Cohomological Approaches to SK_1 and SK_2 of Central Simple Algebras

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Abstract. We discuss several constructions of homomorphisms from SK_1 and SK_2 of central simple algebras to subquotients of Galois cohomology groups.

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To A. Suslin

Contents

Introduction

Given a simple algebra A with centre F, the group $SK_i(A)$ is defined for $i = 1, 2$ as the kernel of the *reduced norm*

$$\mathrm{Nrd}_i : K_i(A) \to K_i(F).$$

The definition of Nrd_1 is classical, and Nrd_2 was defined by Suslin in [47, Cor. 5.7]. For further reference, let us recall these definitions in a uniform way: let X be the Severi-Brauer variety of A. After Quillen [42, Th. 8.4], there is an isomorphism

$$\bigoplus_{r=0}^{d-1} K_i(A^{\otimes r}) \xrightarrow{\sim} K_i(X) \quad (d = \deg(A))$$

for any $i \geq 0$. The reduced norm is then given by the composition

$$K_i(A) \to K_i(X) \to H^0(X, \mathcal{K}_i) \xleftarrow{\sim} K_i(F)$$

where the right isomorphism is obvious for $i = 1$ and is due to Suslin [47, Cor. 5.6] for $i = 2$.

Of course, this definition also makes sense for $i = 0$: in this case, Nrd_0 is simply multiplication by the index of A:

$$K_0(A) \simeq \mathbf{Z} \xrightarrow{\mathrm{ind}(A)} \mathbf{Z} \simeq K_0(F)$$

and $SK_0(A) = 0$.

[For $i > 2$, a reduced norm satisfying reasonable properties cannot exist (Rost, Merkurjev [33, p. 81, Prop. 4]): the right generalisation is in the framework of motivic cohomology, see [22].]

The groups $SK_1(A)$ and $SK_2(A)$ remain mysterious and are known only in very special cases. Here are a few elementary properties they enjoy:

(1) $SK_i(A)$ is Morita-invariant.
(2) $\mathrm{ind}(A)SK_i(A) = 0$ (from Morita invariance, reduce to the case where A is division, and then use a transfer argument thanks to a maximal commutative subfield of A).
(3) The cup-product $K_1(F) \otimes K_1(A) \to K_2(A)$ induces a map

$$K_1(F) \otimes SK_1(A) \to SK_2(A).$$

(4) Let v be a discrete valuation of rank 1 on F, with residue field k, and assume that A spreads as an Azumaya algebra $\mathcal{A}$ over the discrete valuation ring $\mathcal{O}_v$. It can be shown that the map $SK_1(\mathcal{A}) \to SK_1(A)$ is surjective and that, if $K_2(\mathcal{O}_v) \to K_2(F)$ is injective, there is a short exact sequence

$$SK_2(\mathcal{A}) \to SK_2(A) \xrightarrow{\partial} SK_1(\mathcal{A}_k)$$

with

$$\partial(\{f\} \cdot x) = v(f)\bar{x}$$

for $f \in F^*$ and $x \in SK_1(A)$.

(5) Let $A(t) = F(t) \otimes_F A$, and similarly $A(x) = F(x) \otimes_F A$ for any closed point $x \in \mathbf{A}^1_F$. Then there is an isomorphism

$$SK_1(A) \xrightarrow{\sim} SK_1(A(t))$$

due to Platonov and an exact sequence

$$0 \to SK_2(A) \to SK_2(A(t)) \to \bigoplus_{x \in \mathbf{A}^1_F} SK_1(A(x)).$$

From (3) and (4), one deduces that $SK_1(A)$ is a direct summand of $SK_2(A(t))$ via the map $x \mapsto \{t\} \cdot x$: in particular, the latter group is nonzero as soon as the former is. More intriguing is the *Calmès symbol*

$$\begin{aligned} cal : \Lambda^2 \left(\frac{K_1(A)}{\operatorname{ind}(A) K_1(A)} \right) &\to SK_2(A) \\ a \wedge b &\mapsto \operatorname{Nrd}(a) \cdot b - a \cdot \operatorname{Nrd}(b). \end{aligned}$$

The image of this symbol is not detected by residues.

Let us now review known results about SK_1 and SK_2. If F is a global field, then $SK_i(A) = 0$ for $i = 1, 2$: this is classical for $i = 1$ as a consequence of class field theory, while for $i = 2$ it is due to Bak and Rehmann using the Merkurjev-Suslin theorem [2]. In the sequel, I concentrate on more general fields F and always assume that the index of A is invertible in F.

0.A. SK_1. The first one to give an example where $SK_1(A) \neq 0$ was Platonov [41]. In his example, F is provided with a discrete valuation of rank 2 and the Brauer group of the second residue field is nontrivial; in particular, $cd(F) \geq 4$. Over general fields, a striking and early result for SK_1 is Wang's theorem:

THEOREM 1 (Wang [58]). *If the index of A is square-free, then $SK_1(A) = 0$.*

The most successful approach to $SK_1(A)$ for other A has been to relate it to Galois cohomology groups. This approach was initiated by Suslin, who (based on Platonov's results) conjectured the existence of a canonical homomorphism

$$SK_1(A) \to H^4(F, \mu_n^{\otimes 3}) / [A] \cdot H^2(F, \mu_n^{\otimes 2})$$

where n is the index of A, supposed to be prime to $\operatorname{char} F$ [49, Conj. 1.16]. In [49], Suslin was only able to partially carry over this project: he had to assume that $\mu_{n^3} \subset F$ and then could only construct twice the expected map, assuming the Bloch-Kato conjecture in degree 3.

The next result in this direction is due to Rost in the case of a biquaternion algebra:

THEOREM 2 (Rost [33, th. 4]). *If A is a biquaternion algebra, there is an exact sequence*

$$0 \to SK_1(A) \to H^4(F, \mathbf{Z}/2) \to H^4(F(Y), \mathbf{Z}/2)$$

where Y is the quadric defined by an 'Albert form' associated to A.

The surprise here is that Rost gets in particular a finer map than the one expected by Suslin, as he does not have to mod out by multiples of $[A]$.
Merkurjev generalised Rost's theorem to the case of a simple algebra of degree 4 but not necessarily of exponent 2:

Theorem 3 (Merkurjev [35, th. 6.6]). *If A has degree 4, there is an exact sequence*

$$0 \to SK_1(A) \to H^4(F, \mathbf{Z}/2)/2[A] \cdot H^2(F, \mathbf{Z}/2) \to H^4(F(Y), \mathbf{Z}/2)$$

where Y is the generalised Severi-Brauer variety $SB(2, A)$, a twisted form of the Grassmannian $G(2, 4)$.

Note that the right map makes sense because $A_{F(Y)}$ has exponent 2.
Merkurjev's exact sequence is obtained from Rost's by descent from $F(Z)$ to F, where $Z = SB(A^{\otimes 2})$. The point is that neither $SK_1(A)$ nor the kernel of the right map in Theorem 3 changes when one passes from F to $F(Z)$.
More recently, Suslin revisited his homomorphism of [49] in [50], where he constructs an (a priori different) homomorphism using motivic cohomology rather than Chern classes in K-theory. He compares it with the one of Rost-Merkurjev and proves the following amazing theorem:

Theorem 4 (Suslin [50, Th. 6]). *For any central simple algebra A of degree 4, there exists a commutative diagram of isomorphisms*

$$\begin{tikzcd}
SK_1(A) \arrow[r, "\varphi", "\sim"'] \arrow[d, equal] & \dfrac{\mathrm{Ker}(H^4(F, \mu_4^{\otimes 3}) \to H^4(F(X), \mu_4^{\otimes 3}))}{[A] \cdot H^2(F, \mu_4^{\otimes 2})} \arrow[d, "\tau'"'] \\
SK_1(A) \arrow[r, "\psi", "\sim"'] & \dfrac{\mathrm{Ker}(H^4(F, \mu_2^{\otimes 3}) \to H^4(F(Y), \mu_2^{\otimes 3}))}{2[A] \cdot H^2(F, \mu_2^{\otimes 2})}
\end{tikzcd}$$

where $X = SB(A)$, $Y = SB(2, A)$, φ is Suslin's homomorphism just mentioned and ψ is Merkurjev's isomorphism from Theorem 3.

0.B. SK_2. Concerning $SK_2(A)$, the first result (over an arbitrary base field) was the following theorem of Rost and Merkurjev:

Theorem 5 (Rost [43], Merkurjev [31]). *For any quaternion algebra A, $SK_2(A) = 0$.*

Rost and Merkurjev used this theorem as a step to prove the Milnor conjecture in degree 3; conversely, this conjecture and techniques of motivic cohomology were used in [21, th. 9.3] to give a very short proof of Theorem 5. We revisit this proof in Remark 7.3, in the spirit of the techniques developed here.
The following theorem is more recent. In view of the still fluctuant status of the Bloch-Kato conjecture for odd primes, we assume its validity in the statement. (See §2.A for the Bloch-Kato conjecture.)

Theorem 6 (Kahn-Levine [22, Cor. 2], Merkurjev-Suslin [38, Th. 2.4]). *Assume the Bloch-Kato conjecture in degree ≤ 3. For any central simple algebra A of square-free index, $SK_2(A) = 0$.*

From Theorems 1 and 6, we get by a well-known dévissage argument a refinement of the elementary property (2) given above: for any A and $i = 1, 2$, $\frac{\operatorname{ind}(A)}{\prod l_i} SK_i(A) = 0$, where the l_i are the distinct primes dividing $\operatorname{ind}(A)$.
On the other hand, Baptiste Calmès gave a version of Rost's theorem 2 for SK_2 of biquaternion algebras:

Theorem 7 (Calmès [5]). *Under the assumptions of Theorem 2, assume further that F contains a separably closed field. Then there is an exact sequence*

$$\operatorname{Ker}(A_0(Z, K_2) \to K_2(F)) \to SK_2(A) \to H^5(F, \mathbf{Z}/2) \to H^5(F(Y), \mathbf{Z}/2)$$

where Z is a hyperplane section of Y.

(Note that in the case of SK_1, the corresponding group $\operatorname{Ker}(A_0(Z, K_1) \to K_1(F))$ is 0 by a difficult theorem of Rost.)
Finally, let us mention the construction of homomorphisms à la Suslin

$$SK_1(A) \to H^4(F, \mathbf{Q}/\mathbf{Z}(3))/[A] \cdot K_2(F) \tag{0.1}$$

$$SK_2(A) \to H^5(F, \mathbf{Q}/\mathbf{Z}(4))/[A] \cdot K_3^M(F) \tag{0.2}$$

in [22, §6.9], using an étale version of the Bloch-Lichtenbaum spectral sequence for the motive associated to A. The second map depends on the Bloch-Kato conjecture in degree 3 and assumes, as in Theorem 7, that F contains a separably closed field. This construction goes back to 1999 (correspondence with M. Levine), although the targets of (0.1) and (0.2) were only determined in [22, Prop. 6.9.1].

0.C. The results. Calmès' proof of Theorem 7 is based in part on the methods of [18]. In this paper, I propose to generalise his construction to arbitrary central simple algebras, with the same technique. The methods will also shed some light on the difference between Suslin's conjecture and the theorems of Rost and Merkurjev. The main new results are the following:

Theorem A. *Let F be a field and A a simple algebra with centre F and index e, supposed to be a power of a prime l different from* $\operatorname{char} F$. *Then, for any divisor r of e, there is a complex*

$$0 \to SK_1(A) \xrightarrow{\sigma_r^1} H^4(F, \mathbf{Q}/\mathbf{Z}(3))/r[A] \cdot K_2(F) \to A^0(Y^{[r]}, H^4_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(3)))$$

where $Y^{[r]}$ is the generalised Severi-Brauer variety $SB(r, A)$ and the groups $A^0(Y^{[r]}, -)$ denote unramified cohomology. If the Bloch-Kato conjecture holds in degree 3 for the prime l, these complexes refine into complexes

$$0 \to SK_1(A) \to H^4(F, \mu_{e/r}^{\otimes 3})/r[A] \cdot H^2(F, \mu_{e/r}^{\otimes 2}) \to A^0(Y^{[r]}, H^4_{\text{ét}}(\mu_{e/r}^{\otimes 3}))).$$

They are exact for $r = 1, 2$ and $e = 4$.

I don't know, and don't conjecture, that these complexes are exact in general. The map of theorem A coincides with those of Rost and Merkurjev, which is the way we get their nontriviality for $l = 2$ [34].

Theorem B. *Let F, A, e and $Y^{[r]}$ be as in Theorem A; assume the Bloch-Kato conjecture in degree ≤ 3 at the prime l and that F contains a separably closed subfield. Then, for any divisor r of e, there is a complex*

$$0 \to SK_2(A) \xrightarrow{\sigma_r^2} H^5(F, \mathbf{Q}/\mathbf{Z}(4))/r[A] \cdot K_3^M(F) \to A^0(Y^{[r]}, H^5_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(4))).$$

If, moreover, the Bloch-Kato conjecture holds in degree 4 for the prime l, these complexes refine into complexes

$$0 \to SK_2(A) \to H^5(F, \mu_{e/r}^{\otimes 4})/r[A] \cdot H^3(F, \mu_{e/r}^{\otimes 3}) \to A^0(Y^{[r]}, H^5_{\text{ét}}(\mu_{e/r}^{\otimes 4}))).$$

For $l = 2$, the maps starting from $SK_2(A)$ are nontrivial in general for $r = 1, 2$ (unless $\operatorname{ind}(A) \leq 2$*).*

Theorem C. *For any smooth F-variety X, define*

$$SK_1(X, A) = \varinjlim \operatorname{Hom}_F(X, \mathbf{SL}_n(A))^{\text{ab}}$$

where $\mathbf{SL}_n(A)$ is the reductive group representing the functor $R \mapsto SL_n(A \otimes_F R)$. Then there exists a natural transformation

$$c_A(X) : SK_1(X, A) \to H^5_{\text{ét}}(X, \mathbf{Z}(3)).$$

Restricted to fields, c_A is the universal invariant with values in $H^5_{\text{ét}}(\mathbf{Z}(3)) \simeq H^4_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(3))$ in the sense of Merkurjev [35].

Loosely speaking, c_A is defined out of the "positive" generator of the group $H^5_{\text{ét}}(\mathbf{SL}_1(A), \mathbf{Z}(3))/H^5_{\text{ét}}(F, \mathbf{Z}(3))$ which turns out to be infinite cyclic, much like the Rost invariant is defined out of the "positive" generator of the infinite cyclic group $H^3_{\text{ét}}(\mathbf{SL}_1(A), \mathbf{Z}(2)) \simeq H^4_{\text{ét}}(B\mathbf{SL}_1(A), \mathbf{Z}(2))$ (see [8, App. B]). This replies [35, Rk. 5.8] in the same way as what was done for the Arason invariant in [8].

Theorem D. *Let K be the function field of $\mathbf{SL}_1(A)$. If* $\operatorname{ind}(A) = 4$*, we have*

$$SK_1(A_K)/SK_1(A) \simeq \mathbf{Z}/2.$$

In Conjecture 10.16 we conjecture that $SK_1(A_K)/SK_1(A)$ is cyclic for any A.

Theorem E. *If* $\exp(A) = 2 < \operatorname{ind}(A)$*, then*

$$\operatorname{Inv}^4(\mathbf{SL}_1(A), H^*(\mathbf{Q}/\mathbf{Z}(* - 1))) \simeq \mathbf{Z}/2$$

where the former group is Merkurjev's group of invariants of $\mathbf{SL}_1(A)$ with values in $H^4(-, \mathbf{Q}/\mathbf{Z}(3))$ [35]*. In particular the invariant of Theorem C is nontrivial in this case, and equals the invariant σ_2^1 of Theorem A.*

Theorems A, B and C were obtained around 2001/2002, except for the exactness and nontriviality statements for $r = 1$, which follow from the work of Suslin [50]. They were presented at the 2002 Talca-Pucón conference on quadratic forms [20]. Theorems A and C are used by Tim Wouters in recent work [60]. Theorems D and E were obtained while revising this paper for publication.

This paper is organised as follows. We set up notation in Section 1. In Section 2, we recall the slice spectral sequences in the case of geometrically cellular varieties. Sections 3 to 5 are technical. In particular, Section 3 recalls the diagrams of exact sequences from [18, §5], trying to keep track of where the Bloch-Kato conjecture is used; we deduce a simple proof of Suslin's theorem [50, Th. 1], as indicated by himself in the introduction of [50] (see Remark 3.2). In Section 6 we get our first main result, Theorem 6.1, which constructs functorial injections sending a part of lower K-theory of some projective homogeneous varieties into a certain subquotient of the Galois cohomology of the base field. We apply this result in Section 7 to twisted flag varieties, thus getting Theorems A and B (see Corollaries 7.4 and 7.5); in Remark 7.3, we revisit the proof of Theorem 5 given in [21]. In Section 8, we push the main result of [22] one step further. In Section 9, we do some preliminary computations on the slice spectral sequences associated to a reductive group G: the main result is that, if G is simple simply connected of inner type A_r for $r \geq 2$, then the complex $\alpha^* c_3(G)$ of [14] is quasi-isomorphic to $\mathbf{Z}[-1]$ (see Theorem 9.5 for a more complete statement). In section 10, the approach of Merkurjev in [35] plays a central rôle: we prove Theorem C, see Theorem 10.7, Theorem D, see Corollary 10.15 and part of Theorem E, see Proposition 10.11. We conclude with some incomplete computations in Section 11 trying to evaluate the group $SK_1(A_K)/SK_1(A)$ in general, where K is the function field of $\mathbf{SL}_1(A)$: see Theorem 11.9 and Corollary 11.10. At the end of this section we complete the proof of Theorem E, see Corollary 11.12.

This paper contains results which are mostly 8 to 9 years old. The main reason why it was delayed so much is that I tried to compare the 3 ways to construct homomorphisms à la Suslin indicated above: in (0.1)–(0.2), Theorems A and B and Theorem C, and to prove their nontriviality in some new cases. In the first version of this work, I wrote that I had been mostly unsuccessful. Since then the situation has changed a bit with Theorems D and E: they were potentially already in the first version, but Wouters' work [60] was an eye-opener for this. The easy comparisons are, for Theorems A and B, with the Rost and Calmès homomorphisms of Theorems 2 and 7, and with the new Suslin homomorphism of Theorem 4. We can now also compare those of Theorems A and C in certain cases as in Theorem E, see also Corollary 10.10 and [60, §4]. A complete comparison of all invariants still seems challenging[1]: I give some comments on these comparison issues in Subsection 7.F and Remark 10.12.

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[1]Including with the first homomorphism of Suslin in [49], a comparison I had initiated in a preliminary version of this paper. (A vestige remains in §11.C.)

1. Notation

If X is a projective homogeneous variety, we denote as in [18] by E_i the étale F-algebra corresponding to the canonical $\mathbf{Z}$-basis of $CH^i(X_s)$ given by Schubert cycles, where $X_s = X \otimes_F F_s$ and F_s is a separable closure of F.
The motivic cohomology groups used in this paper are (mostly) the Hom groups in Voevodsky's category $DM^{\mathrm{eff}}_{-,\text{ét}}(F)$ of [54, §3.3] (étale topology). In particular, the exponential characteristic p of F is inverted in this category by [54, Prop. 3.3.3 2)], so that those groups are $\mathbf{Z}[1/p]$-modules. Very occasionally we shall use Hom groups in the category $DM^{\mathrm{eff}}_{-}(F)$ (Nisnevich topology).
Let $(\mathbf{Q}/\mathbf{Z})' = \bigoplus_{l\neq p} \mathbf{Q}_l/\mathbf{Z}_l$. We abbreviate the étale cohomology groups $H^i_{\text{ét}}(X, (\mathbf{Q}/\mathbf{Z})'(j))$ with the notation $H^i(X, j)$.
Unless otherwise specified, all cohomology groups appearing are étale cohomology groups, with the exception of cycle cohomology groups in the sense of Rost [44]. The latter are denoted by $A^p(X, M_q)$, where M_* is the relevant cycle module. By Gersten's conjecture [44, Cor. 6.5], these groups are canonically isomorphic to the Zariski cohomology groups $H^p_{\mathrm{Zar}}(X, \mathcal{M}_q)$, where $\mathcal{M}_q$ is the Zariski sheaf on X associated to M_q; we shall occasionally but rarely use this isomorphism, implicitly or explicitly.

2. Motivic cohomology of smooth geometrically cellular varieties updated

2.A. The Bloch-Kato conjecture and the Beilinson-Lichtenbaum conjecture. At the referee's request, I recall these two conjectures and their equivalence:

2.1. Conjecture (Milnor, Bloch, Kato). *Let $n \geq 0$, $m \geq 1$ be two integers. Then, for any field F of characteristic not dividing m, the "norm residue symbol"*

$$K^M_n(F)/m \to H^n(F, \mu_m^{\otimes n})$$

(first defined by Tate in [52]*) is bijective.*

2.2. Conjecture (Suslin-Voevodsky). *Let $n \geq 0$, $m \geq 1$, $i \in \mathbf{Z}$ be three integers. Then, for any field F of characteristic not dividing m and any smooth F-scheme X, the change of topology map*

$$H^i_{\mathrm{Nis}}(X, \mathbf{Z}/m(n)) \to H^i_{\text{ét}}(X, \mathbf{Z}/m(n))$$

is bijective for $i \leq n$ and injective for $i = n+1$, where $\mathbf{Z}/m(n)$ is the mod m version of the n-th motivic complex of Suslin-Voevodsky.

Conjecture 2.2 appears in [51] where (among other places like [54]) the complexes $\mathbf{Z}(n)$ are introduced. It therefore cannot be literally attributed to Beilinson and Lichtenbaum, although it is indeed a common part of conjectures they made in the eighties on the properties of the still conjectural complexes $\mathbf{Z}(n)$. Voevodsky observed in [56] that the special case $X = \operatorname{Spec} F$, $i = n$ of Conjecture 2.2 is a reformulation of Conjecture 2.1. Conversely:

2.3. THEOREM ([51, 10], see also [19]). *Conjecture 2.1 (for the pair (n,m)) implies Conjecture 2.2 (for the triples (n,m,i)).*

We shall actually use in this paper the following variant of Conjecture 2.2 with integral coefficients:

2.4. PROPOSITION. *Conjecture 2.2 for m a power of a prime l is equivalent to the following: let $n \geq 0$, $i \in \mathbf{Z}$ be two integers. Then, for any field F of characteristic $\neq l$ and any smooth F-scheme X, the change of topology map*

$$H^i_{\mathrm{Nis}}(X, \mathbf{Z}(n)) \to H^i_{\text{ét}}(X, \mathbf{Z}(n))$$

is bijective for $i \leq n+1$ and injective for $i = n+2$ after localising at l.

The equivalence is an easy consequence of the fact that the map in Proposition 2.4 is an isomorphism after tensoring with $\mathbf{Q}$ for any $i \in \mathbf{Z}$ [53, Prop. 5.28].
The special case $X = \operatorname{Spec} F$, $i = n+1$ of Proposition 2.4 enunciates that $H^{n+1}_{\text{ét}}(F, \mathbf{Z}(n)) \otimes \mathbf{Z}_{(l)} = 0$: this is called "Hilbert's theorem 90 in degree n" and is actually equivalent (for all F) to the above conjectures.
At the time of writing, the status of Conjecture 2.1 is as follows. For $n = 0$ it is trivial, for $n = 1$ it is Kummer theory ($\iff$ Hilbert's theorem 90), for $n = 2$ it is the Merkurjev-Suslin theorem [36], for m a power of 2 it is due to Voevodsky [56]. In general it seems now to be fully proven as a combination of works by several authors, merging in [57] (see [59] for an overview).
In this paper, we use these conjectures for $n = 2$ (*resp.* $n = 3$) when dealing with SK_1 (*resp.* SK_2) and $\mathbf{Q}/\mathbf{Z}$ coefficients, and for $n = 3$ (*resp.* $n = 4$) when dealing with SK_1 (*resp.* SK_2) and finite coefficients.

2.B. THE SLICE SPECTRAL SEQUENCES. In [18], we constructed spectral sequences for the étale motivic cohomology of smooth geometrically cellular varieties. These results were limited in two respects:

(1) the ground field F was assumed to be of characteristic 0;
(2) the spectral sequences had a strange abutment, which was nevertheless sufficient for applications.

The results of [14] solved both issues. The first one was due to the fact that [18] worked with motives with compact support in Voevodsky's triangulated category of motives [54], which are known to be geometric only in characteristic 0: indeed, it was shown that the motive with compact supports of a cellular variety X is a pure Tate motive in the sense of [14], from which it was deduced by duality that the motive of X (without supports) is also pure Tate if X is smooth. In [14, Prop. 4.11], we prove directly that, over any field, the motive of X is pure Tate if X is smooth and cellular.
The second issue was more subtle and is discussed in [14, Remark 6.3]. The short answer is that by considering a different filtration than the one used in [18], one gets the "right" spectral sequence.
We summarize this discussion by stating the following theorem, which follows from [14, (3.2) and Prop. 4.11] and replaces [18, Th. 4.4]:

2.5. THEOREM. *Let X be a smooth, equidimensional, geometrically cellular variety over a perfect field F. For all $n \geq 0$, there is a spectral sequence $E(X,n)$:*

$$E_2^{p,q}(X,n) = H^{p-q}_{\text{ét}}(F, CH^q(X_s) \otimes \mathbf{Z}(n-q)) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathbf{Z}(n)). \tag{2.1}$$

Note that, by cellularity, each $CH^q(X_s)$ is a permutation Galois module. These spectral sequences have the following properties:

(i) NATURALITY. (2.1) *is covariant in F and contravariant in X (varying among smooth, equidimensional, geometrically cellular varieties) under any maps (even finite correspondences).*

(ii) PRODUCTS. *There are pairings of spectral sequences*

$$E_r^{p,q}(X,m) \times E_r^{p',q'}(X,n) \to E_r^{p+p',q+q'}(X,m+n)$$

which coincide with the usual cup-product on the E_2-terms and the abutments.

(iii) TRANSFER. *For any finite extension E/F and any $n \geq 0$, there is a morphism of spectral sequences*

$$E_r^{p,q}(X_E,n) \to E_r^{p,q}(X,n)$$

which coincides with the usual transfer on the E_2-terms and the abutment.

(iv) COVARIANCE FOR CLOSED EQUIDIMENSIONAL IMMERSIONS. *For any closed immersion $i : Y \hookrightarrow X$ of pure codimension c, where X and Y are smooth, geometrically cellular, there is a morphism of spectral sequences*

$$E_r^{p-c,q-c}(Y,n-c) \xrightarrow{i_*} E_r^{p,q}(X,n)$$

"abutting" to the Gysin homomorphisms

$$H^{p+q-2c}_{\text{ét}}(Y, \mathbf{Z}(n-c)) \xrightarrow{i_*} H^{p+q}_{\text{ét}}(X, \mathbf{Z}(n)).$$

If X is split, then (2.1) *degenerates at E_2.*

The only nonobvious point in this theorem is (ii) (products). In [14, p. 915], it is claimed that there are pairings of slice spectral sequences for the tensor product of two arbitrary motives M and N. This is not true in general: I thank Evgeny Shinder for pointing out this issue. However, these pairings certainly exist if M or N is a mixed Tate motive: the argument is essentially the same as the one that proves that the Künneth maps of [14, Cor. 1.6] are isomorphisms in this case [14, Lemma 4.8]. For the reader's convenience, we outline the construction. We take the notation of [14]:

Given the way the slice spectral sequence is constructed in [14, §3] (bottom of p. 914), to get a morphism of filtrations, we need to get morphisms

$$\nu_{\leq q+q'}(M \otimes M') \to \nu_{\leq q} M \otimes \nu_{\leq q'} M'$$

for two motives M, M' and two integers q, q'.

From the canonical maps $M \to \nu_{\leq q} M$ and $M' \to \nu_{\leq q'} M'$, we get a morphism

$$M \otimes M' \to \nu_{\leq q} M \otimes \nu_{\leq q'} M'$$

and we would like to prove that its composition with $\nu^{>q+q'}(M\otimes M') \to M\otimes M'$ is 0. This will be true provided

$$\nu^{>q+q'}(\nu_{\leq q}M \otimes \nu_{\leq q'}M') = \underline{\mathrm{Hom}}(\mathbf{Z}(q+q'+1), \nu_{\leq q}M \otimes \nu_{\leq q'}M')(q+q'+1) = 0.$$

This is false in general (for example $M = M' = h_1(C)$, $q = q' = 0$, where C is a curve of genus > 0 over an algebraically closed field), but it is true if M or M' is a mixed Tate motive. Indeed, we may reduce to $M = \mathbf{Z}(a)$ for some integer a. Then

$$\nu_{\leq q}M = \begin{cases} 0 & \text{if } q < a \\ \mathbf{Z}(a) & \text{if } q \geq a \end{cases}$$

hence $\underline{\mathrm{Hom}}(\mathbf{Z}(q+q'+1), \nu_{\leq q}M \otimes \nu_{\leq q'}M') = 0$ if $q < a$, and if $q \geq a$ we get

$$\begin{aligned}\underline{\mathrm{Hom}}&(\mathbf{Z}(q+q'+1), \nu_{\leq q}M \otimes \nu_{\leq q'}M') \\ &= \underline{\mathrm{Hom}}(\mathbf{Z}(q+q'+1), \mathbf{Z}(a) \otimes \nu_{\leq q'}M') \\ &= \underline{\mathrm{Hom}}(\mathbf{Z}(q+q'+1-a), \nu_{\leq q'}M') = 0\end{aligned}$$

because $q + q' + 1 - a > q'$.
Dealing with the spectral sequences for étale motivic cohomology, it will suffice that M or N is geometrically mixed Tate in the sense of [14, §5] to have these products.

2.6. *Remark.* As stressed in §1, the spectral sequences of Theorem 2.5 are spectral sequences of $\mathbf{Z}[1/p]$-modules, where p is the exponential characteristic of F. Thus all results of this paper are "away from p". It is nevertheless possible to extend the methods to p-algebras in characteristic p, at some cost: this is briefly discussed in Appendix A. I am grateful to Tim Wouters for a discussion leading to this observation.

2.C. VANISHING OF E_2-TERMS. Since this issue may be confusing, we include here an estimate in the case of the spectral sequences (2.1) and of the coniveau spectral sequences, which will be used in the next section (compare [18, p. 161]). It shows that these two spectral sequences live in somewhat complementary regions of the E_2-plane.

2.7. PROPOSITION. *a) In the spectral sequence* (2.1), *we have* $E_2^{a,b}(X,n) = 0$ *in the following cases:*
(ai) $a \leq b$, $b \geq n-1$, *except* $a = b = n$.
(aii) $a = n+1$ *under the Bloch-Kato conjecture in degree* $n-b$.
Moreover, $E_2^{a,b}(X,n)$ *is uniquely divisible for* $a \leq b$ *and* $b < n-1$.
b) Let X *be a smooth variety. In the coniveau spectral sequence for étale motivic cohomology*

$$E_1^{a,b} = \bigoplus_{x\in X^{(a)}} H^{b-a}(k(x), \mathbf{Z}(n-a)) \Rightarrow H^{a+b}(X, \mathbf{Z}(n))$$

we have $E_1^{a,b} = 0$ *in the following cases:*

(bi) *$a \geq b$, $a \geq n-1$, except $a = b = n$.*
(bii) *$b = n+1$ under the Bloch-Kato conjecture in degree $n-a$.*
Moreover, $E_2^{a,b}(X,n)$ is uniquely divisible for $a \geq b$ and $a < n-1$.
Finally, for $b = n$, the natural map

$$A^a(X, K_n^M)[1/p] \to E_2^{a,n}$$

is surjective under the Bloch-Kato conjecture in degrees $\leq n-a$, and bijective under the Bloch-Kato conjecture in degrees $\leq n-a+1$.

Proof. For (ai), we use that $E_2^{a,b}(X,n) = H_{\text{ét}}^{a-b}(F, CH^b(X_s) \otimes \mathbf{Z}(n-b)) \simeq H_{\text{ét}}^{a-b-1}(F, CH^b(X_s) \otimes \mathbf{Q}/\mathbf{Z}(n-b))$ for $n-b<0$ (by definition of $\mathbf{Z}_{\text{ét}}(n-b)$ for $n-b<0$, see [14, Def. 3.1]), and also that $\mathbf{Z}(0) = \mathbf{Z}$ and $\mathbf{Z}(1) = \mathbb{G}_m[-1]$. (aii) follows from Hilbert 90 in degree $n-b$ (see §2.A after Proposition 2.4). The proofs of (bi) and (bii) are similar. The divisibility claims reduce to the unique divisibility of $H_{\text{ét}}^i(K, \mathbf{Z}(r))$ for $i \leq 0$ ($r > 0$, K/F a function field): this is obvious for $i < 0$, while for $i = 0$ we may reduce to finitely generated fields as in [17, proof of Th. 3.1 a)]. Finally, the last claim follows from a diagram chase in the comparison map between the Gersten complexes for Nisnevich and étale cohomology with $\mathbf{Z}(n)$ coefficients. □

3. Weight 3 and weight 4 étale motivic cohomology

In this section, we examine in more detail the diagrams obtained in [18] by mixing the slice and coniveau spectral sequences, and expand the results in weight 4. In order to stress the irrelevance of Gersten's conjecture, we replace the notation $H^p(X, \mathcal{H}^q)$ or $H^p(X, \mathcal{K}_q)$ used in [18] by the notation $A^p(X, H^q)$ or $A^p(X, K_q)$ (see §1).

3.A. Weight 3. Let X be a projective homogeneous F-variety. In [18, §5.4], we drew a commutative diagram with some exactness properties, by mixing the coniveau spectral sequence and the spectral sequence of [18, Th. 4.4] for étale motivic cohomology in weight 3. We can now use the spectral sequence (2.1) to get the same diagram over any perfect field. To get the diagram of [18, §5.4], we made the blanket assumption in [18] that all groups were localised at 2, because calculations relied on the Bloch-Kato conjecture in degree 3, which was only proven for $l = 2$.
In this paper, we are also interested in making the dependence on this conjecture explicit. How much exactness remains in this diagram if we don't wish to use it in degree 3? Using Proposition 2.7, we see that at least the following part of the diagram of [18, §5.4] remains exact by only using the Bloch-Kato conjecture in degree ≤ 2 (= the Merkurjev-Suslin theorem): the exponential characteristic p is implicitly inverted in this diagram as well as in the next one,

(3.2).

$$
\begin{tikzcd}
 & H^4(F,\mathbf{Z}(3)) \arrow[d] & \\
0 \to A^1(X,K_3^M) \arrow[r] & H^4(X,\mathbf{Z}(3)) \arrow[r] \arrow[d] & A^0(X,H^4(\mathbf{Z}(3))) \\
 & K_2(E_1) \arrow[d, "d_2^{3,1}(3)"] & \\
A^0(X,H^4(\mathbf{Z}(3))) \arrow[d] & H^4(F,3) \arrow[d] \arrow[dr, "\eta^4"] & \\
A^2(X,K_3^M) \arrow[r] \arrow[dr, "\xi^4"] & H^5(X,\mathbf{Z}(3)) \arrow[r] \arrow[d] & A^0(X,H^4(3)) \arrow[d] \\
 & E_2^* & CH^3(X) \arrow[d] \\
 & & H^6(X,\mathbf{Z}(3)).
\end{tikzcd}
\tag{3.1}
$$

The group $A^0(X,H^4(\mathbf{Z}(3)))$, which appears twice in this diagram, is of course torsion, as well as $H^4(F,\mathbf{Z}(3))$, and their l-primary components are 0 under the Bloch-Kato conjecture in degree 3 for the prime l.

3.B. Weight 4. In weight 4, we cannot avoid using the Bloch-Kato conjecture in degree 3. There is a commutative diagram, which was only written down in a special case in [18]:

$$
\begin{tikzcd}
H^5(X,\mathbf{Z}(4)) \to K_3(E_2)_{\mathrm{ind}} \arrow[dr, "d_3^{3,2}(4)"] & K_3^M(E_1) \arrow[d, "d_2^{4,1}(4)"] & \\
A^0(X,H^5(\mathbf{Z}(4))) \arrow[d] & H^5(F,4) \arrow[d] \arrow[dr, "\eta^5"] & \\
A^2(X,K_4^M) \arrow[r] \arrow[dr, "\xi^5"] & H^6(X,\mathbf{Z}(4)) \arrow[r] \arrow[d] & A^0(X,H^5(4)) \arrow[d] \\
 & K_2(E_2) \arrow[dl, "d_3^{4,3}(4)"'] \arrow[d, "d_2^{4,3}(4)"] & H^3(X,K_4^M) \arrow[d] \\
H^6(F,4) & H^4(E_1,3) & H^7(X,\mathbf{Z}(4)).
\end{tikzcd}
\tag{3.2}
$$

In this diagram, the differentials appearing correspond to the spectral sequence (2.1) in weight 4. The path snaking from $A^0(X, H^5(\mathbf{Z}(4)))$ to $H^7(X, \mathbf{Z}(4))$ is exact (it comes from the coniveau spectral sequence for weight 4 étale motivic cohomology: see Proposition 2.7). The differential $d_3^{4,3}(4)$ is only defined on the kernel of $d_2^{4,3}(4)$ and the differential $d_3^{3,2}(4)$ takes values in the cokernel of $d_2^{3,2}(4)$. The column is a complex, exact at $H^6(X, \mathbf{Z}(4))$; its exactness properties at $H^5(F, 4)$ and $K_2(E_2)$ involve the differentials d_3 in an obvious sense.
All these exactness properties depend on the Bloch-Kato conjecture in degree i for any field E and any $i \leq 3$, and also on Hilbert's theorem 90 in degree i under the same conditions (which follows from the Bloch-Kato conjecture, see §2.A).
The map η^5 is the natural map from the Galois cohomology of the ground field to the unramified cohomology of X.

3.C. The groups $\overline{\mathrm{Ker}}\ \eta^4$ and $\overline{\mathrm{Ker}}\ \eta^5$.

3.1. Definition. For $i = 1, 2$, we denote by $\overline{\mathrm{Ker}}\ \eta^{i+3}$ the homology of the complex

$$K^M_{i+1}(E_1) \xrightarrow{d_2^{i+2,1}(X,i+2)} H^{i+3}(F, i+2) \xrightarrow{\eta^{i+3}} A^0(X, H^{i+3}(i+2)).$$

Diagram (3.1) yields an exact sequence

$$A^0(X, H^4(\mathbf{Z}(3))) \to \operatorname{Ker} \xi^4 \to \overline{\mathrm{Ker}}\ \eta^4 \to 0$$

hence an isomorphism

$$\operatorname{Ker} \xi^4 \xrightarrow{\sim} \overline{\mathrm{Ker}}\ \eta^4 \tag{3.3}$$

under the Bloch-Kato conjecture in degree ≤ 3.
If F contains an algebraically closed subfield, then $K_3(E_2)_{\mathrm{ind}}$ is divisible and the differential $d_3^{3,2}(4)$ is 0 since it is a priori torsion [18, Prop. 4.6]. Then diagram (3.2) yields an exact sequence

$$A^0(X, H^5(\mathbf{Z}(4))) \to \operatorname{Ker} \xi^5 \to \overline{\mathrm{Ker}}\ \eta^5 \to 0$$

under the Bloch-Kato conjecture in degree ≤ 3 and an isomorphism

$$\operatorname{Ker} \xi^5 \xrightarrow{\sim} \overline{\mathrm{Ker}}\ \eta^5 \tag{3.4}$$

under the Bloch-Kato conjecture in degree ≤ 4.

3.2. *Remark.* Let us recover Suslin's theorem [50, Th. 1] from (3.3). The point is simply that the coniveau spectral sequence for Nisnevich motivic cohomology yields an isomorphism

$$A^2(X, K_3^M) \xrightarrow{\sim} H^5_{\mathrm{Nis}}(X, \mathbf{Z}(3))$$

(*cf.* [50, Lemma 9]). The differential $d_2^{3,1}(3)$ was computed in [18, Th. 7.1] for Severi-Brauer varieties.

4. Some K-cohomology groups

4.A. $A^1(X,K_3)$ and $A^0(X,K_3)$. Recall from [18, Prop. 4.5] that

$$A^i(X,K_3^M) \xrightarrow{\sim} A^i(X,K_3) \text{ for } i>0. \tag{4.1}$$

For $A^1(X,K_3)$, we have:

4.1. Proposition. *Let X be a projective homogeneous variety over F, and K/F a regular extension. Under the Bloch-Kato conjecture in degree 3, the map*

$$A^1(X,K_3)\to A^1(X_K,K_3)$$

has p-primary torsion kernel, where p is the exponential characteristic of F. More precisely, the kernel of this map is torsion and its l-primary part vanishes for $l\neq p$ if the Bloch-Kato conjecture holds at the prime l in degree 3.

Proof. Up to passing to its perfect closure, we may assume F perfect. By Diagram (3.1) and (4.1), there is a canonical map

$$A^1(X,K_3)\to K_2(E_1)$$

where E_1 is a certain étale F-algebra associated to X, whose kernel is contained in $H^4_{\text{ét}}(F,\mathbf{Z}(3))$: hence the l-primary part of this kernel vanishes under the condition in Proposition 4.1. The result now follows from [47, th. 3.6]. □

Let still X be a projective homogeneous F-variety. As in [18, §5.1], for all $i\geq 0$ we write E_i for the étale F-algebra determined by the Galois-permutation basis of $CH^i(X_s)$ given by Schubert cycles (see §1).

4.2. Theorem. *a) For $i\leq 2$, the map $K_i(F)\to A^0(X,K_i)$ is bijective.*
b) Under the Bloch-Kato conjecture in degree 3, the cokernel of the homomorphism

$$K_3(F)\to A^0(X,K_3)$$

is torsion, and its prime-to-the-characteristic part is

(1) *finite if F is finitely generated over its prime subfield;*
(2) 0 *in the following cases:*
  (i) *F contains a separably closed subfield;*
  (ii) *the map $CH^1(X_{E_1})\to CH^1(X_s)$ is surjective.*

More precisely, under the Bloch-Kato conjecture in degree 3 for the prime l, the above is true after localisation at l.

Proof. a) is well-known and is quoted for reference purposes: it is obvious for $i=0,1$ (since X is proper geometrically connected), and for $i=2$ it is a theorem of Suslin [47, Cor. 5.6].
b) After [17, Th. 3 a)] (see also [27, Th. 16.4]), the homomorphism $K_3^M(K)\to K_3(K)$ is injective for any field K. Consider the commutative diagram with

exact rows

$$\begin{tikzcd}
0 \arrow[r] & K_3^M(F) \arrow[r] \arrow[d] & K_3(F) \arrow[r] \arrow[d] & K_3(F)_{\mathrm{ind}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^0(X, K_3^M) \arrow[r] & A^0(X, K_3) \arrow[r] & A^0(X, K_3^{\mathrm{ind}}).
\end{tikzcd}$$

As X is a rational variety, the right vertical map is bijective [8, lemma 6.2]. It therefore suffices to prove the claims of theorem 4.2 for the left vertical map. Let us first assume F perfect: then we can use Theorem 2.5. Mixing the weight 3 coniveau spectral sequence for étale motivic cohomology with the spectral sequence (2.1) in weight 3, we get modulo the Bloch-Kato conjecture in degree 3 the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & K_3^M(F) \arrow[d] \arrow[dr, "\alpha"] & & \\
0 \arrow[r] & A^1(X, H^2(\mathbf{Z}(3))) \arrow[r] \arrow[dr, "\beta"] & H^3(X, \mathbf{Z}(3)) \arrow[r] \arrow[d] & A^0(X, K_3^M) \arrow[r] & 0 \\
 & & K_3(E_1)_{\mathrm{ind}} \arrow[d] & & \\
 & & 0. & &
\end{tikzcd}$$

For the reader's convenience, let us explain where the Bloch-Kato conjecture in degree 3 is necessary. The weight 3 spectral sequence (2.1) gives a priori an exact sequence

$$\begin{aligned}
H^0(E_1, \mathbf{Z}(2)) \xrightarrow{d_2^{1,1}(X,3)} H^3(F, \mathbf{Z}(3)) \to H^3(X, \mathbf{Z}(3)) \\
\to H^1(E_1, \mathbf{Z}(2)) \to H^4(F, \mathbf{Z}(3)).
\end{aligned}$$

Recall that all groups are étale cohomology groups here. The group $H^0(E_1, \mathbf{Z}(2))$ is conjecturally 0; it is uniquely divisible in any case, see proof of Proposition 2.7. Since the differential $d_2^{1,1}(X,3)$ is torsion (proof as in [18, Prop. 4.6]), it must be 0. The identification of $H^1(E_1, \mathbf{Z}(2))$ with $K_3(E_1)_{\mathrm{ind}}$ only depends on the Merkurjev-Suslin theorem. On the other hand, the bijectivity of $K_3^M(F) \to H^3(F, \mathbf{Z}(3))$ and the vanishing of $H^4(F, \mathbf{Z}(3))$ depend on the Bloch-Kato conjecture in degree 3. This takes care of the vertical exact sequence. Similarly, the Bloch-Kato conjecture in degree 3 is necessary to identify the last term of the horizontal exact sequence (stemming from the coniveau spectral sequence) with $A^0(X, K_3^M)$.

The diagram above gives an isomorphism

$$\operatorname{Coker}\alpha \simeq \operatorname{Coker}\beta.$$

Let us show that $\operatorname{Coker}\beta$ is m-torsion for some $m > 0$. The group $K_3(E_1)_{\mathrm{ind}}$ appearing in the diagram is really

$$H^0(F, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2)))$$

via Shapiro's lemma, the isomorphism $H^1(K, \mathbf{Z}(2)) \simeq K_3(K)_{\mathrm{ind}}$ for any field and Galois descent for $K_3(K)_{\mathrm{ind}}$ [37, 23]. A standard computation shows that the corestriction map

$$H^0(E_1, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2))) \xrightarrow{\mathrm{Cor}} H^0(F, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2)))$$

is split surjective. On the other hand, since $CH^1(X_s)$ is finitely generated, there exists a finite extension E/F such that $CH^1(X_E) \to CH^1(X_s)$ is surjective. Without loss of generality, we may assume that E contains all the residue fields of the étale algebra E_1. A transfer argument then shows that the map $CH^1(X_{E_1}) \to CH^1(X_s)$ has cokernel killed by some integer $m > 0$. Hence the composition

$$\begin{aligned} CH^1(X_{E_1}) \otimes H^1(E_1, \mathbf{Z}(2)) &\to CH^1(X_s) \otimes H^1(E_1, \mathbf{Z}(2)) \\ &\xrightarrow{\sim} CH^1(X_s) \otimes H^0(E_1, H^1(F_s, \mathbf{Z}(2))) \\ &\simeq H^0(E_1, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2))) \end{aligned}$$

has cokernel killed by m, and the same holds for the composition

$$\begin{aligned} CH^1(X_{E_1}) \otimes H^1(E_1, \mathbf{Z}(2)) &\to H^0(E_1, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2))) \\ &\xrightarrow{\mathrm{Cor}} H^0(F, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2))). \end{aligned}$$

But this composition factors via cup-product as

$$\begin{aligned} CH^1(X_{E_1}) \otimes H^1(E_1, \mathbf{Z}(2)) &= A^1(X_{E_1}, \mathcal{H}^2(\mathbf{Z}(1))) \otimes H^1(E_1, \mathbf{Z}(2)) \\ &\to A^1(X_{E_1}, \mathcal{H}^2(\mathbf{Z}(3))) \xrightarrow{\mathrm{Cor}} A^1(X, \mathcal{H}^2(\mathbf{Z}(3))) \\ &\xrightarrow{\beta} H^0(F, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2))) \end{aligned}$$

which proves the claim.

Coming back the the case where F is not necessarily perfect, let F' be its perfect (radicial?) closure and α' the map α "viewed over F'". Then a transfer argument shows that the natural map $\operatorname{Coker}\alpha \to \operatorname{Coker}\alpha'$ has p-primary torsion kernel and cokernel, where p is the exponential characteristic of F. In particular, $\operatorname{Coker}\alpha$ is torsion, and its prime-to-p part is killed by some m.

The integer m equals 1 provided $CH^1(X_{E_1}) \to CH^1(X_s)$ is surjective, which proves 2) (ii) in Theorem 4.2. In general, the map

$$K_3(F_0)_{\mathrm{ind}}/m \to K_3(F)_{\mathrm{ind}}/m$$

is bijective, where F_0 is the field of constants of F [37, 23]. If F_0 is separably closed, then $K_3(F_0)_{\mathrm{ind}}/m = 0$ (ibid.), which proves 2) (i); if F is finitely generated, then F_0 is a finite field or a number field with ring of integers A and $K_3(F_0)_{\mathrm{ind}}$ is a quotient of $K_3(A)$; in both cases it is finitely generated, which proves 1). □

4.3. *Example.* X is a conic curve. Then $\operatorname{Coker}\beta$ is isomorphic to the cokernel of the map

$$\bigoplus_{x\in X^{(1)}} K_3(F(x))_{\mathrm{ind}} \xrightarrow{(N_{F(x)/F})} K_3(F)_{\mathrm{ind}}.$$

Even in the case $F = \mathbf{Q}$, $K_3(\mathbf{Q})_{\mathrm{ind}} \simeq \mathbf{Z}/24$, I am not able either to produce an example where this map is not onto, or to prove that it is always onto. As a first try, one might restrict to points of degree 2 on X. To have an idea of how complex the situation is, the reader may refer to [15, §8]. In particular, Theorem 8.1 (iv) of *loc. cit.* shows that the map is onto provided X has a quadratic splitting field of the form $\mathbf{Q}(\sqrt{-p})$, where p is prime and $\equiv -1 \pmod 8$. If X corresponds to the Hilbert symbol (a,b), with a,b two coprime integers, the theorem of the arithmetic progression shows that there are infinitely many $p \equiv -1 \pmod 8$ such that $p \nmid ab$ and $\left(\dfrac{-p}{l}\right) = -1$ for all primes $l \mid ab$. Since $-p$ is a square in $\mathbf{Q}_2$, this implies that $(a,b)_{\mathbf{Q}(\sqrt{-p})} = 0$ if and only if $(a,b)_{\mathbf{Q}_2} = 0$. Thus the above map is surjective if $X(\mathbf{Q}_2) \neq \emptyset$, but I don't know the answer in the other case.

4.B. $A^i(X, K_4^M)$ AND $A^i(X, K_4)$.

4.4. THEOREM. *a) For any smooth variety X, the natural map*

$$\varphi_i : A^i(X, K_4^M) \to A^i(X, K_4)$$

is bijective for $i \geq 3$ and surjective for $i = 2$ with kernel killed by 2.
b) Suppose that F contains a separably closed subfield. Then φ_2 is bijective.

Proof. a) By definition, both groups are cohomology groups of the respective Gersten complexes

$$\dots \to \bigoplus_{x\in X^{(i)}} K_{4-i}^M(F(x)) \to \dots$$

$$\dots \to \bigoplus_{x\in X^{(i)}} K_{4-i}(F(x)) \to \dots$$

Therefore, Theorem 4.4 is obvious for $i \geq 3$, and φ_2 is surjective. Using the Adams operations on algebraic K-theory, we see that, for any field K, the exact sequence

$$0 \to K_3^M(K) \to K_3(K) \to K_3(K)_{\mathrm{ind}} \to 0$$

is split up to 2-torsion. It follows that $2\operatorname{Ker}\varphi_2 = 0$.
b) We have an exact sequence

$$\bigoplus_{x\in X^{(1)}} K_3(F(x))_{\mathrm{ind}} \xrightarrow{\psi} A^2(X, K_4^M) \xrightarrow{\varphi_2} A^2(X, K_4) \to 0.$$

By assumption, each group $K_3(F(x))_{\mathrm{ind}}$ is divisible (compare the proof of Theorem 4.2). Since their images in $A^2(X, K_4^M)$ are killed by 2, they are 0. □

4.5. *Remark.* I don't know if the condition on F is necessary for the bijectivity of φ_2. Note that ψ factors through the group $A^1(X, H^2(\mathbf{Z}(3)))$ appearing in the proof of Theorem 4.2.

5. An approximation of cycle cohomology

Let M_* be a cycle module in the sense of Rost [44] and let X be projective homogeneous. There are cup-products

$$CH^p(X) \otimes M_{q-p}(F) \to A^p(X, M_q). \tag{5.1}$$

which are isomorphisms when X is split, by [8, Prop. 3.7].
Assume now that X is not necessarily split. Let Y be a splitting variety for X: if $X_s = G_s/P$ where G is a semi-simple F-algebraic group and P is a parabolic subgroup of G_s, we may take Y such that $Y_s = G_s/B$ for B a Borel subgroup contained in P. Then $X_{F(y)}$ is cellular for any point $y \in Y$. It is possible to define a map

$$A^p(X, M_q) \xrightarrow{\tilde\xi^{p,q}} A^0(Y_{E_p}, M_{q-p}) \tag{5.2}$$

which is an isomorphism after tensoring with $\mathbf{Q}$ and corresponds to the inverse of (5.1) when X is split. When $q - p \leq 2$ and $M_* = K_*^M$, this map refines into a map

$$A^p(X, K_q^M) \xrightarrow{\xi^{p,q}} K_{q-p}^M(E_p) \tag{5.3}$$

thanks to Suslin's theorem [47, Cor. 5.6] for $q - p = 2$ and trivially for $q - p = 0, 1$. In this paper, we shall only construct such a map in the substantially simpler inner case where all algebras E_p are split, which is sufficient for our needs.
We note that, if X is split, the functor $K \mapsto CH^p(X_K)$ from field extensions of F to abelian groups is constant, with finitely generated free value. When X is arbitrary, we shall authorise ourselves of this to denote by $CH^p(X_s)$ the common value of $CH^p(X_K)$ for all splitting fields K of X.
For Y a splitting variety of X as above, consider the Rost spectral sequence [44, §8]

$$E_2^{p,q} = A^p(Y, R^q\pi_*M_*) \Rightarrow A^{p+q}(X \times Y, M_*)$$

where π is the projection $X \times Y \to Y$ and the $R^q\pi_*M_*$ are the higher direct images of M_* in the sense of Rost [44, §7]. Using the fact that (5.1) is an isomorphism in the split case, we get canonical isomorphisms

$$R^q\pi_*M_* = CH^q(X_s) \otimes M_{*-q}$$

hence an edge homomorphism

$$A^p(X \times Y, M_q) \to E_2^{0,p} = CH^p(X_s) \otimes A^0(Y, M_{q-p}).$$

In the inner case, the composition of this map with the obvious map $A^p(X, M_q) \to A^p(X \times Y, M_q)$ is the desired map $\widetilde{\xi}^{p,q}$ of (5.2).
In the special case $M_* = K_*^M$, a functoriality argument shows that the map $\xi^{2,3}$ (*resp.* $\xi^{2,4}$) of (5.3) coincides with the map ξ^4 of Diagram (3.1) (*resp.* with the map ξ^5 of Diagram (3.2)).

6. A general K-theoretic construction

Let X be projective homogeneous, and let K be a splitting field for X such that K/F is geometrically rational (for example, take for K the function field of the corresponding full flag variety, see beginning of §5). We assume as in the previous section that the associated algebras E_p are split: this is probably not essential. We write $K_*(X)^{(i)}$ for the coniveau filtration on $K_*(X)$, and $K_*(X)^{(i/i+1)}$ for its successive quotients.

6.A. The first steps of the coniveau filtration.

6.1. Theorem. *For $i \leq 2$,*
a) The map

$$K_i(F) \oplus K_i(X)^{(1)} \to K_i(X)$$

is an isomorphism.
b) The maps

$$\begin{aligned}\operatorname{Ker}(K_i(X)^{(2)} \to K_i(X_K)^{(2)}) \to \operatorname{Ker}(K_i(X)^{(1)} &\to K_i(X_K)^{(1)}) \\ &\to \operatorname{Ker}(K_i(X) \to K_i(X_K))\end{aligned}$$

are isomorphisms. (For $i = 2$, we assume the Bloch-Kato conjecture in degree 3 for the torsion primes of X.)
c) There are canonical monomorphisms

$$\operatorname{Ker}(K_i(X)^{(2/3)} \to K_i(X_K)^{(2/3)}) \hookrightarrow \overline{\operatorname{Ker}}\, \eta^{i+3}$$

where $\overline{\operatorname{Ker}}\, \eta^{i+3}$ was introduced in Definition 3.1. (If $i = 2$, we assume the Bloch-Kato conjecture in degree 3 for the torsion primes of X, and also that F contains a separably closed field.) These homomorphisms are contravariant in X.

Proof. a) By Theorem 4.2 a), the composition

$$K_i(F) \to K_i(X) \to A^0(X, K_i)$$

is bijective; hence this composition yields a splitting to the exact sequence

$$0 \to K_i(X)^{(1)} \to K_i(X) \to A^0(X, K_i).$$

b) It suffices to show that the maps $K_i(X)^{(j/j+1)} \to K_i(X_K)^{(j/j+1)}$ are injective for $j = 0, 1$. For $j = 0$, this is clear from a) (reapplying Theorem 4.2 a)).
For $j = 1$, by the (Brown-Gersten-)Quillen spectral sequence it suffices to show that the map

$$A^1(X, K_{i+1}) \to A^1(X_K, K_{i+1})$$

is injective. For $i = 0$, the statement (concerning Pic) is classical; for $i = 1$, it follows from [32, Theorem] and for $i = 2$ it follows from Proposition 4.1.
c) The BGQ spectral sequence gives a map

$$K_i(X)^{(2/3)} \xrightarrow{\sim} E_\infty^{2,-i-2} \hookrightarrow \operatorname{Coker}(A^0(X, K_{i+1}) \xrightarrow{d_2^{0,-i-1}} A^2(X, K_{i+2})).$$

The differential $d_2^{0,-i-1}$ is 0 by Theorem 4.2. Therefore, we get an injection

$$\operatorname{Ker}(K_i(X)^{(2/3)} \to K_i(X_K)^{(2/3)}) \hookrightarrow \operatorname{Ker}(A^2(X, K_{i+2}) \to A^2(X_K, K_{i+2})).$$

Clearly, the right-hand-side kernel is equal to $\operatorname{Ker} \xi^{2,i+2}$, where $\xi^{2,i+2}$ is the map defined in the previous section. As observed at the end of this section, this map coincides with the map ξ^{i+3} of diagrams (3.1) and (3.2) (for $i = 1, 2$; similarly for $i = 0$). The result then follows from (3.3) and (3.4) (and their analogue for $i = 0$). $\square$

6.B. The reduced norm and projective homogeneous varieties.

6.2. Proposition. *Let B be a central simple F-algebra, and let $\mathcal{F}$ be a locally free sheaf on X, provided with an action of B. For $i \leq 2$, consider the map*

$$u_{\mathcal{F}} : K_i(B) \to K_i(X)$$

induced by the exact functor

$$\begin{aligned} P(B) &\to P(X) \\ M &\mapsto \mathcal{F} \otimes_B M \end{aligned} \tag{6.1}$$

where $P(B)$ (resp. $P(X)$) denotes the category of finitely generated [projective] B-modules (resp. of locally free $\mathcal{O}_X$-sheaves of finite rank).
a) The composition

$$K_i(B) \xrightarrow{u_{\mathcal{F}}} K_i(X) \to A^0(X, K_i) \xleftarrow{\sim} K_i(F)$$

equals $\operatorname{rk}_B(\mathcal{F}) \operatorname{Nrd}_B$, *where* $\operatorname{rk}_B(\mathcal{F}) := \dfrac{\operatorname{rk}(\mathcal{F})}{\deg(B)}$.
b) The map

$$\widetilde{u}_{\mathcal{F}} : K_i(B) \to K_i(X)$$

defined by $x \mapsto u_{\mathcal{F}}(X) - \operatorname{rk}_B(\mathcal{F}) \operatorname{Nrd}_B(x)$ has image contained in $K_i(X)^{(1)}$. The composition

$$K_i(B) \xrightarrow{\widetilde{u}_{\mathcal{F}}} K_i(X)^{(1)} \to A^1(X, K_{i+1}) \xrightarrow{\xi^{1,i+1}} K_i(E_1) = CH^1(X) \otimes K_i(F)$$

where $\xi^{1,i+1}$ is as in Section 5, equals $c_1(\mathcal{F}) \otimes \operatorname{Nrd}_B$.

Proof. Observe that Nrd_B is characterised by the commutation of the diagram

$$\begin{array}{ccc} K_i(B_L) & \xrightarrow{\sim} & K_i(L) \\ \uparrow & & \uparrow \\ K_i(B) & \xrightarrow{\operatorname{Nrd}_B} & K_i(F) \end{array}$$

for any extension L/F that splits B and such that $L = F(Y)$, where Y is a smooth projective geometrically rational F-variety and the upper isomorphism is given by Morita theory. Indeed, this diagram then refines to a diagram of the form

$$\begin{tikzcd} A^0(Y, K_i(B \otimes_F \mathcal{O}_Y)) \arrow[r, "\sim"] & A^0(Y, K_i) \\ K_i(B) \arrow[u] \arrow[r, "\mathrm{Nrd}_B"] & K_i(F) \arrow[u, "\wr"'] \end{tikzcd}$$

see [47, Cor. 5.6] for the right vertical isomorphism.

It is therefore sufficient to check Proposition 6.2 after extending scalars to $L = K(Y)$, where Y is the Severi-Brauer variety of B. Thus, we may assume X and B split.

By Morita, $u_{\mathcal{F}}$ then corresponds to the map $K_i(F) \to K_i(X)$ given by cup-product with $[\mathcal{F} \otimes_B S] \in K_0(X)$, where S is a simple B-module. a) is now obvious, the first statement of b) follows, and the second one is also obvious since $\xi^{i,1}$ commutes with products in the split case. □

From Proposition 6.2 and Theorem 6.1 a), it follows that the restriction of $u_{\mathcal{F}}$ and $\widetilde{u}_{\mathcal{F}}$ to $SK_i(B)$ induce the same map: $SK_i(B) \to K_i(X)^{(2)}$, that we shall still denote by $u_{\mathcal{F}}$. If L/F is chosen as in the proof of Proposition 6.2, then clearly the composition $SK_i(B) \to K_i(X)^{(2)} \to K_i(X_L)^{(2)}$ is 0. This yields:

6.3. Definition. Let L/F be a geometrically rational extension splitting both X and B. We denote by $\sigma^i_{\mathcal{F}} : SK_i(B) \to \overline{\mathrm{Ker}}\eta^{i+3}$ the composition

$$SK_i(B) \xrightarrow{u_{\mathcal{F}}} \mathrm{Ker}(K_i(X)^{(2/3)} \to K_i(X_L)^{(2/3)}) \hookrightarrow \overline{\mathrm{Ker}}\,\eta^{i+3}$$

where the second map is that of Theorem 6.1 c).

7. Twisted flag varieties

In this section, we define maps from $SK_i(A)$ to Galois cohomology as promised in Theorems A and B. We use the results of the previous section. In order to get these maps, it is enough to deal with generalised Severi-Brauer varieties (twisted Grassmannians); however, we start with the apparently greater generality of twisted flag varieties. The reason for doing this is the hope to be able to compare the various maps with each other in the future, see Subsection 7.F.

7.A. K-theory of twisted flag varieties. Let A be a simple algebra of degree d, with centre F. For $\underline{r} = (r_1, \dots, r_k)$ with $d \geq r_1 > \dots > r_k \geq 0$, let $Y^{[\underline{r}]} = SB(\underline{r}; A)$ be the twist of the flag variety $G(r_1, \dots, r_k; d)$ by a 1-cocycle defining A: its function field is generic among extensions K/F such that A_K acquires a chain $I_1 \supset \dots \supset I_k$ of left ideals of respective K-dimensions $dr_1, \dots, dr_k$. If $\underline{s}$ is a subset of $\underline{r}$, there is an obvious projection

$$Y^{[\underline{r}]} \to Y^{[\underline{s}]}.$$

The variety $Y^{[\underline{r}]}$ carries a chain of locally free sheaves

$$A_{Y^{[\underline{r}]}} \twoheadrightarrow \mathcal{J}_{r_1} \twoheadrightarrow \dots \twoheadrightarrow \mathcal{J}_{r_k} \tag{7.1}$$

where $A_{Y^{[\underline{r}]}}$ is the constant sheaf with value A: if A is split, (7.1) corresponds by Morita theory to the tautological flag $\mathbf{A}^d_{Y^{[\underline{r}]}} \twoheadrightarrow V_{r_1} \dots \twoheadrightarrow V_{r_k}$ on $G(r_1, \dots, r_k; d)$ ($\mathcal{J}_{r_j}$ is the quotient of $\operatorname{End}(\mathbf{A}^d)_{Y^{[\underline{r}]}}$ by the sheaf of ideals consisting of endomorphisms vanishing on $\operatorname{Ker}(\mathbf{A}^d_{Y^{[\underline{r}]}} \to V_{r_j})$).

There is an action of A on this chain. More generally, for any partition $\alpha = (\alpha_1, \dots, \alpha_m)$ of $|\alpha| = \sum \alpha_i$ with $\alpha_1 \geq \dots \geq \alpha_m \geq 0$, with associated Schur functor S^α, the sheaf $S^\alpha(V_{r_j})$ on $G(r_1, \dots, r_k; d)$ defines by faithfully flat descent a sheaf $S^\alpha(\mathcal{J}_{r_j})$ of $A^{\otimes|\alpha|}$-algebras on $Y^{[\underline{r}]}$ [26, §4].

By Levine-Srinivas-Weyman [26, Th. 4.6], we have an isomorphism

$$\bigoplus_\alpha K_*(A^{\otimes|\alpha|}) \xrightarrow{(u_\alpha)} K_*(Y^{[\underline{r}]}) \tag{7.2}$$

where $\alpha = (\alpha^1, \dots, \alpha^k)$ is a family of partitions, with $0 \leq \alpha^j_i \leq r_i - r_{i+1}$, $|\alpha| = \sum |\alpha^j|$ and u_α is induced by the exact functor

$$\begin{aligned} P(A^{|\alpha|}) &\to P(Y^{[\underline{r}]}) \\ M &\mapsto S^\alpha(\mathcal{J}) \otimes_{A^{|\alpha|}} M \end{aligned}$$

with $S^\alpha(\mathcal{J}) = S^{\alpha^1}(\mathcal{J}_1) \otimes \dots \otimes S^{\alpha_k}(\mathcal{J}_k)$. Actually our choice of generators is not the one of [26], but rather the same as in Panin [40, Th. 7.1], who proves the same results by a different method.

7.B. Maps from SK_i to Galois cohomology. We now apply Definition 6.3 with $\mathcal{F} = \mathcal{J}_{r_j}$ for each j: in the above notation, this corresponds to the case $\alpha^{j'} = 0$ for $j' \neq j$ and $\alpha^j = (1, 0, \dots)$. We find maps

$$\sigma^i_{r_j} : SK_i(A) \to \overline{\operatorname{Ker}} \eta^{i+1}_{Y^{[\underline{r}]}}. \tag{7.3}$$

We now proceed to compute the differential $d_2^{i+2,1}(Y^{[\underline{r}]}, i+2)$ involved in Definition 3.1. Using the multiplicativity of (2.1) (Th. 2.5 (ii)), we reduce to computing the differential $d_2^{1,1}(Y^{[\underline{r}]}, 1)$ (*cf.* [18, lemma 6.1]). We have an exact sequence [18, 5.2]

$$CH^1(Y_s^{[\underline{r}]})^{G_F} \xrightarrow{d_2^{1,1}(Y^{[\underline{r}]},1)} Br(F) \to Br(Y^{[\underline{r}]}).$$

The group $CH^1(Y_s^{[\underline{r}]})$ has a basis consisting of the first Chern classes of the bundles V_{r_j}: in particular, G_F acts trivially on it. For $j \in [1, k]$, write $Y^{[r_j]}$ for the twisted Grassmannian (generalised Severi-Brauer variety) corresponding to r_j. Then we have a commutative diagram

$$\begin{CD} CH^1(Y_s^{[\underline{r}]}) @>{d_2^{1,1}(Y^{[\underline{r}]},1)}>> Br(F) @>>> Br(Y^{[\underline{r}]}) \\ @AAA @| @AAA \\ \mathbf{Z} = CH^1(Y_s^{[r_j]}) @>{d_2^{1,1}(Y^{[r_j]},1)}>> Br(F) @>>> Br(Y^{[r_j]}). \end{CD} \tag{7.4}$$

This shows that $CH^1(Y_s^{[\underline{r}]})$ is generated by the images of the maps $CH^1(Y_s^{[r_j]}) \to CH^1(Y_s^{[\underline{r}]})$ for $j = 1, \dots, k$, and thus there is no loss of generality in assuming $k = 1$ for the computation of the differential, which we do now. Let us simplify the notation by writing r for r_j. We have the following

7.1. LEMMA ([39, Cor. 2.7]). $\operatorname{Ker}(Br(F) \to Br(Y^{[r]})) = \langle r[A] \rangle$. □

Hence we get $d_2^{1,1}(Y^{[r]}, 1)(1) = r[A]$ (up to a unit), and therefore from Diagram (7.4):

$$d_2^{1,1}(Y^{[\underline{r}]}, 1)(V_{r_j}) = r_j[A] \text{ (up to a unit).}$$

We conclude:

7.2. COROLLARY. *a) The maps* (7.3) *give rise to commutative diagrams of complexes (*$i = 1, 2$*):*

$$\begin{CD}
0 @>>> SK_i(A) @>{\sigma^i_{r_j}}>> \dfrac{H^{i+4}(F, \mathbf{Z}(i+2))}{gcd(r_j)[A] \cdot H^{i+1}(F, \mathbf{Z}(i+1))} @>>> A^0(Y^{[\underline{r}]}), H^{i+4}(\mathbf{Z}(i+2)))\\
@. @| @AAA @A{p^*}AA\\
0 @>>> SK_i(A) @>{\sigma^i_{r_j}}>> \dfrac{H^{i+4}(F, \mathbf{Z}(i+2))}{r_j[A] \cdot H^{i+1}(F, \mathbf{Z}(i+1))} @>>> A^0(Y^{[r_j]}), H^{i+4}(\mathbf{Z}(i+2)))
\end{CD}$$

where $Y^{[r_j]} = SB(r_j, A)$ *is the generalised Severi-Brauer variety of ideals of rank* r_j*, and the middle vertical map is the natural surjection.*
b) If $j = k$ *and* r_k *divides the other* r_j*, then both vertical maps are isomorphisms.*

Proof. The only thing to remain proven is b). The generic fibre of $p : Y^{[\underline{r}]} \to Y^{[r_k]}$ is then easily seen to be the split flag variety $G(r_1 - r_k, \dots, r_{k-1} - r_k; d)$; in particular it is rational and the claim follows. □

7.3. *Remark.* By construction, this homomorphism for $i = 2$ factors through an injection

$$SK_2(A) \hookrightarrow K_2(Y^{[\underline{r}]})^{(2)}.$$

If A is a quaternion algebra, the only choice for $Y^{[\underline{r}]}$ is the conic corresponding to A and $K_2(Y^{[\underline{r}]})^{(2)} = 0$. This is a variant of the proof of Theorem 5 given in [21].

As seen above, for $i = 1$, the definition of $\sigma^i_{r_j}$ only involves the Merkurjev-Suslin theorem, while for $i = 2$ it involves the Bloch-Kato conjecture in degree 3 (for the primes dividing d). If we are ready to grant the Bloch-Kato conjecture one degree further, we get a refinement of these maps:

7.4. COROLLARY. *Assume the Bloch-Kato conjecture in degree* $i + 2$ *(*$i = 1, 2$*). Assume also for simplicity that* r_j *divides* d*. The the complexes on the bottom row of Corollary 7.2 refine into complexes*

(7.5) $\quad SK_1(A) \to H^4(F, \mu_{d/r_j}^{\otimes 3})/r_j[A] \cdot H^2(F, \mu_{d/r_j}^{\otimes 2}) \to A^0(Y^{[r_j]}, H^4(\mu_{d/r_j}^{\otimes 3}))$

$$(7.6)\quad SK_2(A) \to H^5(F,\mu_{d/r_j}^{\otimes 4})/r_j[A]\cdot H^3(F,\mu_{d/r_j}^{\otimes 3}) \to A^0(Y^{[r_j]}, H^5(\mu_{d/r_j}^{\otimes 4})).$$

Proof. Use the fact that $d/r \operatorname{Ker}\eta^i = 0$ (transfer argument), and that the map $H^4(F,\mu_{d/r_j}^{\otimes 3}) \to H^4(F,\mathbf{Q}/\mathbf{Z}(3)) = H^5(F,\mathbf{Z}(3))$ (*resp.* the map $H^5(F,\mu_{d/r_j}^{\otimes 4}) \to H^5(F,\mathbf{Q}/\mathbf{Z}(4)) = H^6(F,\mathbf{Z}(4))$) is injective under the Bloch-Kato conjecture in degree 3 (*resp.* 4). □

7.C. Examples: maps à la Suslin and à la Rost-Merkurjev. The case of Suslin corresponds to $r_j = 1$ for any A. More precisely, the way Suslin constructs his map in [50, §3] shows that it coincides with the one here for $r_j = 1$, compare Remark 3.2. Similarly, the cases of Rost-Merkurjev correspond to $d = 4$, $r_j = 2$. Using the work of Calmès [5, §2.5], one can check that in the case of a biquaternion algebra we get back Rost's map for SK_1 (*resp.* Calmès' map for SK_2). This implies:

7.5. Corollary. *a) For $i = 1$, the bottom sequence in Corollary 7.2 is exact for $r_j = 1, 2$ and $\deg(A) = 4$.*
b) The maps σ_1^1 and σ_2^1 are nonzero in general if $4 \mid \operatorname{ind}(A)$.

Proof. a) Let us first assume $r_j = 1$. Then, as explained above, the map σ_1^1 coincides with Suslin's map in [50, §3], and the exactness is loc. cit., Th. 3. Suppose now that $r_j = 2$. If A is a biquaternion algebra, the exactness is Rost's theorem [33, Th. 4]. If $\exp(A) = 4$, we reduce to the biquaternion case by the same argument as in [35, proof of Th. 6.6].
b) This follows from a) by a standard argument, *cf.* [34]. □

7.D. Some properties of the maps σ_r^i. For simplicity, we replace r_j by r; we still assume that r divides d.

7.6. Lemma. *If $r = d$, the maps* (7.5) *and* (7.6) *are* 0.

Proof. In this case the variety $Y^{[r]}$ has a rational point, hence the two kernels are 0. (Alternately, the coefficients of the cohomology groups involved in Corollary 7.4 are 0!) □

7.7. Proposition. *Let $a \in F^*$. Then, for all $r \mid d$, the diagram*

$$\begin{array}{ccc} SK_1(A) & \xrightarrow{\sigma_r^1} & H^4(F,\mu_{d/r}^{\otimes 3})/r[A]\cdot H^2(F,\mu_{d/r}^{\otimes 2}) \\ {\scriptstyle \cdot\{a\}}\downarrow & & {\scriptstyle \cdot\{a\}}\downarrow \\ SK_2(A) & \xrightarrow{\sigma_r^2} & H^5(F,\mu_{d/r}^{\otimes 4})/r[A]\cdot H^3(F,\mu_{d/r}^{\otimes 3}) \end{array}$$

commutes, where the vertical maps are cup-product by $\{a\}$ and the horizontal maps are those of (7.5) *and* (7.6).

Proof. Since the spectral sequences of [18, Th. 4.4] are multiplicative, it suffices to chek that the diagram

$$\begin{CD}
SK_1(A) @>{\sigma_r^1}>> \operatorname{Ker}\xi^4_{Y^{[r]}}\\
@V{\cdot\{a\}}VV @V{\cdot\{a\}}VV\\
SK_2(A) @>{\sigma_r^2}>> \operatorname{Ker}\xi^5_{Y^{[r]}}
\end{CD}$$

commutes. This in turn reduces to the compatibility of the BGQ spectral sequence and the isomorphisms (7.2) with products. □

Similarly:

7.8. Proposition. *Let A be a discrete valuation F-algebra, with quotient field K and residue field E. Then the diagrams*

$$\begin{CD}
SK_2(A_K) @>{\sigma_r^2}>> H^5(K,\mu_{d/r}^{\otimes 4})/r[A]\cdot H^3(K,\mu_{d/r}^{\otimes 3})\\
@V{\partial}VV @V{\partial}VV\\
SK_1(A_E) @>{\sigma_r^1}>> H^4(E,\mu_{d/r}^{\otimes 3})/r[A]\cdot H^2(E,\mu_{d/r}^{\otimes 2})
\end{CD}$$

commutes, where the homomorphisms ∂ are induced by the residue maps in K-theory and Galois cohomology respectively.

Proof. Similar. □

Using Corollary 7.5 b), Proposition 7.7 and Proposition 7.8, we find that σ_1^2 and σ_2^2 are nontrivial when $4 \mid \operatorname{ind}(A)$.

7.E. A refinement. In this subsection, where we keep the previous notation, we assume that A is a division algebra, d is a power of a prime l and $r[A]=0$: for r strictly dividing d, this is possible if and only if the exponent ε of A is smaller than d (and then we may choose for r any l-power between ε and d/l). Then we can compute $K_1(X)^{(1/2)}$ and extend the map

$$SK_i(A)\to K_i(X)^{(2)}$$

of the previous section to a map

$$K_i(A)\to K_i(X)^{(2)}.$$

This approach corresponds to that of Rost in the case where A is a biquaternion algebra [33].
Let H be the class of a hyperplane section in $K_0(Y^{[r]})$.

7.9. Proposition. *For $i\le 2$,*
a) The composition

$$K_i(F)\xrightarrow{\cdot H} K_i(Y^{[r]})^{(1)}\to A^1(Y^{[r]},K_{i+1})\xrightarrow{\xi^{1,i+1}} K_i(F)$$

is the identity.
b) The induced map

$$K_i(F) \to K_i(Y^{[r]})^{(1/2)}$$

is an isomorphism.
c) Let $\mathcal{J}$ be the tautological bundle on $Y^{[r]}$. Then the image of the map

$$\begin{aligned}\Phi^{[r]} : K_i(A) &\to K_i(Y^{[r]})^{(1)}\\ x &\mapsto \widetilde{u}_{\mathcal{J}}(x) - \mathrm{Nrd}(x)\cdot H\end{aligned}$$

(see Proposition 6.2 b)) sits into $K_i(X)^{(2)}$.

Proof. By Lemma 7.1, the map

$$CH^1(Y^{[r]}) \to CH^1(Y^{[r]}_s)$$

is bijective. In particular, $c_1(H) = h$ in $CH^1(Y^{[r]}_s)$. We then get a) by multiplicativity. b) follows from a) and the fact that the maps

$$K_i(Y^{[r]})^{(1/2)} \to H^1(Y^{[r]}, K_{i+1}) \xrightarrow{\xi^{1,i+1}} K_i(F)$$

are injective. c) follows immediately from a). □

7.F. THE COMPARISON ISSUE. For $s \mid r \mid d$, let $Y^{[r,s]} = SB(r,s,A)$ be as in 7.A with the two projections

$$\begin{tikzcd} & Y^{[r,s]} \arrow[dl, "p_r"] \arrow[dr, "p_s"] & \\ Y^{[r]} & & Y^{[s]}. \end{tikzcd}$$

We have corresponding diagrams ($i = 1, 2$)

$$\begin{tikzcd} & \operatorname{Ker}\eta^{i+3}_{Y^{[r]}} \arrow[dr, "p_r^*"] & \\ SK_i(A) \arrow[ur, "\sigma_r^i"] \arrow[dr, "\sigma_s^i"] & & \operatorname{Ker}\eta^{i+3}_{Y^{[r,s]}} \\ & \operatorname{Ker}\eta^{i+3}_{Y^{[s]}} \arrow[ur, "p_s^*"] & \end{tikzcd}$$

The comparison issue is to know whether this diagram commutes: if this is the case, then the maps σ_r^i and σ_s^i are compatible in an obvious sense thanks to Corollary 7.2 b). In view of Theorem 6.1 c), this commutation is equivalent to the commutation of the diagram

$$\begin{tikzcd} & K_i(Y^{[r]})^{(2)} \arrow[dr, "p_r^*"] & \\ SK_i(A) \arrow[ur, "u_{\mathcal{J}_r}"] \arrow[dr, "u_{\mathcal{J}_s}"'] & & K_i(Y^{[r,s]})^{(2)} \\ & K_i(Y^{[s]})^{(2)} \arrow[ur, "p_s^*"'] & \end{tikzcd}$$

or to the vanishing of the map

$$u_{\mathcal{J}_r} - u_{\mathcal{J}_s} : SK_i(A) \to K_i(Y^{[r,s]})^{(2)}.$$

We may also consider the sheaf $\mathcal{I}_{r,s} = \operatorname{Ker}(\mathcal{I}_r \to \mathcal{I}_s)$; then the above amounts to the vanishing of the map

$$u_{\mathcal{I}_{r,s}} : K_i(A) \to K_i(Y^{[r,s]})$$

on the subgroup $SK_i(A)$. In [50, Th. 4], Suslin obtains this commutation (or vanishing) for $(s, r, d) = (1, 2, 4)$ in a very sophisticated and roundabout way. I have no idea how to prove it in general.

8. Motivic cohomology of some Severi-Brauer varieties

In this section, unlike in the rest of the paper, we write $H^*(X, \mathbf{Z}(n))$ (*resp.* $H^*_{\text{ét}}(X, \mathbf{Z}(n))$) for motivic cohomology of some smooth variety X computed in the Nisnevich (*resp.* étale) topology. We also use Zariski cohomology with coefficients into sheafified étale cohomology groups instead of cycle cohomology, as those are the groups that come naturally.

8.1. Theorem. *Let A have prime index l, and let X be its Severi-Brauer variety. Let $\mathbf{Z}_A$ be the Nisnevich sheaf with transfers defined in* [22, 5.3]. *Let $n \geq 0$, and assume the Bloch-Kato conjecture in degrees $\leq n+1$. Then:*
a) There is an exact sequence

$$0 \to H^n(F, \mathbf{Z}_A(n)) \xrightarrow{\text{Nrd}} H^n(F, \mathbf{Z}(n)) \xrightarrow{\cdot[A]} H^{n+3}_{\text{ét}}(F, \mathbf{Z}(n+1)) \\ \to H^0(X, \mathcal{H}^{n+3}_{\text{ét}}(\mathbf{Z}(n+1))) \to 0.$$

b) There is a cross of exact sequences

$$\begin{array}{ccc}
 & 0 & \\
 & \downarrow & \\
 & H^1(X,\mathcal{H}^{n+3}_{\text{ét}}(\mathbf{Z}(n+1))) & \\
 & \downarrow & \\
0\to H^{n+4}_{\text{ét}}(F,\mathbf{Z}(n+1))\to & H^{n+4}(X,\bar{\mathbf{Z}}(n+1)) & \to H^0(X,\mathcal{H}^{n+2}_{\text{ét}}(\mathbf{Z}(n)) \\
 & \downarrow & \downarrow{\scriptstyle\cdot[A]} \\
 & H^0(X,\mathcal{H}^{n+4}_{\text{ét}}(\mathbf{Z}(n+1))) & H^{n+5}_{\text{ét}}(F,\mathbf{Z}(n+1))
\end{array}$$

where $\bar{\mathbf{Z}}(n)$ is the cone of the morphism $\mathbf{Z}(n)\to R\alpha_*\alpha^*\mathbf{Z}(n)$, with α the projection of the big étale site onto the big Nisnevich site.

Proof. This is an extension of [22, Th. 8.1.4 and 8.2.2], and it is proven by the same method. The exact sequence of a) is part 2 of Theorem 8.1.4 of loc. cit. (where the differential is identified with the cup-product with $[A]$ in 8.2), except that in [22, Th. 8.1.4 (2)], the last term is $H^{n+3}_{\text{ét}}(F(X),\mathbf{Z}(n+3))$ and there is no surjectivity claimed.
To prove a) and b) we look at the spectral sequence (8.4) of [22]. Let $d=\dim X(=l-1)$. In the proof of Theorem 8.1.4 and in 8.2, the following was established:

- $E_2^{p,q}=0$ for $-q\notin[0,d]$, $p<d-1$, $p=d$ or $(p,q)=(d-1,-d)$.
- The differential

$$d_2:\operatorname{Coker}(H^n(F,\mathbf{Z}_A(n))\to H^n_{\text{ét}}(F,\mathbf{Z}(n)))\simeq E_2^{d-1,1-d}\\ \to E_2^{d+1,-d}\simeq H^{n+3}_{\text{ét}}(F,\mathbf{Z}(n+1))$$

  is injective, and induced by the cup-product $H^n_{\text{ét}}(F,\mathbf{Z}(n))\xrightarrow{\cdot[A]} H^{n+3}_{\text{ét}}(F,\mathbf{Z}(n+1))$.

The abutment of this spectral sequence on the diagonal $p+q=N$ is

$$\operatorname{Hom}(\mathbf{Z}(d)[2d],\bar{M}(X)(n+1)[n+2+N])$$

computed in $DM^{\text{eff}}(F)$, where

$$\bar{M}(X)=cone(M(X)\to R\alpha_*\alpha^*M(X)).$$

Note that $\bar{M}(X)(n+1)\simeq M(X)\otimes\bar{\mathbf{Z}}(n+1)$ (by a projection formula). Hence the abutment may be rewritten (by Poincaré duality)

$$H^{n+2+N}(X,\bar{\mathbf{Z}}(n+1)).$$

The Bloch-Kato conjecture in degree $n+1$ identifies $\bar{\mathbf{Z}}(n+1)$ with $\tau_{>n+2}(R\alpha_*\alpha^*\mathbf{Z}(n))$. The hypercohomology spectral sequence then gives

$$H^{n+2+N}(X,\bar{\mathbf{Z}}(n+1))=0 \text{ for } N\le 0$$
$$H^{n+3}(X,\bar{\mathbf{Z}}(n+1))\simeq H^0(X,\mathcal{H}^{n+3}_{\text{ét}}(\mathbf{Z}(n+1)))$$

and for $N=2$ an exact sequence

$$0\to H^1(X,\mathcal{H}^{n+3}_{\text{ét}}(\mathbf{Z}(n+1)))\to H^{n+4}(X,\bar{\mathbf{Z}}(n+1))\\ \to H^0(X,\mathcal{H}^{n+4}_{\text{ét}}(\mathbf{Z}(n+1))).$$

Consider the differentials $d_2^{d-1,q}: E_2^{d-1,q}\to E_2^{d+1,q-1}$ for $-q\le d-1$. We have

$$E_2^{p,q}=\operatorname{Hom}(\mathbf{Z},\bar{\mathbf{Z}}_{A^{\otimes(-q+1)}}(n+1-d-q)[n+2-2d+p-q])$$

where $\bar{\mathbf{Z}}_{A^{\otimes(-q+1)}}=cone(\mathbf{Z}_{A^{\otimes(-q+1)}}\to R\alpha_*\alpha^*\mathbf{Z}_{A^{\otimes(-q+1)}})$. Therefore

$$E_2^{d-1,q}=\operatorname{Hom}(\mathbf{Z},\bar{\mathbf{Z}}_{A^{\otimes(-q+1)}}(n+1-d-q)[n+1-d-q])\\ =\operatorname{Coker}(H^{n+1-d-q}(F,\mathbf{Z}_A(n+1-d-q))\to H^{n+1-d-q}(F,\mathbf{Z}(n+1-d-q)))$$

and

$$E_2^{d+1,q-1}=\operatorname{Hom}(\mathbf{Z},\bar{\mathbf{Z}}_{A^{\otimes(-q+2)}}(n+2-d-q)[n+4-d-q])\\ =H^{n+4-d-q}_{\text{ét}}(F,\mathbf{Z}(n+2-d-q))).$$

The computation of [22, 8.2] identifies $d_2^{d-1,q}$ with the map induced by cup-product by $[A]$. By the above, we get that $d_2^{d-1,q}$ is *injective*. The computation of [22, 8.2] also identifies $d_2^{d+1,q-1}$ with the cup-product by $[A]$. This gives both a) and b). □

9. Étale motivic cohomology of reductive groups

9.A. The slice spectral sequence for a reductive group. Let X be a smooth F-variety. There are spectral sequences [14, (3.1), (3.2)], similar to those of Theorem 2.5:

(9.1) $$E_2^{p,q}(X,n)_{\text{Nis}}=\operatorname{Hom}_{DM^{\text{eff}}_-(F)}(c_q(X),\mathbf{Z}(n-q)[p-q])\Rightarrow H^{p+q}_{\text{Nis}}(X,\mathbf{Z}(n))$$

(9.2)
$$E_2^{p,q}(X,n)_{\text{ét}}=\operatorname{Hom}_{DM^{\text{eff}}_{-,\text{ét}}(F)}(\alpha^*c_q(X),\mathbf{Z}(n-q)[p-q])\Rightarrow H^{p+q}_{\text{ét}}(X,\mathbf{Z}(n))$$

where $c_q(X)$ are complexes of Nisnevich sheaves with transfers associated to X (canonically in the derived category) and α is the projection from the étale site of smooth F-varieties to the Nisnevich site. These spectral sequences have the same formal properties as (2.1): transfers, and products if the motive of X is mixed Tate (*resp.* geometrically mixed Tate), *cf.* discussion in the proof of Th. 2.5 (ii).

Let $X=G$ be a connected reductive group over F, with maximal torus T defined over F. Set $Y=G/T$. Assume first G and T split. In [14, Prop. 9.3], it was shown that $c_q(G)$ is dual, in the derived category, to the complex of constant Nisnevich sheaves $c^q(G)$ (denoted by $K(G,q)$ in *loc. cit.*) given by

$$(9.3)\quad 0 \to \Lambda^q(T^*) \to \Lambda^{q-1}(T^*) \otimes CH^1(Y) \to \dots \\ \dots \to T^* \otimes CH^{q-1}(Y) \to CH^q(Y) \to 0$$

in which T^* is the group of characters of T, $CH^q(Y)$ is in degree 0 and the maps are induced by intersection products and the characteristic map $\gamma : T^* \to CH^1(X)$ (compare [8, 3.14]). Thus (9.1) may be rewritten in this case as

$$E_2^{p,q}(G,n)_{\mathrm{Nis}} = H^{p-q}_{\mathrm{Nis}}(F, c^q(G) \otimes \mathbf{Z}(n-q)) \Rightarrow H^{p+q}_{\mathrm{Nis}}(G, \mathbf{Z}(n)).$$

Since $c^q(G)$ is concentrated in degrees ≤ 0, $c^q(G) \otimes \mathbf{Z}(n-q)$ is concentrated in degrees $\leq n-q$ and $E_2^{p,q}(G,n)_{\mathrm{Nis}} = 0$ for $p > n$. We also have $E_2^{p,q}(G,n)_{\mathrm{Nis}} = 0$ for $q > n$, since $\mathbf{Z}(n-q) = 0$ in this case. For $(p,q) = (n,n)$ this yields

9.1. Lemma (*cf.* Grothendieck [13, p. 21, Rem. 2]). *If G is split, we have isomorphisms $E_2^{n,n}(G,n)_{\mathrm{Nis}} \simeq E_\infty^{p,q}(G,n)_{\mathrm{Nis}} \simeq H^{2n}(G,\mathbf{Z}(n))$, hence an exact sequence*

$$T^* \otimes CH^{n-1}(Y) \to CH^n(Y) \to CH^n(G) \to 0.$$

We shall also use:

9.2. Lemma. *Suppose G split, simply connected and absolutely simple. Then, for all $n > 0$, $CH^n(G)$ is killed by $(n-1)!$ and by the torsion index t_G of G* [7, §5]. *In particular, $CH^i(G) = 0$ for $i = 1, 2$. If G is of type A_r or C_r, $CH^i(G) = 0$ for all $i > 0$.*

Proof. The first fact follows from $K_0(G) = \mathbf{Z}$, *cf.* [8, Proof of Prop. 3.20 (iii)]. For the second one, Demazure proves in [7, Prop. 5] that the cokernels of the characteristic maps $\gamma^n : \mathbf{S}^n(T^*) \to CH^n(Y)$ are killed by t_G: the claim then follows from Lemma 9.1 and a small diagram chase. The last fact follows from [7, Lemme 5], which says that $t_G = 1$ for G of type A_r or C_r. (This also follows from Suslin [48, Th. 2.7 and 2.12].) □

We now relax the assumption that G is split, and would like to study the spectral sequences (9.2). If we knew that

$$(9.4)\qquad \alpha^* c_q(G) \simeq c_q(G_s)$$

in the derived category of complexes of étale sheaves (or G_F-modules), this would allow us to rewrite (9.2) in the form

$$E_2^{p,q}(G,n)_{\text{ét}} = H^{p-q}_{\text{ét}}(F, c^q(G_s) \otimes \mathbf{Z}(n-q)) \Rightarrow H^{p+q}_{\text{ét}}(G, \mathbf{Z}(n))$$

as for the split case, in the Nisnevich topology.

I don't know how to prove (9.4), but at least the proof of [14, Prop. 9.3] shows that the two complexes have isomorphic cohomology sheaves. Hence they are quasi-isomorphic at least in the case where the cohomology of $c^p(G_s)$ is concentrated in at most one degree. We shall therefore make-do with (9.2) and be saved by the fact that, for low values of q and for the groups G we are interested in, the latter fact is true. For simplicity, we shall write

$$\operatorname{Hom}_{DM^{\text{eff}}_{-,\text{ét}}(F)}(\alpha^* c_q(G), \mathbf{Z}(n-q)[p-q]) = \operatorname{Ext}^{p-q}_{\text{ét}}(\alpha^* c_q(G), \mathbf{Z}(n-q)).$$

We always have $c^0(G_s) = CH^0(Y_s) = \mathbf{Z}^{\pi_0(G_s)}$. Suppose that G is semi-simple, simply connected. Then c is bijective and one finds [8]

$$c^1(G_s) = 0 \tag{9.5}$$

$$c^2(G_s) = \mathbf{S}^2(T_s^*)^W[1] \tag{9.6}$$

where W is the Weyl group of G_s. If G is absolutely simple, then $\operatorname{rk} \mathbf{S}^2(T_s^*)^W = 1$ (with trivial Galois action).

We note that the unit section of G splits off from (9.2) spectral sequences

$$\widetilde{E}_2^{p,q}(G,n) \Rightarrow \widetilde{H}^{p+q}_{\text{ét}}(G, \mathbf{Z}(n))$$

with

$$\widetilde{E}_2^{p,q}(G,n) = \begin{cases} \operatorname{Ext}^{p-q}_{\text{ét}}(\alpha^* c_q(G), \mathbf{Z}(n-q)) & \text{for } q > 0 \\ 0 & \text{for } q = 0 \end{cases}$$

and $H^{p+q}_{\text{ét}}(G, \mathbf{Z}(n)) = H^{p+q}_{\text{ét}}(F, \mathbf{Z}(n)) \oplus \widetilde{H}^{p+q}_{\text{ét}}(G, \mathbf{Z}(n))$ via the unit section. These spectral sequences are modules over (9.2).

From the above spectral sequence in weight 3, the corresponding coniveau spectral sequence, (9.5) and (9.6), we get a commutative diagram analogous to (3.1):

(9.7)

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
0 \arrow[r] & A^2(G, K_3^M) \arrow[r] & \widetilde{H}^5(G, \mathbf{Z}(3)) \arrow[r] \arrow[d] & \widetilde{A}^0(G, H^4(3)) \arrow[d, "\alpha"'] \\
 & & \operatorname{Ext}^{-1}_{\text{ét}}(\alpha^* c_3(G), \mathbf{Z}) \arrow[d, "\tilde{d}_2^{2,3}(G,3)"'] & CH^3(G) \arrow[d] \\
 & H^2(F, \mathbb{G}_m \otimes \mathbf{S}^2(T_s^*)^W) \arrow[r, "\sim"] & \operatorname{Ext}^2_{\text{ét}}(\alpha^* c_2(G), \mathbf{Z}(1)) \arrow[d] & \widetilde{H}^6(G, \mathbf{Z}(3)) \\
 & & \widetilde{H}^6(G, \mathbf{Z}(3)) \arrow[d] & \\
 & & \operatorname{Ext}^0_{\text{ét}}(\alpha^* c_3(G), \mathbf{Z}) &
\end{tikzcd}$$

In this diagram, the column and the row forking downwards are both exact. The groups marked with a $\tilde{}$ are, as above, the direct summands of the corresponding groups without a $\tilde{}$ defined by the unit section of G.

9.B. An invariant computation. In this subsection, we want to compute $\mathbf{S}^3(T_s^*)^W$ when G is absolutely simple simply connected. We start with the case of type A_r. It is then convenient to think of G_s as $\mathbf{SL}_{r+1}$ embedded into $\mathbf{GL}_{r+1}$. The maximal torus T_s of G_s is then a subtorus of a maximal torus S of $\mathbf{GL}_{r+1}$, conjugate to its canonical maximal subtorus. The character group S^* is free of rank $r+1$, with basis $(e_1,\dots,e_{r+1})$, and T_s^* is the quotient of S^* by $\mathbf{Z}\sigma_1$, with $\sigma_1=\sum e_i$.
The Weyl group W of G_s coincides with that of $\mathbf{GL}_{r+1}$; it is isomorphic to $\mathfrak{S}_{r+1}$ and permutes the e_i. Let σ_i be the i-th symmetric function in the e_i: by the symmetric functions theorem, we have

$$\mathbf{S}(S^*)^W=\mathbf{Z}[\sigma_1,\dots,\sigma_{r+1}].$$

It is clear that the sequence

$$0\to\sigma_1\mathbf{S}(S^*)\to\mathbf{S}(S^*)\to\mathbf{S}(T_s^*)\to 0 \tag{9.8}$$

is exact.

9.3. Lemma. *If $r\geq 2$, the map $\mathbf{S}^3(S^*)^W\to\mathbf{S}^3(T_s^*)^W$ is surjective; $\mathbf{S}^3(T_s^*)^W$ is free of rank 1, with basis the image $\bar\sigma_3$ of σ_3. If $r=1$, $\mathbf{S}^3(T_s^*)^W=0$.*

Proof. Suppose first $r\geq 2$. In view of (9.8), for the first assertion it suffices to check that $H^1(W,\mathbf{S}^2(S^*))=0$. A basis of $\mathbf{S}^2(S^*)$ is given by $(e_1^2,\dots,e_{r+1}^2,e_1e_2,\dots)$. The group W permutes the squares and the rectangular products transitively; the isotropy group of e_1^2 is $\mathfrak{S}_r$ while the isotropy group of e_1e_2 is $\mathfrak{S}_{r-1}$. By Shapiro lemma, we get

$$H^1(W,\mathbf{S}^2(S^*))\simeq H^1(\mathfrak{S}_r,\mathbf{Z})\oplus H^1(\mathfrak{S}_{r-1},\mathbf{Z})=0.$$

For the second assertion, we use (9.8) again and get an exact sequence (thanks to the symmetric functions theorem)

$$0\to\sigma_1\langle\sigma_1^2,\sigma_2\rangle\to\langle\sigma_1^3,\sigma_1\sigma_2,\sigma_3\rangle\to\mathbf{S}(T_s^*)^W\to 0.$$

If $r=1$, the same calculation gives the result. □

In the other cases, an application of the theory of exponents [4, V.6.2, Prop. 3 and tables of Ch. VI] gives

9.4. Lemma. *If G is not of type A_r, $\mathbf{S}^3(T_s^*)^W=0$.* □

9.C. Some facts about the $c^q(G_s)$. Part a) of the following theorem is a version of S. Gille's theorem [11, th. 1.5][2]:

9.5. Theorem. *Let G be semi-simple and simply connected. Then:*
a) For $q\geq 3$, $H^r(c^q(G_s))=0$ for $r=-q,-q+1$, and $H^{-q+2}(c^q(G_s))$ is torsion-free.
b) Suppose G simple. For $q=3$, $H^{-1}(c^3(G_s))\simeq\mathbf{S}^3(T_s^*)^W$ and $H^0(c^3(G_s))\simeq CH^3(G_s)$.
c) If G is simple of type A_r, with $r\geq 2$, then $c^3(G_s)\simeq\mathbf{Z}(\chi)[1]$, generated by

[2]For $q=3$ and G of type A_r, it was obtained in 2001/2002. The general case was inspired by Gille's work.

$\bar\sigma_3$ (see Lemma 9.3) where χ is the quadratic character of G_F corresponding to its (possibly trivial) outer action on the Dynkin diagram of G. If G is of type A_1, $c^3(G_s)=0$. If G is not of type A_r, $c^3(G_s)=CH^3(G_s)[0]$.

Proof. a) For two split reductive groups G,H and $n\geq 0$, we have the Künneth formula

$$c^n(G\times H)\simeq \bigoplus_{p+q=n} c^p(G)\overset{L}{\otimes} c^q(H) \tag{9.9}$$

in the derived category [14, Lemma 4.8], since $M(G)$ and $M(H)$ are mixed Tate motives. Thus we may assume G to be simple. Consider now the commutative diagram

$$\begin{tikzcd}
 & & \Lambda^{q-2}(T_s^*)\otimes\mathbf{S}^2(T_s^*)^W \arrow[r, "e"] \arrow[d] & \Lambda^{q-3}(T_s^*)\otimes\mathbf{S}^2(T_s^*)^W\otimes T_s^* \arrow[d, "f"'] \\
\Lambda^q(T_s^*) \arrow[r] \arrow[d, "||"] & \Lambda^{q-1}(T_s^*)\otimes T_s^* \arrow[r] \arrow[d, "1\otimes\gamma"', "\wr"] & \Lambda^{q-2}(T_s^*)\otimes\mathbf{S}^2(T_s^*) \arrow[r] \arrow[d, "1\otimes\gamma^2"'] & \Lambda^{q-3}(T_s^*)\otimes\mathbf{S}^3(T_s^*) \arrow[d, "1\otimes\gamma^3"'] \\
\Lambda^q(T_s^*) \arrow[r] & \Lambda^{q-1}(T_s^*)\otimes CH^1(Y_s) \arrow[r] & \Lambda^{q-2}(T_s^*)\otimes CH^2(Y_s) \arrow[r] & \Lambda^{q-3}(T_s^*)\otimes CH^3(Y_s)
\end{tikzcd}$$

where the bottom row is the beginning of $c^q(G_s)$, the middle row is the q-th Koszul complex for T_s^*, γ^i are induced by the characteristic map, the top row is $\mathbf{S}^2(T_s^*)^W$ tensored with the beginning of the $(q-2)$-nd Koszul complex for T_s^*, the middle column is obtained by tensoring the exact sequence of free abelian groups

$$0\to\mathbf{S}^2(T_s^*)^W\to\mathbf{S}^2(T_s^*)\to CH^2(Y_s)\to 0$$

with $\Lambda^{q-2}(T_s^*)$ and, finally, f is induced by the product $\mathbf{S}^2(T_s^*)^W\otimes T_s^*\to\mathbf{S}^3(T_s^*)$. The middle row is universally exact as the Koszul complex of a free module, and the middle column is (split) short exact.
Since G is simple, $\mathbf{S}^2(T_s^*)^W$ is a rank 1 direct summand of $\mathbf{S}^2(T_s^*)$, which implies that f is injective and remains so after tensoring with $\mathbf{Z}/m$ for any m. The same is true for e by the acyclicity of Koszul complexes. A diagram chase then gives the result.
b) For $q=3$, let us rewrite part of the above diagram, for clarity:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^3(T_s^*) \arrow[r] \arrow[d, "||"'] & \Lambda^2(T_s^*)\otimes T_s^* \arrow[r] \arrow[d, "1\otimes\gamma"'] & T_s^*\otimes\mathbf{S}^2(T_s^*) \arrow[r] \arrow[d, "1\otimes\gamma^2"'] & \mathbf{S}^3(T_s^*) \arrow[r] \arrow[d, "\gamma^3"'] & 0 \\
0 \arrow[r] & \Lambda^3(T_s^*) \arrow[r] & \Lambda^2(T_s^*)\otimes CH^1(Y_s) \arrow[r] & T_s^*\otimes CH^2(Y_s) \arrow[r] & CH^3(Y_s) \arrow[r] & 0.
\end{tikzcd}$$

The two left vertical maps are isomorphisms; by (9.6), $1\otimes\gamma^2$ is surjective, with kernel $T_s^*\otimes\mathbf{S}^2(T_s^*)^W$; also, by [7, p. 292, Cor. 2] $\operatorname{Ker}\gamma^3$ is the $\mathbf{Q}$-span of

$T_s^*\mathbf{S}^2(T_s^*)^W + \mathbf{S}^3(T_s^*)^W$ in $\mathbf{S}^3(T_s)^*$. Using Lemma 9.1, it follows that

$$H^i(c^3(G_s)) = \begin{cases} 0 & \text{for } i=-3\\ \operatorname{Ker}\varphi & \text{for } i=-2\\ \operatorname{Coker}\varphi & \text{for } i=-1\\ CH^3(G_s) & \text{for } i=0\end{cases}$$

where φ is the map

$$T_s^*\otimes \mathbf{S}^2(T_s^*)^W \to \langle T_s^*\mathbf{S}^2(T_s^*)^W + \mathbf{S}^3(T_s^*)^W\rangle_{\mathbf{Q}},$$

$\langle - \rangle_{\mathbf{Q}}$ denoting the $\mathbf{Q}$-span. We have seen in a) that $\operatorname{Ker}\varphi = 0$ and $\operatorname{Coker}\varphi$ is torsion-free. We may factor φ as a composition

$$T_s^*\otimes \mathbf{S}^2(T_s^*)^W \xrightarrow{\tilde\varphi} T_s^*\mathbf{S}^2(T_s^*)^W + \mathbf{S}^3(T_s^*)^W \hookrightarrow \langle T_s^*\mathbf{S}^2(T_s^*)^W + \mathbf{S}^3(T_s^*)^W\rangle_{\mathbf{Q}}.$$

Thus $\operatorname{Coker}\varphi$ is an extension of the finite group

$$\frac{\langle T_s^*\mathbf{S}^2(T_s^*)^W + \mathbf{S}^3(T_s^*)^W\rangle_{\mathbf{Q}}}{T_s^*\mathbf{S}^2(T_s^*)^W + \mathbf{S}^3(T_s^*)^W}$$

by a group isomorphic to $\mathbf{S}^3(T_s^*)^W/\mathbf{S}^3(T_s^*)^W \cap T_s^*\mathbf{S}^2(T_s^*)^W$; but

$$\mathbf{S}^3(T_s^*)^W \cap T_s^*\mathbf{S}^2(T_s^*)^W \subseteq (T_s^*\mathbf{S}^2(T_s^*)^W)^W = T_s^{*W}\mathbf{S}^2(T_s^*)^W = 0.$$

Thus, the map $\mathbf{S}^3(T_s^*)^W \to \operatorname{Coker}\tilde\varphi$ is bijective. To conclude, we use the fact that $\mathbf{S}^3(T_s^*)^W$ is pure in $\mathbf{S}^3(T_s^*)$ (the quotient is torsion-free), which follows from Lemmas 9.3 and 9.4: since $\operatorname{Coker}\varphi$ is torsion-free, this implies that it is isomorphic to $\mathbf{S}^3(T_s^*)^W$.
c) now follows from b), Lemmas 9.3, 9.4 and 9.2. For G of type A_r with $r \geq 2$, the claim on the Galois action follows from the well-known fact that the nontrivial outer automorphism of the Dynkin diagram of G_s maps $\bar e_i$ to $-\bar e_{r+1-i}$, where $\bar e_i$ is the image of e_i in T_s^*. □

Here is a complement to Theorem 9.5:

9.6. Lemma. *Let $r \geq 2$, and consider the embedding $\iota : \mathbf{SL}_{r+1} \hookrightarrow \mathbf{SL}_{r+2}$ given by $u \mapsto \left(\begin{smallmatrix} u & 0\\ 0 & 1\end{smallmatrix}\right)$. Then the induced morphism $\iota^* : c^i(\mathbf{SL}_{r+2}) \to c^i(\mathbf{SL}_{r+1})$ is a quasi-isomorphism for $i = 2, 3$.*

Proof. Let T_{r+1}, T_{r+2} be the diagonal tori of $\mathbf{SL}_{r+1}$ and $\mathbf{SL}_{r+2}$ respectively. It suffices to check that $\mathbf{S}^i(T_{r+2}^*)^{\mathfrak{S}_{r+2}} \xrightarrow{\sim} \mathbf{S}^i(T_{r+1}^*)^{\mathfrak{S}_{r+1}}$ for $i = 2, 3$. This follows from the computations in the proof of Lemma 9.3. □

9.7. *Remark.* For G of type C_r, $CH^i(G_s) = 0$ for all $i > 0$, and for general G, $CH^3(G_s)$ is a 2-torsion group (see Lemma 9.2). Marlin computed $CH^*(G_s)$ for G of type B_r, D_r, G_2 or F_4 in [29]: he finds $CH^3(G_s) = \mathbf{Z}/2$ in each case. I don't know the value of $CH^3(G_s)$ for G of type E_6, E_7, E_8: is it also $\mathbf{Z}/2$?

10. The generic element

In this section we prove Theorem C, see (10.2), (10.3) and Theorem 10.7, Theorem D, see Corollary 10.15, and part of Theorem E, see Proposition 10.11.

10.A. The cohomological generic element. Let G be an absolutely simple simply connected group. From Theorem 9.5 and Diagram (9.7), we first deduce:

10.1. Corollary. *If G is not of inner type A_r for $r \geq 2$, we have $A^2(G, K_3^M) = \widetilde{H}^5(G, \mathbf{Z}(3)) = 0$; the group $\widetilde{A}^0(G, H^4(3))$ is isomorphic to the kernel of the étale motivic cycle map $CH^3(G) \to H^6(G, \mathbf{Z}(3))$ (hence is at most $\mathbf{Z}/2$ except perhaps for types E_6, E_7, E_8, see Remark 9.7).*

Proof. All claims follow from the diagram and the fact that we have $H^{-1}(F, c^3(G_s)) = 0$ in these cases (note that obviously

$$\operatorname{Ker}(CH^3(G) \to H^6(G, \mathbf{Z}(3)) = \operatorname{Ker}(CH^3(G) \to \widetilde{H}^6(G, \mathbf{Z}(3))).$$

□

10.2. Proposition. *If G is of inner type A_r with $r \geq 2$, the map α in Diagram (9.7) is 0.*

Proof. We have $G = \mathbf{SL}_1(A)$ for some central simple algebra A. If $CH^3(G) = 0$, there is nothing to prove; by Merkurjev [35, Prop. 4.3], this happens if and only if $\operatorname{ind}(A)$ is odd. Suppose now $\operatorname{ind}(A)$ even. If A is a quaternion algebra, we have $\widetilde{A}^0(G, H^4(3)) = 0$ by [35, Lemma 5.1]. In general, we proceed as in [35, proof of Prop. 4.3]. Note that α really comes from a map $\alpha' : A^0(G, H^4(3)) \to CH^3(G)$ and that $\alpha = 0$ if and only if $\alpha' = 0$. Let K/F be a field extension such that $\operatorname{ind}(A_K) = 2$, so that $A_K = M_n(Q)$ for some quaternion division algebra Q over K and $G_K = \mathbf{SL}_n(Q)$. Set $H = \mathbf{SL}_1(Q)$ and $X = G_K/H$. By loc. cit., the generic fibre of the projection $G_K \to X$ is H_E, with $E = K(X)$. We then have a commutative diagram

$$\begin{tikzcd}
A^0(G, H^4(3)) \arrow[r] \arrow[d, "\alpha'"'] & A^0(G_K, H^4(3)) \arrow[r] \arrow[d, "\alpha'"'] & A^0(H_E, H^4(3)) \arrow[d, "\alpha'=0"'] \\
CH^3(G) \arrow[r] & CH^3(G_K) \arrow[r] & CH^3(H_E)
\end{tikzcd}$$

and the bottom horizontal maps are isomorphisms by loc. cit. (see [35, Rk 4.4]). □

From now on, we suppose G of inner type A_r with $r \geq 2$, *i.e.* $\deg(A) > 2$ if $G = \mathbf{SL}_1(A)$. Then $H^{-1}(F, c^3(G_s))$ is canonically isomorphic to $\mathbf{Z}$, $H^2(F, \mathbb{G}_m \otimes \mathbf{S}^2(T_s^*)^W) \simeq Br(F)$ and $H^0(F, c^3(G_s)) = 0$. For the reader's convenience, let us redraw Diagram (9.7) in this case, taking Proposition 10.2 into account:

$$
\begin{tikzcd}
 & 0 \arrow[d] & \\
0 \to A^2(G,K_3^M) \arrow[r] & \widetilde{H}^5(G,\mathbf{Z}(3)) \to \widetilde{A}^0(G,H^4(3)) \to 0 \arrow[d] & \\
 & \mathbf{Z} \arrow[d, "\tilde{d}_2^{2,3}(G,3)"'] & \\
 & Br(F) \arrow[d] & \\
 & \widetilde{H}^6(G,\mathbf{Z}(3)) \arrow[d] & \\
 & 0 &
\end{tikzcd}
\tag{10.1}
$$

Since $A^0(G,H^4(3))$ and $Br(F)$ are torsion, we recover Merkurjev's result that $A^2(G,K_3^M) = A^2(G,K_3)$ is infinite cyclic [35, Lemma 5.7]. We also find

10.3. Theorem. *The group $\widetilde{H}^5(G,\mathbf{Z}(3))$ is infinite cyclic and the group $\widetilde{A}^0(G,H^4(3))$ is cyclic of order $(\widetilde{H}^5(G,\mathbf{Z}(3)) : A^2(G,K_3^M))$.*

10.4. Definition. Let $G = \mathbf{SL}_1(A)$. We denote by c_A the "positive" generator of $\widetilde{H}^5(G,\mathbf{Z}(3)) \subset H^5(G,\mathbf{Z}(3))$, that is, the generator that maps to a positive multiple of $1 \in \mathbf{Z}$, and by $\bar{c}_A$ its image in $\widetilde{A}^0(G,H^4(3)) \subset A^0(G,H^4(3))$ ($\bar{c}_A$ generates $\widetilde{A}^0(G,H^4(3))$).

10.5. Lemma. *Let still $G = \mathbf{SL}_1(A)$, and let $p_1, p_2, \mu : G \times_F G \to G$ be repectively the first projection, the second projection and the multiplication map. Then*

$$\mu^* c_A = p_1^* c_A + p_2^* c_A.$$

Proof. Since $\widetilde{H}^5(G,\mathbf{Z}(3)) \to H^{-1}(F, c^3(G_s))$ is injective for any group G, it is sufficient to show that the maps μ^* and $p_1^* + p_2^*$ from $c^3(G_s)$ to $c^3(G_s \times_{F_s} G_s)$ are equal.[3]

The Künneth formula (9.9) gives an isomorphism

$$c^3(G_s) \oplus c^3(G_s) \xrightarrow{\sim} c^3(G_s \times_{F_s} G_s)$$

induced by $p_1^* \oplus p_2^*$, since $c^1(G_s) = 0$.

Let $C = c^3(G_s)$. The inclusion $\iota_1 : G \times \{1\} \to G \times G$ induces a map $\iota_1^* : C \oplus C \to C$; since $p_1 \circ \iota_1 = Id$ and $p_1 \circ \iota_1$ is the trivial map, ι_1^* is the first

[3]Note that morphisms between reductive groups preserving the unit sections act on the spectral sequences (9.2) by preserving the spectral sequences $\widetilde{E}_r^{p,q}$. This applies to μ and to the maps ι_1 and ι_2 further below.

projection. Similarly, $\iota_2 : \{1\} \times G \to G \times G$ induces the second projection. We conclude that $\mu^* : C \to C \oplus C$ is the diagonal map, using the left and right unit formulas $\mu \circ \iota_1 = \mu \circ \iota_2 = Id$. □

Let X be a smooth F-variety. To any morphism $f : X \to \mathbf{SL}_1(A)$, we associate the pull-back of c_A:

$$c_A(f) = f^* c_A \in H^5(X, \mathbf{Z}(3)).$$

Lemma 10.5 shows that we have

$$c_A(fg) = c_A(f) + c_A(g)$$

for two maps f, g, where $fg := \mu \circ (f, g)$.
Recall that $\deg(A) > 2$. Consider the embedding $\iota_n : \mathbf{SL}_1(A) \hookrightarrow \mathbf{SL}_n(A)$ given by $u \mapsto \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Noting that $\mathbf{SL}_n(A) = \mathbf{SL}_1(M_n(A))$, Lemma 9.6 shows that

$$c_{M_n(A)}(\iota_n) = c_A.$$

In particular, $\iota_n^* : \widetilde{H}^5(\mathbf{SL}_n(A), \mathbf{Z}(3)) \to \widetilde{H}^5(\mathbf{SL}_1(A), \mathbf{Z}(3))$ is an isomorphism. So, if f is a morphism from X to $\mathbf{SL}_n(A)$, we may define $c_A(f) = (\iota_n^*)^{-1} c_{M_n(A)}(f)$, and this definition is "stable". We record this as:

10.6. Proposition. *If* $\deg(A) > 2$*, the maps*

$$\widetilde{H}^5(\mathbf{SL}_n(A), \mathbf{Z}(3)) \to \widetilde{H}^5(\mathbf{SL}_1(A), \mathbf{Z}(3))$$
$$\widetilde{A}^0(\mathbf{SL}_n(A), H^5(\mathbf{Z}(3))) \to \widetilde{A}^0(\mathbf{SL}_1(A), H^5(\mathbf{Z}(3)))$$

induced by the inclusion $\mathbf{SL}_1(A) \hookrightarrow \mathbf{SL}_n(A)$ *are isomorphisms.* □

In particular, suppose $X = \operatorname{Spec} R$ affine. Then $\operatorname{Hom}_F(X, \mathbf{SL}_n(A)) = SL_n(A \otimes_F R)$. Define $SL(A \otimes_F R) = \varinjlim SL_n(A \otimes_F R)$ as usual, and

$$SK_1(X, A) = SL(A \otimes_F R)^{\mathrm{ab}}.$$

For X smooth in general, we may similarly define

$$SL(X, A) = \varinjlim \operatorname{Hom}_F(X, \mathbf{SL}_n(A)), \quad SK_1(X, A) = SL(X, A)^{\mathrm{ab}}.$$

The above discussion then yields a homomorphism

$$SK_1(X, A) \to H^5(X, \mathbf{Z}(3)) \tag{10.2}$$

which is contravariant in X.
In particular, for $X = \operatorname{Spec} L$, L/F a function field, we get a homomorphism

$$\bar{c}_A(L) : SK_1(A_L) \to H^5(L, \mathbf{Z}(3)) \xleftarrow{\sim} H^4(L, 3). \tag{10.3}$$

The following theorem was (embarrassingly) pointed out by Philippe Gille, whom I thank here.

10.7. Theorem. *In* (10.3)*,* $L \mapsto \bar{c}_A(L)$ *defines the universal invariant of* $\mathbf{SL}_1(A)$ *of degree* 4 *with values in* $H^4(3)$*, in the sense of Merkurjev* [35, Def. 2.1].

Proof. Let G be an algebraic group and let M be a cycle module of bounded exponent as in [35, p. 133]. By [35, Th. 2.3], we have an isomorphism

$$\mathrm{Inv}^d(G, M) \xrightarrow{\sim} A^0(G, M_d)_{\mathrm{mult}}, \quad d \in \mathbf{Z}$$

induced by evaluation on the generic point of G, where the left group is the group of invariants of G with values in M_d as in [35, Def. 2.1] and the right group is the multipicative part of $A^0(G, M_d)$ as in [35, 1.3].
We cannot apply this directly to $M_d(K) = H^d(K, d-1)$, which is not of bounded exponent. However, any cycle module M_* such that $M_d(K)$ is torsion prime to $\operatorname{char} F$ for any $d \in \mathbf{Z}$ and any function field K/F may be written as the filtering direct limit of its torsion sub-cycle modules ${}_mM_*$, $m \geq 1$. Then the maps

$$\varinjlim \mathrm{Inv}^d(G, {}_mM) \to \mathrm{Inv}^d(G, M)_{\mathrm{tors}}$$
$$\varinjlim A^0(G, {}_mM_d)_{\mathrm{mult}} \to A^0(G, M_d)_{\mathrm{mult}}$$

are bijective, so that [35, Th. 2.3] extends to an isomorphism

$$\mathrm{Inv}^d(G, M)_{\mathrm{tors}} \xrightarrow{\sim} A^0(G, M_d)_{\mathrm{mult}} \tag{10.4}$$

for any torsion cycle module M (excluding the characteristic of F) and any $d \in \mathbf{Z}$.
In the case $G = \mathbf{SL}_1(A)$, any invariant of G, evaluated at a function field K, factors through $G(K)^{\mathrm{ab}} = SK_1(A_K)$, which is of exponent bounded by $\operatorname{ind}(A)$ (see introduction), so any invariant is a torsion invariant.
(This argument extends to any simply connected semisimple group G by [35, Cor. 2.6] and a transfer argument. On the other hand, $\mathrm{Inv}^1(\mathbb{G}_m, K_*^M) = \mathbf{Z}$ as the construction of [35, beg. of 2.3] shows.)
Thanks to Theorem 10.3, the only thing which remains to be proven is that $\widetilde{A}^0(G, H^4(3)) = A^0(G, H^4(3))_{\mathrm{mult}}$ (notation as in [35, 1.3]): this follows from Lemma 10.5. □

10.8. *Remark.* The above proof yields a little more: if $eSK_1(A_K) = 0$ for all K/F, then $e\,\mathrm{Inv}^d(\mathbf{SL}_1(A), M) = 0$ for any cycle module M and any $d \in \mathbf{Z}$. In particular, $e\widetilde{A}^0(G, H^4(3)) = 0$. This will be amplified in Lemma 10.13 below.

A delicate issue is the behaviour of c_A and $\bar{c}_A$ under extension of scalars: in other words, the universal invariant of Theorem 10.7 might cease to be universal after extending the base field. This is directly related to the differential $\widetilde{d}_2^{2,3}(G, 3)$ in Diagram (10.1). Here is at least one case where this does not happen:

10.9. Lemma. *Let L/F be an extension such that* $\exp(A_L) = \exp(A)$. *Then*

$$\widetilde{H}^5(G, \mathbf{Z}(3)) \xrightarrow{\sim} \widetilde{H}^5(G_L, \mathbf{Z}(3))$$
$$\widetilde{A}^0(G, H^5(\mathbf{Z}(3))) \twoheadrightarrow \widetilde{A}^0(G_L, H^5(\mathbf{Z}(3))).$$

In particular, the image of c_A (resp. $\bar{c}_A$) under extension of scalars equals c_{A_L} (resp. $\bar{c}_{A_L}$).

Proof. We shall show in Corollary 11.3 that $\widetilde{d}_2^{2,3}(G,3)(1)$ is a multiple of $[A] \in Br(F)$. The claim then follows from a diagram chase with (10.1). $\square$

The following corollary to Theorem 10.7 is a special case of [60, Prop. 4.1].

10.10. COROLLARY. *Assume A of degree $d = l^n$ (l prime) and of exponent $\varepsilon < d$. Let r be such that $\varepsilon \mid r \mid d/l$. Then there is an integer $m(A,r)$ such that*

$$\sigma_r^1 = m(A,r)\bar{c}_A$$

where σ_r^1 is the invariant in (7.3) *(see § 7.D).* $\square$

As in [60, proof of Prop. 4.3], one might learn more on $m(A,r)$ by considering the generic algebra of degree d and exponent ε. We shall content ourselves with

10.11. PROPOSITION (*cf.* [34]). *For $\varepsilon = 2 < \operatorname{ind}(A)$, $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3)) \neq 0$ and $m(A,2)$ is odd.*

Proof. 1) If A is a biquaternion algebra, the Rost invariant of Theorem 2 is nontrivial [34, proof of Cor.] and, by Remark 10.8 and the remark after Theorem 6 in the introduction, $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3))$ is cyclic of order ≤ 2. Hence this group is cyclic of order 2 and the Rost invariant coincides with $\bar{c}_A$ (recovering [35, Th. 5.4]). Thus $m(A,2) = 1$ in this case.
2) If $\operatorname{ind}(A) = 4$, let D be the division algebra similar to A, so that $A = M_n(D)$ for some $n \geq 1$. By Morita invariance of algebraic K-theory, the invariant σ_2^1 is the same for A and D. On the other hand, Proposition 10.6 yields an isomorphism

$$\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3)) \xrightarrow{\sim} \widetilde{A}^0(\mathbf{SL}_1(D), H^4(3))$$

so 1) extends to this case.
3) In general, let $L = F(SB(4,A))$, so that $\operatorname{ind}(A_L) = 4$. By 2), $\bar{c}_{A_L} \neq 0$, hence $\bar{c}_A \neq 0$ by Lemma 10.9. Since σ_r^1 commutes with any extension of scalars by construction, we have $m(A,2) = m(A_L,2)$ in $\mathbf{Z}/2$, which shows that $m(A,2)$ is odd. $\square$

We shall show in Corollary 11.12 that actually $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3)) \simeq \mathbf{Z}/2$ in Proposition 10.11.

10.12. *Remark.* Let r be a divisor of $d = \deg(A)$. Let us write $H^4(3)/r[A]$ for the degree 4 part of the cycle module given by

$$K \mapsto H^n(K, n-1)/r[A]$$
$$:= \operatorname{Coker}(H^{n-2}(K, \mu_r^{\otimes n-2}) \xrightarrow{\cdot r[A]} H^n(K, \mathbf{Q}/\mathbf{Z}(n-1))).$$

It is tempting to conjecture that the map

$$A^0(\mathbf{SL}_1(A), H^4(3))_{\text{mult}} \to A^0(\mathbf{SL}_1(A), H^4(3)/r[A])_{\text{mult}}$$

is surjective, which would provide a relationship between the invariant c_A and the invariant σ_r^1 of Corollary 7.2 in general.[4] However, since $A^0(-)_{\text{mult}}$ is left

[4]Since this was written, Wouters has resolved this question in the negative, [60, Prop. 4.2].

exact rather than right exact, this does not look straightforward at all. A description of the kernel of cup-product with $r[A]$ seems a major issue to solve (*cf.* (3.3)).

10.B. THE K-THEORETIC GENERIC ELEMENT. In the universal case $X = \mathbf{SL}_1(A)$, we may write $SK_1(\mathbf{SL}_1(A), A) = SK_1(A) \oplus \widetilde{SK}_1(\mathbf{SL}_1(A), A)$ using the unit section of $\mathbf{SL}_1(A)$. The induced morphism

$$\widetilde{SK}_1(\mathbf{SL}_1(A), A) \to \widetilde{H}^5(\mathbf{SL}_1(A), \mathbf{Z}(3))$$

is surjective, hence split surjective since $\widetilde{H}^5(\mathbf{SL}_1(A), \mathbf{Z}(3)) = \mathbf{Z}$. An explicit splitting sends c_A to the class of the inclusions $\iota_n : \mathbf{SL}_1(A) \hookrightarrow \mathbf{SL}_n(A)$.

10.13. LEMMA. *a) For any smooth F-variety Y, the map*

$$H^0(Y, SK_1(\mathcal{O}_Y \otimes_F A)) \to SK_1(F(Y) \otimes_F A)$$

is surjective; the image of $c_{F(Y)\otimes_F A}$ is contained in $A^0(Y, H^4(3))$.
b) For $Y = \mathbf{SL}_1(A)$ and $K = F(Y)$, the map c_{A_K} induces a surjection

$$SK_1(A_K)/SK_1(A) \twoheadrightarrow \widetilde{A}^0(\mathbf{SL}_1(A), H^4(3)) \tag{10.5}$$

sending the generic element to $\bar{c}_A$.

Proof. The first assertion of a) is classical (Rost, *cf.* [6, p. 38]), and the second one follows from this and the construction of c_A. For b), let $\eta = \operatorname{Spec} K$ be the generic point of $\mathbf{SL}_1(A)$. It defines an element $\bar{\eta} \in SK_1(A_K)$: the *generic element*. By construction, we have

$$c_{A_K}(\bar{\eta}) = \bar{c}_A$$

from which b) follows. □

We want to better understand the map (10.5). This is possible if A is biquaternion:

10.14. THEOREM. *If A is a biquaternion algebra,* (10.5) *is an isomorphism and both sides are isomorphic to* $\mathbf{Z}/2$.

Proof. By Lemma 10.9 and Proposition 10.11, we have a commutative diagram of injections

$$\begin{tikzcd} 0 \arrow[r] & SK_1(A) \arrow[r, "\bar{c}_A"] \arrow[d, "a"'] & H^5(F, \mathbf{Z}(3)) \arrow[d, "b"'] \\ 0 \arrow[r] & SK_1(A_K) \arrow[r, "\bar{c}_{A_K}"] & H^5(K, \mathbf{Z}(3)). \end{tikzcd}$$

Since $\mathbf{SL}_1(A)$ has a rational point, a and b have compatible retractions and this diagram induces a third injection

$$0 \longrightarrow SK_1(A_K)/SK_1(A) \longrightarrow H^5(K, \mathbf{Z}(3))/H^5(F, \mathbf{Z}(3)) \tag{10.6}$$

which is obviously also induced by (10.5). This proves the first claim. The second one follows from [35, Th. 5.4] (or part 1) of the proof of Proposition 10.11). □

10.15. Corollary. *If* $\operatorname{ind}(A) = 4$*, then* $SK_1(A_K)/SK_1(A) \simeq \mathbf{Z}/2$*.*

Proof. If A is biquaternion, this follows from Theorem 10.14. In general, let $L = F(SB(A^{\otimes 2}))$. By [35, Prop. 6.3], the maps $SK_1(A) \to SK_1(A_L)$ and $SK_1(A_K) \to SK_1(A_{KL})$ are isomorphisms, so we are reduced to the biquaternion case. □

In an earlier version of this paper, I had conjectured that (10.5) is always an isomorphism. In the light of the proof of Theorem 10.14, this seems a bit optimistic unless all primes factors of $\operatorname{ind}(A)$ occur at most with exponent 2. In general, a computation of $c^q(\mathbf{SL}_1(A))$ for all $q > 1$ will yield higher cohomological invariants for $SK_1(A)$. A still optimistic but more reasonable conjecture is that these future invariants will detect all of $SK_1(A)$. Based on this expectation, we propose

10.16. Conjecture. *If* $K = F(\mathbf{SL}_1(A))$*, the group* $SK_1(A_K)/SK_1(A)$ *is cyclic, generated by the generic element.*

10.17. *Remark.* The homomorphism

$$c_A : \operatorname{Hom}(\mathbf{SL}_1(A), \mathbf{SL}_1(A)) \to H^5(\mathbf{SL}_1(A), \mathbf{Z}(3))$$

also behaves well with respect to composition: for $f \in \operatorname{Hom}(\mathbf{SL}_1(A), \mathbf{SL}_1(A))$, we have $c_A(f) \in \widetilde{H}^5(\mathbf{SL}_1(A), \mathbf{Z}(3))$ if and only if $f(1) = 1$. If this is the case, set $c_A(f) = n(f)c_A$. Then, clearly, $n(g \circ f) = n(g)n(f)$. Can one describe this "degree" map in a more naïve fashion?

11. Some computations

We now try and evaluate the groups $SK_1(A_K)/SK_1(A)$, where K is the function field of $\mathbf{SL}_1(A)$, and $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3))$: our main results in this direction are Theorem 11.9 and Corollaries 11.10 and 11.12, the latter completing the proof of Theorem E. Unfortunately we are not able to prove the nontriviality of either of these groups when $\operatorname{ind}(A)$ is odd (not squarefree) by the present methods.
We assume that $n = \deg(A)$ is of the form l^m, l prime.

11.A. Comparing some quotients.

First we have already noted:

11.1. Lemma. $|\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3))| \le \operatorname{ind}(A)/l$*.*

Proof. This follows from Lemma 10.13 b) and the fact that $SK_1(A_K)$ has exponent $\le \operatorname{ind}(A)/l$. □

See Corollary 11.12 for a refinement of this lemma when A is of exponent l.
Let $G = \mathbf{SL}_1(A)$. We note the isomorphisms

$$A^2(G, K_3^M) \xrightarrow{\sim} A^2(G, K_3)$$
$$K_2(F) \xrightarrow{\sim} A^0(G, K_2).$$

The first one is trivial and the second one is [8, Cor. B.3]. By the second one, the BGQ spectral sequence yields an injection

$$K_1(G)^{(2/3)} \hookrightarrow A^2(G, K_3). \tag{11.1}$$

11.2. PROPOSITION. *If G is split, with $r \geq 2$, the maps $\widetilde{H}^5(G, \mathbf{Z}(3)) \to \mathbf{Z}$ and $A^2(G, K_3^M) \to \widetilde{H}^5(G, \mathbf{Z}(3))$ from* (10.1) *are both bijective. The same is true of the map* (11.1).

Proof. Mixing the coniveau spectral sequence for *Nisnevich* motivic cohomology with the slice spectral sequence (9.1) (also for Nisnevich motivic cohomology) yields a diagram similar to (10.1) and mapping to it:

$$\begin{tikzcd} A^2(G, K_3^M) \arrow[r, "\sim"] & \widetilde{H}^5_{\mathrm{Zar}}(G, \mathbf{Z}(3)) \arrow[d, "\wr"] \\ & \mathbf{Z} \end{tikzcd} \tag{11.2}$$

This proves the first two claims of Proposition 11.2 at once. For the last one, we notice that if G is split then all its Chow groups are 0 by Lemma 9.2, hence all differentials leaving from $A^2(G, K_3)$ in the BGQ spectral sequence vanish. □

Note that the horizontal map in (11.2) is an isomorphism for any G, whether split or not.

11.3. COROLLARY. *In Diagram* (10.1) *for $G = \mathbf{SL}_1(A)$, we have*

$$\widetilde{d}_2^{2,3}(G, 3)(1) = t[A]$$

for some integer t, where $[A]$ is the class of A in $Br(F)$. In particular, $(\mathbf{Z} : \widetilde{H}^5(G, \mathbf{Z}(3)))$ divides the exponent of $[A]$.

Proof. Let K be the function field of the Severi-Brauer variety of A. Then A splits over K. The first statement now follows from Proposition 11.2 and Amitsur's theorem [1] that $\mathrm{Ker}(Br(F) \to Br(K)) = \langle [A] \rangle$. The second statement is obvious. □

11.4. COROLLARY. *In general,*

$$\begin{aligned} (\mathbf{Z} : A^2(G, K_3^M)) = (A^2(G_s, K_3^M) &: A^2(G, K_3^M)) \\ &\mid (K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}). \end{aligned}$$

Proof. This follows immediately from Proposition 11.2. □

The following diagram is a little more precise and may be helpful to the reader ($G = \mathbf{SL}_1(A)$):

$$\begin{tikzcd} & \frac{SK_1(A_K)}{SK_1(A)} \arrow[r, two heads] & \widetilde{A}^0(G, H^4(3)) & \frac{K_1(G_s)^{(2/3)}}{K_1(G)^{(2/3)}} \arrow[d, "\text{onto}"'] \\ 0 \arrow[rr] & & \frac{\widetilde{H}^5(G, \mathbf{Z}(3))}{A^2(G, K_3^M)} \arrow[u, "\wr"] \arrow[r] & \mathbf{Z}/A^2(G, K_3^M) \arrow[r] & \mathbf{Z}/t\mathbf{Z} \arrow[r] & 0 \end{tikzcd} \tag{11.3}$$

where t is as in Corollary 11.3.

11.B. THE MAP $Br(F) \to \widetilde{H}^6(G, \mathbf{Z}(3))$. In order to better understand the differential $\widetilde{d}_2^{2,3}(G, 3)$ in the future, we note:

11.5. PROPOSITION. *Let $G = \mathbf{SL}_1(A)$.*
a) We have an exact sequence

$$0 \to A^1(G, H^4(3)) \to \widetilde{H}^6(G, \mathbf{Z}(3))/CH^3(G) \to \widetilde{A}^0(G, H^5(3)).$$

b) The composition

$$Br(F) \to \widetilde{H}^6(G, \mathbf{Z}(3)) \to \widetilde{H}^6(G, \mathbf{Z}(3))/CH^3(G) \to \widetilde{A}^0(G, H^5(3))$$

from Diagram (10.1) *is* 0*, and so is the map* $\widetilde{H}^6(G, \mathbf{Z}(3))/CH^3(G) \to \widetilde{A}^0(G, H^5(3))$*. Hence we have in fact an exact sequence*

$$0 \to CH^3(G) \to \widetilde{H}^6(G, \mathbf{Z}(3)) \to A^1(G, H^4(3)) \to 0.$$

Proof. a) follows from the coniveau spectral sequence for the étale motivic cohomology of G. b) The second vanishing follows from the first, since $Br(F) \to \widetilde{H}^6(G, \mathbf{Z}(3))$ is surjective. For the first vanishing, given the definition of the homomorphism $Br(F) \to \widetilde{H}^6(G, \mathbf{Z}(3))$, it suffices to show that the map $\alpha^* c_i(V) \to \alpha^* c_i(G)$ induces 0 on homology sheaves for $i = 1, 2, 3$ if V is a suitable open subset V of G.
Let B be a Borel subgroup containing $T_s \subset G_s$. Consider the big cell $\bar{U}_0 \subset G_s/B$: it is an affine space, hence all its Chow groups are 0. Observe that U_0 is defined over a finite extension of F, hence it has only a finite number of Galois conjugates: then their intersection $\bar{U}$ is defined over F, and its geometric Chow groups are still 0. Let U be the inverse image of $\bar{U}$ in Y_s: then U is defined over F and all its geometric Chow groups are 0. Hence, for all $p > 0$, the étale complex $\alpha^* c_p(U)$ is concentrated in degrees < 0.
We now take for V the inverse image of U (viewed as an open subset of Y) in G. As in [14, Prop. 9.3], we have for all $N \geq 0$ a spectral sequence

$$E_1^{p,q}(V_s) = H^q(c_{N-p}(U_s)) \otimes \Lambda^p(T_s^*) \Rightarrow H^{p+q}(c_N(V_s))$$

which maps to the corresponding spectral sequence $E_r^{p,q}(G_s)$ for G_s (that yields the complexes (9.3)). For $N > 0$, we have $E_1^{p,q}(G_s) = 0$ for $q \neq 0$ and $E_1^{p,q}(V_s) = 0$ for $q = 0$, hence all maps $H^i(c_N(V_s)) \to H^i(c_N(G_s))$ are 0. This completes the proof of b). □

11.C. A Chern class computation. We use Gillet's convention for higher Chern classes [12].

11.6. Lemma. *For a smooth variety X, consider the higher Chern class*

$$c_{3,1} : K_1(X) \to A^2(X, K_3).$$

Then $2d_2^{0,-2} = 0$ and the diagram

$$\begin{tikzcd}
K_1(X)^{(2)} \arrow[r, "c_{3,1}"] \arrow[d] & A^2(X,K_3) & A^2(X,K_3)/d_2^{0,-2}A^0(X,K_2) \arrow[l, "2"'] \\
K_1(X)^{(2/3)} \arrow[rr, "\sim"] & & E_\infty^{2,-3} \arrow[u]
\end{tikzcd}$$

commutes, where $d_2^{0,-2}$ and $E_\infty^{2,-3}$ are relative to the BGQ spectral sequence for X.

Proof. The BGQ spectral sequence for X may be considered as the coniveau spectral sequence for X relative to algebraic K-theory. For a given $i \geq 0$, consider the corresponding coniveau spectral sequence $'E_r^{p,q}$ relative to $U \mapsto H^*(U, \mathcal{K}_i)$ (for U running through open subsets of X). By [12, pp. 239–240], the i-th Chern class C_i defines a morphism of spectral sequences $E_r^{p,q} \to {}'E_r^{p,q}$ $(r \geq 1)$ converging to the higher Chern classes $c_{i,-p-q} : K_{-p-q}(X) \to H^{p+q+i}(X, \mathcal{K}_i)$.

The group $'E_1^{p,q}$ is 0 for $q \neq -i$ and $'E_1^{p,-i} = \bigoplus_{x \in X^{(p)}} K_{i-p}(F(x))$. Hence $'E_2^{p,q} = 0$ for $q \neq -i$ and $'E_2^{p,-i} = H^p(X, \mathcal{K}_i) = {}'E_\infty^{p,-i}$. By [12, Th. 3.9], the map from $E_1^{p,-i}$ to $'E_1^{p,-i}$ induced by C_i equals $\frac{(-1)^p(i-1)!}{(i-p-1)!} c_{i-p,i-p}$ on each summand $K_{i-p}(F(x))$. In particular, for $i = 3$, $c_{1,1}$ is the identity for fields and we get a commutative diagram

$$\begin{tikzcd}
E_2^{0,-2} \arrow[r, "d_2^{0,-2}"] \arrow[d] & E_2^{2,-3} \arrow[d, "2"'] \\
0 \arrow[r] & {}'E_2^{2,-3} = E_2^{2,-3}
\end{tikzcd}$$

which proves the first claim of the lemma; the second one follows from the morphism of spectral sequences. □

11.D. Some computations, continued. The group $A^1(G, H^4(3))$ of Proposition 11.5 is mysterious and would require a further analysis: we shall refrain from starting it in this paper and will concentrate on computing the index $(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)})$, which can be done in some interesting cases.

For this, we may try and look at the map $K_1(G) \to K_1(G_s)$ and use the results of Levine [25] and Suslin [48]. In particular, we have an isomorphism [25, Th. 4.3]

$$K_1(G) \simeq K_1(F) \oplus \bigoplus_{i=1}^{r} K_0(A^{\otimes i})$$

where $r = \operatorname{rk} G = \deg A - 1$. If G (equivalently A) is split, the summand $K_0(A^{\otimes i}) \simeq \mathbf{Z}$ is generated by the class of $\Lambda^i(\rho_r)$, where ρ_r is the standard representation of $G = \mathbf{SL}_{r+1}$ into $\mathbf{GL}_{r+1}$. While Levine thinks of ρ_r as a representation, Suslin thinks of it as the generic matrix and denotes it by α_{r+1}: the two viewpoints are of course equivalent.

If we pass to the separable closure, we get a commutative diagram

$$\begin{array}{ccc} K_1(G)^{(2/3)} & \xrightarrow{\gamma_3} & A^2(G,K_3) \\ \downarrow & & \downarrow \\ K_1(G_s)^{(2/3)} & \xrightarrow{\gamma_3} & A^2(G_s,K_3) \simeq \mathbf{Z}. \end{array}$$

11.7. Lemma. *Suppose $G = \mathbf{SL}_n$, with $n = r+1$.*
a) All $[\Lambda^i(\rho_r)]$ belong to $K_1(G)^{(1)}$ and the image of $[\Lambda^i(\rho_r)]$ in $A^1(G,K_2) = \mathbf{Z}$ is $\binom{n-2}{i-1}$.
b) For all i, $[\Lambda^i(\rho_r)] - \binom{n-2}{i-1}[\rho_r] \in K_1(G)^{(2)}$ and its image in $A^2(G,K_3) = \mathbf{Z}$ is $\binom{n-3}{i-2}$.

Proof. (It may not be the most direct, but it works.) For the first assertion of a), we need to show that $[\Lambda^i(\rho_r)]_{|F(SL_n))} = 0$ or, which amounts to the same, that $\Lambda^i(\alpha_n)$ is a product of commutators, where α_n is the generic matrix with determinant 1. For this, it suffices to see that $\det \Lambda^i(\alpha_n) = 1$. But, for any matrix u, $\det \Lambda^i(u)$ is a certain power of $\det(u)$, hence the claim.

For the second assertion of a) and for b), we first do a Chern class computation. Let $\bar\gamma_j = \gamma_j([\rho_r]) = \gamma_j([\alpha_n])$, where γ_j is the j-th gamma operation in K-theory. Note the formula (*cf.* [48, p. 65])

$$\sum [\Lambda^i(\alpha_n)]u^i = \sum \bar\gamma_i u^i(1+u)^{n-i}.$$

Also, from [46, 1.3.4 a) p. 277 and Remark p. 297] (see also [45, IV.6]), we find

$$c_{2,1}(\bar\gamma_j) = \begin{cases} 0 & \text{for } j > 2 \\ -c_{2,1}(\alpha_n) & \text{for } j = 2 \\ c_{2,1}(\alpha_n) & \text{for } j = 1 \end{cases}$$

and

$$c_{3,1}(\bar\gamma_j) = \begin{cases} 0 & \text{for } j > 3 \\ 2c_{3,1}(\alpha_n) & \text{for } j = 3 \\ -3c_{3,1}(\alpha_n) & \text{for } j = 2 \\ c_{3,1}(\alpha_n) & \text{for } j = 1 \end{cases}$$

from which we deduce

$$\begin{aligned} (11.4) \quad \sum c_{2,1}([\Lambda^i(\alpha_n)])u^i &= c_{2,1}(\alpha_n)(u(1+u)^{n-1} - u^2(1+u)^{n-2}) \\ &= c_{2,1}(\alpha_n)u(1+u)^{n-2} = c_{2,1}(\alpha_n)\sum \binom{n-2}{i-1}u^i =: c_{2,1}(\alpha_n)\varphi(u) \end{aligned}$$

and

$$\sum c_{3,1}([\Lambda^i(\alpha_n)])u^i$$
$$= c_{3,1}(\alpha_n)(u(1+u)^{n-1} - 3u^2(1+u)^{n-2} + 2u^3(1+u)^{n-3})$$
$$= c_{3,1}(\alpha_n)u(1+u)^{n-3}(1+u)$$

hence

$$\sum c_{3,1}([\Lambda^i(\alpha_n)])u^i - c_{3,1}(\alpha_n)\varphi(u) = -2c_{3,1}(\alpha_n)u^2(1+u)^{n-3} = -2c_{3,1}(\alpha_n)\sum\binom{n-3}{i-2}u^i. \tag{11.5}$$

We now use the fact that, for $i \geq 1$, $A^i(\mathbf{SL}_n, K_{i+1})$ is generated by $c_{i+1,1}([\alpha_n])$ [48, Th. 2.9]. By an analogue of Lemma 11.6, the edge homomorphism $K_1(X)^{(1)} \to A^1(X, K_2)$ of the BGQ spectral sequence coincides with $-c_{2,1}$ for any smooth variety X. With (11.4), this proves the second part of a) and the first part of b). Then the second part of b) follows from Lemma 11.6 and (11.5). □

Let G not be necessarily split anymore. Let e_i be the positive generator of the summand $K_0(A^{\otimes i})$: $e_i \mapsto \operatorname{ind}(A^{\otimes i})[\Lambda^i(\alpha_n)]$. Lemma 11.7 shows that $\frac{\operatorname{ind}(A)}{\operatorname{ind}(A^{\otimes i})}e_i - \binom{n-2}{i-1}e_1 \in K_1(G)^{(2)}$ and that its image in $A^2(G_s, K_3) = \mathbf{Z}$ is $\operatorname{ind}(A)\binom{n-3}{i-2}$.

11.8. Lemma. *$v_l(\binom{n-2}{i-1}) = v_l(i)$. (Recall that $n = l^m$.)*

Proof. For an integer e, let $s_l(e)$ be the sum of the digits of e written in base l. It is well-known that

$$v_l\binom{a}{b} = \frac{s_l(b) + s_l(a-b) - s_l(a)}{l-1}.$$

Clearly, we have $s_l(l^m-2) = m(l-1)-1$. Let $t = v_l(i)$ and write $i-1 = \sum a_j l^j$, with $0 \leq a_j \leq l-1$, $a_j = l-1$ for $j < t$ and $a_t < l-1$. Then $l^m - i - 1 = \sum b_j l^j$ with $b_j = l-1$ for $j < t$, $b_t = l-2-a_t$ and $b_j = l-1-a_j$ for $t < j \leq m$. Hence

$$s_l(i-1) + s_l(l^m-i-1) - s_l(l^m-2) =$$
$$2t(l-1) + (m-t)(l-1) - 1 - (m(l-1)-1) = t(l-1).$$

□

11.9. Theorem. *We have*

$$(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}) = \begin{cases} \inf(l^{2t}\operatorname{ind}(A^{\otimes l^t})) & \text{if } l > 2 \\ \inf(l^{2t-1}\operatorname{ind}(A^{\otimes l^t})) & \text{if } l = 2. \end{cases}$$

Proof. Since the index $(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)})$ *a priori* divides $\mathrm{ind}(A)$ (transfer argument), to evaluate it we may tensor both groups with $\mathbf{Z}_l$, as well as $A^2(G, K_3)$ and $A^2(G_s, K_3)$. Note also that, since $K_1(G)^{(1/2)} \hookrightarrow A^1(G, K_2) \simeq \mathbf{Z}$ is torsion-free, $x \in K_1(G)^{(1)} \otimes \mathbf{Z}_l$ and $mx \in K_1(G)^{(2)} \otimes \mathbf{Z}_l$ for some $m \in \mathbf{Z}_l - \{0\}$ implies $x \in K_1(G)^{(2)} \otimes \mathbf{Z}_l$. This will allow us to divide freely by l-units below.
By Lemma 11.7, we have

$$\frac{n}{\mathrm{ind}(A^{\otimes i})\binom{n-2}{i-1}}e_i - e_1 \mapsto n\frac{\binom{n-3}{i-2}}{\binom{n-2}{i-1}} = \frac{n(i-1)}{n-2} \tag{11.6}$$

under the composite map $K_1(G)^{(2/3)} \otimes \mathbf{Z}_l \to K_1(G_s)^{(2/3)} \otimes \mathbf{Z}_l \xrightarrow{\sim} A^2(G_s, K_3) \otimes \mathbf{Z}_l \xrightarrow{\sim} \mathbf{Z}_l$ (note that the coefficient of e_i is an l-integer by Lemma 11.8).
Let $x = \sum \lambda_i e_i \in K_1(G)^{(2)} \otimes \mathbf{Z}_l$ (with $\lambda_i \in \mathbf{Z}_l$). In $\mathbf{Q}_l$, write

$$\lambda_i = \mu_i \frac{n}{\mathrm{ind}(A^{\otimes i})\binom{n-2}{i-1}}$$

so that

$$x = \sum \mu_i \left(\frac{n}{\mathrm{ind}(A^{\otimes i})\binom{n-2}{i-1}}e_i - e_1\right) + \sum \mu_i e_1 \tag{11.7}$$

hence $x \in K_1(G)^{(2)} \otimes \mathbf{Z}_l$ if and only if $\sum \mu_i = 0$. Note that

$$x \mapsto \sum \mu_i \frac{n(i-1)}{n-2} = \sum \mu_i \frac{n(i-1)}{n-2} + \frac{n}{n-2}\sum \mu_i = \sum i\mu_i \frac{n}{n-2}.$$

Since $v_l(\mu_i) \geq -v_l\left(\frac{n}{\mathrm{ind}(A^{\otimes i})\binom{n-2}{i-1}}\right)$, we have

$$v_l(i\mu_i \frac{n}{n-2}) \geq \begin{cases} 2v_l(i) + v_l(\mathrm{ind}(A^{\otimes i})) & \text{if } l > 2 \\ 2v_l(i) + v_l(\mathrm{ind}(A^{\otimes i})) - 1 & \text{if } l = 2 \end{cases}$$

(see Lemma 11.8).
This proves the inequality $\geq$ in Theorem 11.9. To get equality, let $s = \inf\{t \mid l^{2t}\mathrm{ind}(A^{\otimes l^t}) \text{ is minimum}\}$. Suppose first that $l > 2$. Choose $\lambda_{l^s} = 1$, $\mu_{2l^s} = -\mu_{l^s}$ and $\lambda_i = 0$ otherwise, and we are done.
Suppose now that $l = 2$. We can then argue as above by taking $\mu_{3\cdot 2^s} = -\mu_{2^s}$ provided $3 \cdot 2^s < n = 2^m$, *i.e.* $s \leq m - 2$; $s = m$ is clearly impossible and $s = m - 1$ may occur only when $2^{2m-3}\mathrm{ind}(A^{\otimes 2^{m-1}}) < 2^m$, *i.e.* when $2^m\mathrm{ind}(A^{\otimes 2^{m-1}}) \leq 4$. This means $m = 1$ or $m = 2$, $\exp(A) = 2$. In the first case we clearly have equality. In the second one we may compute directly

$$2e_2 - e_1 \mapsto 2$$
$$e_3 - e_1 \mapsto 4$$

which shows that $(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}) = 2$. So equality still holds in this case. $\square$

11.10. Corollary. *a) If* $\mathrm{ind}(A) = \exp(A)$*, then* $(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}) = \mathrm{ind}(A)$.
b) Suppose $\exp(A) = l$*. If* $l > 2$ *we have*

$$(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}) = \begin{cases} l & \text{if } \mathrm{ind}(A) = l \\ l^2 & \text{if } \mathrm{ind}(A) > l \end{cases}$$

while if $l = 2$ *we always have* $(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}) = 2$.

Proof. a) is obvious, since in this case necessarily $\mathrm{ind}(A^{\otimes l^t}) = l^{q-t}$ for all $t \leq q$, if $\mathrm{ind}(A) = l^q$. For b), we have $(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}) = \inf(\mathrm{ind}(A), l^2)$ (for $l = 2$) or $\inf(\mathrm{ind}(A), 2)$ (for $l = 2$) and the result immediately follows. $\square$

11.11. *Remark.* An easier computation gives $(K_1(G_s)^{(1/2)} : K_1(G)^{(1/2)}) = lcm(i \cdot \mathrm{ind}(A^{\otimes i})) = \mathrm{ind}(A)$. Since $A^1(G, K_2) \xrightarrow{\sim} A^1(G_s, K_2)$ [8, Cor. B.3], this yields $(A^1(G, K_2) : K_1(G)^{(1/2)}) = \mathrm{ind}(A)$.

The first part of the following corollary was (embarrassingly) pointed out by Wouters [60, 2.4 (c)]:

11.12. Corollary. *If* A *is of exponent* l*, then* $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3))$ *is cyclic of order dividing* 2 *if* $l = 2$ *and dividing* l^2 *if* $l > 2$*. If moreover* $l = 2$ *and* $\mathrm{ind}(A) > 2$*, then* $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3)) \simeq \mathbf{Z}/2$ *and the invariants* c_A *of Theorem 10.7 and* σ_2^1 *of § 7.D coincide. In general*

$$|\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3))| \leq \begin{cases} \exp(A)^2 & \text{if } l \text{ is odd} \\ \exp(A)^2/2 & \text{if } l = 2. \end{cases}$$

Proof. The first statement follows from Corollary 11.10, Diagram (11.3) and Theorem 10.3. The second one then follows from Proposition 10.11. The last one follows from taking $l^t = \exp(A)$ in Theorem 11.9. $\square$

11.13. *Question.* Let l be odd. Is it true that $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3)) \simeq \mathbf{Z}/l$ if A is of exponent l and index $> l$?

Appendix A. A cancellation theorem over imperfect fields

A.1. Theorem. *Let* F *be a field and* $M, N \in DM_-^{\mathrm{eff}}(F)$ *where* N *is a mixed Tate motive (see* [14, Def. 4.1]*). Then the map* $- \otimes \mathbf{Z}(1)$ *induces an isomorphism*

$$\mathrm{Hom}_{DM}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{DM}(M(1), N(1)).$$

Proof. It is enough to prove this for $M = C_*(X)[i]$, X a smooth variety and $i \in \mathbf{Z}$, and $N = \mathbf{Z}(n)$, $n \geq 0$. By [54, Prop. 3.2.3] and [55], the left hand side is functorially isomorphic to Bloch's higher Chow group $CH^n(X, 2n+i)$. By [30, Th. 15.12] (projective bundle formula in DM), the right hand side is a direct summand of $CH^{n+1}(X \times \mathbf{P}^1, 2n+2+i)$. By the projective bundle formula for higher Chow groups ([3, Th. 7.1], [24, Cor. 5.4]), the latter decomposes as a direct sum

$$CH^{n+1}(X \times \mathbf{P}^1, 2n+2+i) \simeq CH^{n+1}(X, 2n+2+i) \oplus CH^n(X, 2n+i).$$

Moreover, the constructions of the projective bundle isomorphisms in [30] and [3, 24] show that the latter two are compatible via the isomorphism between motivic cohomology and higher Chow groups in [55]. This proves the theorem. □

Theorem A.1 is sufficient to extend to imperfect fields the construction of the slice spectral sequences in the form of (9.1), *i.e.* for motivic cohomology computed in the Nisnevich topology (= Bloch's higher Chow groups). It is not sufficient, however, to obtain a version of the étale spectral sequences of (9.2) which is interesting at p, since p is automatically inverted in $DM^{\text{eff}}_{-,\text{ét}}(F)$ (see Remark 2.6). In order to achieve this, one may presumably proceed by working directly on Bloch's cycle complexes, as follows:
By the work of Geisser-Levine [9], the étale hypercohomology of Bloch's cycle complexes provides an interesting theory modulo p. The first thing to do is to find a version of the slice filtration directly on the cycle complexes of a given smooth F-variety X: this can be achieved by using the "homotopy coniveau filtration" (which is at the basis of the construction of the Bloch-Lichtenbaum spectral sequence), see [28] and [22, §4].
This will give spectral sequences comparable to those of Theorem 2.5 and (9.2). The issue is then to identify the E_2-terms. This can presumably be done by a slightly tedious imitation of the computations in [14] and §9, where the tediousness comes from the fact that one is limited to work with smooth varieties rather than general motives.
In the course of the computation, the following ingredients will certainly appear: étale versions of the localisation theorem for higher Chow groups (see e.g. the proof of [14, Prop. 4.11]) and of Bloch's projective bundle theorem. They should be obtained much as in [16, Th. 4.2 and Th. 5.1]. Hopefully a partial purity statement similar to [16, Th. 4.2] will be sufficient for the applications. We leave this programme to the interested reader.

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