Discrete Mathematics and Theoretical Computer Science **AC**, 2003, 345–358

Non Uniform Random Walks

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Given $\varepsilon_i \in [0,1)$ for each $1 < i < n$, a particle performs the following random walk on $\{1,2,\ldots,n\}$:

If the particle is at n, it chooses a point uniformly at random (u.a.r.) from $\{1,\ldots,n-1\}$. If the current position of the particle is m ($1 < m < n$), with probability ε_m it decides to go back, in which case it chooses a point u.a.r. from $\{m+1,\ldots,n\}$. With probability $1-\varepsilon_m$ it decides to go forward, in which case it chooses a point u.a.r. from $\{1,\ldots,m-1\}$. The particle moves to the selected point.

What is the expected time taken by the particle to reach 1 *if it starts the walk at* n?

Apart from being a natural variant of the classical one dimensional random walk, variants and special cases of this problem arise in Theoretical Computer Science [1, 2, 3, 6]. In this paper we study this problem and observe interesting properties of this walk. First we show that the expected number of times the particle visits i (before getting absorbed at 1) is the same when the walk is started at j, for all $j > i$. Then we show that for the following parameterized family of ε's: $\varepsilon_i = \frac{n-i}{n-i+\alpha\cdot(i-1)}$, $1 < i < n$ where α does not depend on i, the expected number of times the particle visits i is the same when the walk is started at j, for all $j < i$. Using these observations we obtain the expected absorption time for this family of ε's. As α varies from infinity to 1, this time goes from $\Theta(\log n)$ to $\Theta(n)$.

Finally we study the behavior of the expected convergence time as a function of ε. It remains an open question to determine whether this quantity increases when all ε's are increased. We give some preliminary results to this effect.

Keywords: Non uniform random walk

1 Introduction

Consider the following random walk performed by a particle on $\{1,2,\ldots,n\}$:

Problem 1.1. *Given $\varepsilon_i \in [0,1)$ for each $1 < i < n$: If the particle is at n, it chooses a point u.a.r. from $\{1,\ldots,n-1\}$. If the particle is at m ($1 < m < n$), with probability ε_m it decides to go back, in which case it chooses a point u.a.r. from $\{m+1,\ldots,n\}$. With probability $1-\varepsilon_m$ it decides to go forward, in which case it chooses a point u.a.r. from $\{1,\ldots,m-1\}$. The particle moves to the selected point.*

What is the expected time taken by the particle to reach 1 *if it starts the walk at* n?

Variants and special case of this problem arise in Theoretical Computer Science. In particular in the analysis of randomized algorithms [3]. A special case of this random walk is the analysis of an algorithm which finds the k-th smallest element in a list. This simple randomized algorithm, which is essentially the best known, corresponds to the walk with all ε's zero. For more motivation for studying this problem from the Computer Science point of view, one can refer to [1, 2, 3, 5, 6].

In this paper we study this random walk and give various results regarding its convergence time.

2 Our Results and Organization

In Section 3 we set up Problem 1.1 in the language of Markov chains. We show that the expected number of times the particle visits i (before getting absorbed at 1) is the same when the walk is started at j, for all $j > i$.

In Section 4 we show that for the following parameterized family of ε's:

$$\varepsilon_i = \frac{n-i}{n-i+\alpha\cdot(i-1)}\ ,\ 1 < i < n$$

where α does not depend on i, the expected number of times the particle visits i is the same when the walk is started at j, for all $j < i$. Using these observations we obtain the expected absorption time for this family of ε's.

For some important special cases for α, we have the following theorem. We defer the statement for general α for the full version. Let X_n denote the time it takes the particle performing the random walk, starting at n, to get absorbed at 1.

Theorem 2.1.

1. *For* $\alpha = \infty$, $E[X_n] = \Theta(\log n)$.
2. *For* $\alpha = n$, $E[X_n] = \Theta(\log n)$.
3. *For* $\alpha = 1$, $E[X_n] = \Theta(n)$.

We prove this theorem in Section 4. In Appendix A we give a general formula for $E[X_n]$ depending on α. It seems hard to get a bound for general ε's. Hence it seems interesting to investigate other natural ε's for which reasonable bounds can be obtained on the convergence time.

In Section 5 we give some preliminary results regarding how the expected absorption time will change if we change ε's. This leads to some interesting questions, some of which remain open.

3 Preliminary Results

In this section we build up the basics and then prove the main lemmata which we will use in the next section to analyze the random walk.

Let the set of states for Problem 1.1 be

$$S := \{1,2,\ldots,n\}.$$

If we consider the transition matrix $P(\varepsilon)$ (we will drop ε when it is clear from the context) whose rows and columns are labeled by the elements of S, the matrix can be characterized by the vector

$$\varepsilon = (0,\varepsilon_2,\varepsilon_3,\ldots\varepsilon_{n-1},0)\,.$$

The matrix $P(\varepsilon)$ is

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1-\varepsilon_2 & 0 & \frac{\varepsilon_2}{n-2} & \cdots & \frac{\varepsilon_2}{n-2} & \frac{\varepsilon_2}{n-2} \\ \frac{1-\varepsilon_3}{2} & \frac{1-\varepsilon_3}{2} & 0 & \cdots & \frac{\varepsilon_3}{n-3} & \frac{\varepsilon_3}{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1-\varepsilon_{n-1}}{n-2} & \frac{1-\varepsilon_{n-1}}{n-2} & \frac{1-\varepsilon_{n-1}}{n-2} & \cdots & 0 & \varepsilon_{n-2} \\ \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 \end{bmatrix}$$

The matrix $P(\varepsilon)$ has the following generic structure

$$\left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline & & & \\ \mathbf{R}(\varepsilon) & & \mathbf{Q}(\varepsilon) & \\ & & & \end{array}\right] \tag{1}$$

Here $R(\varepsilon) = \left[1-\varepsilon_2, \frac{1-\varepsilon_3}{2}, \ldots, \frac{1-\varepsilon_{n-1}}{n-2}, \frac{1}{n-1}\right]^t$, and $Q(\varepsilon)$ is the following sub-matrix of $P(\varepsilon)$

$$\begin{bmatrix} 0 & \frac{\varepsilon_2}{n-2} & \cdots & \frac{\varepsilon_2}{n-2} & \frac{\varepsilon_2}{n-2} \\ \frac{1-\varepsilon_3}{2} & 0 & \cdots & \frac{\varepsilon_3}{n-3} & \frac{\varepsilon_3}{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1-\varepsilon_{n-1}}{n-2} & \frac{1-\varepsilon_{n-1}}{n-2} & \cdots & 0 & \varepsilon_{n-2} \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 \end{bmatrix}$$

Henceforth we will drop the ε's and just refer to these matrices as P, Q, R.

The following proposition tells us that the particle of Problem 1.1 will reach state 1 with probability 1.

Proposition 3.1. *[4] Let $i \in \{2, \ldots, n\}$ and $\varepsilon_j < 1$, for all $1 < j < n$. Then the probability that a particle starting at position i will be at state 1 after k steps, tends to 1 as k tends to infinity.*

Proof. Let $i \in \{2, \ldots, n\}$. The probability that the particle is in state 1 after first step is $\frac{1-\varepsilon_i}{i-1}$. Let $p = \min_j \frac{1-\varepsilon_j}{j-1}$. (Notice $p > 0$). Hence the probability that starting at i the particle is not at 1 in at most k steps is at most $(1-p)^k$, which goes to zero as k goes to infinity. □

Corollary 3.2. *Let P be the transition matrix as above with its elements labeled by p_{ij}, then*

$$p_{ij}^{(k)} \to 0 \quad 2 \le i, j \le n$$

$$p_{i1}^{(k)} \to 1 \quad 1 < i \le n$$

as $k \to \infty$.

Let Q be the sub-matrix of P excluding the first row and first column. Using the corollary above we get an interesting characterization for matrix Q (sub-matrix of P).

Proposition 3.3. *[4]*

$$(I-Q)^{-1} = \sum_{k=0}^{\infty} Q^k$$

Proof. Consider the identity

$$(I-Q)\cdot(I+Q+Q^2+\cdots+Q^k) = I - Q^{k+1}.$$

By previous corollary we have that Q^{k+1} tends to the zero matrix as k tends to infinity. Hence for sufficiently large k, $\det(I-Q^{k+1}) \neq 0$. But

$$\det(I-Q^{k+1}) = \det(I-Q)\cdot\det(I+Q+\cdots+Q^k)$$

Thus $\det(I-Q) \neq 0$, and hence

$$I+Q+Q^2+\cdots+Q^k = (I-Q)^{-1}(I-Q^{k+1})$$

Taking limits both sides as $k \to \infty$, we have that

$$\begin{aligned} \lim_{k\to\infty} \sum_{j=0}^{k} Q^j &= (I-Q)^{-1} \cdot \lim_{k\to\infty}(I-Q^{k+1}) \\ &= (I-Q)^{-1} \end{aligned}$$

□

The next result gives an intuitive interpretation to the entries of the matrix $N := (I-Q)^{-1}$.

Lemma 3.4. *Let Y_{ij} ($i, j \in \{2,\ldots,n\}$) be the random variable indicating the number of times the particle is in state j if it starts at state i. Then $E[Y_{ij}] = n_{ij}$.*

Proof. Define a $0/1$ random variable for $k \geq 0$, such that

$$Z_{ij}^{(k)} := \begin{cases} 1 & \text{if particle starting at } i \text{ is at} \\ & \text{state } j \text{ after } k \text{ steps} \\ 0 & \text{otherwise} \end{cases}$$

Then $Y_{ij} = \sum_{k=0}^{\infty} Z_{ij}^{(k)}$. Hence by linearity of expectation

$$E[Y_{ij}] = \sum_{k=0}^{\infty} E[Z_{ij}^{(k)}].$$

But $E[Z_{ij}^{(k)}] = Pr[Z_{ij}^{(k)} = 1] = p_{ij}^{(k)}$. Hence $E[Y_{ij}] = \sum_{k=0}^{\infty} p_{ij}^{(k)}$ which by Proposition 3.3 is n_{ij}. □

The following lemma is an interesting characteristic of Problem 1.1 and is not true in general. This is one of the main insights in this paper about the random walk.

Lemma 3.5.

$$n_{j_1 i} = n_{j_2 i}$$

where $i, j_1, j_2 \in \{2,\dots,n\}$ and $j_1, j_2 > i$.

Proof. Let $U := (I - Q)$. Then

$$N = \frac{\text{Adjoint}(U)}{\det(U)}$$

Let U_{lk} denote the matrix obtained from U by removing the l-th row and k-th column. Then

$$n_{j_1 i} = (-1)^{i+j_1} \frac{\det(U_{ij_1})}{\det(U)}$$

$$n_{j_2 i} = (-1)^{i+j_2} \frac{\det(U_{ij_2})}{\det(U)}$$

We will show that

$$(-1)^{i+j_1}\det(U_{ij_1}) = (-1)^{i+j_2}\det(U_{ij_2})$$

Assume $j_1 < j_2$. In U we first arrange that j_2-th column is immediately after column j_1. Call the new matrix V. Thus V_{ij_1} and V_{ij_2} (defined similar to U_{ij_1}, U_{ij_2}) differ only at their j_1-th column. Also note that

$$\det(U_{ij_1}) = (-1)^{j_2 - j_1 - 1}\det(V_{ij_1}) \quad \text{and} \quad \det(U_{ij_2}) = \det(V_{ij_2})$$

Let S_{ij_1} be the matrix obtained from V_{ij_1} by replacing its j_1-th column by $R\backslash\{i\}$ (R without its i-th element). (See (1), the generic structure of matrix P, for definition of R.) Let T_{ij_1} be the matrix obtained from V_{ij_1} by adding to its j_1-th column the vector $R\backslash\{i\}$. Then

$$\det(V_{ij_1}) + \det(S_{ij_1}) = \det(T_{ij_1})$$

We claim that $\det(S_{ij_1}) = 0$. This will be proved later. Now since $\det(V_{ij_1}) = \det(T_{ij_1})$, we concentrate on $\det(T_{ij_1})$. To the j_1-th column of T_{ij_1}, we add the remaining columns of T_{ij_1}. The determinant will not change. The j_1-th column of the new matrix would become (-1) times that of matrix V_{ij_2}. Hence we have that $\det(V_{ij_1}) = -\det(V_{ij_2})$. Thus $\det(U_{ij_1}) = (-1)^{j_2-j_1}\det(U_{ij_2})$ and hence

$$(-1)^{i+j_1}\det(U_{ij_1}) = (-1)^{i+j_2}\det(U_{ij_2})$$

Notice that till this point we have not used the fact that we have a particular Markov chain (of Problem 1.1) in hand. This fact is crucially used in the proof of the claim.

To prove the claim, consider W, the sub-matrix of S_{ij_1}, formed by the intersection of its rows $R_i, \dots, R_{n-2}$ (since S_{ij_1} is a $(n-2)\times(n-2)$ matrix) and columns $C_1, \dots, C_i, C_{j_1}$. Let the columns of the new matrix be

$$\widetilde{C}_1, \widetilde{C}_2, \dots, \widetilde{C}_i, \widetilde{C}_{j_1}$$

Then we have from the definition of S_{ij_1} that

$$\widetilde{C}_1 = \widetilde{C}_2 = \dots = \widetilde{C}_i = -\widetilde{C}_{j_1}$$

Thus W has rank 1. Now add $-C_{j_1}$ in S_{ij_1} to columns $C_1, \dots, C_i$. Call this new matrix Z. Then $\det(Z) = \det(S_{ij_1})$. But Z is a matrix with all the entries in the submatrix formed by the first i columns and the last $n-i-1$ rows are 0. Hence $\det(Z) = \det(S_{ij_1}) = 0$. Hence the lemma follows. $\square$

4 Analyzing the Problem

In this section we analyze Problem 1.1 for an infinite family of ε's.

Let X_i be a random variable which denotes the number of steps taken by the particle from i to 1, $(1 < i < n)$. Then we have

$$E[X_i] = 1 + \frac{(1-\varepsilon_i)}{i-1} \cdot \sum_{j=2}^{i-1} E[X_j] + \frac{\varepsilon_i}{n-i} \cdot \sum_{j=i+1}^{n} E[X_j]$$

$$E[X_n] = 1 + \sum_{j=1}^{n-1} \frac{E[X_j]}{n-1}$$

We can write this in the form

$$(I-Q)\cdot x = b$$

where Q is the matrix defined in the previous section,

$$x = [E[X_2], \ldots, E[X_n]]^t$$

and

$$b = [1,1,\cdots,1]^t \in \mathbb{R}^{n-1}.$$

The problem now reduces to finding the matrix $N = (I-Q)^{-1}$. In what follows we assume that $n \geq 4$. Now we present the analysis for the case when

$$\varepsilon_i = \frac{n-i}{n-i+\alpha\cdot(i-1)}\ , 1 < i < n$$

where α does not depend on i. First we need an important lemma which will enable us to analyze the problem for these values of ε_i.

Lemma 4.1. *If the ε_i's are given by the following equation*

$$\varepsilon_i = \frac{n-i}{n-i+\alpha\cdot(i-1)}\ , 1 < i < n$$

where α *does not depend on i, then*

$$n_{j_1 i} = n_{j_2 i}$$

where $i, j_1, j_2 \in \{2,\ldots,n\}$ *and* $j_1, j_2 < i$.

Proof. The proof is almost same as the proof of Lemma 3.5, except for the proof of the claim. Here assume that $j_1 < j_2 < i$ and arrange such that j_1 is just before j_2. The proof of the claim follows with slight difference. The difference is that to show that the determinant of the matrix Z is zero, we use the fact that $\frac{1-\varepsilon_i}{i-1} = \alpha \cdot \frac{\varepsilon_i}{n-i}$. The matrix W is formed by the intersection of its rows $R_1, \ldots, R_i$ and columns $C_{j_1}, C_i, \ldots, C_{n-2}$. Then the columns of W will be

$$\widetilde{C}_{j_1}, \widetilde{C}_i, \ldots, \widetilde{C}_{n-2}.$$

Then we have from the definition of S_{ij_1} and the fact that

$$\frac{1-\varepsilon_i}{i-1} = \alpha \cdot \frac{\varepsilon_i}{n-i},$$

where α does not depend on i.

$$\widetilde{C_i} = \widetilde{C_{i+1}} = \ldots = \widetilde{C_{n-2}} = -\alpha \cdot \widetilde{C_{j_1}}.$$

Again W has rank 1. Now add $-\alpha \cdot C_{j_1}$ in S_{ij_1} to columns $C_i, \ldots, C_{n-2}$. Call this new matrix Z. Then $\det(Z) = \det(S_{ij_1})$. But Z is a matrix with all the entries in the submatrix formed by the first i rows and the last $n-i-1$ columns are 0. Hence $\det(Z) = \det(S_{ij_1}) = 0$. Hence the lemma follows. □

Assuming the results of Lemma 4.1 and Lemma 3.5, it is easy to see that the j-th column of matrix N will have at most three distinct elements. All the entries in a particular column above the diagonal element will take the same value, those below the diagonal element will take the same value and the diagonal element itself. Hence the problem reduces to finding solution to a 3×3 system of linear equations.

A general formula can now be worked out (depending on the value of α), we do so it for three illustrative values of α. (Here we label the rows and columns of N by $\{2, \ldots, n\}$.) The reader is referred to the Appendix A for the general formula.

The case $\alpha = \infty$

This would imply that $\varepsilon_i = 0$ for all $1 < i < n$. In this case the matrix $U = I - Q$ becomes

$$u_{ij} = \begin{cases} 1 & \text{if } i = j; \\ -\frac{1}{i-1} & \text{if } i > j; \\ 0 & \text{otherwise.} \end{cases}$$

for $2 \leq i, j \leq n$.

Theorem 4.2. *The inverse of the matrix described above is N defined by*

$$n_{ij} = \begin{cases} 1 & \text{if } i = j; \\ \frac{1}{j} & \text{if } i > j; \\ 0 & \text{otherwise.} \end{cases}$$

for $2 \leq i, j \leq n$.

Hence from this we conclude that

$$\begin{aligned} E[X_n] &= \sum_{i=1}^{n-1} \frac{1}{i} \\ &= \Theta(\log n). \end{aligned}$$

For an alternative proof of this fact the reader may go through [3]. Notice that the lower bound does not follow from it in an obvious manner.

The Case $\alpha = 1$

In this case $1 - \varepsilon_i = 1 - \frac{n-i}{n-1} = \frac{i-1}{n-1}$. Thus $\frac{\varepsilon_i}{n-i} = \frac{1}{n-1}$ and $\frac{1-\varepsilon_i}{i-1} = \frac{1}{n-1}$. In this case the matrix $U = I - Q$ becomes

$$u_{ij} = \begin{cases} 1 & \text{if } i = j; \\ -\frac{1}{n-1} & \text{otherwise.} \end{cases}$$

for $2 \le i, j \le n$.

Theorem 4.3. *The inverse of the matrix described above is N defined by*

$$n_{ij} = \begin{cases} \frac{2(n-1)}{n} & \text{if } i = j; \\ \frac{n-1}{n} & \text{otherwise.} \end{cases}$$

for $2 \le i, j \le n$.

Thus in this case we have that

$$\begin{aligned} E[X_n] &= (n-2)\left(\frac{n-1}{n}\right) + 2\left(\frac{n-1}{n}\right) \\ &= n-1 \\ &= \Theta(n). \end{aligned}$$

Hence if we do not restrict ε's, the expected time might be as large as $\Theta(n)$.

The Case $\alpha = n$

In this case $1-\varepsilon_i = \frac{n(i-1)}{i(n-1)}$ and thus $\left(\frac{1-\varepsilon_i}{i-1}\right) = n\left(\frac{\varepsilon_i}{n-i}\right)$. In this case the matrix $U = I - Q$ becomes

$$u_{ij} = \begin{cases} 1 & \text{if } i = j; \\ -\frac{n}{i(n-1)} & \text{if } i > j; \\ -\frac{1}{i(n-1)} & \text{if } i < j. \end{cases}$$

for $2 \le i, j \le n$. Define

$$\begin{aligned} \Delta_j &= \left(\frac{-n^2j^2 - n^2j + 2nj^2 - j^2 - n + j}{j(n-1)}\right) \\ &= -\left(\frac{j(n-1)((n+1)(j+1) - 2j) + n}{j(n-1)}\right). \end{aligned}$$

Theorem 4.4. *The inverse of the matrix U described above is N defined by*

$$n_{ij} = \begin{cases} \frac{4(n-1)}{3n-2} & \text{if } i = j = 2; \\ \frac{2(n-1)}{3n-2} & \text{if } j = 2, i > 2; \\ \frac{n-1}{n^2-n+1} & \text{if } 2 \le i < n, j = n; \\ \frac{n^2-1}{n^2-n+1} & \text{if } i = j = n; \\ -\frac{n}{\Delta_j} & \text{if } i < j, 2 < j < n; \\ -\frac{nj+2n-j}{\Delta_j} & \text{if } i = j, 2 < j < n; \\ -\frac{2n-1}{\Delta_j} & \text{if } i > j, 2 < j < n. \end{cases}$$

for $2 \le i, j \le n$.

To calculate $E[X_n]$, we have that

$$\begin{aligned}
E[X_n] &= \sum_{j=2}^{n} n_{nj} \\
&= \frac{2\cdot(n-1)}{3n-2} + \frac{n^2-1}{n^2-n+1} + \sum_{j=3}^{n-1}\left(\frac{(2n-1)\,j(n-1)}{j(n-1)((n+1)(j+1)-2j)+n}\right) \\
&< 1+2+\sum_{j=3}^{n-1}\left(\frac{(2n-1)\,j(n-1)}{j(n-1)((n+1)(j+1)-2j)}\right) \\
&= 3+\sum_{j=3}^{n-1}\left(\frac{2n-1}{(n+1)(j+1)-2j}\right) \\
&< 3+\sum_{j=3}^{n-1}\left(\frac{2n+2}{(n+1)(j+1)-2j}\right) \\
&= 3+\sum_{j=3}^{n-1}\left(\frac{2}{j+1-\frac{2j}{n+1}}\right) \\
&< 2+\sum_{j=1}^{n-1}\frac{2}{j} \\
&= O(\log n).
\end{aligned}$$

Hence $E[X_n] = O(\log n)$. In fact a more careful analysis will show that $E[X_n] = \Theta(\log n)$. This completes the proof of Theorem 2.1.

5 Monotonicity

In the previous section we showed how to analyze Problem 1.1 for an infinite family (with parameter α) of instances. Analyzing the problem for an arbitrary vector ε seems hard. One way to get around this is the following. First, analyze the problem for a *general* enough family of ε's. Then given any vector ε for the problem, find two vectors $\varepsilon^{(lower)}$ and $\varepsilon^{(upper)}$ from the analyzed family, such that ε is dominated in both directions by these two. Then use the expected time of absorption for $\varepsilon^{(lower)}$ and $\varepsilon^{(upper)}$ as lower and upper bounds for ε respectively. But to be able to do this we have to answer in affirmative the following question:

Problem 5.1. *Let $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ be two given vectors for Problem 1.1. Then if $\varepsilon^{(2)} \le \varepsilon^{(1)}$ ($\varepsilon_i^{(2)} \le \varepsilon_i^{(1)}$ for all $1 < i < n$), is it true that the expected number of steps required by the particle with $\varepsilon^{(1)}$, be more than that required with $\varepsilon^{(2)}$?*

It is not clear what the answer to this question is and seems difficult to resolve. Here we give some results which shed some light on this problem.

Let X_m^k denote the position of the particle starting at m after k steps. It is natural to ask if there are natural restrictions on ε's for which $E\left[X_m^k\right] \ge E\left[X_m^{(k+1)}\right]$. The following theorem gives one such condition.

Theorem 5.2. *Given $\varepsilon_i \le \frac{i}{n+1}$ for $1 < i < n$. Then for any $m \in \{2, \dots, n\}$*

$$E\left[X_m^k\right] \ge E\left[X_m^{(k+1)}\right]$$

for all $k \ge 0$.

Proof. Assume $n > X_m^k > 1$, else we have nothing to prove. By definition we have that

$$\begin{aligned} E\left[X_m^{(k+1)} | X_m^{(k)}\right] &= \frac{1-\varepsilon_{X_m^{(k)}}}{X_m^{(k)}-1}\left(1+\cdots+(X_m^{(k)}-1)\right) + \frac{\varepsilon_{X_m^{(k)}}}{n-X_m^{(k)}}\left((X_m^{(k)}+1)+\cdots+n\right) \\ &= \left(\frac{1-\varepsilon_{X_m^{(k)}}}{2}\right) X_m^{(k)} + \frac{\varepsilon_{X_m^{(k)}}}{2}\left(n+X_m^{(k)}+1\right) \\ &\le X_m^{(k)}. \end{aligned}$$

Here the last inequality follows from the hypothesis of the theorem. Hence

$$E\left[X_m^{(k+1)}\right] = E\left[E\left[X_m^{(k+1)} | X_m^{(k)}\right]\right] \le E\left[X_m^{(k)}\right].$$

□

Hence we have that the expected position of the particle never increases, no matter where it starts from, if ε's satisfy the condition of the above theorem. Define the suffix vector for π, $\sigma(\pi)$ such that

$$\sigma(\pi)(i) := \sum_{j=i}^{n} \pi(j).$$

For a probability vector π, $\sigma(\pi)(i)$ measures the probability that the particle's position is at least i. We can show that this random variable satisfies a certain monotonicity property.

Theorem 5.3. *Given a vector $\pi \in \mathbb{R}^n$ all of whose entries are positive, and $\varepsilon^{(1)}, \varepsilon^{(2)}$, such that $\varepsilon^{(1)} \ge \varepsilon^{(2)}$. Let $P(\varepsilon^{(1)})$ and $P(\varepsilon^{(2)})$ be the transition matrices. Then*

$$\sum_{i=1}^{n} \sigma(\pi \cdot P(\varepsilon^{(1)}))(i) \ge \sum_{i=1}^{n} \sigma(\pi \cdot P(\varepsilon^{(2)}))(i)$$

Proof. Let $\rho = \pi \cdot P(\varepsilon^{(1)})$. Then

$$\rho(i) = \sum_{j=1}^{n} \pi(j) \cdot p_{ji}(\varepsilon^{(1)}).$$

$$\sigma(\rho)(k) = \sum_{i=k}^{n} \sum_{j=1}^{n} \pi(j) \cdot p_{ji}(\varepsilon^{(1)})$$

and

$$\begin{aligned}\sum_{k=1}^{n}\sigma(\rho)(k) &= \sum_{k=1}^{n}\sum_{i=k}^{n}\sum_{j=1}^{n}\pi(j)\cdot p_{ji}(\varepsilon^{(1)}) \\ &= \sum_{j=1}^{n}\pi(j)\left(\sum_{k=1}^{n}\sum_{i=k}^{n}p_{ji}(\varepsilon^{(1)})\right)\end{aligned}$$

We will show that for all j

$$\sum_{k=1}^{n}\sum_{i=k}^{n}p_{ji}(\varepsilon^{(1)}) \geq \sum_{k=1}^{n}\sum_{i=k}^{n}p_{ji}(\varepsilon^{(2)}).$$

For $j = 1, n$, since $p_{ji}(\varepsilon^{(1)}) = p_{ji}(\varepsilon^{(2)})$ the sums above are the same. Now consider $1 < j < n$. Then

$$\begin{aligned}\sum_{k=1}^{n}\sum_{i=k}^{n}p_{ji}(\varepsilon^{(1)}) &= \left(\frac{1-\varepsilon_j^{(1)}}{j-1}\right)\cdot(1+2+\cdots+j-1) \\ &+ \left(\frac{\varepsilon_j^{(1)}}{n-j}\right)\cdot((j+1)+(j+2)+\cdots+n) \\ &= \left(\frac{1-\varepsilon_j^{(1)}}{2}\right)\cdot j+\left(\frac{\varepsilon_j^{(1)}}{2}\right)\cdot(n+j+1) \\ &= \frac{1}{2}\cdot(j+n\varepsilon_j^{(1)}+\varepsilon_j^{(1)})\end{aligned}$$

Hence now the theorem follows trivially. □

Remark. The above theorem has a stronger form that is true. It can be shown that

$$\sigma(\rho)(k) := \sum_{i=k}^{n}\sum_{j=1}^{n}\pi(j)\cdot p_{ji}(\varepsilon)$$

is a non-decreasing function of ε, for all k.

A problem similar to Problem 5.1, which seems important to resolve, is the following

Problem 5.4. *Let $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ be two given characteristic vectors for Problem 1.1. Then if $\varepsilon^{(2)} \leq \varepsilon^{(1)}$ (term-wise), is it true that the expected position of the particle after k steps, for each $k \geq 0$, of the particle with $\varepsilon^{(1)}$, be more than that with $\varepsilon^{(2)}$?*

Using standard path coupling arguments, we can show affirmatively Problem 5.1, but the ε's are quite restricted. We omit the details.

6 Conclusion

The main contribution of this work is to propose a new random walk special cases of which arise in Theoretical Computer Science. We analyze this problem for an infinite natural family. Many problems remain open.

1. Obtain intuitive proofs of Lemmata 3.5 and 4.1. These Lemmata seem to be the crux of the matter hence it is natural to develop a deeper understanding here.

2. Resolution to Problems 5.1 and 5.4 seems important from the point of view of furthering our understanding as well as to allow the application of the results to more general settings.

3. We leave as an open problem to analyze (asymptotically) the expected absorption time for more general ε-s. To start with when all ε-s are a constant, say $1/2$.

4. Results about the distribution of X_n also seem interesting. For instance, how concentrated is X_n about $E[X_n]$?

Acknowledgements

I would like to thank Sundar Vishwanathan for suggesting to study this problem. I would also like to thank Russ Lyons, Dana Randall and Prasad Tetali for useful discussions and comments on earlier drafts of this paper.

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Appendix

A Analysis for general α

For given ε-s satisfying the condition of Lemma 4.1, it follows from Lemmata 3.5, 4.1 that $N := (I-Q)^{-1}$ looks like the following

$$\begin{bmatrix} y_2 & x_3 & x_4 & \cdots & x_n \\ z_2 & y_3 & x_4 & \cdots & x_n \\ z_2 & z_3 & y_4 & \cdots & x_n \\ z_2 & z_3 & z_4 & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_2 & z_3 & z_4 & \cdots & x_n \\ z_2 & z_3 & z_4 & \cdots & y_n \end{bmatrix}$$

The goal is to find the x_i, y_i, z_i by noticing that $(I-Q)\cdot N = I$, thus reducing this task to solving a 3×3 system of linear equations. Once we have N, it follows that $E[X_n] = \sum_{j=2}^{n-1} z_j + y_n$. Hence we concentrate on calculating the quantities in this summation. Let $\beta := \frac{1}{\alpha}$. For $\lambda_2 := 1-\varepsilon_2$, define A_2 as below.

$$A_2 := \begin{bmatrix} 1 & -2 \\ 1 & -\beta(n-2)\lambda_2 \end{bmatrix}$$

Then it is clear from the fact that $(I-Q)\cdot N = I$, that

$$A_2 \cdot \begin{bmatrix} y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Its easy to verify that

$$A_2^{-1} = \begin{bmatrix} \frac{\beta(n-2)\lambda_2}{\beta\lambda_2 n - 2\beta\lambda_2 - 2} & \frac{-2}{\beta\lambda_2 n - 2\beta\lambda_2 - 2} \\ \frac{1}{\beta\lambda_2 n - 2\beta\lambda_2 - 2} & \frac{-1}{\beta\lambda_2 n - 2\beta\lambda_2 - 2} \end{bmatrix}.$$

Hence $z_2 = \frac{-1}{\beta\lambda_2 n - 2\beta\lambda_2 - 2}$.

Similarly for A_n defined as:

$$A_n := \begin{bmatrix} -\frac{n-2}{n-1} & 1 \\ 1+\beta & -1 \end{bmatrix}, A_n \cdot \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

It follows that

$$A_n^{-1} = \begin{bmatrix} \frac{n-1}{1+\beta n - \beta} & \frac{n-1}{1+\beta n - \beta} \\ \frac{(1+\beta)(n-1)}{1+\beta n - \beta} & \frac{n-2}{1+\beta n - \beta} \end{bmatrix}.$$

Hence $y_n = \frac{(1+\beta)(n-1)}{1+\beta n - \beta}$.

For $2 < j < n$, define

$$A_j := \begin{bmatrix} -\lambda_j(j-2) & 1 & -\lambda_j\beta(n-j) \\ -j+2 & -1 & j-1 \\ 1+\beta(n-j) & -\beta & \beta(n-j) \end{bmatrix}.$$

Again it is true that

$$A_j \cdot \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Define Δ_j to be

$$\lambda_j j\beta n - 2\lambda_j\beta j^2 + 6\beta\lambda_j j - 3\beta\lambda_j n - 2\beta\lambda_j + 2j\beta n - 2\beta j^2 + j\lambda_j\beta^2 n - 3\beta n + 3\beta j + 2\lambda_j\beta^2 n - 2\lambda_j\beta^2 j + j - 1 - \beta^2 n^2\lambda_j.$$

The inverse of A_j can be computed as follows.

$$\begin{aligned}
A_j^{-1}(1,1) &= \frac{\beta(-n+2j-1)}{\Delta_j} \\
A_j^{-1}(1,2) &= -\frac{\beta(n-j-\beta\lambda_j n+\beta\lambda_j j)}{\Delta_j} \\
A_j^{-1}(1,3) &= \frac{j-1-\beta\lambda_j n+\beta\lambda_j j}{\Delta_j} \\
A_j^{-1}(2,1) &= -\frac{-2j\beta n+2\beta j^2+3\beta n-3\beta j-j+1}{\Delta_j} \\
A_j^{-1}(2,2) &= \frac{\beta\lambda_j\left(-jn+j^2+3n-3j+\beta n^2-2j\beta n+bj^2\right)}{\Delta_j} \\
A_j^{-1}(2,3) &= -\frac{\lambda_j\left(-j^2+3j-2-j\beta n+\beta j^2+2\beta n-2\beta j\right)}{\Delta_j} \\
A_j^{-1}(3,1) &= \frac{-2\beta+1+\beta n}{\Delta_j} \\
A_j^{-1}(3,2) &= -\frac{\beta\lambda_j j-2\beta\lambda_j-1-\beta n+\beta j}{\Delta_j} \\
A_j^{-1}(3,3) &= \frac{\lambda_j j-2\lambda_j+j-2}{\Delta_j}
\end{aligned}$$

This means that for $2 < j < n$, $z_j = \frac{-2\beta+1+\beta n}{\Delta_j}$.

Now we are ready to state the expected absorption time of the particle starting at n in terms of the ε-s.

$$E[X_n] = \sum_{j=2}^{n-1} z_j + y_n = \frac{-1}{\beta\lambda_2 n-2\beta\lambda_2-2} + \sum_{j=3}^{n-1}\frac{(-2\beta+1+\beta n)}{\Delta_j} + \frac{(1+\beta)(n-1)}{1+\beta n-\beta}.$$