$ and $i'_U$ the closed immersions $\underline{U} \otimes k = \underline{U^h} \otimes k \to \underline{U^h}$ and $\underline{U'} \otimes k = \underline{U'^h} \otimes k \to \underline{U'^h}$, and by $j_U$, $j^\circ_U$, $j'_U$ the open immersions $U^h_{\mathrm{triv}} \to \underline{U^h}$, $(U^{h\circ})_{\mathrm{triv}} \to \underline{U^h}$ and $U'^h_{\mathrm{triv}} \to \underline{U'^h}$. Similarly, as above, we denote by $\Theta_U$ and $\Theta'_U$ the operations $C^* i_{U*} i^*_U j_{U*} C^*$ and $C^* i'_{U*} i'^*_U j'_{U*} C^*$.

Since the derived direct images of $\mathbb{Z}/p^n\mathbb{Z}(r)'$ by the left morphisms in the first lines of the diagrams (4.7.1) and (4.7.2) are again $\mathbb{Z}/p^n\mathbb{Z}(r)'$ ([I2] Theorem 7.4), we see that the left and middle vertical morphisms in the diagram in Proposition 4.5.3 are induced by sheafifying the following morphisms of presheaves on $\underline{X}_{\text{ét}}$:

$$\Gamma(\underline{U}, \Theta(\Lambda)) \overset{\text{q.i.}}{\to} \Gamma(\underline{U^h}, \Theta_U(\Lambda)) \overset{\text{q.i.}}{\to} \Gamma(\underline{U^h}, \Theta_U(\overline{\mathcal{S}}_n(r)_{U,Z})) \leftarrow \Gamma(\underline{U}, i_* \mathcal{S}^\sim_n(r)_{X,Z})$$

$$\Gamma(\underline{U}, \Theta(\varepsilon_* C^*(\Lambda))) \overset{\text{q.i}}{\to} \Gamma(\underline{U^h}, \Theta_U(\varepsilon_{U*} C^*(\Lambda))) \overset{\text{q.i.}}{\to}$$
$$\Gamma(\underline{U^h}, \Theta_U(\varepsilon_{U*} C^*(\overline{\mathcal{S}}_n(r)_{U^\circ, Z^\circ}))) \leftarrow \Gamma(\underline{U}, i_* \mathcal{S}^\sim_n(r)_{X^\circ, Z^\circ})$$

Here $\Lambda = \mathbb{Z}/p^n\mathbb{Z}(r)'$ and q.i. means a quasi-isomorphism. See (4.7.1) and (4.7.2) for $\varepsilon$ and $\varepsilon_U$. Let $K^\bullet$, $K^\bullet_U$ and $L^\bullet_U$ be the cokernels of the injective homomorphisms $\Lambda_{X_{\mathrm{triv}}} \to \varepsilon_* C^*(\Lambda_{(X_{\mathrm{triv}})^\circ})$, $\Lambda_{U^h_{\mathrm{triv}}} \to \varepsilon_{U*} C^*(\Lambda_{(U^h_{\mathrm{triv}})^\circ})$, and $\overline{\mathcal{S}}_n(r)_{U,Z} \to \varepsilon_{U*} C^*(\overline{\mathcal{S}}_n(r)_{U^\circ, Z^\circ})$ (Lemma 4.7.5). We have a natural injective homomorphism from the first line to the second one. Taking its quotient and using (4.7.12), we obtain

$$\Gamma(\underline{U}, \Theta(K^\bullet)) \overset{\text{q.i.}}{\to} \Gamma(\underline{U^h}, \Theta_U(K^\bullet_U)) \overset{\text{q.i.}}{\to} \Gamma(\underline{U^h}, \Theta_U(L^\bullet_U)) \leftarrow \Gamma(\underline{U}, i_* \mathcal{S}^\approx_n(r)_{X',Z'}[-1]) \tag{4.8.1}$$

We have quasi-isomorphisms $K^\bullet \leftarrow \tau_{\leq 1} K^\bullet \to \mathcal{H}^1(K^\bullet)[-1] \cong \Lambda(-1)_{X'_{\mathrm{triv}}}[-1]$ ([I2]Theorem 7.4). Hence varying $U$ and sheafifying, we obtain the required morphism $\alpha$ (4.5.5).

We can apply the same argument to $\varepsilon' \colon ((X'_{\mathrm{triv}})^\circ)_{\text{Két}} \to (X'_{\mathrm{triv}})_{\text{ét}}$, $\varepsilon'_U \colon ((U'^h_{\mathrm{triv}})^\circ)_{\text{Két}} \to (U'^h_{\mathrm{triv}})_{\text{ét}}$ and the resolutions $\Lambda_{U'^h_{\mathrm{triv}}} \to \overline{\mathcal{S}}_n(r)_{U',Z'}$ and $\Lambda_{(U'^h_{\mathrm{triv}})^\circ} \to \overline{\mathcal{S}}_n(r)_{U'^\circ, Z'^\circ}$. We define $K'^\bullet$, $K'^\bullet_U$ and $L'^\bullet_U$ to be the cokernels of the injective homomorphisms $\Lambda_{X'_{\mathrm{triv}}} \to \varepsilon'_* C^*(\Lambda_{(X'_{\mathrm{triv}})^\circ})$, $\Lambda_{U'^h_{\mathrm{triv}}} \to \varepsilon'_{U*} C^*(\Lambda_{(U'^h_{\mathrm{triv}})^\circ})$ and $\overline{\mathcal{S}}_n(r)_{U',Z'} \to \varepsilon'_{U*} C^*(\overline{\mathcal{S}}_n(r)_{U'^\circ, Z'^\circ})$ (Lemma 4.7.8). Then using (4.7.11), we obtain

$$\Gamma(\underline{U'}, \Theta'(K'^\bullet)) \overset{\text{q.i.}}{\to} \Gamma(\underline{U'^h}, \Theta'_U(K'^\bullet_U)) \overset{\text{q.i.}}{\to} \Gamma(\underline{U'^h}, \Theta'_U(L'^\bullet_U)) \leftarrow \Gamma(\underline{U'}, i'_* \mathcal{S}^\approx_n(r)_{X',Z'}[-1]) \tag{4.8.2}$$

By [I2]Theorem 7.4 and [Na] Theorem (5.1), we have quasi-isomorphisms $K'^\bullet \leftarrow \tau_{\leq 1} K'^\bullet \to \mathcal{H}^1(K'^\bullet)[-1] \cong \Lambda(-1)_{X'_{\mathrm{triv}}}[-1]$. Hence, varying $U$ and sheafifying, we obtain the required $\beta$. We have a natural map from (4.8.1) to (4.8.2) and hence the two maps $\alpha$ and $\beta$ coincide.

Let us compare $\beta$ with the map (4.5.4). By (4.7.10), we have a morphism $\overline{\mathcal{S}}_n(r-1)_{U',Z'}[-1] \to L'^\bullet_U$ and the last map of (4.8.2) factors through

$\Gamma(\underline{U'^h}, \Theta'_U(\overline{\mathcal{S}}_n(r)_{U',Z'}[-1]))$. We have the following commutative diagram of complexes on $(U'^h_{\rm triv})_{\rm \acute{e}t}$:

$$\begin{array}{ccccc} K'^{\bullet}_U & \overset{\rm q.i.}{\leftarrow} & \tau_{\le 1}K'^{\bullet}_U & \overset{\rm q.i.}{\to} & \mathbb{Z}/p^n(r)'(-1)[-1] \\ {\rm q.i.}\downarrow & & {\rm q.i.}\downarrow & & \| \\ L'^{\bullet}_U & \overset{\rm q.i.}{\leftarrow} & \tau_{\le 1}L'^{\bullet}_U & \overset{\rm q.i.}{\to} & \mathbb{Z}/p^n(r)'(-1)[-1] \\ \uparrow & & \uparrow & & (*)\uparrow \\ \overline{\mathcal{S}}_n(r-1)_{U',Z'}[-1] & \overset{\rm q.i.}{\leftarrow} & \tau_{\le 1}(\overline{\mathcal{S}}_n(r-1)_{U',Z'}[-1]) & \cong & \mathbb{Z}/p^n(r-1)'[-1] \end{array}$$

Here the morphism $(*)$ is the composite

(4.8.3)

$$\mathbb{Z}/p^n(r-1)' \cong \mathcal{H}^0(\overline{\mathcal{S}}_n(r-1)_{U',Z'}) \to \mathcal{H}^1(L'^{\bullet}_U) \overset{\sim}{\leftarrow} \mathcal{H}^1(K'^{\bullet}_U) \cong \mathbb{Z}/p^n(r)'(-1)$$

Hence to prove the coincidence of $\beta$ and (4.5.4), it suffices to prove the following:

Proposition 4.8.4. *The map* (4.8.3) *is the natural map.*

*Proof.* By the definition of $K'^{\bullet}_U$ and $L'^{\bullet}_U$, the map (4.8.3) coincides with the composite of

$$\mathbb{Z}/p^n(r-1)' \cong \mathcal{H}^0(\overline{\mathcal{S}}_n(r-1)_{U',Z'}) \overset{\cong}{\leftarrow} \mathcal{H}^1(\overline{\mathcal{S}}_n(r)^{\circ}_{U',Z'}) \to R^1\varepsilon'_{U*}\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}}$$
$$\overset{\cong}{\leftarrow} R^1\varepsilon'_{U*}\mathbb{Z}/p^n(r)' \overset{(**)}{\cong} \mathbb{Z}/p^n(r)'(-1)$$

where the second isomorphism is defined by (4.7.10). Note that all sheaves appearing above are ind locally constant. Let $I$ be the kernel of the surjection $G'^{\circ}_U \to G'_U$, which is canonically isomorphic to $\hat{\mathbb{Z}}(1)$. Then we have a natural isomorphism $H^1(I, \mathbb{Z}/p^n(r)') \cong {\rm Hom}(\hat{\mathbb{Z}}(1), \mathbb{Z}/p^n(r)') \cong \mathbb{Z}/p^n(r)'(-1)$, and it is compatible with the isomorphism $(**)$ above. Hence we may replace $R^1\varepsilon'_{U*}(-)$ with $H^1(I, -)$ regarding locally constant sheaves on $((U'^h_{\rm triv})^{\circ})_{\rm K\acute{e}t}$ and $(U'^h_{\rm triv})_{\rm \acute{e}t}$ as $G'^{\circ}_U$ and $G'_U$-modules. Let $\alpha \in \mathbb{Z}/p^n(r-1)' \cong \mathcal{H}^0(\overline{\mathcal{S}}_n(r-1)_{U',Z'})$. Since $F^*_{Z_n}(t) = t^p$, we see that the image of $\alpha$ in $\mathcal{H}^1(\overline{\mathcal{S}}_n(r)^{\circ}_{U',Z'})$ is the class of $(\alpha \cdot d\log(t), 0)$. Choose an isomorphism $M_{\widetilde{\eta'^{\circ \rm ur}}} \cong \widetilde{\mathbb{N}} \oplus (\mathcal{K}'^{\rm ur})^*$ as in the definition of $\overline{D'}^{\circ}$ in §4.7. Then the pair $(t, \{(1/p^n, 1)\}_{n\in\mathbb{N}})$ defines an element of $Q'^{\circ}$ and we denote its image under $Q'^{\circ} \to \Gamma(\overline{D'}^{\circ}_n, M) \to \Gamma(\overline{E'^{\circ}}_n, M)$ by $[\underline{t}]$. Since the images of $t$ and $[\underline{t}]$ in $\Gamma(\overline{U'}^{\circ}_n, M)$ coincide, there exists a unique $u \in \Gamma(\overline{E'^{\circ}}_n, 1 + J_{\overline{E'^{\circ}}_n})$ such that $u \cdot t = [\underline{t}]$ in $\Gamma(\overline{E'^{\circ}}_n, M)$. We have $\varphi(u) = u^p$ and $d\log(u) = -d\log(t)$. Hence $(\alpha \cdot d\log(t), 0) \in (\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}})^1$ is the image of $-\alpha \cdot \log(u) \in (\overline{\mathcal{S}}_n(r)_{U'^{\circ},Z'^{\circ}})^0$ by the differential map. Hence the image of $(\alpha \cdot d\log(t), 0)$ in $H^1(I, \mathbb{Z}/p^n\mathbb{Z}(r)')$ is given by the cocycle $\sigma \mapsto -(\sigma(-\alpha\log(u)) - (-\alpha\log(u))) = \alpha \cdot \log(\sigma([\underline{t}])[\underline{t}]^{-1})$. This completes the proof because $I \cong \hat{\mathbb{Z}}(1) \to \mathbb{Z}/p^n(1)' \subset \Gamma(\overline{E'^{\circ}}_n, J_{\overline{E'^{\circ}}_n})$ is given by $\sigma \mapsto \log(\sigma([\underline{t}])[\underline{t}]^{-1})$. □

In the case that $X$ does not have a global embedding into $Z$ as in the beginning of §4.7, we choose a strict étale covering $X^0 \to X$, and $X^0 \hookrightarrow Z^0$, $\underline{Z'^0} \subset$

$\underline{Z}^0$, and $\{F_{Z_n^0}: Z_n^0 \to Z_n^0\}$ satisfying the conditions in the beginning of §4.7. Such a covering and an embedding exist by a similar argument as the proof of Proposition 4.2.1 (1). From this embedding, we can construct $X^\bullet \hookrightarrow Z^\bullet$ and $\underline{Z'^\bullet} \subset \underline{Z^\bullet}$ as in Proposition 4.2.2 endowed with $\{F_{Z_n^\bullet}\}$. We can verify that $X^\nu \hookrightarrow Z^\nu$, $\underline{Z'^\nu} \subset \underline{Z^\nu}$ and $\{F_{Z_n^\nu}\}$ satisfy the conditions in the beginning of §4.7 for each $\nu \in \mathbb{N}$. By applying the above argument to each level, we can construct $\alpha$ and $\beta$ on $(\underline{X^\bullet})_{\text{ét}}$, which coincide with each other, and show that $\beta$ coincides with (4.5.4) on $(\underline{X^\bullet})_{\text{ét}}$. Note that our construction does not depend on $\{t_1, \ldots, t_d, t\}$ chosen in the beginning of §4.7. By taking $R\theta_*$ for the morphism of topoi $\theta\colon (\underline{X^\bullet})^\sim_{\text{ét}} \to (\underline{X})^\sim_{\text{ét}}$, we obtain Proposition 4.5.3 for a general $X$.

## §4.9. Proof of Theorem 4.1.2.

We will prove Theorem 4.1.2 by the induction on the number of elements of $I$. In the case that $I$ is empty, the theorem is nothing but Theorem 3.2.2. Assume that $I$ is non-empty, choose $i_0 \in I$ and we define $(X, M)$, $(X, M^\circ)$ and $(X', M')$ as in the beginning of §4.3. As the induction hypothesis, we assume that Theorem 4.1.2 is true for $(X, M)$ and $(X', M')$.

By Lemma 4.4.8, Lemma 4.4.9 and Proposition 4.5.2, for an integer $r \geq 2\dim(X_K)$, the comparison maps $B_{\text{st}} \otimes_{\mathbb{Q}_p} V_i^q(r) \to B_{\text{st}} \otimes_{K_0} D_i^q(r)$ $(i = 1, 2)$ and $B_{\text{st}} \otimes_{\mathbb{Q}_p} V_3^q(r-1) \to B_{\text{st}} \otimes_{K_0} D_3^q(r-1)$ are compatible with the Gysin exact sequences (4.5.1) and (4.4.1). Since the comparison maps are isomorphisms for $(X, M)$ and $(X', M')$ for every $q$ by the induction hypothesis, we see that the comparison map for $(X, M^\circ)$ is also an isomorphism for every $q$. Furthermore, by Lemma 4.4.6, the comparison maps above tensored with $B_{\text{dR}}$ over $B_{\text{st}}$ send $Fil^i$ to $Fil^i$ and are compatible with the Gysin exact sequences (4.5.1) and (4.4.5). By the induction hypothesis, the comparison maps tensored with $B_{\text{dR}}$ are filtered isomorphisms for $(X, M)$ and $(X', M')$. Hence by five lemma, it also holds for $(X, M^\circ)$. Thus we see that Theorem 4.1.2 is true for $(X^\circ, M^\circ)$.

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