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Lecture notes

Ten lectures on weighted pluripotential theory

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Abstract

These are notes from a ten lecture course given to a general audience of PhD students at the University of Padova October 17-28, 2011. The goal is to present some basic notions in potential theory and weighted potential theory in the complex plane $\mathbb{C}$ (lectures 1-5) with an eye towards developing pluripotential theory and weighted pluripotential theory in $\mathbb{C}^N$, $N > 1$ (lectures 6-9), culminating in some very recent results in the subject (lecture 10).

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1 Subharmonic functions and potential theory in $\mathbb{C}$.

To motivate the definition of subharmonic functions on domains in the complex plane, we begin with their analogue on the real line $\mathbb{R}$. A twice-differentiable function $h : I \to \mathbb{R}$ on an open interval $I \subset \mathbb{R}$ is linear if and only if $h''(x) = 0$ on I. A twice-differentiable function $g : I \to \mathbb{R}$ on an open interval $I \subset \mathbb{R}$ is convex if and only if $g''(x) \geq 0$ on I. The relation between these classes of functions is as follows: if $g \leq h$ at the endpoints of any subinterval $I' \subset I$, then $g \leq h$ on I'. Of course, the notion of convexity does not require any differentiability.

In $\mathbb{C} = \mathbb{R}^2$ with variables $z = x + iy$, let $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ be the Laplacian operator. Recall that a twice-differentiable function $h : D \to \mathbb{R}$ on a domain $D \subset \mathbb{C}$ is harmonic in D if $\Delta h = 0$ there. Here is our first definition of subharmonic:

Definition 1.1. A function $u : D \to \mathbb{R}$ is *subharmonic* (shm) in a domain $D \subset \mathbb{C}$ if u is uppersemicontinuous (usc) in D and for any subdomain $D' \subset\subset D$ and any h harmonic on a neighborhood of $\overline{D}'$, if $u \leq h$ on $\partial D'$ then $u \leq h$ on D'.

Recall u is usc on D means that for each $a \in \mathbb{R}$, the set $\{z \in D : u(z) < a\}$ is open; for such a function and a compact subset K of D one can find a decreasing sequence of continuous functions $\{u_j\}$ with $u_j \downarrow u$ on K (cf., Theorem 2.1.3 of [25]). There is an analogous notion of lowersemicontinuous (lsc): v is lsc on D means that for each $a \in \mathbb{R}$, the set $\{z \in D : v(z) > a\}$ is open; equivalently, $u = -v$ is usc. Thus a function is continuous on D if and only if u is usc and lsc on D. If $D = \mathbb{C}$ and $u(z) = -1$ for $|z| < 1$ while $u(z) = 0$ for $|z| \geq 1$, then u is usc. For completeness, we say a function $v : D \to \mathbb{R}$ is *superharmonic* in D if $u = -v$ is shm there.

A second, equivalent definition of shm is the following:

Definition 1.2. A function $u : D \to \mathbb{R}$ is *subharmonic* in a domain $D \subset \mathbb{C}$ if u is usc in D and u satisfies a subaveraging property in D: for each $z_0 \in D$ and $r > 0$ with $B(z_0, r) := \{z : |z - z_0| < r\} \subset D$,

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta. \tag{1}$$

A *harmonic* function h on D satisfies a *mean-value* property: for each $z_0 \in D$ and $r > 0$ with $B(z_0, r) \subset D$,

$$h(z_0) = \frac{1}{2\pi} \int_0^{2\pi} h(z_0 + re^{i\theta}) d\theta. \tag{2}$$

Moreover, $\Delta h = 0$ in D. We recall that if h is harmonic in a domain D and continuous in $\overline{D}$, if $h \leq M$ on ∂D then $h \leq M$ in D (maximum principle); also, since $-h$ is harmonic, harmonic functions satisfy a minimum principle as well. From our second definition, we will see that shm functions satisfy a maximum principle.

Proposition 1.1. *Let u be usc in a domain $D \subset \mathbb{C}$ and satisfy (1). Then*

1. *if $u(z_0) = \sup_{z \in D} u(z)$ for some $z_0 \in D$, then $u(z) \equiv u(z_0)$;*
2. *if D is bounded and $\limsup_{z \to \zeta} u(z) \leq M$ for all $\zeta \in \partial D$, then $u \leq M$ in D.*

Proof. For (1), let $U = \{z \in D : u(z) = u(z_0)\}$. Then $U \neq \emptyset$ and $D \setminus U = \{z \in D : u(z) < u(z_0)\}$ is open by usc of u. Hence U is closed. Using property (1), we show U is open. If $w \in U$ then for any $r > 0$ with $B(w, r) \subset D$,

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} u(w) d\theta = u(w)$$

hence equality holds. Since $u(w + re^{i\theta}) \leq u(w)$, we must have that $u(w + re^{i\theta}) = u(w)$ for almost all θ for all $r > 0$ with $B(w, r) \subset D$. To complete the proof that U is open, we observe that, again by usc, if $u(w + r_0 e^{i\theta_0}) < u(w)$ for some point $w + r_0 e^{i\theta_0}$ in D, then the inequality $u(w') < u(w)$ persists for all points w' in an open neighborhood of $w + r_0 e^{i\theta_0}$. This contradicts the equality $u(w + re^{i\theta}) = u(w)$ for almost all θ for all $r > 0$ with $B(w, r) \subset D$.

For (2), the extension of u to ∂D via $u(\zeta) := \limsup_{z \to \zeta} u(z)$ if $\zeta \in \partial D$ gives an usc function on the compact set $\overline{D}$. From the exercises u attains its maximum value in $\overline{D}$ at some point w. If $w \in \partial D$, by hypothesis $u \leq u(w) \leq M$ in D. If $w \in D$, by (1) u is constant on D and hence on $\overline{D}$ so $u \leq M$ in D. □

We prove the equivalence of Definitions 1.1 and 1.2: To show that the second definition implies the first, it clearly suffices to check the domination property in the first definition on disks $B(z_0, r) \subset D$. If h is harmonic on a neighborhood of $\overline{B}(z_0, r)$ and $u \leq h$ on $\partial B(z_0, r)$, then by (1) and (2) $u - h$ satisfies (1). Furthermore, $u - h$ is usc (why?); hence, by Proposition 1.1, $u - h \leq 0$ on $\partial B(z_0, r)$ implies $u - h \leq 0$ on $B(z_0, r)$.

For the converse, we recall the solution of the Dirichlet problem in the unit disk $B := B(0, 1)$. Let f be a continuous, real-valued function on ∂B. We seek a harmonic function h in B, $h \in C(\overline{B})$, with $h = f$ on ∂B. This is achieved by writing down the Poisson integral formula:

$$P_{f,B}(z) := h(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} f(e^{i\theta}) d\theta.$$

Note that

$$P_{f,B}(0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta$$

is the mean value of f over ∂B. A formula can easily be given for the solution of the Dirichlet problem with boundary data f in any disk $B(z_0,r)$ and we will use the notation $P_{f,B(z_0,r)}$ for such a function.

Given u usc satisfying (1), since u is usc, on $\partial B(z_0,r)$ we can find a decreasing sequence of continuous functions f_j with $f_j \downarrow u$ there. The functions $h_j(z) := P_{f_j,B(z_0,r)}(z)$ then form a decreasing sequence of harmonic functions in $B(z_0,r)$. Then $u \le f_j$ on $\partial B(z_0,r)$ implies that $u \le h_j$ on $B(z_0,r)$. Hence

$$u(z_0) \le \lim_{j\to\infty} h_j(z_0) = \lim_{j\to\infty}\Big(\frac{1}{2\pi}\int_0^{2\pi} h_j(z_0+re^{i\theta})d\theta\Big)$$

$$= \frac{1}{2\pi}\int_0^{2\pi}[\lim_{j\to\infty} h_j(z_0+re^{i\theta})]d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})d\theta$$

by monotone convergence.

The canonical examples of shm functions are those of the form $u = \log|f|$ where $f \in \mathcal{O}(D)$ (the holomorphic functions on D). The class of shm functions on a domain D, denoted $SH(D)$, forms a convex cone; i.e., if $u, v \in SH(D)$ and $\alpha, \beta \ge 0$, then $\alpha u + \beta v \in SH(D)$. The maximum $\max(u,v)$ of two shm functions in D is shm in D, and one can "glue" shm functions (see exercise 6). Thus shm functions are very flexible to work with as opposed to holomorphic or harmonic functions. The limit function $u(z) := \lim_{n\to\infty} u_n(z)$ of a decreasing sequence $\{u_n\} \subset SH(D)$ is shm in D (we may have $u \equiv -\infty$); while for any family $\{v_\alpha\} \subset SH(D)$ (resp., sequence $\{v_n\} \subset SH(D)$) which is uniformly bounded above on any compact subset of D, the functions

$$v(z) := \sup_\alpha v_\alpha(z) \text{ and } w(z) := \limsup_{n\to\infty} v_n(z)$$

are "nearly" shm: the usc regularizations

$$v^*(z) := \limsup_{\zeta\to z} v(\zeta) \text{ and } w^*(z) := \limsup_{\zeta\to z} w(\zeta)$$

are shm in D. Finally, if ϕ is a real-valued, convex increasing function of a real variable, and u is shm in D, then so is $\phi \circ u$.

We will use the complex differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \text{ and } \frac{\partial}{\partial \bar z} := \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

For a function u, $\partial u := \frac{\partial u}{\partial z}dz$ and $\overline{\partial} u := \frac{\partial}{\partial \bar z}d\bar z$ where $dz = dx + idy$ and $d\bar z = dx - idy$. We let

$$d = \partial + \overline{\partial},\ d^c = i(\overline{\partial} - \partial), \text{ so } dd^c = 2i\partial\overline{\partial}.$$

Thus for $u \in C^2(D)$, $dd^c u = \Delta u dx \wedge dy$ and u is shm if and only if the Laplacian Δu is a nonnegative function on D. In this notation, a complex-valued function $f : D \to \mathbb{C}$ is holomorphic in D if $f \in C^1(D)$ and $\frac{\partial f}{\partial \bar z} = 0$ in D; this is easily seen to be equivalent, writing $f = u + iv$, to the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

We can smooth a shm function u by convolving with a regularizing kernel $\chi(z) = \chi(|z|) \ge 0$ with $\chi \in C_0^\infty(\mathbb{C})$ ($C^\infty-$functions with compact support) and $\int_{\mathbb{C}} \chi dm = 1$ (here dm is Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$); i.e., if supp$\chi \subset B(0,r)$,

$$(u * \chi)(z) := \int_{\mathbb{C}} u(z-\zeta)\chi(\zeta)dm(\zeta)$$

is shm and C^∞ on $\{z \in D : \mathrm{dist}(z,\partial D) < r\}$. (See exercise 12 for more on regularizing kernels). The regularity follows via a change of variables:

$$(u * \chi)(z) = \int_{\mathbb{C}} u(\zeta)\chi(z-\zeta)dm(\zeta);$$

differentiating under the integral sign, we see that $u * \chi$ is as differentiable as χ. The subharmonicity follows from Fubini's theorem:

$$\frac{1}{2\pi}\int_0^{2\pi}(u*\chi)(z_0+re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}} u(z_0+re^{i\theta}-\zeta)\chi(\zeta)dm(\zeta)d\theta$$

$$= \int_{\mathbb{C}}\chi(\zeta)\Big(\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta}-\zeta)d\theta\Big)dm(\zeta)$$

$$\ge \int_{\mathbb{C}}\chi(\zeta)u(z_0-\zeta)dm(\zeta) = (u*\chi)(z_0).$$

We claim that given u shm in a domain D, we can find a decreasing sequence $\{u_j\}$ of smooth shm functions with $\Delta u_j \ge 0$ defined on $\{z \in D : \mathrm{dist}(z,\partial D) > 1/j\}$ and $\lim_j u_j = u$ in D. For example, if supp$\chi \subset B(0,1)$, we can take $u_j = u * \chi_{1/j}$ where $\chi_{1/j}(z) := j^2\chi(jz)$. This will allow us to first verify properties of smooth shm functions and then pass to the limit. It remains to show that $u_j = u * \chi_{1/j}$ decrease to u on D as $j \to \infty$. We proceed in several steps, each one being interesting in itself.

1. *A radial function $u(z)=u(|z|)=u(r)$ on a disk $B(0,R)$ is shm if and only if $r\to u(r)$ is a convex, increasing function of* $\log r$.
   Note since $v(z)=\log|z|$ is shm in $\mathbb{C}$ and $f\circ v$ is shm for f convex and increasing, the "if" direction is proved. For the converse, if $u=u(r)$ is shm, then u is increasing by the maximum principle Proposition 1.1. The convexity is less obvious; a relatively painless way to verify it goes as follows: given r_1,r_2 between 0 and R, choose constants a,b so that
$$a+b\log r_1=u(r_1) \text{ and } a+b\log r_2=u(r_2).$$
   Note that $r\to\log r$ is harmonic for $r>0$. Thus $u(r)-[a+b\log r]$ is shm on the annulus $B(0,r_2)-\overline{B}(0,r_1)$. Applying the maximum principle, we see that
$$u(r)\leq a+b\log r \text{ on } B(0,r_2)-\overline{B}(0,r_1).$$
   Thus for $r_1\leq r\leq r_2$, writing $\log r=(1-t)\log r_1+t\log r_2$ for some $0\leq t\leq 1$, we have
$$\begin{aligned}u(r)\leq a+b\log r&=(1-t)[a+b\log r_1]+t[a+b\log r_2]\\&=(1-t)u(r_1)+tu(r_2).\end{aligned}$$
2. *For $u(z)$ shm on a disk $B(0,R)$, the function*
$$M_u(r):=\frac{1}{2\pi}\int_0^{2\pi}u(re^{i\theta})d\theta$$
   is a convex, increasing function of $\log r$ *and* $\lim_{r\to 0^-}M_u(r)=u(0)$.
   This is left as an exercise for the reader (hint: use Fubini).
3. $u_j=u*\chi_{1/j}$ *decrease to u on D as $j\to\infty$.*
   We have
$$\begin{aligned}u_j(\zeta)&=\int_{\mathbb{C}}u(\zeta-z)\chi_{1/j}(z)dm(z)\\&=\int_0^{2\pi}\int_0^{1/j}u(\zeta-re^{it})\chi_{1/j}(re^{it})rdrdt \text{ (why?)}\\&=\int_0^{2\pi}\int_0^1 v(\frac{s}{j}e^{it})\chi(s)sdsdt\\&=\int_0^1(\int_0^{2\pi}v(\frac{s}{j}e^{it})dt)\chi(s)sds\end{aligned}$$
   where we let $s=rj$ and $v(z):=u(\zeta-z)$. By (2), $\int_0^{2\pi}v(\frac{s}{j}e^{it})dt$ decreases to $2\pi v(0)=2\pi u(\zeta)$ as $j\uparrow\infty$; thus by monotone convergence, $u_j(\zeta)$ decreases to $2\pi\int_0^1 u(\zeta)\chi(s)sds=u(\zeta)$ (why?).

We remark that the occurrence of the combination $a+b\log r$ in step (1) is very natural: see also exercise 10 and Proposition 1.3 below.

Corollary 1.2. *If u,v are shm on D and $u=v$ a.e. then $u\equiv v$.*

Proof. Since $u=v$ a.e., $u_j=u*\chi_{1/j}\equiv v*\chi_{1/j}=v_j$. The result follows since $u_j\downarrow u$ and $v_j\downarrow v$. □

We can solve the Dirichlet problem on more general bounded domains $D\subset\mathbb{C}$ with reasonable boundaries; i.e., we can construct h satisfying $\Delta h=0$ in D and $h=f$ on ∂D, one forms the envelope
$$U(0;f)(z):=\sup\{v(z):v\in SH(D):\limsup_{z\to\zeta}v(z)\leq f(\zeta)$$
$$\text{for all } \zeta\in\partial D\}.$$
This family of all $v\in SH(D)$ satisfying $\limsup_{z\to\zeta}v(z)\leq f(\zeta)$ for all $\zeta\in\partial D$ is a *Perron* family: for any such v and any disk $\tilde{B}\subset D$, the function $\tilde{v}$ defined as v in $D\setminus\tilde{B}$ and as $P_{v|_{\partial\tilde{B}},\tilde{B}}$ in $\tilde{B}$ is in the family (and is harmonic in $\tilde{B}$). This follows from the Gluing lemma – see exercise 6. To show $U(0;f)$ is harmonic in D, it suffices to show harmonicity on any disk $\tilde{B}\subset D$. We return to this issue in the next section.

Subharmonic functions need not be twice-differentiable, let alone continuous. Thus we need a way of interpreting derivatives, in particular, the Laplacian, in a generalized sense. A *distribution* $\mathcal{L}$ in one real variable is a linear functional on the vector space $C_0^\infty(\mathbb{R})$ of test functions, i.e., C^∞ functions on $\mathbb{R}$ with compact support. Standard examples include, for any $\psi\in C(\mathbb{R})$, the distribution $\mathcal{L}_\psi$ of integration with respect to ψ:
$$\mathcal{L}_\psi(f):=\int f(x)\psi(x)dx;$$

and the distribution $\mathcal{L}(f) := f(0)$, known as the *delta function*: we often write $\delta_0(f) = f(0)$. More generally, for any $x \in \mathbb{R}$, $\delta_x(f) := f(x)$ is the *delta function at* x. These delta functions are examples of *positive* distributions: $\mathcal{L}$ is positive if $f \geq 0$ implies $\mathcal{L}(f) \geq 0$ for $f \in C_0^\infty(\mathbb{R})$. It turns out that *a positive distribution is a positive measure*; in particular, δ_x is represented by a point mass at the point x. If $\psi \in C(\mathbb{R})$ is a nonnegative function, then $\mathcal{L}_\psi$ is a positive distribution (and $\psi(x)dx$ is a positive measure).

We define the derivative $\mathcal{L}'$ of a distribution $\mathcal{L}$ by $\mathcal{L}'(f) := -\mathcal{L}(f')$. The reader may check that if $\mathcal{L} = \mathcal{L}_g$ for a C^1 function g, then $\mathcal{L}'_g = \mathcal{L}_{g'}$. We can also multiply a distribution by a smooth (C^∞) function: since, clearly, for $g, h \in C(\mathbb{R})$ and $f \in C_0(\mathbb{R})$ (continuous functions on $\mathbb{R}$ with compact support) we have

$$\int f(x)[g(x)h(x)]dx = \int [f(x)g(x)]h(x)dx,$$

we then define, for a distribution $\mathcal{L}$ and a smooth function g, the new distribution $g \cdot \mathcal{L}$ via

$$(g \cdot \mathcal{L})(f) := \mathcal{L}(gf).$$

Convergence of a sequence $\{\mathcal{L}^{(n)}\}$ of distributions is akin to, but easier than, weak-* convergence of a sequence of measures: $\mathcal{L}^{(n)} \to \mathcal{L}$ as distributions if $\mathcal{L}^{(n)}(\phi) \to \mathcal{L}(\phi)$ for all $\phi \in C_0^\infty(\mathbb{R})$. All these notions are easily extended to higher (real) dimensions; of particular interest to us is the case of $\mathbb{R}^2 = \mathbb{C}$. We include some optional exercises on distributions in Appendix B at the end of these notes.

Using some standard multivariate calculus, we prove a fundamental result on the Laplace operator in $\mathbb{R}^2 = \mathbb{C}$. Recall that a function $u : D \to \mathbb{R}$ is *locally integrable* on D if for each compact set $K \subset D$, $\int_K |u(z)|dm(z) < +\infty$.

Proposition 1.3. *$E(z) := \frac{1}{2\pi}\log|z|$ is a fundamental solution for Δ: we have $\Delta(\frac{1}{2\pi}\log|z|) = \delta_0$, the unit point mass at the origin, in the sense of distributions.*

Proof. To this end, fix $\phi \in C_0^\infty(D)$ where D is a neighborhood of the origin. We want to show that

$$\int_D \Delta\phi(z) \cdot E(z)dm(z) = \phi(0).$$

We make use of a standard multivariate calculus result, sometimes known as a *Green's identity*: let u, v be twice-differentiable functions defined in a neighborhood of the closure $\overline{\Omega}$ of a bounded, open set Ω with C^1-boundary. Then

$$\int_\Omega (u\Delta v - v\Delta u)dm = \int_{\partial\Omega} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})ds \tag{3}$$

where ds denotes arclength measure on $\partial\Omega$. Apply (3) to the functions ϕ, E in $D_\epsilon := \{z \in D : |z| > \epsilon\}$ to obtain

$$\int_{D_\epsilon} [\Delta\phi(z) \cdot E(z) - \Delta E(z) \cdot \phi(z)]dm(z) = \int_{D_\epsilon} \Delta\phi(z) \cdot E(z)dm(z)$$

$$= \int_{\partial D_\epsilon} [E\frac{\partial \phi}{\partial n} - \phi\frac{\partial E}{\partial n}]ds = -\int_{\partial B(0,\epsilon)} [E\frac{\partial \phi}{\partial n} - \phi\frac{\partial E}{\partial n}]ds.$$

The area integral tends to $\int_D \Delta\phi(z) \cdot E(z)dm(z)$ as $\epsilon \to 0$ since E is locally integrable. Since $\epsilon\log\epsilon \to 0$ as $\epsilon \to 0$,

$$\int_{\partial B(0,\epsilon)} E\frac{\partial \phi}{\partial n}ds \to 0$$

and

$$-\int_{\partial B(0,\epsilon)} \phi\frac{\partial E}{\partial n}ds = \frac{1}{2\pi}\int_0^{2\pi} \phi(\epsilon e^{i\theta})\frac{1}{\epsilon}\epsilon d\theta \to \phi(0).$$

□

We remark that (3) is the same as

$$\int_\Omega (udd^c v - vdd^c u) = \int_{\partial\Omega} (ud^c v - vd^c u)$$

which follows from Stokes theorem. Note that $d^c u = \frac{\partial u}{\partial n}ds$ (see also exercise 1).

Since the function $u(z) = \log|z|$ is locally integrable, it follows that given a positive measure μ of finite total mass and, say, compact support, one can form the convolution

$$V_\mu(z) := -p_\mu(z) := (u * \mu)(z) := \int_{\mathbb{C}} \log|z - \zeta|d\mu(\zeta).$$

This yields a shm function V_μ on $\mathbb{C}$; and since δ_0 acts as the identity under convolution (why?)

$$\Delta V_\mu = \Delta(u * \mu) = \Delta u * \mu = 2\pi\delta_0 * \mu = 2\pi\mu.$$

Note that V_μ is *harmonic* on $\mathbb{C} \setminus \operatorname{supp}\mu$. We call p_μ the *logarithmic potential function* of μ. The notation for the superharmonic function p_μ is standard; to emphasize the difference with the subharmonic function $-p_\mu$, we have introduced the notation V_μ. If μ is a probability measure, i.e., $\mu(\mathbb{C}) = 1$, then V_μ is in the class

$$L(\mathbb{C}) := \{u \text{ shm on } \mathbb{C},\ u(z) - \log|z| = 0(1),\ |z| \to \infty\}. \tag{4}$$

of global shm functions of at most logarithmic growth. We will see the importance of this collection of shm functions, and its *plurisubharmonic* generalization, throughout this course. We next give an important continuity property of logarithmc potentials.

Proposition 1.4. *Let μ be a positive measure of finite total mass and compact support K and let*

$$V_\mu(z) := \int_{\mathbb{C}} \log|z-\zeta| d\mu(\zeta).$$

For $z_0 \in K$,

$$\liminf_{z\to z_0} V_\mu(z) = \liminf_{z\to z_0,\ z\in K} V_\mu(z).$$

In particular, if $V_\mu|_K$ is continuous, then V_μ is continuous on $\mathbb{C}$; and if $V_\mu \geq M$ on K, then $V_\mu \geq M$ on $\mathbb{C}$.

Proof. First, if $V_\mu(z_0) = -\infty$ the result is clear by usc of V_μ. If $V_\mu(z_0) > -\infty$ then μ puts no mass on the point $\{z_0\}$ (why?); hence, given $\epsilon > 0$ we can find $r > 0$ with $\mu(B(z_0, r)) < \epsilon$. Now given $z \in \mathbb{C} \setminus K$, take a point $z' \in K$ such that $|z - z'| = \min_{w\in K} |z - w|$. Then for any $w \in K$ we have

$$\frac{|z'-w|}{|z-w|} \leq \frac{|z'-z| + |z-w|}{|z-w|} \leq 2$$

and

$$\begin{aligned} V_\mu(z) &= V_\mu(z') - \int_K \log\frac{|z'-w|}{|z-w|} d\mu(w) \\ &\geq V_\mu(z') - \epsilon\log 2 - \int_{K\setminus B(z_0,r)} \log\frac{|z'-w|}{|z-w|} d\mu(w). \end{aligned}$$

Now as $z \to z_0$ clearly $z' \to z_0$ so that

$$\liminf_{z\to z_0} V_\mu(z) \geq \liminf_{z'\to z_0,\ z'\in K} V_\mu(z') - \epsilon\log 2.$$

The last statement is left for the exercises. □

Two standard examples of functions V_μ are the following:

1. If $\mu = \delta_0$, then $V_\mu(z) = \log|z|$.
2. If $\mu = \frac{1}{2\pi}d\theta$ on $|z| = 1$, then $V_\mu(z) = \log^+|z| := \max[\log|z|, 0]$.

A useful result, which generalizes to $\mathbb{C}^N$ for $N > 1$, is the *comparison principle*.

Proposition 1.5. *Let u, v be shm and locally bounded in a bounded domain $D \subset \mathbb{C}$. Suppose* $\liminf_{z\to\zeta}[u(z) - v(z)] \geq 0$ *for all $\zeta \in \partial D$. Then*

$$\int_{\{u<v\}} dd^c v \leq \int_{\{u<v\}} dd^c u. \tag{5}$$

Proof. We verify the result in the case where $u, v \in C^2(D) \cap C^1(\overline{D})$ and $u = v$ on ∂D. In this case, we may assume $D = \{u < v\}$. Then $d^c(u - v) = \frac{\partial(u-v)}{\partial n} ds$ and $\frac{\partial(u-v)}{\partial n} \geq 0$ on ∂D (see exercise 1 below). Stokes' theorem gives

$$\int_{\{u<v\}} dd^c(u-v) = \int_D dd^c(u-v) = \int_{\partial D} d^c(u-v) \geq 0.$$

The general case requires some approximation. □

Note this result says that *harmonic* functions have "minimal" Laplacian (indeed, 0!) among shm functions. In section 7, we discuss an analogue of this in $\mathbb{C}^N$, $N > 1$ where "dd^c" is replaced by the *complex Monge-Ampère operator*, "$(dd^c\,\cdot)^N$." Using Proposition 1.5 we can prove a type of domination principle for subharmonic functions.

Proposition 1.6. *Let u, v be shm and locally bounded in a bounded domain $D \subset \mathbb{C}$. Suppose* $\liminf_{z\to\zeta}[v(z) - u(z)] \geq 0$ *for all $\zeta \in \partial D$ and assume that*

$$dd^c u \geq dd^c v \text{ in } D.$$

Then $v \geq u$ in D.

Proof. Again, we verify the result in the case where $u, v \in C^2(D) \cap C^1(\overline{D})$ and $v \geq u$ on ∂D. Assume not, i.e., suppose $\{z \in D : u(z) > v(z)\} \neq \emptyset$. For $\epsilon, \delta > 0$ small, we have

$$u(z) + \epsilon|z|^2 - \delta < u(z) \text{ in } \overline{D},$$

and we can choose such ϵ, δ such that

$$S := \{z \in D : u(z) + \epsilon|z|^2 - \delta > v(z)\} \neq \emptyset.$$

In our setting, S is open; in the general case, S still has positive Lebesgue measure by Corollary 1.2. By Proposition 1.5

$$\int_S dd^c(u + \epsilon|z|^2 - \delta) \leq \int_S dd^c v.$$

By hypothesis, $\int_S dd^c v \leq \int_S dd^c u$. On the other hand, since S has positive Lebesgue measure, $\int_S dd^c|z|^2 > 0$ and

$$\int_S dd^c(u + \epsilon|z|^2 - \delta) = \int_S dd^c u + \epsilon \int_S dd^c|z|^2 > \int_S dd^c u,$$

a contradiction. □

Exercises.

1. Let $\rho(z) = |z|^2 - 1$. Show that, on the unit circle $T = \{z = e^{i\theta} : \theta \in [0, 2\pi]\}$, $d^c\rho = 2d\theta$ and, writing $d^c\rho = adx + bdy$, show that $a = -2y$ and $b = 2x$. In particular, the coefficients $< a, b > = < -2y, 2x >$ give a tangent vector to T at each point. (More generally, if $D = \{z \in \mathbb{C} : \rho(z) < 0\}$ is a bounded domain with C^1 boundary where ρ is a C^1 function on a neighborhood of $\overline{D}$ and $\nabla\rho \neq 0$ on ∂D, then the coefficient functions of $d^c\rho$ at $p \in \partial D$ define a tangent vector to ∂D at p and $d^c\rho = \frac{\partial\rho}{\partial n} ds$ with $\frac{\partial\rho}{\partial n} \geq 0$ on ∂D).
2. Verify that if $u \in C^2(D)$ then $dd^c u = \Delta u dx \wedge dy$ in D.
3. Suppose $u : D \to \mathbb{R}$ is usc; i.e., for each $a \in \mathbb{R}$, the set $\{z \in D : u(z) < a\}$ is open. Show that
   (a) For each $z \in D$, $\limsup_{\zeta \to z} u(\zeta) \leq u(z)$ (this is equivalent to usc of u in D).
   (b) For each $K \subset D$ compact, $M := \sup_{z \in K} u(z) < \infty$ and there exists $z_0 \in K$ with $u(z_0) = M$.
4. Use part (a) of the previous exercise and the subaveraging property to show that if u is shm in D, then for each $z \in D$, $\limsup_{\zeta \to z} u(\zeta) = u(z)$.
5. An exercise on convolutions on $\mathbb{R}$:
   (a) Let $f(x) = e^{-x^2}$ and $g(x) = e^{-2x^2}$, Compute $f * g$. (*Hint: You may use the fact that* $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.)
   (b) More generally, let $f_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ for $t > 0$, Prove that this family of functions acts as a *one-parameter subgroup* in the sense that, for $s, t > 0$
   $$f_t * f_s = f_{t+s}.$$
6. *Gluing shm functions*. Let u, v be shm in open sets U, V where $U \subset V$ and assume that $\limsup_{\zeta \to z} u(\zeta) \leq v(z)$ for $z \in V \cap \partial U$. Show that the function w defined to be $w = \max(u, v)$ in U and $w = v$ in $V \setminus U$ is shm in V.
7. In this exercise, you will show that a shm function $u \not\equiv -\infty$ on a domain D is locally integrable on D.
   (a) Verify that it suffices to show for all $z \in D$ there exists $r = r(z) > 0$ with $\int_{B(z,r)} |u(\zeta)| dm(\zeta) < +\infty$.
   (b) Let P denote the set of points $z \in D$ with this property. Show P is both open and closed.
   (c) Show that $u = -\infty$ on $D \setminus P$ to conclude the proof (why?).
8. Prove that if $u : D \to \mathbb{R}$ is shm on the domain D, then
   $$P := \{z \in D : u(z) = -\infty\}$$
   is a $G_\delta-$set, i.e., a countable intersection of open sets.
9. Use the fact mentioned that for a function $u \not\equiv c$ shm in a ball $B(0, R)$, the mean value over circles,
   $$r \to M_u(r) := \frac{1}{2\pi} \int_o^{2\pi} u(r(e^{i\theta}) d\theta$$
   is a convex increasing function of r, to show that if u is shm in $\mathbb{C}$ and $u(z) = o(\log|z|)$ as $|z| \to \infty$, then u must be constant. Thus functions in the class $L(\mathbb{C})$ are of "minimal" growth.
10. Show that if $u(z) = u(|z|)$ is a radial function which is harmonic in an annulus $A = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ where $r_1 > 0$ and $r_2 \leq +\infty$, then u is of the form
    $$u(z) = a + b \log|z|$$
    for some $a, b \in \mathbb{R}$. (*Hint: Write* Δu *in polar coordinates*).
11. Verify the claims in the last sentence of Proposition 1.4.
12. (*Optional*). For those unfamiliar with regularizing kernels, we mention and leave as exercises the following general results for $u : D \to \mathbb{R}$.
    (a) If $u \in C(D)$, on any compact subset $K \subset D$ we have $u_j = u * \chi_{1/j} \to u$ uniformly on K as $j \to \infty$.
    (b) If $u \in L^p_{loc}(D)$ with $1 \leq p < \infty$, we have $u_j \to u$ in $L^p_{loc}(D)$.

2 Logarithmic energy, transfinite diameter and applications.

Now let $K \subset \mathbb{C}$ be compact and let $\mathcal{M}(K)$ denote the convex set of probability measures on K. For $\mu \in \mathcal{M}(K)$ define the *logarithmic energy*

$$I(\mu) := \int_K \int_K \log \frac{1}{|z-\zeta|} d\mu(z) d\mu(\zeta) = \int_K p_\mu(z) d\mu(z).$$

Consider the *energy minimization problem*: minimize $I(\mu)$ over all $\mu \in \mathcal{M}(K)$. It turns out that either $\inf_{\mu \in \mathcal{M}(K)} I(\mu) =: I(\mu_K) < +\infty$ for a unique $\mu_K \in \mathcal{M}(K)$ or else $I(\mu) = +\infty$ for all $\mu \in \mathcal{M}(K)$.

We remark that you've likely seen a (real) three-dimensional version of an analogous problem in *Newtonian* potential theory: thinking in terms of electrostatics, given a compact set K (conductor) in $\mathbb{R}^3$, we want to minimize the Newtonial potential energy

$$N(\mu) := \int_K \int_K \frac{1}{|\mathbf{x}-\mathbf{y}|} d\mu(\mathbf{x}) d\mu(\mathbf{y})$$

over all probability measures (positive charges of total charge one) on K. The difference between the formulas for $I(\mu)$ in $\mathbb{C} = \mathbb{R}^2$ and $N(\mu)$ in $\mathbb{R}^3$ is explained by the fact that whereas $\frac{1}{2\pi} \log|z|$ is a fundamental solution of the Laplacian Δ in two (real) dimensions as we saw in Proposition 1.3, up to a dimensional constant, $E(\mathbf{x}) = \frac{1}{|\mathbf{x}|}$ is a fundamental solution of the Laplacian Δ in three (real) dimensions.

The existence of an energy-minimizing measure $\mu_K \in \mathcal{M}(K)$ is standard: let $M := \inf_{\mu \in \mathcal{M}(K)} I(\mu)$ and take a sequence $\{\mu_n\} \in \mathcal{M}(K)$ with $\lim_{n\to\infty} I(\mu_n) = M$. There exists a subsequence, which we still label as $\{\mu_n\}$ for simplicity, which converges weak-* to a measure $\mu \in \mathcal{M}(K)$ (why?) and thus by definition, $I(\mu) \geq M$. We claim that

$$\liminf_{n\to\infty} I(\mu_n) \geq I(\mu). \tag{6}$$

Given (6), we have $I(\mu) \leq \liminf_{n\to\infty} I(\mu_n) = M$ and hence $I(\mu) = M$. The proof of (6), which is left to the exercises, follows from weak-* convergence of $\mu_n \times \mu_n$ to $\mu \times \mu$ and lowersemicontinuity of $z \to \log \frac{1}{|z-\zeta|}$.

The uniqueness follows, e.g., from a convexity property of the function $\mu \to I(\mu)$. We state without proof the key element (cf., [26] Lemma I.1.8).

Proposition 2.1. *For μ a signed measure with compact support and total mass 0; i.e., $\int_{\mathbb{C}} d\mu = 0$, $I(\mu) \geq 0$ with equality if and only if μ is the zero measure.*

Corollary 2.2. *For a compact set K, the functional $\mu \to I(\mu)$ is convex on $\mathcal{M}(K)$. Hence if $\inf_{\mu \in \mathcal{M}(K)} I(\mu) := M < +\infty$ and if $\mu_1, \mu_2 \in \mathcal{M}(K)$ satisfy $I(\mu_1) = I(\mu_2) = M$, then $\mu_1 = \mu_2$.*

Proof. It suffices to show that

$$I(\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2) \leq \frac{1}{2} I(\mu_1) + \frac{1}{2} I(\mu_2) \tag{7}$$

(midpoint convexity) since $\mu \to I(\mu)$ is uppersemicontinuous (exercise). We introduce the temporary notation

$$< \mu, \nu > = \int_{\mathbb{C}} p_\mu d\nu = \int_{\mathbb{C}} p_\nu d\mu.$$

Note that for any $c \in \mathbb{R}$,

$$I(c\mu) = c^2 I(\mu). \tag{8}$$

Now

$$I(\mu_1 + \mu_2) = I(\mu_1) + I(\mu_2) + 2 < \mu_1, \mu_2 > \tag{9}$$

and

$$I(\mu_1 - \mu_2) = I(\mu_1) + I(\mu_2) - 2 < \mu_1, \mu_2 > \geq 0 \tag{10}$$

by the previous proposition. Thus

$$2 < \mu_1, \mu_2 > \leq I(\mu_1) + I(\mu_2);$$

plugging this into (9) gives

$$I(\mu_1 + \mu_2) \leq 2[I(\mu_1) + I(\mu_2)].$$

Replacing μ_1, μ_2 by $\mu_1/2, \mu_2/2$ and using (8) gives (7).

For the uniqueness of the energy minimizing measure, if $I(\mu_1) = I(\mu_2) = M$, by (7), $I(\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2) \leq M$ and hence, since $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 \in \mathcal{M}(K)$, we have $I(\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2) = M$. From (9), (8) and (10) we have

$$I(\mu_1 - \mu_2) = 2[I(\mu_1) + I(\mu_2)] - I(\mu_1 + \mu_2) = 0.$$

But $I(\mu_1 - \mu_2) \geq 0$ from (10) and the result follows from Proposition 2.1. □

We will give a characterization of the energy-minimizing measure μ_K for compact sets K with $\inf_{\mu\in\mathcal{M}(K)} I(\mu) < +\infty$ in Theorem 2.6. First, we show that the energy minimization problem is related to the following discretized version: for each $n = 1, 2, \ldots$

$$\delta_n(K) := \max_{z_0,\ldots,z_n\in K} \prod_{j<k} |z_j - z_k|^{1/\binom{n+1}{2}}$$

is called the $n-th$ order diameter of K. With this notation, $\delta_1(K) = \max_{z_0,z_1\in K} |z_0 - z_1|$ is the "ordinary" diameter of K. Note that

$$VDM(z_0,\ldots,z_n) = \det[z_i^j]_{i,j=0,1,\ldots,n} = \prod_{j<k}(z_j - z_k)$$

$$= \det \begin{bmatrix} 1 & z_0 & \ldots & z_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & \ldots & z_n^n \end{bmatrix}$$

is a classical Vandermonde determinant; the basis monomials $1, z, \ldots, z^n$ for the space of polynomials of degree at most n are evaluated at the points $z_0, \ldots, z_n$.

If, for example, $\lambda_0, \lambda_1, \lambda_2 \in K$ are points which achieve $\delta_2(K)$, we have

$$[\delta_2(K)]^3 = |\lambda_0 - \lambda_1| \cdot |\lambda_1 - \lambda_2| \cdot |\lambda_0 - \lambda_2| \le \delta_1(K)^3$$

so that $\delta_2(K) \le \delta_1(K)$. More generally, the sequence of numbers $\{\delta_n(K)\}$ is decreasing (exercise 7) and hence the limit

$$\lim_{n\to\infty} \big[\max_{\lambda_i\in K} |VDM(\lambda_0,\ldots,\lambda_n)|\big]^{1/\binom{n+1}{2}} := \delta(K) \tag{11}$$

exists and is called the transfinite diameter of K. Points $\lambda_0, \ldots, \lambda_n \in K$ for which

$$|VDM(\lambda_0,\ldots,\lambda_n)| = |\det \begin{bmatrix} 1 & \lambda_0 & \ldots & \lambda_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \ldots & \lambda_n^n \end{bmatrix} |$$

is maximal are called *Fekete points of order n*. The quantity $\delta(K)$ in (11) coincides with $e^{-I(\mu_K)}$ when $\delta(K) > 0$.

Proposition 2.3. *For $K \subset \mathbb{C}$ compact with $\delta(K) > 0$,*

$$e^{-I(\mu_K)} = \delta(K).$$

Proof. To show

$$e^{-I(\mu_K)} \le \delta(K), \tag{12}$$

we begin by forming the function

$$F_n(z_0,\ldots,z_n) := \sum_{0\le i<j\le n} \log \frac{1}{|z_i - z_j|}$$

on K^{n+1} and we observe that for Fekete points $\lambda_0, \ldots, \lambda_n$ of order n for K,

$$F_n(\lambda_0,\ldots,\lambda_n) = \binom{n+1}{2} \log \frac{1}{\delta_n(K)} = \min_{z_0,\ldots,z_n\in K} F_n(z_0,\ldots,z_n).$$

Thus we have

$$\binom{n+1}{2} I(\mu_K) = \int_K \cdots \int_K F_n(z_0,\ldots,z_n) d\mu_K(z_0) \cdots d\mu_K(z_n)$$

$$\ge \binom{n+1}{2} \log \frac{1}{\delta_n(K)}$$

since μ_K is a probability measure. This gives (12).

For the reverse inequality, let μ be any weak-* limit of the sequence of Fekete measures

$$\mu_n := \frac{1}{n+1} \sum_{j=0}^n \delta_{\lambda_j}$$

(question: why does such a limit exist?). Then $\mu \in \mathcal{M}(K)$ (why?) and

$$I(\mu) = \int_K \int_K \log \frac{1}{|z-\zeta|} d\mu(z) d\mu(\zeta)$$

$$= \lim_{M\to\infty} \int_K \int_K \min[M, \log \frac{1}{|z-\zeta|}] d\mu(z) d\mu(\zeta)$$

$$= \lim_{M\to\infty}\lim_{n\to\infty}\int_K\int_K \min[M,\log\frac{1}{|z-\zeta|}]d\mu_n(z)d\mu_n(\zeta)$$

$$\leq \lim_{M\to\infty}\lim_{n\to\infty}\Big(\frac{2}{(n+1)^2}\binom{n+1}{2}\log\frac{1}{\delta_n(K)}+\frac{M}{n+1}\Big)=\log\frac{1}{\delta(K)}.$$

Thus from (12) we have shown that

$$I(\mu)\leq \log\frac{1}{\delta(K)}\leq I(\mu_K).$$

But $I(\mu_K)=\inf_{\nu\in\mathcal{M}(K)}I(\nu)$ and the proposition is proved. □

As an example, for the unit circle $T=\{z:|z|=1\}$, clearly the $(n+1)-$st roots of unity $1,\omega:=e^{2\pi i/(n+1)},\omega^2,...,\omega^n$ or any rotation of these points forms a set of Fekete points of order n; and the weak-* limit of these Fekete measures is normalized arclength $d\mu_T:=\frac{1}{2\pi}d\theta$. Note that the same conclusions hold for the closed unit disk $\overline{D}:=\{z:|z|\leq 1\}$. Indeed, Fekete points for a compact set K always lie on the *outer boundary* of K; i.e., on the boundary of the unbounded component of $\mathbb{C}\setminus K$ (why?).

Note as a consequence of the uniqueness of the energy minimizing measure μ_K, we have proved that if $\delta(K)>0$, any sequence of Fekete measures $\{\mu_n\}$ converges weak-* to μ_K (see also Proposition 4.8). Thus the support of μ_K is in the outer boundary of K. It turns out that

$$\mu_K=\frac{1}{2\pi}\Delta V_K^*=\frac{1}{2\pi}dd^cV_K^* \text{ where} \tag{13}$$

$$V_K(z)=\sup\{u(z):u\in L(\mathbb{C}),\ u\leq 0 \text{ on } K\}. \tag{14}$$

and $V_K^*(z):=\limsup_{\zeta\to z}V_K(\zeta)\in L^+(\mathbb{C})$. Recall that

$$L(\mathbb{C})=\{u \text{ shm on } \mathbb{C},\ u(z)-\log|z|=0(1),\ |z|\to\infty\};$$

here, the subclass

$$L^+(\mathbb{C}):=\{u\in L(\mathbb{C}):u(z)\geq \log^+|z|+C\}$$

where $C=C(u)$. Clearly we can replace $\log^+|z|$ by $\frac{1}{2}\log(1+|z|^2)$. We discuss this "upper envelope" in the next section; and we will see in Section 4 that

$$V_K(z)=\sup\{\frac{1}{deg(p)}\log|p(z)|:||p||_K:=\sup_K|p|\leq 1\}. \tag{15}$$

For the unit circle T we have

$$V_T(z)=\max[\log|z|,0] \text{ and } \mu_T=\frac{1}{2\pi}d\theta.$$

Note that $V_T=V_T^*$ (why?).

Given a set $E\subset\mathbb{C}$, we say the set E is a *polar* set if $I(\mu)=+\infty$ for every finite Borel measure μ with compact support in E. It turns out this is equivalent to the existence of a function u shm, $u\not\equiv-\infty$, with $E\subset\{u(z)=-\infty\}$. You showed in exercise 8 of section 1 that the (polar) set of points where a shm function takes the value $-\infty$ is a G_δ set; a theorem of Deny shows a type of converse: given a polar set P which is a $G_\delta-$set, there exists a shm function u in $\mathbb{C}$ with $P=\{z\in\mathbb{C}:u(z)=-\infty\}$.

Using the second (equivalent) definition of polar set, from the fact that $u(z)=\log|f(z)|$ is shm if f is holomorphic it follows that any discrete set in $\mathbb{C}$ is polar. We can give a direct proof that any bounded countable set is polar, as follows: let $S=\{a_j\}\subset D$ where D is a disk. Let $M_j:=\max_{z\in\overline{D}}\log|z-a_j|$. Fix any point $p\in D\setminus S$, and choose $\epsilon_j>0$ and sufficiently small so that $\sum_j\epsilon_j<+\infty$ and

$$\sum_j\epsilon_j[\log|p-a_j|-M_j]>-\infty.$$

Then

$$u(z):=\sum_j\epsilon_j[\log|z-a_j|-M_j]$$

is shm in D (why?), $u(a_j)=-\infty$ for all j, and $u\not\equiv-\infty$ since $u(p)>-\infty$.

A compact set K is polar precisely when $I(\mu)=+\infty$ for all $\mu\in\mathcal{M}(K)$. In this case, $V_K^*\equiv+\infty$; if K is not polar, we have $V_K^*\in L^+(\mathbb{C})$ (see Proposition 3.2). If a property **P** holds on a set S except perhaps for a polar subset of S, we say **P** holds q.e. (quasi-everywhere) on S. We will indicate in the next section the importance of detecting polarity of a set.

Proposition 2.4. *If μ is a finite Borel measure with compact support and $I(\mu)<\infty$, then $\mu(E)=0$ for each Borel polar set E. In particular, every Borel polar set has Lebesgue measure zero.*

Proof. If E is a Borel set with $\mu(E)>0$, we show E is not polar. To this end, take $K\subset E$ compact with $\mu(K)>0$.The measure $\tilde{\mu}:=\mu|_K$ is a finite Borel measure with compact support. Setting d :=diam(suppμ), we have

$$I(\tilde{\mu})=\int_K\int_K\log\frac{d}{|z-\zeta|}d\mu(z)d\mu(\zeta)-\mu(K)^2\log d$$

$$\leq\int_{\mathbb{C}}\int_{\mathbb{C}}\log\frac{d}{|z-\zeta|}d\mu(z)d\mu(\zeta)-\mu(K)^2\log d$$

$$= I(\mu) + \mu(\mathbb{C})^2 \log d - \mu(K)^2 \log d < \infty.$$

For the second statement, it thus suffices to show that for any $r > 0$, $d\mu := dm|_{B(0,r)}$ satisfies $I(\mu) < \infty$ where $dm =$ Lebesgue measure (why?). Now for $z \in B(0,r)$,

$$p_\mu(z) = \int_{B(0,r)} \log \frac{2r}{|z-\zeta|} dm(\zeta) - \pi r^2 \log(2r)$$
$$\leq \int_0^{2\pi} \int_0^{2r} (\log \frac{2r}{\rho}) \rho d\rho d\theta - \pi r^2 \log(2r) \text{ (why?)}$$
$$= 2\pi r^2 - \pi r^2 \log(2r).$$

Hence $I(\mu) \leq \big(2\pi r^2 - \pi r^2 \log(2r)\big) \cdot \pi r^2 < \infty.$ □

Remark 1. Note that the fact that polar sets E have Lebesgue measure zero follows immediately from Corollary 1.2 and the definition of polar as the existence of a shm function $u \not\equiv -\infty$ with $E \subset \{u(z) = -\infty\}$.

Corollary 2.5. *A countable union of Borel polar sets is polar.*

We come to the characterization of the equilibrium measure μ_K for a nonpolar compact set K. This is one of the main results in potential theory and is known as Frostman's theorem.

Theorem 2.6. **[Frostman]** *Let $K \subset \mathbb{C}$ with $I(\mu_K) < +\infty$. Then*

1. *$p_{\mu_K}(z) \leq I(\mu_K)$ for all $z \in \mathbb{C}$; and*
2. *$p_{\mu_K}(z) = I(\mu_K)$ q.e. on K.*

Proof. For each $n = 1, 2, \ldots$ let

$$K_n := \{z \in K : p_{\mu_K}(z) \leq I(\mu_K) - 1/n\} \text{ and}$$
$$L_n := \{z \in \text{supp}\mu_K : p_{\mu_K}(z) > I(\mu_K) + 1/n\}.$$

We will verify two items:

1. K_n is polar for each $n = 1, 2, \ldots$ and
2. $L_n = \emptyset$ for each $n = 1, 2, \ldots$

Given these two items, the second one implies that $p_{\mu_K}(z) \leq I(\mu_K)$ on suppμ_K and hence on $\mathbb{C}$ by Proposition 1.4. This is (1) of the theorem. Next, setting $E := \cup_{n=1}^\infty K_n$, the first item and Corollary 2.5 imply that E is a polar set; moreover we have $p_{\mu_K}(z) = I(\mu_K)$ on $K \setminus E$.

We prove item (1) by contradiction. Thus we suppose K_n is not polar for some n so we can find $\mu \in \mathcal{M}(K_n)$ with $I(\mu) < +\infty$. We have $I(\mu_K) = \int_K p_{\mu_K} d\mu_K$ so that we can find $z_0 \in$ supp(μ_K) with $p_{\mu_K}(z_0) \geq I(\mu_K)$; by lsc of p_{μ_K}, there exists $r > 0$ with $p_{\mu_K} > I(\mu_K) - \frac{1}{2n}$ on $\overline{B}(z_0, r)$. Thus $K_n \cap \overline{B}(z_0, r) = \emptyset$; also, we note that $a := \mu_K(\overline{B}(z_0, r)) > 0$ since $z_0 \in$ supp(μ_K). We next define a *signed* measure σ on K by setting

$$\sigma = \mu \text{ on } K_n; \ \sigma = -\mu_K/a \text{ on } \overline{B}(z_0, r).$$

Since $I(\mu)$, $I(\mu_K) < +\infty$, clearly $I(|\sigma|) < +\infty$. For each $t \in (0, a)$, the measure $\mu_t := \mu_K + t\sigma$ is *positive* and, indeed, $\mu_t \in \mathcal{M}(K)$ for such t (why?). We estimate the difference $I(\mu_t) - I(\mu_K)$:

$$I(\mu_t) - I(\mu_K) = I(\mu_K + t\sigma) - I(\mu_K)$$
$$= 2t \int_K \int_K \log \frac{1}{|z-\zeta|} d\mu_K(\zeta) d\sigma(z) + t^2 I(\sigma)$$
$$= 2t \int_K p_{\mu_K}(z) d\sigma(z) + 0(t^2)$$
$$= 2t \Big(\int_{K_n} p_{\mu_K}(z) d\mu(z) - \frac{1}{a} \int_{\overline{B}(z_0, r)} p_{\mu_K}(z) d\mu_K(z) + 0(t) \Big)$$
$$\leq 2t \big([I(\mu_K) - 1/n] - [I(\mu_K) - \frac{1}{2n}] + 0(t) \big).$$

Thus $I(\mu_t) < I(\mu_K)$ for t sufficiently small, contradicting the minimality of $I(\mu_K)$.

We prove item (2) by contradiction. Suppose $L_n \neq \emptyset$ for some n and take $z_0 \in L_n$; hence $p_{\mu_K}(z_0) > I(\mu_K) + 1/n$. By lsc of p_{μ_K}, there exists $r > 0$ with $p_{\mu_K}(z) > I(\mu_K) + 1/n$ on $\overline{B}(z_0, r)$. Also, since $z_0 \in$ suppμ_K, $m := \mu_K(\overline{B}(z_0, r)) > 0$. By item (1) and Proposition 2.4, $\mu_K(K_n) = 0$ for each n so that $p_{\mu_K} \geq I(\mu_K)$ μ_K−a.e. on K. Thus

$$I(\mu_K) = \int_K p_{\mu_K} d\mu_K = \int_{\overline{B}(z_0, r)} p_{\mu_K} d\mu_K + \int_{K \setminus \overline{B}(z_0, r)} p_{\mu_K} d\mu_K$$
$$\geq (I(\mu_K) + 1/n)m + I(\mu_K)(1 - m) > I(\mu_K)$$

which is a contradiction.

□

There is an important result that will be useful in the weighted setting and which will generalize to the several complex variable setting. We will refer to it as a *global domination principle*; we will use it in the next section together with Frostman's theorem to relate V_K^* with p_{μ_K}.

Proposition 2.7. *Let $u \in L(\mathbb{C})$ and $v \in L^+(\mathbb{C})$ and suppose $u \leq v$ a.e.-$dd^c v$. Then $u \leq v$ on $\mathbb{C}$.*

Proof. We give the proof in case u, v are continuous and indicate modifications in the general case. Suppose the result is false; i.e., there exists $z_0 \in \mathbb{C}$ with $u(z_0) > v(z_0)$. Since $v \in L^+(\mathbb{C})$, by adding a constant to u, v we may assume $v(z) \geq \frac{1}{2}\log(1+|z|^2)$ in $\mathbb{C}$. By exercise 6 below, $\Delta\left[\frac{1}{2}\log(1+|z|^2)\right] > 0$ on $\mathbb{C}$. Fix $\delta, \ \epsilon > 0$ with $\delta < \epsilon/2$ in such a way that the set

$$S := \{z \in \mathbb{C} : u(z) + \frac{\delta}{2}\log(1+|z|^2) > (1+\epsilon)v(z)\}$$

contains z_0. In our setting, S is open; in the general case, by Corollary 1.2, S has positive Lebesgue measure. Moreover, since $\delta < \epsilon$ and $v \geq \frac{1}{2}\log(1+|z|^2)$, S is bounded. By Proposition 1.5, we conclude that

$$\int_S dd^c[u(z) + \frac{\delta}{2}\log(1+|z|^2)] \leq \int_S dd^c(1+\epsilon)v(z).$$

But $\int_S dd^c \frac{\delta}{2}\log(1+|z|^2) > 0$ since S has positive Lebesgue measure, so

$$(1+\epsilon)\int_S dd^c v > 0.$$

By hypothesis, for almost all points in $\mathrm{supp}(dd^c v) \cap S$, we have

$$(1+\epsilon)v(z) \leq u(z) + \frac{\delta}{2}\log(1+|z|^2) \leq v(z) + \frac{\delta}{2}\log(1+|z|^2),$$

i.e., $v(z) \leq \frac{1}{4}\log(1+|z|^2)$ since $\delta < \epsilon/2$. This contradicts the normalization $v \geq \frac{1}{2}\log(1+|z|^2)$. □

Remark 2. Note some hypothesis on v stronger than $v \in L(\mathbb{C})$ is necessary, since, e.g., $u(z) = \log|z|$ and $v(z) = \log|z| + c$ satisfy the hypothesis but not the conclusion if $c < 0$. However, one can show that if $v \in L^+(\mathbb{C})$ and $\nu = dd^c v$, then $I(\nu) < \infty$; and one can weaken the hypothesis $v \in L^+(\mathbb{C})$ to $v \in L(\mathbb{C})$ with $I(\nu) < \infty$ (cf., [26] Theorem 3.2 in Chapter II).

As an application of Frostman's theorem, we discuss a classical result of Brolin from complex dynamics (cf., Theorem 6.5.8 of [25]). The set-up begins with a polynomial $p(z)$ of degree $d > 1$ in $\mathbb{C}$. Writing $p^n = p \circ \cdots \circ p$ for the $n-$th iterate of p, the *Fatou set* or *attracting basin of* ∞ is the set

$$F := \{z \in \mathbb{C} : p^n(z) \to \infty \text{ as } n \to \infty\}$$

and the *Julia set* J is the boundary of F. Two standard examples are $p(z) = z^2$ (or $p(z) = z^d$ for any $d > 1$) in which case $F = \{z : |z| > 1\}$ and $J = \{z : |z| = 1\}$; and $p(z) = z^2 - 2$, in which case $F = \{z : z \notin [-2, 2]\}$ and $J = [-2, 2]$.

Note that in the case of $p(z) = z^d$ where $J = \{z : |z| = 1\}$ and $d\mu_J = \frac{1}{2\pi}d\theta$, we have $\mathrm{supp}\mu_J = J$ and $I(\mu_J) = 0$. More generally, for a *monic* polynomial $p(z) = z^d + \cdots$, the Julia set J is nonpolar; $\mathrm{supp}\mu_J = J$; and $I(\mu_J) = 0$. We refer the reader to [25], section 6.5 for verification of these facts. We show that we can recover μ_J via a pre-image process.

Theorem 2.8. **[Brolin]** *Fix $w \in J$ and define the sequence of discrete probability measures $\{\mu_n\}$ on J via*

$$\mu_n = \frac{1}{d^n}\sum_{p^n(z_j)=w} \delta_{z_j}.$$

Then $\mu_n \to \mu_J$ weak-.*

Proof. Note that $w \in J$ implies that $z_j \in J$ if $p^n(z_j) = w$ (exercise). Let

$$V_{\mu_n}(z) = \int_J \log|z - \zeta| d\mu_n(\zeta).$$

Writing $p^n(z) - w = \prod_{j=1}^{d^n}(z - z_j)$, we have

$$V_{\mu_n}(z) = \frac{1}{d^n}\sum_{j=1}^{d^n}\log|z - z_j| = \frac{1}{d^n}\log|p^n(z) - w|.$$

For $z \in J$, the points $\{p^n(z)\}$ and hence $\{p^n(z) - w\}$ remain bounded so we conclude that

$$\limsup_{n\to\infty} V_{\mu_n}(z) \leq 0 \text{ for } z \in J. \tag{16}$$

Now if $\{\mu_{n_j}\}$ is a subsequence of $\{\mu_n\}$, since $V_{\mu_J} = -p_{\mu_J} \geq I(\mu_J) = 0$ by Frostman's theorem, from Fatou's lemma and Fubini's theorem we have

$$\int_J [\limsup_{j\to\infty} V_{\mu_{n_j}}(z)]d\mu_J(z) \geq \limsup_{j\to\infty} \int_J V_{\mu_{n_j}}(z)d\mu_J(z)$$

$$= \limsup_{j\to\infty} \int_J V_{\mu_J}(z)d\mu_{n_j}(z) \geq 0.$$

From (16), we conclude that

$$\limsup_{j\to\infty} V_{\mu_{n_j}} = 0 \ \mu_J - a.e. \text{ on } J. \tag{17}$$

Recall that supp$\mu_J = J$. We use this fact to complete the proof by contradiction: suppose $\mu_n \not\to \mu_J$ weak-*. Then there exists a subsequence $\{\mu_{n_j}\}$ of $\{\mu_n\}$, a function $\phi \in C(J)$, and $\epsilon > 0$ with

$$|\int_J \phi d\mu_{n_j} - \int_J \phi d\mu_J| \geq \epsilon \tag{18}$$

for all j. Take a further subsequence, which we still denote by $\{\mu_{n_j}\}$, which converges weak-* to a measure $\mu \in \mathcal{M}(J)$. An argument similar to that used to prove (6) shows that

$$\limsup_{j\to\infty} V_{\mu_{n_j}}(z) \leq V_\mu(z) \text{ for } z \in \mathbb{C}.$$

Then (17) shows that $V_\mu(z) \geq 0$ μ_J-a.e on J. Since supp$\mu_J = J$ and V_μ is usc, we have $V_\mu(z) \geq 0$ on J. Thus

$$I(\mu) = \int_J [-V_\mu(z)]d\mu(z) \leq 0 = I(\mu_J).$$

By uniqueness of the energy minimizing measure, $\mu = \mu_J$. This contradicts (18). $\square$

Let $\mathcal{P}_n$ denote the vector space of holomorphic polynomials of degree at most n. For a compact set $K \subset \mathbb{C}$ and a measure ν on K, we say that the pair (K,ν) satisfies the Bernstein-Markov inequality for holomorphic polynomials in $\mathbb{C}$ if, given $\epsilon > 0$, there exists a constant $\tilde{M} = \tilde{M}(\epsilon)$ such that for all $n = 1,2,...$ and all $p_n \in \mathcal{P}_n$

$$||p_n||_K \leq \tilde{M}(1+\epsilon)^n ||p_n||_{L^2(\nu)}. \tag{19}$$

Equivalently, for all $p_n \in \mathcal{P}_n$,

$$||p_n||_K \leq M_n ||p_n||_{L^2(\nu)} \text{ with } \limsup_{n\to\infty} M_n^{1/n} = 1.$$

Thus there is a strong comparability between L^2 and L^∞ norms. We will see in section 9 that *any* compact set K admits a measure ν with (K,ν) satisfying a Bernstein-Markov inequality. For now, we observe that one can recover the transfinite diameter $\delta(K)$ in an L^2-fashion with such a measure.

Theorem 2.9. *Let K be compact and let (K,ν) satisfy a Bernstein-Markov inequality for holomorphic polynomials. Then*

$$\lim_{n\to\infty} Z_n^{1/n^2} = \delta(K)$$

where

$$Z_n = Z_n(K,\nu) := \tag{20}$$

$$\int_{K^{n+1}} |VDM(\lambda_0,...,\lambda_n)|^2 d\nu(\lambda_0)\cdots d\nu(\lambda_n).$$

We will see the utility of this result, and generalizations of it, later on. The quantity Z_n is called the $n-$th *free energy* of (K,ν).

Exercises.

1. Prove (6) using weak-* convergence of $\mu_n \times \mu_n$ to $\mu \times \mu$ and lowersemicontinuity of $z \to \log \frac{1}{|z-\zeta|}$. (*Hint: If you have trouble, see the start of the proof of Proposition 5.2 in section 5.*)
2. Use Proposition 2.4 to verify Corollary 2.5: a countable union of Borel polar sets is polar.
3. Show that if $\{K_n\}$ are compact sets in $\mathbb{C}$ with $K_{n+1} \subset K_n$ for all n, then $\lim_{n\to\infty} I(\mu_{K_n}) = I(\mu_K)$ where $K = \cap_n K_n$. (*Hint: Use (6).*)
4. Verify the claim in the proof of Theorem 2.8 that $w \in J$ implies that $z_j \in J$ if $p^n(z_j) = w$.
5. Generally Fekete points of order n for a compact set K are not unique. In the case of the interval $[-1,1] \subset \mathbb{R} \subset \mathbb{C}$, they are unique. Find explicitly Fekete points $z_0 < z_1 < z_2 < z_3$ of order 3 for $[-1,1]$.
6. Compute $\Delta\left(\frac{1}{2}\log(1+|z|^2)\right)$.

7. Verify that $\delta_{n+1}(K) \leq \delta_n(K)$ for $n = 1, 2, \ldots$ for any compact set $K \subset \mathbb{C}$. Conclude that the limit in (11) exists.
8. Using the previous exercise, and observing that the function $VDM(\lambda_0, \ldots, \lambda_n)$ is a holomorphic polynomial of degree at most n in each variable, prove Theorem 2.9. (*Hint: Apply the Bernstein-Markov property repeatedly*).
9. *Extra Credit: Polar sets and energy.*
   (a) Find an example of a probability measure ν with compact support such that $I(\nu) < +\infty$ but ν puts no mass on polar sets.
   (b) Prove Proposition 2.7 under the weaker hypothesis on ν that ν puts no mass on polar sets (instead of $I(\nu) < +\infty$).

3 Upper envelopes, extremal subharmonic functions and applications.

In the first section, we claimed that for any family $\{v_\alpha\} \subset SH(D)$ which is uniformly bounded above on any compact subset of D, the function

$$v(z) := \sup_\alpha v_\alpha(z)$$

is "nearly" shm in the sense that the usc regularization

$$v^*(z) := \limsup_{\zeta \to z} v(\zeta)$$

is shm in D. This fact is fairly straightforward (exercise 2). The following simple example shows that the set

$$\{z \in D : v(z) < v^*(z)\}$$

need not be empty: let $D = B(0,1)$, let $\{v_\alpha\} = \{u_n\}$ where $u_n(z) = \frac{1}{n}\log|z|$; then, in $B(0,1)$, clearly $u(z) = \sup_n u_n(z) = 0$ for $0 < |z| < 1$ but $u(0) = -\infty$. Here, $u^*(z) \equiv 0$ and

$$\{z \in B(0,1) : u(z) < u^*(z)\} = \{0\}$$

which is admittedly "small". In general the *Brelot-Cartan Theorem* says that the set $\{z \in D : v(z) < v^*(z)\}$, called a *negligible set*, is always polar.

We remark that the converse of the Brelot-Cartan theorem is true: a polar set is negilgible. We prove this for bounded polar sets E. For such a set, by definition, there exists u shm in a domain D containing E, $u \not\equiv -\infty$, with $E \subset \{z \in D : u(z) = -\infty\}$. On $D' \subset\subset D$ with $E \subset D'$, we can assume $u < 0$ (why?). Now take $\{v_\alpha\} = \{\alpha u\}$ for $0 < \alpha < 1$. We leave it as an exercise to see that for $v(z) := \sup_\alpha v_\alpha(z)$,

$$\{z \in D' : u(z) = -\infty\} = \{z \in D' : v(z) < v^*(z)\}. \tag{21}$$

Before we return to our general upper envelope constructions, we mention a beautiful and very general result of Choquet: if $\{v_\alpha\}$ is a family of real-valued functions defined on a separable metric space X which is uniformly bounded above on any compact subset of X, then one can extract a *countable* subfamily $\{u_n\} \subset \{v_\alpha\}$ with the property that

$$\big(\sup_\alpha v_\alpha\big)^* = \big(\sup_n u_n\big)^*.$$

We note that if each v_α is continuous (or even lsc; i.e., $-v_\alpha$ is usc), then we can even have $\sup_\alpha v_\alpha = \sup_n u_n$ (exercise 1). Now recall given a bounded domain D with reasonable boundary and $f \in C(\partial D)$ we formed the Perron envelope

$$U(0;f)(z) := \sup\{v(z) : v \in SH(D) : \limsup_{z\to\zeta} v(z) \le f(\zeta)$$
$$\text{for all } \zeta \in \partial D\}.$$

Claim: *$U(0;f)$ is harmonic in D.*
To prove the claim, we show $U(0;f)$ is harmonic on any disk $B \subset D$. To this end, we first note that since any shm v is a decreasing limit of smooth shm functions, we can assume that each v is continuous in D.

1. We can then recover $U(0;f)$ as an upper envelope of a countable family of continuous functions $\{u_n\}$; by replacing u_n by $v_n := \max[u_1,...,u_n]$ we have $U(0;f)$ as an increasing sequence of continuous shm functions $\{v_n\}$.
2. Replace each v_n by its Poisson modification $\tilde{v}_n$ on B. Then, on B, $U(0;f)$ is the monotone, increasing limit of *harmonic* functions.
3. By Harnack's theorem (a monotone limit of harmonic functions in B either converges to a harmonic function or is identically $\pm\infty$), $U(0;f)$ is harmonic in B.

As another example of this type of argument, recall for $K \subset \mathbb{C}$ compact, we defined

$$V_K(z) = \sup\{u(z) : u \in L(\mathbb{C}),\ u \le 0 \text{ on } K\}.$$

This is again a Perron envelope, for any such u and any disk $B \subset \mathbb{C} \setminus K$, the function $\tilde{u}$ defined as u in $\mathbb{C} \setminus B$ and as $P_{u|_{\partial B},B}$ in B is in the family. An appropriate modification of the above argument shows that, provided V_K is locally bounded above, we have that V_K is harmonic outside of K (modulo topological issues). In this case, since $V_K = 0$ on K and $\{V_K < V_K^*\}$ is negligible and hence polar, $V_K^* = 0$ q.e. on K.

We can *almost* show that (13) holds in this setting; i.e., $\mu_K = \frac{1}{2\pi}\Delta V_K^*$. From Theorem 2.6, $p_{\mu_K} \le I(\mu_K)$ on $\mathbb{C}$ so that $-[p_{\mu_K} - I(\mu_K)] \in L^+(\mathbb{C})$. Moreover, $p_{\mu_K} = I(\mu_K)$ q.e on K so that

$$V_K^* = -[p_{\mu_K} - I(\mu_K)] \text{ q.e. on } K.$$

By the domination principle Proposition 2.7,

$$V_K^* \le -[p_{\mu_K} - I(\mu_K)] \text{ on } \mathbb{C}.$$

Both functions V_K^* and $-[p_{\mu_K} - I(\mu_K)]$ are harmonic outside of K. *If we knew that* $V_K^* \in L^+(\mathbb{C})$ using Remark 2 we could apply the domination principle Proposition 2.7 in the other direction to conclude that

$$V_K^* = -[p_{\mu_K} - I(\mu_K)] \text{ on } \mathbb{C}$$

and hence we would have (13). We will work towards verifying the italicized statement. Often the notation g_K is used for V_K^*, the *Green function* for K: it is characterized (uniquely) as the shm function in $\mathbb{C}$ which is in $L^+(\mathbb{C})$; harmonic in $\mathbb{C}\setminus K$; and equals 0 q.e. on K. We say K has a *classical Green function* if $g_K = 0$ on all of K.

To see when V_K is locally bounded above (Proposition 3.2), we first state and prove a very useful and general result, known as *Hartogs lemma*.

Lemma 3.1. *Let $\{u_j\}$ be a family of shm functions on a domain $D \subset \mathbb{C}$ which are locally uniformly bounded above in D. Suppose there exists $M < +\infty$ with*

$$\limsup_{j\to\infty} u_j(z) \le M \text{ for all } z \in D.$$

Given $\epsilon > 0$ and $K \subset D$ compact, there exists $j_0 = j_0(\epsilon, K)$ such that for $j \ge j_0$,

$$\sup_{z\in K} u_j(z) \le M + \epsilon.$$

Proof. Let $u(z) := \limsup_{j\to\infty} u_j(z)$ and $v_n(z) := \sup_{j\ge n} u_j(z)$. Then $v_n \downarrow u$. The functions v_n^* are shm and decrease pointwise to a shm function v on D. By the Brelot-Cartan theorem, $v_n = v_n^*$ q.e. and since a countable union of polar sets is polar (Corollary 2.5), $v = u$ q.e. Hence the shm functions v and u^* are equal q.e. and therefore a.e.; by Corollary 1.2 $v = u^*$ on D. Since $\{v_n^*\}$ form a decreasing sequence of shm functions with $v_n^* \le M$, by Dini's theorem, on any compact set $K \subset D$ the sequence $\{v_n^*\}$ converges uniformly to v and $v \le M$ on K. Since $u_n \le v_n \le v_n^*$, the result follows. □

We saw that for the unit circle $T = \{z \in \mathbb{C} : |z| = 1\}$ we have $V_T(z) = V_T^*(z) = \max[\log|z|, 0]$; hence for the closed unit disk $B = \overline{B}(0,1) = \{z \in \mathbb{C} : |z| \le 1\}$ we have $V_B(z) = V_B^*(z) = \max[\log|z|, 0]$ (why?). More generally, if $B = \overline{B}(a,r) = \{z \in \mathbb{C} : |z-a| \le r\}$ we have $V_B(z) = V_B^*(z) = \max[\log|z-a|/r, 0]$. If $K = \overline{B}(a,r)\cup\{p\}$ where $p \notin \overline{B}(a,r)$, $V_K^*(z) = V_B^*(z) = \max[\log|z-a|/r, 0]$ and $\{z : V_K(z) < V_K^*(z)\} = \{p\}$.

Proposition 3.2. *Let $K \subset \mathbb{C}$ be compact. Either $V_K^* \equiv +\infty$, which occurs if K is polar, or else we have $V_K^* \in L^+(\mathbb{C})$.*

Proof. If V_K is locally bounded above, on a disk B, e.g., the unit disk, $V_K \le M$; i.e., for all $u \in L(\mathbb{C})$ with $u \le 0$ on K, we have $u - M \le 0$ on B so that $u - M \le V_B$ in $\mathbb{C}$ and hence $V_K \le M + V_B$ in $\mathbb{C}$. Hence $V_K^* \in L(\mathbb{C})$.

If V_K is not locally bounded above, we claim that $P := \{z \in \mathbb{C} : V_K(z) < +\infty\}$ is polar; hence $V_K^* \equiv +\infty$. Since $V_K = 0$ on K, this shows, in particular, that K is polar. Thus assume V_K is not locally bounded above. Then there is a closed disk B and sequence $\{u_j\} \subset L(\mathbb{C})$ with $u_j \le 0$ on K such that $M_j := \sup_B u_j \ge j$ for $j = 1, 2, \ldots$ It follows that

$$u_j(z) - M_j \le V_B(z),\ z \in \mathbb{C},\ j = 1, 2, \ldots$$

We claim that from Hartogs lemma, there exists $z_0 \in \mathbb{C}$ with

$$\delta := \limsup_{j\to\infty} \exp(u_j(z_0) - M_j) > 0.$$

For if not, $\limsup_{j\to\infty} \exp(u_j(z) - M_j) \le 0$ for all $z \in \mathbb{C}$. Hartogs lemma implies, e.g., that $\exp(u_j(z) - M_j) \le 1/2$ for $z \in B$ and all j sufficiently large. But this contradicts the definition of $M_j := \sup_B u_j$.

Choose a subsequence $\{u_{j_k}\}$ so that

$$\delta = \lim_{k\to\infty} \exp(u_{j_k}(z_0) - M_{j_k}) \text{ and } M_{j_k} \ge 2^k$$

and define

$$w(z) := \sum_{k=1}^{\infty} 2^{-k}[u_{j_k}(z) - M_{j_k}]. \tag{22}$$

Check that $w(z_0) > -\infty$ (so $w \not\equiv -\infty$); w is shm in $\mathbb{C}$ (why?); and, indeed, $w \in L(\mathbb{C})$. We claim that $w = -\infty$ on P. For if $V_K(z) = M < +\infty$, we have $u_{j_k}(z) \le M$ for all k and hence

$$\sum_k 2^{-k} u_{j_k}(z) < +\infty.$$

Thus

$$w(z) \le \sum_k 2^{-k} u_{j_k}(z) - \sum_k 1 = -\infty.$$

Hence P is polar.

□

Polar sets are removable sets for certain classes of functions. Recall the Riemann removable singularity theorem: if f is holomorphic in a punctured disk $B \setminus \{p\}$ and $|f|$ is bounded near p, then f can be defined at p to be holomorphic in B. In particular, the same result applies to harmonic functions, and even locally bounded above shm functions. More generally, the "size" of the removable set can be bigger but not too big: it can be a polar set.

Proposition 3.3. *Let u be shm on $D \setminus P$ where D is a bounded domain and P is a polar set. Suppose u is locally bounded above near P. Then u has a unique shm extension to D.*

Proof. We extend u to D by setting

$$u(z) := \limsup_{\zeta \to z,\ \zeta \in D \setminus P} u(\zeta).$$

Clearly this extension is usc in D. To see that u is shm in D, take any relatively compact subdomain D' in D and a harmonic function h on $\overline{D}'$ with $u \leq h$ on $\partial D'$. There exists v shm in $\mathbb{C}$ with $v = -\infty$ on P. For $\epsilon > 0$, $u - h + \epsilon v$ is shm on $D' \setminus P$ and equals $-\infty$ on $D' \cap P$; hence it is shm on D'. By the maximum principle,

$$u - h + \epsilon v \leq \sup_{\partial D'} \epsilon v \text{ on } D'.$$

Let $\epsilon \to 0$ to conclude $u \leq h$ on $D' \setminus P$. Since P has measure zero, from Corollary 1.2, $u \leq h$ on P.

Uniqueness also follows from Corollary 1.2: two shm functions which agree a.e. are identical. □

Corollary 3.4. *Let h be harmonic on $D \setminus P$ where D is a bounded domain and P is a polar set. Suppose $|h|$ is locally bounded near P. Then h has a unique harmonic extension to D.*

How "big" can polar sets be? We saw that polar sets must have Lebesgue measure zero, and indeed, a polar set must have zero Hausdorff dimension so it can't be too big. On the other hand, we saw that countable sets are polar; but there do exist uncountable polar sets. Examples can be constructed from certain generalized Cantor sets. We refer the reader to [25].

There is a notion of "thinness" of a set, which is very closely related to polarity. Recall from exercise 4 of section 1, if u is shm in D, then for each $z \in D$, $\limsup_{\zeta \to z} u(\zeta) = u(z)$. Let $S \subset \mathbb{C}$ and $z \in \overline{S \setminus \{z\}}$. We say that S is *thin* at z if there exists u shm on a neighborhood of z_0 with

$$\limsup_{\zeta \to z,\ \zeta \in S \setminus \{z\}} u(\zeta) < u(z).$$

(For consistency, if $\zeta \notin \overline{S \setminus \{z\}}$, we say that S is thin at ζ). It can be shown that *an F_σ polar set S is thin at each point, and, conversely, a set S which is thin at every point of itself must be polar.* We refer the reader to section 3.8 of [25] for details.

Exercises.

1. Let $\{v_\alpha\}$ be a family of real-valued lsc functions defined on a separable metric space X which is uniformly bounded above on any compact subset of X. Show that $v(x) := \sup_\alpha v_\alpha(x)$ is lsc and that one can extract a *countable* subfamily $\{u_n\} \subset \{v_\alpha\}$ with the property that
$$\sup_\alpha v_\alpha = \sup_n u_n.$$
(*Hint*: Look at the set $\{(x,t) \in X \times \mathbb{R} : v(x) > t\}$ and use the Lindelöf property of X).
2. Let $\{v_\alpha\} \subset SH(D)$ be uniformly bounded above on any compact subset of D and define $v(z) := \sup_\alpha v_\alpha(z)$. Show that $v^*(z) := \limsup_{\zeta \to z} v(\zeta)$ is shm in D.
3. Verify equation (21).
4. Given a bounded domain D, and a point $z_0 \in D$, define
$$G(z; z_0) := \sup\{u(z) : u \in SH(D),\ u \leq 0,$$
$$u(z) - \log|z - z_0| \text{ bounded as } z \to z_0\}$$
the *Green function* for D with pole at z_0. Show that $G(z; z_0)$ is harmonic in $D \setminus \{z_0\}$.
5. Find a formula for $G(z; z_0)$ if $D = B(0,1)$ and $|z_0| < 1$. (*Hint: First do the case $z_0 = 0$ and for $z_0 \neq 0$ find a holomorphic self-map of $B(0,1)$ taking z_0 to* 0).
6. Given a bounded domain D and a subset $E \subset D$, define
$$\omega(z, E, D) := \sup\{u(z) : u \in SH(D),\ u \leq 0,\ u|_E \leq -1\},$$
the *relative extremal function* for E relative to D. Show that if $\omega^*(z, E, D) \not\equiv 0$ then $\omega^*(z, E, D)$ is harmonic in $D \setminus \overline{E}$.
7. Find a formula for $\omega(z, E, D)$ if $D = B(0,R)$ and $E = B(0,r)$, for $r < R$.
8. Prove the *two-constants theorem*: for $E \subset D$, if u is shm in D satisfies $u \leq M$ in D and $u \leq m < M$ on E, then for $z \in D$,
$$u(z) \leq M(1 + \omega^*(z, E, D)) - m\omega^*(z, E, D).$$
(*Remark: If you apply this result to $u = \log|f|$ where f is holomorphic in D, $|f| \leq M'$ on D and $|f| \leq m'$ on E you get a generalization of the "three-circles" theorem from complex analysis.*)
9. Verify the "why?" in the proof of Proposition 3.2; i.e., prove the shm of w in equation 22.
10. Prove Corollary 3.4.

4 Polynomial approximation and interpolation in $\mathbb{C}$.

There is a close relation between the smoothness of a function f and the speed at which f may be approximated by polynomials. To state results of this type we introduce, for any continuous complex-valued function f on any compact set K in the plane $\mathbb{C}$, the *approximation numbers*

$$d_n = d_n(f,K) \equiv \inf\{||f - p_n||_K : p_n \in \mathcal{P}_n\},$$

where recall $\mathcal{P}_n$ is the vector space of complex polynomials in z of degree at most n. The Weierstrass approximation theorem states that $\lim_{n\to\infty} d_n = 0$ for any continuous function f on $[-1,1]$, and it is natural to ask for additional conditions on f which guarantee that d_n converges rapidly to zero. A beautiful result of this type is the classical theorem of Bernstein, which states that f extends to a holomorphic function on an open neighborhood of $[-1,1]$ in $\mathbb{C}$ if and only if d_n satisfies an exponential decay estimate

$$d_n \le C\rho^n \quad \text{for some constants } C > 0 \text{ and } \rho \in (0,1).$$

In fact, a sharp version of the Bernstein theorem relates the constant ρ to the size of the open neighborhood of $[-1,1]$ to which f can be extended. Walsh [27] later gave an important extension of the Bernstein theorem in which the interval $[-1,1]$ is replaced by certain compact subsets of $\mathbb{C}$. The theorems of Bernstein and Walsh serve as a link between the classical ideas of approximation theory and some higher-dimensional problems concerning holomorphic functions of several complex variables.

An elementary approach to the theorems of Bernstein and Walsh is to regard them as statements about the error in truncating geometrically convergent series expansions. As the simplest example, consider first the closed unit disk $\overline{\Delta} = \{z : |z| \le 1\}$ in $\mathbb{C}$, and suppose that f is holomorphic on a neighborhood of $\overline{\Delta}$. To be specific, we assume that f is holomorphic on the open disk $\{z : |z| < R\}$, where $R > 1$, and we ask to what extent the size of the radius R determines the rate of decay of the approximation numbers $d_n(f,\overline{\Delta})$. To study this, we recall that the Taylor expansion $\sum a_k z^k$ for f about the origin converges absolutely and uniformly on compact subsets of $\{z : |z| < R\}$ to f. Applying the Cauchy estimates to f on $\{z : |z| < r\}$, where $1 < r < R$, we obtain $|a_n| \le M/r^n$ with $M = \sup\{|f(z)| : |z| \le r\}$. Letting $p_n(z) = \sum_{k=0}^n a_k z^k$ be the n-th Taylor polynomial for f, it follows that $d_n(f,\overline{\Delta}) \le ||f - p_n||_{\overline{\Delta}} \le \dfrac{M}{r^n(r-1)}$. This implies that $\limsup_{n\to\infty} d_n(f,\overline{\Delta})^{1/n} \le 1/r$, and we may now let $r \uparrow R$ to conclude that

$$\limsup_{n\to\infty} d_n(f,\overline{\Delta})^{1/n} \le 1/R.$$

This proves the following equivalence in one direction.

Theorem 4.1. *Let f be continuous on $\overline{\Delta} = \{z \in \mathbb{C} : |z| \le 1\}$, and $R > 1$. Then*

$$\limsup_{n\to\infty} d_n(f,\overline{\Delta})^{1/n} \le 1/R \tag{23}$$

if and only if f is the restriction to $\overline{\Delta}$ of a function holomorphic in $\{z \in \mathbb{C} : |z| < R\}$.

Proof. We have already proved "if". To prove "only if" we will use the fact that any polynomial $p(z)$ satisfies the *Bernstein-Walsh inequality*

$$|p(z)| \le ||p||_{\overline{\Delta}}\, \rho^{\deg\, p}, \qquad |z| \le \rho; \tag{24}$$

this estimate follows from applying Lemma 1 below, with $g_{\overline{\Delta}}(z) \equiv \log|z|$, so for the moment we assume (24) and complete the proof of the theorem. Let f be a continuous function on $\overline{\Delta}$ such that (23) holds; we will show that if p_n is a polynomial of degree $\le n$ satisfying $d_n = ||f - p_n||_{\overline{\Delta}}$, then the series $p_0 + \sum_1^\infty (p_n - p_{n-1})$ converges uniformly on compact subsets of $\{z : |z| < R\}$ to a holomorphic function F which agrees with f on $\overline{\Delta}$. To do this, we choose R' with $1 < R' < R$; by hypothesis the polynomials p_n satisfy

$$||f - p_n||_{\overline{\Delta}} \le \frac{M}{R'^n}, \qquad n = 0,1,2,..., \tag{25}$$

for some $M > 0$. We now let $1 < \rho < R'$, and apply (24) to the polynomial $p_n - p_{n-1}$ to obtain

$$\sup_{|z|\le\rho} |p_n(z) - p_{n-1}(z)| \le \rho^n ||p_n - p_{n-1}||_{\overline{\Delta}} \le \rho^n(||p_n - f||_{\overline{\Delta}} + ||f - p_{n-1}||_{\overline{\Delta}})$$

$$\le \rho^n \frac{M(1+R')}{R'^n}.$$

Since ρ and R' were arbitrary numbers satisfying $1 < \rho < R' < R$, we conclude that $p_0 + \sum_1^\infty (p_n - p_{n-1})$ is locally uniformly Cauchy on $\{z : |z| < R\}$, and hence converges locally uniformly on $\{z : |z| < R\}$ to a holomorphic function F; from (25) we see that $F \equiv f$ on $\overline{\Delta}$, so the theorem is proved. □

For more general compact sets $K \subset \mathbb{C}$, we will see the importance of the function V_K from (14). We begin with a lemma.

Lemma 4.2. (Bernstein-Walsh property) *Let K be a compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected. Suppose that $\mathbb{C} \setminus K$ has a classical Green function g_K; i.e., there is a continuous function $g_K : \mathbb{C} \to [0,+\infty)$ which is identically equal to zero on K, harmonic on $\mathbb{C} \setminus K$, and has a logarithmic singularity at infinity in the sense that $g_K(z) - \log|z|$ is harmonic at infinity. Then*

$$g_K(z) \equiv \max\left\{0, \sup_p \left\{\frac{1}{\deg p}\log|p(z)|\right\}\right\}, \tag{26}$$

where the supremum is taken over all non-constant polynomials p such that $||p||_K \leq 1$. *In particular,* $g_K = V_K$ *and, if* $R > 1$ *and*

$$D_R \equiv \{z : V_K(z) < \log R\}, \tag{27}$$

then

$$|p(z)| \leq ||p||_K R^{\deg p}, \qquad z \in D_R. \tag{28}$$

The topological condition that $\mathbb{C} \setminus K$ is connected is equivalent to K being *polynomially convex*: this means that $K = \widehat{K}$ where

$$\widehat{K} \equiv \{z \in \mathbb{C} : |p(z)| \leq ||p||_K, \ p \text{ polynomial}\}$$

is the *polynomial hull* of K (see the exercises). Note that using (15), i.e., the right-hand-side of (26), we have

$$V_K = V_{\widehat{K}}.$$

A compact set with V_K continuous, equivalently, $V_K = V_K^*$, is called *regular*. Any compact set K can be approximated from the outside by regular compacta; i.e., one can find $\{K_j\}$ regular with $K_{j+1} \subset K_j$ and $\cap_j K_j = K$. We can take, e.g., $K_j = \{z \in \mathbb{C} : \text{dist}(z,K) \leq 1/j\}$. The fact that each K_j is regular can be seen by recalling from section 3 that for a closed unit disk $B = \overline{B}(a,r) = \{z \in \mathbb{C} : |z-a| \leq r\}$ we have $V_B(z) = V_B^*(z) = \max[\log|z-a|/r, 0]$. Now each $z \in K_j$ belongs to a closed ball $\tilde{B} := \overline{B}(a,1/j) \subset K_j$ and since clearly $V_{K_j}(z) \leq V_{\tilde{B}}(z) = V_{\tilde{B}}^*(z) = 0$, we have $V_{K_j}^* = 0$ on K_j so that $V_{K_j}^* \leq V_{K_j}$ (why?) and hence equality holds.

It is easy to prove a weak form of (26). In fact, if p is any nonconstant polynomial such that $||p||_K \leq 1$, then the function $V \equiv \frac{1}{\deg p}\log|p| - g_K$ is subharmonic on $\mathbb{C} \setminus K$, bounded at ∞, and continuously assumes nonpositive values on ∂K. By the maximum principle we have $V \leq 0$ on $\mathbb{C} \cup \{\infty\} - K$, which proves that $g_K(z)$ is greater than or equal to the right side of (26). To show that $g_K(z)$ is actually equal to the right side of (26), we will construct a *sequence* of monic polynomials $\{p_n(z) = z^n + \cdots\}$ with $\deg p_n = n$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{|p_n(z)|}{||p_n||_K}\right) = g_K(z)$$

locally uniformly on $\mathbb{C} \cup \{\infty\} - K$ (cf., [27], section 4.4); for example, from (2.3) a sequence of *Fekete polynomials* $p_n(z) = \prod_{j=1}^n (z - z_{nj})$ where $z_{n1}, ..., z_{nn}$ is a set of Fekete points of order $n-1$ for K will do since the corresponding sequence of Fekete measures $\{\mu_n\}$ converges weak-* to μ_K. Note as a consequence, we have proved the following.

Corollary 4.3. *Let K be a regular compact set in* $\mathbb{C}$. *Then the functions*

$$V_K^{(n)}(z) := \max\left\{0, \sup_p \left\{\frac{1}{\deg p} \log|p(z), \ p \in \mathcal{P}_n|\right\}\right\}$$

converge uniformly to V_K *on* $\mathbb{C}$.

We remark that for a general compact set $K \subset \mathbb{C}$, if one minimizes the supremum norm on K of monic polynomials of degree n; i.e., one takes

$$\tau_n(K) := \inf\{||p_n||_K : p_n(z) = z^n + \cdots\},$$

then

$$\lim_{n\to\infty} \tau_n(K)^{1/n} = \inf_{n\geq 1} \tau_n(K)^{1/n} = \delta(K). \tag{29}$$

Thus the *Chebyshev constant* $\lim_{n\to\infty} \tau_n(K)^{1/n}$ of K coincides with the transfinite diameter. A monic polynomial t_n with $||t_n||_K = \tau_n(K)$ is called a *Chebyshev polynomial* for K; such a polynomial exists (and is unique if K has at least n points). We omit the proof but we can easily give one inequality: taking a Fekete polynomial $p_n(z) = \prod_{j=1}^n (z - z_{nj})$, by definition we have $||t_n||_K \leq ||p_n||_K$; but then for *any* $z \in K$, the $(n+1)-$tuple $z, z_{n1}, ..., z_{nn}$ is a candidate for a set of Fekete points of order $n+1$ for K. Thus

$$|p_n(z)| \cdot \delta_n(K)^{\binom{n}{2}} = \prod_{j=1}^n |z - z_{nj}| \prod_{j<k} |z_{nj} - z_{nk}| \leq \delta_{n+1}(K)^{\binom{n+1}{2}}$$

and since $\delta_{n+1}(K) \leq \delta_n(K)$ (exercise 7 in section 2), we have

$$||t_n||_K \leq ||p_n||_K \leq \frac{\delta_n(K)^{\binom{n+1}{2}}}{\delta_n(K)^{\binom{n}{2}}} = \delta_n(K)^n$$

giving

$$\limsup_{n\to\infty} \tau_n(K)^{1/n} \leq \delta(K).$$

Note we have also proved that $\lim_{n\to\infty} ||p_n||_K^{1/n} \leq \delta(K)$ for the Fekete polynomials p_n.

Theorem 4.4. (Walsh) *Let K be a compact subset of the plane such that* $\mathbb{C} \setminus K$ *is connected and has a Green's function* g_K. *Let* $R > 1$, *and define* D_R *by (27). Let f be continuous on K. Then*

$$\limsup_{n\to\infty} d_n(f,K)^{1/n} \leq 1/R$$

if and only if f is the restriction to K of a function holomorphic in D_R.

To prove "only if" in this theorem we repeat the proof after the statement of Theorem 23, using the Bernstein-Walsh inequality (28). The proof of the "if" direction we are about to outline is one of the simplest to give, yet the most difficult to generalize; it uses polynomial interpolation to construct good approximators. The key ingredient we need is the Hermite remainder formula for interpolation of a holomorphic function of one variable. Let $z_1, \dots z_n$ be n distinct points in the plane and let f be a function which is defined at these points. The polynomials $l_j(z) = \prod_{k\neq j}(z-z_k)/\prod_{k\neq j}(z_j-z_k)$, $j=1,\dots,n$, are polynomials of degree $n-1$ with $l_j(z_k)=\delta_{j,k}$, called the *fundamental Lagrange interpolating polynomials*, or FLIP's, associated to $z_1,\dots,z_n$. We remark that we can also write

$$l_j(z) = \frac{VDM(z_1,\dots,z_{j-1},z,z_{j+1},\dots,z_n)}{VDM(z_1,\dots,z_n)} \text{ (why?)}$$

and this form of a FLIP will generalize to $\mathbb{C}^N$, $N>1$. Then the polynomial $p(z)=\sum_{j=1}^n f(z_j)l_j(z)$ is the unique polynomial of degree $n-1$ satisfying $p(z_j)=f(z_j)$, $j=1,\dots,n$; we call it the *Lagrange interpolating polynomial,* or LIP, associated to $f, z_1,\dots,z_n$. Suppose now that Γ is a rectifiable Jordan curve such that the points $z_1,\dots,z_n$ are inside Γ, and f is holomorphic inside and on Γ. We can estimate the error in our approximation of f by p at points inside Γ using the following formula.

Lemma 4.5. (Hermite Remainder Formula) *For any z inside Γ,*

$$f(z)-p(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(z)}{\omega(t)}\frac{f(t)}{(t-z)}dt,$$

where $\omega(z)=\prod_{k=1}^n(z-z_k)$.

Proof. The function

$$\tilde{p}(z) \equiv \frac{1}{2\pi i}\int_\Gamma \left[\frac{\omega(t)-\omega(z)}{t-z}\right]\frac{f(t)}{\omega(t)}dt$$

is clearly a polynomial of degree $\le n-1$. Using the Cauchy integral formula for f, we see that

$$f(z)-\tilde{p}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(z)}{\omega(t)}\frac{f(t)}{(t-z)}dt \tag{30}$$

for z inside Γ. In particular, for each k we have $f(z_k)-\tilde{p}(z_k)=0$, and hence $\tilde{p}=p$. Now the lemma follows from (30). □

The proof of the "if" direction in Theorem 4.4 can now be completed using Lagrange interpolating polynomials for f at Fekete points of K and the Hermite remainder formula (exercise 4). We next give a fundamental result of Walsh. Let $\{z_{nj}\}$, $j=0,\dots,n$; $n=1,2,\dots$ be an array of points. For each f defined in a neighborhood of this array, we can form the sequence of LIP's $\{p_n\}$ associated to f. We write $p_n = L_n f$ to denote the degree and the dependence on f; i.e., $L_n f$ is the LIP of degree n associated to $f, z_{n0},\dots,z_{nn}$. Let $\omega_n(z) := \prod_{j=0}^n (z-z_{nj})$.

Theorem 4.6. *Let $K\subset\mathbb{C}$ be compact and regular with $\mathbb{C}\setminus K$ connected. Let $\{z_{nj}\}$ be an array of points in K. Then for any f which is holomorphic in a neighborhood of K, we have $L_n f \rightrightarrows f$ on K if and only if*

$$\lim_{n\to\infty} |\omega_n(z)|^{\frac{1}{n+1}} = \delta(K)\cdot e^{V_K(z)} \tag{31}$$

uniformly on compact subsets of $\mathbb{C}\setminus K$.

Condition (31) is equivalent to

$$\lim_{n\to\infty} ||\omega_n||_K^{1/n+1} = \delta(K).$$

We will call the array $\{z_{nj}\}$ "good" – meaning good for polynomial interpolation of holomorphic functions – if condition (31) holds. To construct arrays satisfying (31), define

$$\Lambda_n \equiv \sup_{z\in K}\sum_{j=0}^n |l_{nj}(z)|$$

the n-th *Lebesgue constant* for the array. This is the norm of the linear operator

$$\mathcal{L}_n : C(K)\to\mathcal{P}_n\subset C(K)$$

defined by $\mathcal{L}_n(f) := L_n f$ where we equip $C(K)$ with the supremum norm (exercise). We observe that, from Theorem 4.4, if the array satisfies

$$\lim_{n\to\infty}\Lambda_n^{1/n} = 1, \tag{32}$$

then (31) holds. To see this, we take f holomorphic on a neighborhood of K, and we show that $L_d f \rightrightarrows f$ on K. To this end, we note that f is holomorphic in D_R for some $R>1$ so by Theorem 4.4 we can find a sequence of polynomials $\{p_n\}$ with $\deg p_n \le n$ and $||f-p_n||_K = 0(1/R^n)$. Since $L_n p_n = p_n$ (why?), we have

$$\begin{aligned} ||f-L_n f||_K &\le ||f-p_n||_K + ||p_n - L_n f||_K \\ &= ||f-p_n||_K + ||L_n(p_n-f)||_K \le (1+\Lambda_n)||f-p_n||_K \end{aligned}$$

and the result follows.

Next, the condition (32) implies that the array is *asymptotically Fekete* in the sense that

$$\lim_{n\to\infty} |VDM(z_{n0},\dots,z_{nn})]^{1/\binom{n+1}{2}} := \delta(K). \tag{33}$$

(cf., [10]). Moreover, on pp. 462-463 in [10], it was observed that for an array $\{z_{nj}\} \subset K$ with

$$|VDM(z_{n0},\dots,z_{nn})| = c_n V_n(K)$$

where

$$0 < c_n < 1,\ \limsup_{n\to\infty} c_n^{1/n} < 1, \text{ and } \lim_{n\to\infty} c_n^{1/l_n} = 1$$

(e.g., $c_n = \nu^n$ for $0 < \nu < 1$), property (33) holds but (32) does not. More precisely, we have the following.

Proposition 4.7. *Let $\{z_{nj}\}_{j=0,\dots,n;\ n=1,2,\dots} \subset K$ be an array of points. Suppose that*

$$\lim_{n\to\infty} \Big(\frac{V_n(K)}{|VDM(z_{n0},\dots,z_{nn})|}\Big)^{1/n} = 1.$$

Then (32) holds.

Proof. The result follows trivially from the observation that if

$$\frac{V_n(K)}{|VDM(z_{n0},\dots,z_{nn})|} \le a(n),$$

then $\Lambda_n \le (n+1)\cdot a(n)$. This observation is a consequence of the fact that each FLIP can be written as

$$l_{nj}(z) \equiv \frac{VDM(z_{n0},\dots,z,\dots,z_{nn})}{VDM(z_{n0},\dots,z_{nn})}$$

so that

$$|l_{nj}(z)| \le a(n)\frac{|VDM(z_{n0},\dots,z,\dots,z_{nn})|}{V_n(K)}.$$

Since $|VDM(z_{n0},\dots,z,\dots,z_{nn})| \le V_n(K)$ for each $z \in K$, we have $||l_{nj}||_K \le a(n)$. □

Indeed, both the conditions (32) and (33) imply that the sequence of discrete measures

$$\mu_n := \frac{1}{n+1}\sum_{j=0}^{n} \delta_{z_{nj}}$$

converge weak-* to μ_K.

Proposition 4.8. *Let $K \subset \mathbb{C}$ be compact with $\delta(K) > 0$. For any array $\{z_{nj}\} \subset K$ satisfying (33), $\mu_n \to \mu_K$ weak-*.*

We will prove a more general version of this result in section 5 (Proposition 5.2). To summarize, we have the following (see [10] for more details).

Proposition 4.9. *Let $K \subset \mathbb{C}$ be compact, regular, and polynomially convex. Consider the following four properties which an array $\{z_{nj}\}_{j=0,\dots,n;\ n=1,2,\dots} \subset K$ may or may not possess:*

1. $\lim_{n\to\infty} \Lambda_n^{1/n} = 1$;
2. $\lim_{n\to\infty} |VDM(z_{n0},\dots,z_{nn})|^{\frac{1}{\binom{n+1}{2}}} = \delta(K)$;
3. $\lim_{n\to\infty} \frac{1}{n+1}\sum_{j=0}^{n} \delta_{z_{nj}} = \mu_K$ *weak-**;
4. $L_n f \rightrightarrows f$ *on K for each f holomorphic on a neighborhood of K.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) *and there are counterexamples to each of the reverse implications.*

We end this section with a construction, due to Edrei and Leja, of a *sequence* of points $\{z_j\}$ in a compact set K with the property that the array $\{z_{nj}\} = \{z_j\}$ satisfies (33) and hence, if K is regular with $\mathbb{C}\setminus K$ connected, (31) holds. Let z_0 be any point in K, and, having chosen $z_1,\dots,z_{n-1} \in K$, we choose $z_n \in K$ such that

$$\max_{z\in K}\prod_{j=0}^{n-1}|z - z_j| = \prod_{j=0}^{n-1}|z_n - z_j|. \tag{34}$$

The proof that (33) holds is outlined in exercise 8.

Exercises.

1. Prove that for $K \subset \mathbb{C}$ compact, $\widehat{K} = K$ if and only if $\mathbb{C}\setminus K$ is connected.

2. For a compact set $K \subset \mathbb{C}$:
   (a) Determine $\widehat{K}$ if $K = \{z : |z| = 1\}$.
   (b) Determine $\widehat{K}$ if $K = \{z : a \leq |z| \leq b\}$ where $0 < a < b$.
   (c) Show that if $K = \widehat{K}$ then $\mathbb{C} \setminus K$ is connected.
   (d) Note that if $\mathbb{C} \setminus K$ is connected, then Runge's theorem states that any f analytic on a neighborhood of K can be uniformly approximated on K by polynomials. (Theorem 4.4 is a quantitative version of this). Use this to prove the converse to (c): if $\mathbb{C} \setminus K$ is connected, then $K = \widehat{K}$. (*Hint*: If $z_0 \in \mathbb{C} \setminus K$, then $K \cup \{z_0\}$ also has connected complement. Take a sequence $z_n \to z_0$ and consider $f_n(z) = \frac{1}{z - z_n}$ which is holomorphic on a neighborhood of $K \cup \{z_0\}$. Now use Runge to find a polynomial p with $|p(z_0)| > \max_{\zeta \in K} |p(\zeta)|$).
3. Suppose that $\mathbb{C} \setminus K$ is connected and has a Green function, and assume that $(\mathbb{C} \cup \{\infty\}) \setminus K$ is simply connected. Prove that for $z \notin K$, $g_K(z) = \log|\phi(z)|$ where ϕ is a conformal map of $(\mathbb{C} \cup \{\infty\}) \setminus K$ onto $\{z : |z| > 1\}$ with $\phi(\infty) = \infty$. Use this result to find $g_{[-1,1]}$.
4. Use the Hermite remainder formula to prove the "if" direction of Theorem 4.4.
5. Prove that the condition
$$\lim_{n\to\infty} |\omega_n(z)|^{\frac{1}{n+1}} = \delta(K) \cdot e^{V_K(z)}$$
uniformly on compact subsets of $\mathbb{C} \setminus K$ is equivalent to
$$\lim_{n\to\infty} ||\omega_n||_K^{\frac{1}{n+1}} = \delta(K).$$
6. Use the Hermite remainder formula to prove the following: given *any* array $\{z_{nj}\}$ in the closed unit disk $\overline{D} = \{z : |z| \leq 1\}$, if f is analytic in $D_R = \{z : |z| < R\}$ where $R > 3$, then $\{L_n f\}$ converge uniformly to f on $\overline{D}$.
7. Use the previous exercise to prove the following: given *any* bounded array $\{z_{nj}\}$ in $\mathbb{C}$, if f is an entire function, then the sequence of LIP's $\{L_n f\}$ converges uniformly on compact subsets of $\mathbb{C}$ to f.
8. Verify that a Leja sequence for K defined in (34) satisfies (33) using the following outline:
   (a) Show for any monic polynomial $p_n(z) = z^n + \cdots$, $||p_n||_K \geq \delta(K)^n$ (you may assume (29)).
   (b) Verify that, for the Leja sequence $\{z_j\}_{j=0,1,\ldots}$,
$$V_{n+1}(K) \geq |VDM(z_0, \ldots, z_n)| \geq ||\omega_n||_K \cdot ||\omega_{n-1}||_K \cdots ||\omega_0||_K$$
   where $\omega_j(z) = \prod_{i=0}^{j}(z - z_i)$.
   (c) Combine parts (a) and (b).
9. *EXTRA extra credit*: Prove that if $K \subset \mathbb{C}$ is not polar, then there exists a regular compact subset $K' \subset K$. This is a deep theorem of Ancona [1].

5 Weighted potential theory in $\mathbb{C}$.

Let $K \subset \mathbb{C}$ be closed and let w be an *admissible* weight function on K: w is a nonnegative, uppersemicontinuous function with $\{z \in K : w(z) > 0\}$ nonpolar; if K is unbounded, we require that w satisfies the growth property $|z|w(z) \to 0$ as $|z| \to \infty$, $z \in K$. We write $Q := -\log w$ and denote the collection of lowersemicontinuous Q of this form as $\mathcal{A}(K)$. Associated to K, Q is a *weighted energy minimization problem*: for a probability measure τ on K, consider the weighted energy

$$I^w(\tau) := \int_K \int_K \log \frac{1}{|z-t|w(z)w(t)} d\tau(t)d\tau(z) = I(\tau) + 2\int_K Q d\tau$$

and find $\inf_\tau I^w(\tau)$ where the infimum is taken over all probability measures τ on K. This is often referred to as *a logarithmic energy minimization in the presence of an external field Q*. The associated discrete problem leads to the *weighted transfinite diameter of K with respect to w*:

$$\delta^w(K) := \lim_{n\to\infty} \big[\max_{\lambda_i \in K} |VDM(\lambda_0, ..., \lambda_n)| w(\lambda_0)^n \cdots w(\lambda_n)^n\big]^{1/\binom{n+1}{2}}. \tag{35}$$

Here $VDM(\zeta_1, ..., \zeta_n) = \det[\zeta_i^{j-1}]_{i,j=1,...,n} = \prod_{j<k}(\zeta_j - \zeta_k)$ is the classical Vandermonde determinant. The proof that the limit exists is similar to the unweighted case and is left as exercise 1. Points $\lambda_0, ..., \lambda_n \in K$ for which

$$|VDM(\lambda_0, ..., \lambda_n)| w(\lambda_0)^n \cdots w(\lambda_n)^n$$
$$= |\det \begin{bmatrix} 1 & \lambda_0 & \dots & \lambda_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^n \end{bmatrix} | \cdot w(\lambda_0)^n \cdots w(\lambda_n)^n$$

is maximal are called *weighted Fekete points of order n*. For future use, we write

$$\delta_n^w(K) := \big[\max_{\lambda_i \in K} |VDM(\lambda_0, ..., \lambda_n)| w(\lambda_0)^n \cdots w(\lambda_n)^n\big]^{1/\binom{n+1}{2}}. \tag{36}$$

We have

$$\inf_\tau I^w(\tau) = -\log \delta^w(K). \tag{37}$$

We also define the *weighted Green function*

$$V_{K,Q}^*(z) := \limsup_{\zeta\to z} V_{K,Q}(\zeta)$$

where

$$V_{K,Q}(z) := \sup\{u(z) : u \in L(\mathbb{C}),\ u \le Q \text{ on } K\}.$$

The case $w \equiv 1$ on K; i.e., $Q \equiv 0$, is the "unweighted" case and we simply write V_K. We have $V_{K,Q}^* \in L^+(\mathbb{C})$ and the measure

$$\mu_{K,Q} := \frac{1}{2\pi}\Delta V_{K,Q}^* = \frac{1}{2\pi} dd^c V_{K,Q}^*,$$

which has compact support, is the *weighted equilibrium measure*: indeed, $\mu_{K,Q}$ is the unique probability measure on K satisfying

$$\inf_\tau I^w(\tau) = I^w(\mu_{K,Q}).$$

We remark that there exists $\eta > 0$ such that the support S_w of $\mu_{K,Q}$ is contained in $\{z \in K : w(z) \ge \eta\}$ (Remark 1.4, p. 27 of [26]). Note if $Q \equiv 0$ we write $\mu_K = \mu_{K,0}$.

For $K \subset \mathbb{C}$ compact, we say K is locally regular if for each $z \in K$ the unweighted Green function for the sets $K \cap \overline{B(z,r)}$, $r > 0$ are continuous. Here $B(z,r)$ denotes the Euclidean disk with center z and radius r. In this one-variable setting, local regularity of K is equivalent to (global) regularity; i.e., $V_K = V_K^*$ is continuous. If K is regular and Q is continuous, then $V_{K,Q}$ is continuous. We have the elementary fact that for such K and Q,

$$V_{K,Q}(z) = V_{K,Q}^*(z) \le Q(z) \text{ on } K. \tag{38}$$

In general, it is known that

$$\text{supp}(\mu_{K,Q}) \subset \{z \in K : V_{K,Q}^*(z) \ge Q(z)\} \tag{39}$$

and that $V_{K,Q}^* = Q$ on $\text{supp}(\mu_{K,Q})$ except perhaps for a polar set (cf., [26]). To prove (39), we use a Perron family argument as in the previous section. Let $S_w := \text{supp}(\mu_{K,Q})$ and $S_w^* := \{z \in K : V_{K,Q}^*(z) \ge Q(z)\}$. Fix $z_0 \in K \setminus S_w^*$. Since $V_{K,Q}^*$ is usc and Q is lsc, we can find a ball $B(z_0, r)$ with

$$\sup_{z\in B(z_0,r)} V_{K,Q}^*(z) < \inf_{z\in B(z_0,r)\cap K} Q(z).$$

We now form $u \in L(\mathbb{C})$ by setting $u = V_{K,Q}^*$ on $\mathbb{C}\setminus B(z_0,r)$ and on $B(z_0,r)$, we replace $V_{K,Q}^*$ by $P_{f,B(z_0,r)}$ with $f = V_{K,Q}^*|_{\partial B(z_0,r)}$. Since clearly $u \le Q$ on K, we have $u \le V_{K,Q}^*$ and hence $u = V_{K,Q}^*$ in $\mathbb{C}$. Thus $\Delta V_{K,Q}^* = \Delta u = 0$ on $B(z_0,r)$; hence $z_0 \notin S_w$.

For an unbounded set K, the condition that $|z|w(z) \to 0$ as $|z| \to \infty$, $z \in K$, translates into

$$Q(z) - \log|z| \to +\infty \text{ as } |z| \to \infty,\ z \in K.$$

Thus, $\{z \in K : V_{K,Q}(z) = Q(z)\}$ is a bounded set; hence $\text{supp}(\mu_{K,Q})$ is compact.

A characterization of the logarithmic potential function $p_{\mu_{K,Q}}$ akin to the Frostman theorem reads as follows:

Proposition 5.1. *If $\mu \in \mathcal{M}(K)$ has compact support and $I(\mu) < +\infty$, and if $p_\mu(z) + Q(z)$ is equal to a constant F q.e. on $\mathrm{supp}(\mu)$ and is greater than or equal to F on K, then $V^*_{K,Q} = -p_\mu + F$ and hence $\mu = \mu_{K,Q}$.*

Proof. We give the proof when $V_{K,Q}$ is continuous. In this case, by (38) $V^*_{K,Q} \le Q$ on K. Since $-p_\mu + F = Q$ q.e. on $\mathrm{supp}(\mu)$, by Proposition 2.7 (and Remark 2), we have $-p_\mu + F \ge V^*_{K,Q}$ on $\mathbb{C}$. But $-p_\mu + F \in L(\mathbb{C})$ and by hypothesis, $-p_\mu + F \le Q$ on K, so $-p_\mu + F \le V_{K,Q} \le V^*_{K,Q}$.

□

The weighted theory introduces new phenomena from the unweighted case. As an elementary example, μ_K puts no mass on the interior of K (indeed, the support of μ_K is the outer boundary of K); but this is not necessarily true in the weighted setting. As a simple but illustrative example, taking K to be the closed unit disk $\{z : |z| \le 1\}$ and $Q(z) = |z|^2$, using Proposition 5.1 one can see that $V_{K,Q} = Q$ on the disk $\{z : |z| \le 1/\sqrt{2}\}$ and $V_{K,Q}(z) = \log|z| + 1/2 - \log(1/\sqrt{2})$ outside this disk (exercise 3). Indeed, taking $K = \mathbb{C}$ and the same weight function $Q(z) = |z|^2$, one obtains the same weighted extremal function $V_{K,Q}$. This last result is a special case of the following: let $Q(z) = Q(|z|) = Q(r)$ be a radially symmetric weight function on $\mathbb{C}$ which is convex on $r > 0$. Let r_0 be the smallest number for which $Q'(r) > 0$ for all $r > r_0$ and let R_0 be the smallest solution of $R_0 Q'(R_0) = 1$. Then $S_w = \{z : r_0 \le |z| \le R_0\}$ and $d\mu_{K,Q}(r) = \frac{1}{2\pi}\big(rQ'(r)\big)' dr d\theta$. This is part of Theorem IV.6.1 of [26].

We will prove the following fact, which says that for any doubly indexed array of points $\{z_k^{(n_j)}\}_{k=1,\dots,n_j;\ j=1,2,\dots}$ in E which is *asymptotically Fekete with respect to the weight* w, the limiting measures

$$d\mu_{n_j} := \frac{1}{n_j}\sum_{k=1}^{n_j} \delta_{z_k^{(n_j)}} \tag{40}$$

have the same weak-* limit, the weighted equilibrium measure $d\mu_{K,Q}$.

Proposition 5.2. *Let $K \subset \mathbb{C}$ be compact and let w be an admissible weight on K. If, for a subsequence of positive integers $\{n_j\}$ with $n_j \uparrow \infty$, the points $z_1^{(n_j)}, \dots, z_{n_j}^{(n_j)} \in K$ are chosen so that*

$$\lim_{j\to\infty}\big[|VDM(z_1^{(n_j)}, \dots, z_{n_j}^{(n_j)})|^2 w(z_1^{(n_j)})^{2n_j} \cdots w(z_{n_j}^{(n_j)})^{2n_j}\big]^{1/n_j^2} = \delta^w(K),$$

then $d\mu_{n_j} \to d\mu_{K,Q}$ weak- where $d\mu_{n_j}$ is defined in (40).*

Proof. Take a subsequence of the measures $\{\mu_{n_j}\}$ which converges weak-* to a probability measure σ on K. We use the same notation for the subsequence and the original sequence. We show that $I^w(\sigma) = -\log\delta^w(K)$; by uniqueness of the weighted energy minimizing measure (37) we will then have $\sigma = \mu_{K,Q}$. First of all, choose continuous admissible weight functions $\{w_m\}$ with $w_m \downarrow w$ (recall w is usc!) and $w_m \ge \alpha_m > 0$ on K and for a real number M let

$$h_{M,m}(z,t) := \min[M, \log\frac{1}{|z-t|w_m(z)w_m(t)}] \le \log\frac{1}{|z-t|w_m(z)w_m(t)}$$

and

$$h_M(z,t) := \min[M, \log\frac{1}{|z-t|w(z)w(t)}] \le \log\frac{1}{|z-t|w(z)w(t)}.$$

Then $h_{M,m} \le h_M$. By the Stone-Weierstrass theorem, every continuous function on $K \times K$ can be uniformly approximated by finite sums of the form $\sum_j f_j(z)g_j(t)$ where f_j, g_j are continuous on K; hence $\mu_{n_j} \times \mu_{n_j} \to \sigma \times \sigma$ and we have

$$I^w(\sigma) = \lim_{M\to\infty}\lim_{m\to\infty}\int_K\int_K h_{M,m}(z,t)d\sigma(z)d\sigma(t)$$

$$= \lim_{M\to\infty}\lim_{m\to\infty}\lim_{j\to\infty}\int_K\int_K h_{M,m}(z,t)d\mu_{n_j}(z)d\mu_{n_j}(t)$$

$$\le \lim_{M\to\infty}\limsup_{j\to\infty}\int_K\int_K h_M(z,t)d\mu_{n_j}(z)d\mu_{n_j}(t)$$

since $h_{M,m} \le h_M$. Now

$$h_M(z_k^{(n_j)}, z_l^{(n_j)}) \le \log\frac{1}{|z_k^{(n_j)} - z_l^{(n_j)}|w(z_k^{(n_j)})w(z_l^{(n_j)})}$$

if $k \ne l$ and hence

$$\int_K\int_K h_M(z,t)d\mu_{n_j}(z)d\mu_{n_j}(t) \le$$

$$\frac{1}{n_j}M + (\frac{1}{n_j^2 - n_j})[\sum_{k\ne l}\log\frac{1}{|z_k^{(n_j)} - z_l^{(n_j)}|w(z_k^{(n_j)})w(z_l^{(n_j)})}].$$

By assumption, given $\epsilon > 0$,

$$(\frac{1}{n_j^2 - n_j})[\sum_{k\neq l} \log \frac{1}{|z_k^{(n_j)} - z_l^{(n_j)}|w(z_k^{(n_j)})w(z_l^{(n_j)})}] \leq -\log[\delta^w(K) - \epsilon]$$

for $j \geq j(\epsilon)$; in particular, $w(z_k^{(n_j)}) > 0$ for such j and hence

$$I^w(\sigma) \leq \lim_{M\to\infty} \limsup_{j\to\infty} \frac{1}{n_j} M - \log[\delta^w(K) - \epsilon] = -\log[\delta^w(K) - \epsilon]$$

for all $\epsilon > 0$; i.e., $I^w(\sigma) = -\log \delta^w(K)$. □

A weighted polynomial on K is a function of the form $w(z)^n p_n(z)$ where p_n is a holomorphic polynomial of degree at most n. As in the unweighted case, the weighted extremal function $V_{K,Q}$ can be obtained by using only polynomials; i.e.,

$$V_{K,Q}(z) = \sup\{\frac{1}{deg(p)} \log |p(z)| : p \text{ polynomial, } ||w^{deg(p)} p||_K \leq 1\}.$$

Let μ be a measure with support in K such that (K, w, μ) satisfies a Bernstein-Markov inequality for weighted polynomials: given $\epsilon > 0$, there exists a constant $M = M(\epsilon)$ such that for all weighted polynomials $w^n p_n$

$$||w^n p_n||_K \leq M(1+\epsilon)^n ||w^n p_n||_{L^2(\mu)}. \tag{41}$$

Equivalently, for all $p_n \in \mathcal{P}_n$,

$$||w^n p_n||_K \leq M_n ||w^n p_n||_{L^2(\mu)} \text{ with } \limsup_{n\to\infty} M_n^{1/n} = 1.$$

In this setting, we will restrict our attention to compact sets K. We have a weighted version of Theorem 2.9.

Theorem 5.3. *Let K be compact and let (K, w, μ) satisfy a Bernstein-Markov inequality for weighted polynomials. Then*

$$\lim_{n\to\infty} Z_n^{1/n^2} = \delta^w(K)$$

where, analogous to (20),

$$Z_n = Z_n(K, w, \mu) :=$$

$$\int_{K^{n+1}} |VDM(\lambda_0, ..., \lambda_n)|^2 w(\lambda_0)^{2n} \cdots w(\lambda_n)^{2n} d\mu(\lambda_0) \cdots d\mu(\lambda_n).$$

Note that the proofs of many of the results in the weighted situation are similar to their analogues in the unweighted case. We will see that in the case of $\mathbb{C}^N$, $N > 1$, the weighted theory is essential to prove results even in the unweighted case.

As an application of weighted potential theory, we consider the theory of incomplete polynomials. For simplicity, we work on the real interval $K = [0, 1]$. Given $0 < \theta < 1$, a $\theta-$incomplete polynomial is a polynomial of the form

$$p_N(x) = \sum_{k=s_N}^{N} a_k x^k$$

where $s_N/N \to \theta$ as $N \to \infty$. Thus such a polynomial is "missing" a fraction θ of its lowest degree terms. Taking $N = \frac{n}{1-\theta}$, we see that these incomplete polynomials are closely related to weighted polynomials $w(x)^n p_n(x)$ where $w(x) = x^{\frac{\theta}{1-\theta}}$. One can prove that $S_w = [\theta^2, 1]$. It turns out that *a continuous function f on $[0, 1]$ is the uniform limit of incomplete polynomials if and only if f vanishes on $[0, \theta^2]$ if and only if f is the uniform limit of weighted polynomials $w(x)^n p_n(x)$*. This is a special case of the general *weighted approximation problem*: given $K \subset \mathbb{C}$ closed and an admissible weight w on K, which $f \in C(K)$ can be uniformly approximated on K by a sequence of weighted polynomials $\{w^n p_n\}$? For details, see Chapter VI, section 1 of [26].

Exercises.

1. Following the "unweighted" proof, verify that the limit
$$\lim_{n\to\infty} \delta_n^w(K) = \delta^w(K)$$
exists for a nonpolar set K and an admissible weight function w on K. Here $\delta_n^w(K)$ is defined in (36).
2. Using the previous exercise, and observing that the function $VDM(\lambda_0, ..., \lambda_n)w(\lambda_0)^n \cdots w(\lambda_n)^n$ is a weighted polynomial of degree at most n in each variable, prove Theorem 5.3.
3. Use Proposition 5.1 to verify for K the closed unit disk $\{z : |z| \leq 1\}$ and $Q(z) = |z|^2$, that $V_{K,Q} = Q$ on the disk $\{z : |z| \leq 1/\sqrt{2}\}$ and $V_{K,Q}(z) = \log |z| + 1/2 - \log(1/\sqrt{2})$ outside this disk.
4. Prove the following weighted version of Corollary 4.3: let K be a regular compact set, let $w = e^{-Q}$ be continuous, and for $n = 1, 2, ...,$ define
$$\Phi_{K,Q,n}(z) := \sup\{|p(z)| : ||w^{\deg p} p||_K \leq 1, \ p \in \mathcal{P}_n\}.$$
Then
$$\frac{1}{n} \log \Phi_{K,Q,n} \to V_{K,Q}$$
locally uniformly on $\mathbb{C}$.

6 Plurisubharmonic functions in $\mathbb{C}^N$, $N > 1$ and the complex Monge-Ampère operator.

Let D be a domain in $\mathbb{C}^N$. A complex-valued function $f : D \to \mathbb{C}$ is called *holomorphic* and we write $f \in \mathcal{O}(D)$ if f is holomorphic in each variable $z_1, ..., z_N$ separately. Apriori, if one assumes that $f \in C^1(D)$, holomorphicity is equivalent to f satisfying the system of partial differential equations

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \ j = 1, ..., N \tag{42}$$

where, for $z_j = x_j + i y_j$,

$$\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right).$$

It turns out that the hypothesis that $f \in C^1(D)$ is superfluous. A *holomorphic mapping* $F : D' \to D$ where D' is a domain in $\mathbb{C}^m$ is a mapping $F = (f_1, ..., f_N)$ where each $f_i : D' \to \mathbb{C}$ is holomorphic. Our main interest is in the class of *plurisubharmonic* (psh) functions: a real-valued function $u : D \to [-\infty, +\infty)$ defined on a domain $D \subset \mathbb{C}^N$ is plurisubharmonic in D and we write $u \in PSH(D)$ if the following two conditions are satisfied:

1. u is uppersemicontinuous on D and
2. $u|_{D \cap l}$ is subharmonic (shm) on components of $D \cap l$ for each complex line (one-dimensional (complex) affine space) l.

Remark 3. It is unknown if (2) implies (1); i.e., it is unknown whether condition (1) is superfluous.

From this definition, and the properties of shm functions on domains in $\mathbb{C}$, many analogous properties follow readily for psh functions. Analogous to the univariate case, smoothing a psh function u by convolving with a *radial* regularizing kernel $\chi(z_1, ..., z_N) = \chi(|z_1|, ..., |z_N|)$ gives a plurisubharmonic function (on a smaller domain), so that given u psh in a domain D, we can find a decreasing sequence $\{u_j\}$ of smooth psh functions, $u_j = u * \chi_{1/j}$ defined on $\{z \in D : \operatorname{dist}(z, \partial D) > 1/j\}$ with $\lim_j u_j = u$ in D. This allows us, as in the subharmonic case, to verify properties for smooth psh functions and then pass to the limit. The class of psh functions on a domain D, denoted $PSH(D)$, forms a convex cone; i.e., if $u, v \in PSH(D)$ and $\alpha, \beta \geq 0$, then $\alpha u + \beta v \in PSH(D)$. The limit function $u(z) := \lim_{n \to \infty} u_n(z)$ of a decreasing sequence $\{u_n\} \subset PSH(D)$ is psh in D (we may have $u \equiv -\infty$); while for any family $\{v_\alpha\} \subset PSH(D)$ (resp., sequence $\{v_n\} \subset PSH(D)$) which is uniformly bounded above on any compact subset of D, the functions

$$v(z) := \sup_\alpha v_\alpha(z) \text{ and } w(z) := \limsup_{n \to \infty} v_n(z)$$

are "nearly" psh: the usc regularizations

$$v^*(z) := \limsup_{\zeta \to z} v(\zeta) \text{ and } w^*(z) := \limsup_{\zeta \to z} w(\zeta)$$

are psh in D. Finally, if ϕ is a real-valued, convex increasing function of a real variable, and u is psh in D, then so is $\phi \circ u$.

Analogous to the univariate case, a set of the form

$$N := \{z \in D : v(z) := \sup_\alpha v_\alpha(z) < v^*(z)\} \tag{43}$$

where $\{v_\alpha\} \subset PSH(D)$ is called a *plurinegligible* set; and $E \subset \mathbb{C}^N$ is *pluripolar* if there exists u psh, $u \not\equiv -\infty$ with $E \subset \{u(z) = -\infty\}$ (we will be more precise about this notion in section 7). The proof that any polar set is negligible in section 3 using (21) carries over to show any pluripolar set is plurinegligible; the converse is true but is a very deep result of Bedford and Taylor [4].

If $u \in C^2(D)$, then u is psh if and only if for each $z \in D$ and vector $a \in \mathbb{C}^N$, the Laplacian of $t \mapsto u(z + ta)$ is nonnegative at $t = 0$; i.e., *the complex Hessian* $[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z)]$ *of* u *is positive semidefinite on* D:

$$\sum_{j,k=1}^N \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) a_j \bar{a}_k \geq 0$$

Exercise 1 will verify this. In particular, the trace of the complex Hessian is nonnegative so that u is $\mathbb{R}^{2N}-$subharmonic. If the complex Hessian is positive *definite* on D, we say that u is *strictly* psh there.

Proposition 6.1. *A function $u : D \to [-\infty, +\infty)$ is psh if and only if for all holomorphic mappings $F : D' \to D$ where $D' \subset \mathbb{C}^m$ either $u \circ F$ is shm in D' (in the $\mathbb{R}^{2m}$ sense) or $u \circ F \equiv -\infty$.*

Proof. If $u \in PSH(D) \cap C^2(D)$, the holomorphicity of $F = (f_1, ..., f_N)$ and the chain rule (use (42) for each f_j) show that the complex Hessian of $u \circ F$ is positive semidefinite in D'; i.e., $u \circ F \in PSH(D')$ (and hence shm in D' in the $\mathbb{R}^{2m}$ sense). For arbitrary $u \in PSH(D)$, take a decreasing sequence $\{u_j\}$ of smooth psh functions, $u_j = u * \chi_{1/j}$ defined on $\{z \in D : \operatorname{dist}(z, \partial D) > 1/j\}$ with $\lim_j u_j = u$ in D and apply the previous result to $\{u_j\}$; then, since a decreasing sequence of psh functions is psh or identically minus infinity, the result follows.

The converse is trivial since one can take the holomorphic maps $t \to a + tb$ for $a \in D$, $b \in \mathbb{C}^n$, and $t \in \mathbb{C}$ with $|t|$ sufficiently small. □

Indeed, it turns out that $u : D \to [-\infty, +\infty)$ *is psh if and only if* $u \circ A$ *is* $\mathbb{R}^{2N}-$*subharmonic in* $A^{-1}(D)$ *for every complex linear isomorphism* $A : \mathbb{C}^N \to \mathbb{C}^N$.

The canonical examples of psh functions are those of the form $u = \log|f|$ where $f \in \mathcal{O}(D)$. In particular, if $p(z) := p(z_1, ..., z_N)$ is a holomorphic polynomial of degree $d \geq 1$, then

$$u(z) := \frac{1}{d} \log|p(z)|$$

is a psh function in all of $\mathbb{C}^N$ with the property that

$$u(z) \leq \log|z| + 0(1) \text{ as } |z| \to \infty.$$

The class

$$L = L(\mathbb{C}^N) := \{u \text{ psh in } \mathbb{C}^N : \ u(z) - \log|z| = 0(1), \ |z| \to \infty\}$$

of psh functions of logarithmic growth, the multivariate analogue of (4), plays an important role in pluripotential theory.

However, unlike logarithmic potential theory in the plane, in which case subharmonic functions are those locally integrable functions u with Laplacian $\Delta u \geq 0$ in the sense of distributions, the differential operator of paramount importance in $\mathbb{C}^N$ if $N > 1$ is a non-linear operator, the so-called *complex Monge-Ampère operator*. We proceed with an introduction to this topic.

If $u \in C^1(D)$, we write the $1-$form

$$du = \sum_{j=1}^{N} \frac{\partial u}{\partial z_j} dz_j + \sum_{j=1}^{N} \frac{\partial u}{\partial \overline{z}_j} d\overline{z}_j =: \partial u + \overline{\partial} u$$

as the sum of a form ∂u of *bidegree* $(1,0)$ and a form $\overline{\partial} u$ of bidegree $(0,1)$ where, recall,

$$\frac{\partial u}{\partial z_j} = \frac{1}{2}(\frac{\partial u}{\partial x_j} - i\frac{\partial u}{\partial y_j}); \ \frac{\partial u}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial u}{\partial x_j} + i\frac{\partial u}{\partial y_j});$$

and we have

$$dz_j = dx_j + idy_j; \ d\overline{z}_j = dx_j - idy_j.$$

For a complex-valued $f \in C^1(D)$, one easily checks that f is holomorphic in D if and only if $\overline{\partial} f = 0$ in D (see also exercise 16 at the end of this section). We also define

$$d^c u := i(\overline{\partial} u - \partial u).$$

Note that if $u \in C^2(D)$,

$$dd^c u = 2i\partial\overline{\partial} u = 2i \sum_{j,k=1}^{N} \frac{\partial^2 u}{\partial z_j \overline{\partial} z_k} dz_j \wedge d\overline{z}_k$$

so that the coefficients of the $2-$form $dd^c u$ form the $N \times N$ complex Hessian matrix

$$H(u) := [\frac{\partial^2 u}{\partial z_j \overline{\partial} z_k}]_{j,k=1}^N,$$

of u. We saw that if $u \in C^2(D)$, then $u \in PSH(D)$ if and only if $H(u)$ is positive semi-definite at each point of D; i.e., $dd^c u$ is a *positive form of bidegree* $(1,1)$; more generally it turns out that if u is only usc and locally integrable on D, then $u \in PSH(D)$ if and only if $dd^c u$ is a *positive current*. For a brief overview of differential forms in $\mathbb{C}^N$ and currents – differential forms with distribution coefficients – see Appendix A. We remark that if $u \in C^2(D) \cap PSH(D)$, then

1. the trace of $H(u)$ is nonnegative – this is (one-fourth) the $\mathbb{R}^{2N}$ Laplacian so that *a psh function* u *is* $\mathbb{R}^{2N}-$*subharmonic*; and
2. the determinant of $H(u)$ is a nonnegative function on u.

Elementary linear algebra shows that

$$(dd^c u)^N := dd^c u \wedge \cdots \wedge dd^c u = c_N \det H(u) dV$$

where $dV = (\frac{1}{2i})^N dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_N \wedge d\overline{z}_N$ is the volume form on $\mathbb{C}^N$ and c_N is a dimensional constant.

For $u \in C^2(D)$, we thus obtain an absolutely continuous measure, $(dd^c u)^N$, the *complex Monge-Ampère measure* associated to u. To elaborate in $\mathbb{C}^2$ with variables (z, w), for a C^1 function u,

$$\partial u := \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial w} dw, \quad \overline{\partial} u := \frac{\partial u}{\partial \overline{z}} d\overline{z} + \frac{\partial u}{\partial \overline{w}} d\overline{w}.$$

For a C^2 function u,

$$dd^c u = 2i\big[\frac{\partial^2 u}{\partial z \partial \overline{z}} dz \wedge d\overline{z} + \frac{\partial^2 u}{\partial w \partial \overline{w}} dw \wedge d\overline{w} + \frac{\partial^2 u}{\partial z \partial \overline{w}} dz \wedge d\overline{w} + \frac{\partial^2 u}{\partial \overline{z} \partial w} d\overline{z} \wedge dw\big]$$

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and

$$(dd^c u)^2 = 16\big[\frac{\partial^2 u}{\partial z\partial \overline{z}}\frac{\partial^2 u}{\partial w\partial \overline{w}} - \frac{\partial^2 u}{\partial z\partial \overline{w}}\frac{\partial^2 u}{\partial w\partial \overline{z}}\big]\frac{i}{2}dz\wedge d\overline{z}\wedge \frac{i}{2}dw\wedge d\overline{w}$$

is indeed a positive constant times the determinant of the complex Hessian of u times the volume form on $\mathbb{C}^2$. Thus if u is also psh, $(dd^c u)^2$ is a positive measure which is absolutely continuous with respect to Lebesgue measure. Note that for real-valued u,

$$\frac{\partial^2 u}{\partial z\partial \overline{w}} = \overline{\frac{\partial^2 u}{\partial w\partial \overline{z}}}.$$

As an elementary example, take $u(z,w) = |z|^2 + |w|^2 = z\overline{z} + w\overline{w}$. Then

$$dd^c u = 2idz\wedge d\overline{z} + 2idw\wedge d\overline{w}$$

and

$$(dd^c u)^2 = 16\cdot \frac{i}{2}dz\wedge d\overline{z}\wedge \frac{i}{2}dw\wedge d\overline{w}.$$

Bedford and Taylor, and, independently, Sadullaev, have shown how to associate a positive measure (not necessarily absolutely continuous) to any locally bounded plurisubharmonic function u in such a way that, among other things, this *Monge-Ampère measure* associated to u, denoted $(dd^c u)^N$, is continuous under decreasing limits – if $\{u_j\}$ form a decreasing sequence of locally bounded psh functions with $u_j \downarrow u$ then

$$(dd^c u_j)^N \to (dd^c u)^N$$

weakly as measures. In particular, since, as with subharmonic functions, given a general psh function u on a domain D, the standard smoothings $u_j := u * \chi_{1/j}$ decrease to u, this gives us a way of (in principle) computing $(dd^c u)^N$. For a general psh function, $dd^c u$ is a positive $(1,1)-$current; i.e., a $(1,1)-$form with distribution coefficients. Hence the wedge product $dd^c u\wedge dd^c u$ does not, apriori, make sense as we would be multiplying distributions or measures. Bedford and Taylor [3] gave an inductive way to define $(dd^c u)^k,\ k = 1,...,N$, for $u\in L^\infty_{loc}(D)\cap PSH(D)$. We give their definition of $(dd^c u)^2$ in $\mathbb{C}^2$ for u psh and locally bounded in D.

We first recall that a psh function u in D is an usc function u in D which is subharmonic on components of $D\cap l$ for complex affine lines l. In particular, u is a locally integrable function in D such that $dd^c u$ is a positive $(1,1)$ current. The derivatives are to be interpreted in the distribution sense and are actually measures; i.e., they act on compactly supported continuous functions. Here, a $(1,1)$ current T on a domain D in $\mathbb{C}^2$ is positive if T applied to $i\beta\wedge\overline{\beta}$ is a positive distribution for all $(1,0)$ forms $\beta = adz + bdw$ with $a,b\in C_0^\infty(D)$ (smooth functions having compact support in D). Writing the action of a current T on a form ψ as $< T,\psi >$, this means that

$$< T,\phi(i\beta\wedge\overline{\beta}) >\geq 0 \quad \text{for all } \phi\in C_0^\infty(D) \text{ with } \phi\geq 0.$$

As an example, take $u(z,w) = \log|z|$. Then the $(1,1)$ current

$$T = dd^c u = i\pi\delta_0(z)dz\wedge d\overline{z}$$

is a current of integration on the complex line $E = \{(z,w) : z = 0\}$. Here we have written $dd^c u$ as a $(1,1)$ form where the coefficient $\delta_0(z)$ is a distribution, the point mass at $z = 0$ in the complex $z-$plane. More generally, if f is holomorphic and $u = \log|f|$, then, locally, $dd^c u$ is the current of integration on the complex hypersurface $\{f = 0\}$. For a discussion of currents and the general definition of positivity, we refer the reader to Klimek [K], section 3.3.

Following [3], we now define $(dd^c u)^2$ for a psh u in D if $u\in L^\infty_{loc}(D)$ using the fact that $dd^c u$ is a positive $(1,1)$ current with measure coefficients. First note that if u were of class C^2, given $\phi\in C_0^\infty(D)$, we have

$$\int_D \phi(dd^c u)^2 = -\int_D d\phi\wedge d^c u\wedge dd^c u \text{ (exercise 15)} \tag{44}$$

$$= \int_D du\wedge d^c\phi\wedge dd^c u = \int_D udd^c\phi\wedge dd^c u$$

since all boundary integrals vanish. The applications of Stokes' theorem are justified if u is smooth; for arbitrary $u\in PSH(D)\cap L^\infty_{loc}(D)$, these formal calculations serve as motivation to *define* $(dd^c u)^2$ as a positive measure (precisely, a positive current of bidegree $(2,2)$ and hence a positive measure) via

$$< (dd^c u)^2,\phi >:= \int_D udd^c\phi\wedge dd^c u.$$

This defines $(dd^c u)^2$ as a $(2,2)$ current (acting on $(0,0)$ forms; i.e., test functions) since $udd^c u$ has measure coefficients. We refer the reader to [3] or [20] (p. 113) for the verification of the positivity of $(dd^c u)^2$.

In some sense, the complex Monge-Ampère measure associated to a locally bounded psh function is a "minimal" Laplacian. Bellman's principle states that if B is a positive semidefinite Hermitian $N\times N$ matrix, then

$$(\det B)^{1/N} = \frac{1}{N}\inf_A \text{trace}(AB)$$

where the infimum is taken over all positive definite Hermitian $N \times N$ matrices A with $\det A = 1$. Hence, given such a matrix $A = [a_{jk}]$, let

$$\Delta_A := \frac{1}{N} \sum_{j,k=1}^{N} a_{jk} \frac{\partial^2}{\partial z_j \partial \overline{z}_k}.$$

Then $(dd^c u)^N = \inf_A [\Delta_A u]^N$ if $u \in C^2(D)$.

Exercises.

1. Verify that for $u \in C^2(D)$, $z \in D$, and $a \in \mathbb{C}^N$ the Laplacian of $t \mapsto u(z + ta)$ (for $t \in \mathbb{C}$ with $z + ta \in D$) is equal to a positive multiple of
$$\sum_{j,k=1}^{N} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) a_j \overline{a}_k.$$
2. Prove that if u is psh in a domain $D \subset \mathbb{C}^N$, then u is shm as a function on a domain in $\mathbb{R}^{2N}$; i.e., u is usc in D and $\Delta u \geq 0$ in the sense of distributions.
3. If $N > 1$, find a function u which is shm in $\mathbb{C}^N = \mathbb{R}^{2N}$ but which is not psh in $\mathbb{C}^N$. Can you find such a u which is *harmonic* in $\mathbb{C}^N = \mathbb{R}^{2N}$?
4. Find a harmonic function h in $\mathbb{R}^2$ and a real linear isomorphism $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h \circ T$ is not subharmonic in $\mathbb{R}^2$.
5. Verify that if ϕ is a real-valued, convex increasing function of a real variable, and $u \in C^2(D)$ is psh in D, then $\phi \circ u$ is psh in D. (Note, in particular, that e^u is psh in D).
6. *Gluing psh functions*. Let u, v be psh in open sets U, V where $U \subset V$ and assume that $\limsup_{\zeta \to z} u(\zeta) \leq v(z)$ for $z \in V \cap \partial U$. Show that the function w defined to be $w = \max(u, v)$ in U and $w = v$ in $V \setminus U$ is psh in V.
7. Let $E = E_1 \times E_2 \subset \mathbb{C} \times \mathbb{C} = \mathbb{C}^2$. Show that E is pluripolar in $\mathbb{C}^2$ if and only if at least one of E_1, E_2 is polar in $\mathbb{C}$.
8. Is $\{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im} z_1 = \mathrm{Im} z_2 = 0\}$ pluripolar? Why or why not?
9. Is $\{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}$ pluripolar? Why or why not?
10. *Extra Credit*. A psh function $u(z_1, ..., z_n)$ is, in particular, shm in each complex variable z_j when all of the others are fixed. Is the converse true?
11. Let $D \subset \mathbb{C}^N = \mathbb{R}^{2N}$ be a bounded, smoothly bounded domain and let ρ be a smooth *defining function* for D: ρ is defined and smooth on a neighborhood of $\overline{D}$; $D = \{z : \rho(z) < 0\}$; and $\nabla \rho \neq 0$ on ∂D.
    (a) Show that $\nabla \rho \neq 0$ on ∂D is equivalent to $d\rho \neq 0$ on ∂D and the tangent space $T_p(\partial D)$ at any point $p \in \partial D$ is given by $\{v \in \mathbb{C}^N : d\rho(v) = 0\}$.
    (b) Show that the coefficient functions of $d^c \rho$ at $p \in \partial D$ define a tangent vector to ∂D at p.
    (c) As an example, take $\rho(z) = |z_1|^2 + \cdots + |z_N|^2 - 1$. Then D is the unit ball. Compute $T_p(\partial D)$ for $p = (1, 0, ..., 0)$ and the coefficient functions of $d^c \rho$ at this point.
12. *An illustrative example*. In $\mathbb{C}^2$, let $u(z, w) = \frac{1}{2} \log(|z|^2 + |w|^2)$. This psh function is smooth away from $(0, 0)$. Prove that
$$(dd^c u)^2 = 0 \text{ on } \mathbb{C}^2 \setminus \{0\}.$$
(Note u is not locally bounded near $(0, 0)$ but it turns out that one can define $(dd^c v)^2$ for psh v with compact singularities and here $(dd^c u)^2 = (2\pi)^2 \delta_{(0,0)}$.)
13. In $\mathbb{C}^2$, let $v(z, w) = \frac{1}{2} \log(|z|^2 + |w|^4)$. This psh function is smooth away from $(0, 0)$. Prove that $(dd^c v)^2 = 0$ on $\mathbb{C}^2 \setminus \{0\}$. (Here, it turns out that $(dd^c v)^2 = 2(2\pi)^2 \delta_{(0,0)}$.)
14. In $\mathbb{C}^2$, let $u(z, w) = |z|^2 + |w|^2$. Compute $(dd^c u)^2$.
15. In (44), verify the equality
$$-\int_D d\phi \wedge d^c u \wedge dd^c u = \int_D du \wedge d^c \phi \wedge dd^c u.$$
16. For a complex-valued $f \in C^1(D)$, write $f = u + iv$ where u, v are real-valued. Show that f is holomorphic in D if and only if $d^c u = dv$ in D.

7 Upper envelopes, extremal plurisubharmonic functions and applications.

There is a special subclass of psh functions which play the role of harmonic functions in classical potential theory, the so-called *maximal* psh functions. We call $u \in PSH(D)$ *maximal* if, for any relative compact subdomain D' and any $v \in PSH(D')$ which is usc on $\overline{D}'$, if $u \geq v$ on $\partial D'$, then $u \geq v$ in D'. If u is harmonic (in the $\mathbb{R}^{2N}$ sense; i.e., $\Delta u \geq 0$), and u is psh, then u is clearly maximal. In this case, (exercise 2) u is *pluriharmonic*; i.e., $dd^c u = 0$ in D, which is equivalent to $u|_{D\cap l}$ is *harmonic* on components of $D\cap l$ for each complex line l. Pluriharmonic functions are very special; locally, such a function is the real part of a holomorphic function. The converse statement, that the real and imaginary parts of a holomorphic function are pluriharmonic, follows from exercise 16 of section 6.

However, maximal psh functions need not even be continuous; indeed, if u is a psh function depending on fewer than N of the variables $z_1,...,z_N$, then u is maximal (why?). In the case where $u \in L^\infty_{loc}(D) \cap PSH(D)$, it is known that u is maximal in D if and only if $(dd^c u)^N = 0$ in D. Thus solutions of a *Dirichlet problem for the complex Monge-Ampère operator* are maximal. We can easily verify the maximality criterion for smooth psh functions.

Proposition 7.1. *Let $u \in C^2(D)$ be psh. If u is maximal in D then* $\det H(u) \equiv 0$ *in D; i.e., $(dd^c u)^N = 0$ in D.*

Proof. Suppose u is maximal in D but $\det H(u) \not\equiv 0$ in D. We can find a point $z_0 \in D$ such that for each $a \in \mathbb{C}^N \setminus \{0\}$

$$\sum_{j,k=1}^N \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z_0) a_j \overline{a}_k > 0.$$

This strict inequality persists for all $z \in \overline{B}(z_0, r)$ for small $r > 0$ (why?). By compactness of $\overline{B}(z_0, r)$ we can find $c > 0$ with

$$\sum_{j,k=1}^N \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) a_j \overline{a}_k \geq c \sum_{j=1}^n |a_j|^2 \tag{45}$$

for $z \in \overline{B}(z_0, r)$ and for each $a \in \mathbb{C}^N \setminus \{0\}$. From (45), the function $v(z)$ defined to be $v(z) = u(z)$ on $D \setminus \overline{B}(z_0, r)$ and $v(z) = u(z) + c(r^2 - |z - z_0|^2)$ on $\overline{B}(z_0, r)$ is psh in D. Moreover, v agrees with u on $\partial\overline{B}(z_0, r)$; and we have $v(z_0) > u(z_0)$, contradicting maximality of u. □

The converse is also true. Note this generalizes the univariate situation where $\det H(u) \equiv 0$ simply says that $\Delta u = 0$.

We now outline the procedure of solving the Dirichlet problem for the complex Monge-Ampère operator in the unit ball B in $\mathbb{C}^N$. Let f be a continuous, real-valued function on ∂B. We seek a psh function u in B, $u \in C(\overline{B})$, with $u = f$ on ∂B and $(dd^c u)^N = 0$ in B. Bedford and Taylor proved existence and uniqueness of the solution u (in the slightly more general setting where B is a so-called *strictly pseudoconvex* domain). We caution the reader that no matter how smooth f is, the solution u is generally not in $C^2(B)$ (although $u \in C^{1,1}(B)$ if $f \in C^2(\partial B)$; see (4) below). To construct u, one forms the Perron-Bremmermann envelope

$$U(0; f)(z) := u(z)$$
$$:= \sup\{v(z) : v \in PSH(B) : \limsup_{z\to\zeta} v(z) \leq f(\zeta) \text{ for all } \zeta \in \partial B\}.$$

The proof that u works consists of the following steps:

1. *$u \in C(B)$ and $u = f$ on ∂B*:

   Proof of (1): We first show $u = u^*$ in B. Take h harmonic (in the $\mathbb{R}^{2N}$−sense) in B with $h = f$ on ∂B; clearly $u \leq h$ in B since each competitor v is shm and satisfies $v \leq h$. It is classical that h is continuous on $\overline{B}$ hence $u^* \leq h$ in $\overline{B}$ so that, in particular, since u^* is psh and satisfies $\limsup_{z\to\zeta} u^*(z) \leq f(\zeta)$ for all $\zeta \in \partial B$, $u^* \leq u$ in B and equality holds.

   Now we show $u^* = f$ on ∂B. Fix $z_0 \in \partial B$ and $\epsilon > 0$ and define, where $< z, z' > := \sum_{j=1^N} z_j \overline{z}'_j$,

   $$v(z) := c\left[\text{Re} < z, z_0 > -1\right] + f(z_0) - \epsilon \in C(\overline{B})$$

   where $c > 0$ is chosen to insure $v \leq f$ on ∂B. Note that v is a competitor for u and, by construction, $v(z_0) = f(z_0) - \epsilon$; thus

   $$\liminf_{z\to z_0} u(z) \geq f(z_0),$$

   yielding the result. Here, the function $b(z) := \text{Re} < z, z_0 > -1$ is a psh *barrier* for ∂B at z_0: $b \in PSH(B) \cap C(\overline{B})$ with $b(z_0) = 0 > b(z)$ for $z \in B$.

2. *u is maximal in B*;

   Proof of (2): If $G \subset\subset B$, v is usc on $\overline{G}$, psh on G and $v \leq u$ on ∂G, then by the gluing lemma for psh functions, the function V defined as $V = \max(u, v)$ in G and $V = u$ in $B \setminus G$ is psh and is a competitor for u; thus, in particular, $v \leq u$ in G.

3. $u \in C(\overline{B})$;

   This is a theorem of J. B. Walsh (cf., Theorem 3.1.4 [20]); it uses the notion of psh barriers but is fairly straighforward; we omit the proof.

4. *If $f \in C^2(\partial B)$, then $u \in C^{1,1}(B)$:*
   This is very clever; it uses *automorphisms* of B to show, e.g., that given $\epsilon > 0$, there exists $C > 0$ such u satisfies an estimate of the form
$$u(z+h) - 2u(z) + u(z-h) \leq C|h|^2$$
   for $|z| \leq 1-\epsilon$ and $|h| \leq \epsilon/2$.
5. $(dd^c u)^N = 0$ on B:
   This is first proved under the assumption that $u \in C^{1,1}(B)$ which follows if $f \in C^{1,1}(\partial B)$. The general case follows by approximating $f \in C(\partial B)$ by a decreasing sequence $f_j \in C^2(\partial B)$, giving rise to a corresponding sequence $\{u_j\}$ which decrease and converge uniformly to u; since $(dd^c u_j)^N = 0$ and the complex Monge-Ampère operator is continuous under decreasing limits, we have $(dd^c u)^N = 0$.

A nice exposition of the details of steps (3)-(5) can be found in chapter 4 of [20]. We remark that it is already easy to see from (1)-(3) that a general maximal psh function is locally a decreasing limit of continuous maximal functions:

Proposition 7.2. *Let u be psh and maximal in a domain $D \subset \mathbb{C}^N$. For any ball B with $\overline{B} \subset D$, there exist $\{u_j\}$ continuous in $\overline{B}$ and psh and maximal in B with $u_j \downarrow u$ in B.*

Proof. By smoothing, we can find $\{v_j\}$ psh and smooth in a neighborhood G of $\overline{B}$ with $G \subset D$ and $v_j \downarrow u$ in G. Now define u_j on G by $u_j = U(0; v_j|_{\partial B})$ in $\overline{B}$ and $u_j = v_j$ in $G \setminus \overline{B}$. □

Here is an interesting example, due to Gamelin, of $f \in C^\infty(\partial B)$ – indeed, here we will have $f \in C^\omega(\partial B)$! – with $u \notin C^2(B)$. Take, for $N = 2$,
$$f(z,w) = (|z|^2 - 1/2)^2 = (|w|^2 - 1/2)^2.$$
Then
$$u(z,w) = \left(\max[0, |z|^2 - 1/2, |w|^2 - 1/2]\right)^2$$
satisfies $(dd^c u)^2 = 0$ in B and $u = f$ on ∂B, but $u \notin C^2(B)$.

Returning to our discussion of maximal psh functions, for a function $u \in PSH(D) \cap C^2(D)$, it is easy to see why $(dd^c u)^N = 0$ implies that u is maximal: at each point $z_0 \in D$, $H(u)$ has a zero eigenvalue; assuming, as we do for simplicity, that $(dd^c u)^{N-1} \neq 0$, we can find an analytic disk through z_0 on which u is harmonic. That is, there exists a holomorphic mapping f from the unit disk in $\mathbb{C}$ into D with $u(0) = z_0$ such that $u \circ f$ is harmonic on D. Any psh function v is subharmonic on this disk; if u dominates v on the boundary of the disk, then u dominates v in the disk. Moreover, we have the following elementary result, generalizing Proposition 1.5 (the *comparison principle*): *Let $u, v \in PSH(D) \cap C^2(\overline{D})$; suppose $u|_{\partial D} = v|_{\partial D}$ and $u \geq v$ in D. Then*
$$\int_D (dd^c u)^N \leq \int_D (dd^c v)^N. \tag{46}$$
We verify this for $N = 2$. We have
$$\int_D [(dd^c v)^2 - (dd^c u)^2] = \int_D (dd^c v - dd^c u) \wedge (dd^c v + dd^c u)$$
$$= \int_{\partial D} d^c(v-u) \wedge (dd^c v + dd^c u).$$
This last integral is nonnegative because $dd^c v + dd^c u$ is a positive $(1,1)-$form and $v - u = a\rho$ where ρ is a defining function for D (see exercise 11 from section 6) and $a \geq 0$. Hence, on ∂D, $d^c(v-u) = ad^c\rho$ and one can show that $ad^c\rho \wedge (dd^c v + dd^c u) = f d\sigma$ where $d\sigma$ is surface area on ∂G and $f \geq 0$. Equation (46) shows why maximal psh functions have "minimal" Monge-Ampère mass.

Given these results on the Dirichlet problem and maximal psh functions, many important notions and results in pluripotential theory can be proved in ways analogous to those in classical logarithmic potential theory. We now describe some *extremal* psh functions modeled on their one-variable counterparts.

Recall the class of plurisubharmonic functions u in $\mathbb{C}^N$ of *logarithmic growth*, i.e., such that $u(z) \leq \log|z| + C$, $|z| \to \infty$ where $C = C(u)$, is called the class $L = L(\mathbb{C}^N)$. The functions $\frac{1}{\deg p} \log|p(z)|$ for a polynomial p clearly belong to L. For any Borel set E, set
$$V_E(z) := \sup\{u(z) : u \in L,\ u \leq 0 \text{ on } E\} \tag{47}$$
and we call $V_E^*(z)$ the *L-extremal function of E*. We generally restrict our attention to compact sets $K \subset \mathbb{C}^N$. The function V_K is lower semicontinuous, but it need not be upper semicontinuous. The proof of Proposition 3.2 carries over to show that the upper semicontinuous regularization
$$V_K^*(z) = \limsup_{\zeta \to z} V_K(\zeta)$$
of V_K is either identically $+\infty$ or else V_K^* is plurisubharmonic. The first case occurs if the set K is too "small"; precisely if K is pluripolar. In the second case, as in the univariate situation, we have $V_K^* \in L^+(\mathbb{C}^N)$ where
$$L^+(\mathbb{C}^N) := \{u \in L(\mathbb{C}^N) : u(z) \geq \log^+|z| + C\}$$

where $C = C(u)$. We say that K is *L-regular* if $V_K = V_K^*$, that is, if V_K is continuous. For example, if $\mathbb{C}^N \setminus K$ is regular with respect to $\mathbb{R}^{2N}$-potential theory, then K is L-regular.

A simple example is a closed Euclidean ball $K = \{z \in \mathbb{C}^N : |z-a| \leq R\}$; in this case, $V_K(z) = V_K^*(z) = \max[0, \log|z-a|/R]$. Let's verify this for $a = 0$ and $R = 1$; i.e., for the closed unit ball $K = \{z \in \mathbb{C}^N : |z| \leq 1\}$, we show $V_K(z) = V_K^*(z) = \log^+|z|$. Clearly $V_K(z) \geq \log^+|z|$ since $\log^+|z| \in L$ and is 0 on K. For the reverse inequality, take any $u \in L$ with $u \leq 0$ on K. For $w \in \mathbb{C}^N$ with $|w| > 1$, the univariate function

$$v(\zeta) := u(w/\zeta) - \log^+ \frac{|w|}{|\zeta|}$$

is shm on the punctured disk $\{\zeta \in \mathbb{C} : 0 < |\zeta| < |w|\}$. Since $u \in L$, v is bounded above as $|\zeta| \to 0$. Thus by Proposition 3.3, v extends to a shm function $\tilde{v}$ on the disk $D := \{\zeta \in \mathbb{C} : |\zeta| < |w|\}$. Since $v = \tilde{v} \leq 0$ on ∂D, by the maximum principle (Proposition 1.1), $v = \tilde{v} \leq 0$ on D. In particular, $v(1) = \tilde{v}(1) = u(w) - \log^+|w| \leq 0$, finishing the proof.

For a product set $K = K_1 \times \cdots \times K_N$ of planar compact sets $K_j \subset \mathbb{C}$, $V_K(z_1,\ldots,z_N) = \max_{j=1,\ldots,N} V_{K_j}(z_j)$. In particular, for a polydisk

$$P := \{(z_1,\ldots,z_N) : |z_j - a_j| \leq r_j,\ j = 1,\ldots,N\},$$

$V_K(z_1,\ldots,z_N) = \max_{j=1,\ldots,N}[0, \log|z_j - a_j|/r_j]$. Any compact set K can be approximated from above by the decreasing sequence of L-regular sets $K_n := \{z : \text{dist}(z,K) \leq 1/n\}$. The fact that each K_n is $L-$regular can be seen as in section 4 by utilizing the fact observed above that a closed Euclidean ball has this property.

As a generalization of the one-variable Green function g_K, we may define

$$\tilde{V}_K(z) := \max\left\{0, \sup_p \left\{\frac{1}{\deg p}\log|p(z)|\right\}\right\} \tag{48}$$

where the supremum is taken over all non-constant polynomials p with $||p||_K \leq 1$. We define the *polynomial hull* of K as

$$\widehat{K} \equiv \{z \in \mathbb{C}^N : |p(z)| \leq ||p||_K,\ p \text{ polynomial}\}.$$

Clearly $\tilde{V}_K = \tilde{V}_{\widehat{K}}$ and if $K = \widehat{K}$ we say K is polynomially convex. It turns out $\widehat{K}$ can just as well be constructed as a "hull" with respect to continuous psh functions; i.e., for D a neighborhood of $\widehat{K}$ (e.g., a sufficiently large ball or all of $\mathbb{C}^N$,

$$\widehat{K} = \widehat{K}_{PSH(D)} := \{z : u(z) \leq \sup_{\zeta \in K} u(\zeta) \text{ for all } u \in PSH(D) \cap C(D)\}.$$

It is known that for compact sets K, the upper envelope

$$V_K(z) := \sup\{u(z) : u \in L,\ u \leq 0 \text{ on } K\}$$

as defined in (47) coincides with that in (48). We sketch a proof of this. An important feature of the proof is the correspondence between psh functions in $L(\mathbb{C}^N)$ and "homogeneous" psh functions in $\mathbb{C}^{N+1}$. We remind the reader of the standard correspondence between polynomials p_d of degree d in N variables and homogeneous polynomials H_d of degree d in $N+1$ variables via

$$p_d(z_1,\ldots,z_N) \mapsto H_d(w_0,\ldots,w_N) := w_0^d p_d(w_1/w_0,\ldots,w_N/w_0).$$

Clearly $\tilde{V}_K(z) \leq V_K(z)$ and to prove the reverse inequality, by approximating K from above we may assume K is L-regular. We consider $h(z,w)$ defined for $(z,w) \in \mathbb{C}^{N+1} = \mathbb{C}^N \times \mathbb{C}$ as follows:

$$h(z,w) := |w| \exp V_K(z/w) \text{ if } w \neq 0;$$

$$h(z,w) := \limsup_{(z',w') \to (z,0)} h(z',w') \text{ if } w = 0.$$

This is a nonnegative *homogeneous* psh function in $\mathbb{C}^{N+1}$; i.e., we have $h(tz,tw) = |t|h(z,w)$ for $t \in \mathbb{C}$. We say that the function $\log h$ is *logarithmically homogeneous*: $\log h(tz,tw) = \log|t| + \log h(z,w)$. Fix a point $(z_0,w_0) \neq (0,0)$ with $z_0/w_0 \notin K$ and fix $0 < \epsilon < 1$. Using the fact that the polynomial hull coincides with the hull with respect to continuous psh functions, it follows that the compact set

$$E := \{(z,w) \in \mathbb{C}^{N+1} : h(z,w) \leq (1-\epsilon)h(z_0,w_0)\}$$

is polynomially convex. Moreover, E is *circled*: $(z,w) \in E$ implies $(e^{it}z, e^{it}w) \in E$ for all real t.

Claim. *Given a compact, circled set $E \subset \mathbb{C}^N$ and a polynomial $p_d = h_d + h_{d-1} + \cdots + h_0$ of degree d written as a sum of homogeneous polynomials, we have $||h_j||_E \leq ||p_d||_E$, $j = 0,\ldots,d$.*

From the Claim, whose proof is left as exercise 8, the polynomial hull of our circled set E is the same as the hull obtained using only homogeneous polynomials. Since $E = \widehat{E}$ and $(z_0,w_0) \notin E$, we can find a homogeneous polynomial h_s of degree s with $|h_s(z_0,w_0)| > ||h_s||_E$. Define

$$p_s(z,w) := \frac{h_s(z,w)}{||h_s||_E} \cdot [(1-\epsilon)h(z_0,w_0)]^s.$$

Then $|p_s(z,w)|^{1/s} \leq |h(z,w)|$ for $(z,w) \in \partial E$ and by homogeneity of $|p_s|^{1/s}$ and h we have $|p_s|^{1/s} \leq h$ in all of $\mathbb{C}^{N+1}$. At (z_0,w_0), we have

$$|p_s(z_0,w_0)|^{1/s} > (1-\epsilon)h(z_0,w_0);$$

since $\epsilon > 0$ was arbitrary, as was the point (z_0, w_0) (provided $z_0/w_0 \notin K$), we get that

$$h(z,w) = \sup_s\{|p_s(z,w)|^{1/s} : p_s \text{ homogeneous of degree } s,\ |p_s|^{1/s} \leq |h|\}.$$

At $w = 1$, we obtain

$$\exp V_K(z) = h(z,1)$$
$$= \sup_s\{|Q_s(z)|^{1/s} : Q_s \text{ of degree } s,\ |Q_s|^{1/s} \leq \exp V_K\}$$

which proves the result (note $V_K \leq 0$ on K).

From now on, we write V_K for the (unregularized) $L-$extremal function of a compact set K and we verify that:

Claim: If K is a nonpluripolar compact set, then V_K^* is maximal in $\mathbb{C}^N \setminus K$; i.e., $(dd^cV_K^*)^N = 0$ in $\mathbb{C}^N \setminus K$. Hence

$$\mu_K := \frac{1}{(2\pi)^N}(dd^cV_K^*)^N \tag{49}$$

is a positive measure on K (indeed, $\mu_K \in \mathcal{M}(K)$) and is called the *extremal measure* for K.

To prove the Claim, we begin with

$$V_K(z) = \sup\{u(z) : u \in L : u \leq 0 \text{ on } K\}.$$

From the the existence on a ball B of a psh function $u \in C(\overline{B})$ with $u = f$ on ∂B and $(dd^cu)^N = 0$ in B together with exercise 6 in section 6 (the gluing lemma for psh functions), we see that the class of $u \in L$ with $u \leq 0$ on K is a Perron-Bremermann family (see step (2) below). Thus:

1. From Choquet's lemma, we can recover V_K as an upper envelope of a countable family $\{u_n\}$; by replacing u_n by $v_n := \max[u_1,...,u_n]$ we have V_K as an increasing sequence of psh functions $\{v_n\}$.
2. Fix a ball $B \subset \mathbb{C}^N \setminus K$ and replace each v_n by its Perron-Bremermann modification $\tilde{v}_n$ on B; i.e., $\tilde{v}_n = v_n$ on $\mathbb{C}^N \setminus B$ and on B, $\tilde{v}_n$ is maximal with boundary values v_n. Then, on B, V_K is the monotone, increasing limit of maximal psh functions; i.e., we have $(dd^c\tilde{v}_n)^N = 0$ on B.
3. By continuity of the complex Monge-Ampère operator under *increasing* limits for locally bounded psh functions (cf., [4]), $(dd^cV_K^*)^N = 0$ in B.

The precise definition of pluripolar is a local one: E is *pluripolar* if for each $z \in E$ there exists a neighborhood U of z and a psh function u in U with $E \cap U \subset \{z \in U : u(z) = -\infty\}$. For example, any analytic subvariety V of $\mathbb{C}^N$ is pluripolar as locally $V = \{f_1 = \cdots = f_m = 0\}$ for holomorphic f_j; whence $u = \log[|f_1|^2 + \cdots + |f_m|^2]$ works. The *first problem of Lelong* was to determine whether (locally) pluripolar sets, as defined above, were *globally* pluripolar; i.e., if E is pluripolar, can one find u psh on a neighborhood of E with $E \subset \{u = -\infty\}$? Indeed, one can; u can be taken to be psh on all of $\mathbb{C}^N$; and we can even find such a $u \in L$. We remark that:

1. Nonpluripolar sets can be small: Take a nonpolar Cantor set $E \subset \mathbb{R} \subset \mathbb{C}$ of Hausdorff dimension 0 (see [25] for a construction). Then $E \times \cdots \times E$ is nonpluripolar in $\mathbb{C}^N$ (in general, $E_1 \times \cdots \times E_j \subset \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_j}$ is nonpluripolar in $\mathbb{C}^{m_1+\cdots+m_j}$ if and only if $E_k \subset \mathbb{C}^{m_k}$ is nonpluripolar for $k = 1,...,j$) and has Hausdorff dimension 0.
2. *Pluripolar sets can be big*: A complex hypersurface $S = \{z : f(z) = 0\}$ for a holomorphic function f is a pluripolar set (take $u = \log|f|$) which has Hausdorff dimension $2N - 2$. Recall that a psh function is, in particular, subharmonic (in the $\mathbb{R}^{2N}$ sense); hence a pluripolar set is Newtonian polar. For such sets is known that the Hausdorff dimension cannot exceed $2N - 2$.
3. *Size doesn't matter*: In $\mathbb{C}^2$, the totally real plane $\mathbb{R}^2 = \{(z_1,z_2) : \mathrm{Im}z_1 = \mathrm{Im}z_2 = 0\}$ is nonpluripolar (why?) but the *complex* plane $\mathbb{C} = \{(z_1,0) : z_1 \in \mathbb{C}\}$ is pluripolar (take $u = \log|z_1|$). Also, there exist C^∞ arcs in $\mathbb{C}^N$ which are not pluripolar; while such a real-analytic arc must be pluripolar (why?).

One can easily construct examples of nonpluripolar sets $E \subset \mathbb{C}^N$ which intersect every affine complex line in finitely many points (hence these intersections are polar in these lines). Indeed, take

$$E := \{(z_1,z_2) \in \mathbb{C}^2 : \mathrm{Im}(z_1 + z_2^2) = \mathrm{Re}(z_1 + z_2 + z_2^2) = 0\}.$$

Then for any complex line $L := \{(z_1,z_2) : a_1z_1 + a_2z_2 = b\}$, $a_1, a_2, b \in \mathbb{C}$, $E \cap L$ is the intersection of two real quadrics and hence consists of at most four points. However, E is a totally real, two-(real)-dimensional submanifold of $\mathbb{C}^2$ and hence – as is the case with $\mathbb{R}^2 = \mathbb{R}^2 + i0 \subset \mathbb{C}^2$ in 3. – is not pluripolar. Thus pluripolarity cannot be detected by "slicing" with complex lines. In this example, E intersects the one-(complex)-dimensional analytic variety $A := \{(z_1,z_2) : z_1 + z_2^2 = 0\}$ in a nonpolar set. Nevertheless, one can construct a nonpluripolar set E in $\mathbb{C}^N$, $N > 1$, which intersects every one-dimensional complex analytic subvariety in a polar set [19].

The *second problem of Lelong* was to decide whether plurinegligible sets (recall (43)) were pluripolar: The positive solution of these problems comes fairly quickly utilizing results of Bedford and Taylor on the *relative capacity* $C(E,D)$ of a subset E of a bounded domain D in $\mathbb{C}^N$. Define, for E a Borel subset of D,

$$C(E,D) := \sup\{\int_E (dd^cu)^N : u \text{ psh in } D,\ 0 \leq u \leq 1 \text{ in } D\}.$$

For E a subset of D, define

$$\omega(z,E,D) := \sup\{u(z) : u \text{ psh in } D,\ u \leq 0 \text{ in } D,\ u|_E \leq -1\}.$$

The usc regularization $\omega^*(z,E,D)$ is called the *relative extremal function* of E relative to D (recall exercise 6 of section 3 for the univariate version of this). Indeed, if D is, e.g., a ball, and $K \subset D$ is compact, it turns out that

$$C(K,D) = \int_K (dd^c\omega^*(z,K,D))^N = \int_D (dd^c\omega^*(z,K,D))^N. \tag{50}$$

As an example, take $K = \{z \in \mathbb{C}^N : |z| \leq r\}$ and $D = \{z \in \mathbb{C}^N : |z| < R\}$ with $R > r$. One can check that

$$\omega(z,K,D) = \frac{\log^+\frac{|z|}{r} - \log\frac{R}{r}}{\log\frac{R}{r}} = \frac{1}{\log(R/r)}[\log^+\frac{|z|}{r} - \log\frac{R}{r}].$$

Thus

$$(dd^c\omega(z,K,D))^N = \frac{1}{(\log(R/r))^N}\cdot(dd^c\log^+\frac{|z|}{r})^N.$$

The function $\log^+\frac{|z|}{r}$ we recognize as the $L-$extremal function V_K of K. Recalling from (49) that $\mu_K := \frac{1}{(2\pi)^N}(dd^cV_K^*)^N$ is a probability measure, using (50) we see that

$$C(K,D) = (\frac{2\pi}{\log(R/r)})^N.$$

Proposition 7.3. *Either $\omega^*(z) = \omega^*(z,E,D) \equiv 0$ in D or else ω^* is a nonconstant psh function in D satisfying $(dd^c\omega^*)^N = 0$ in $D \setminus \overline{E}$. We have $\omega^* \equiv 0$ if and only if E is pluripolar.*

Proof. If $\omega^*(z^0) = 0$ at some point $z^0 \in D$, then $\omega^* \equiv 0$ in D by the maximum principle for shm functions on domains in $\mathbb{R}^{2N}$. Hence $\omega(z,E,D) = 0$ a.e. in D. Fix a point z' with $\omega(z',E,D) = 0$ and take a sequence of psh functions u_j in D with $u_j \leq 0$ in D, $u_j|_E \leq -1$, and $u_j(z') \geq -1/2^j$. Then $u(z) := \sum u_j(z)$ is psh in D (the partial sums form a decreasing sequence of psh functions) with $u(z') \geq -1$ (so $u \not\equiv -\infty$) and $u|_E = -\infty$; thus E is pluripolar.

Conversely, if E is pluripolar, there exists u psh in D with $u|_E = -\infty$; since D is bounded we may assume $u \leq 0$ in D. Then $\epsilon u \leq \omega(z,E,D)$ in D for all $\epsilon > 0$ which implies that $\omega(z,E,D) = 0$ for $z \in D$ where $u(z) \neq -\infty$. Since pluripolar sets have measure zero (why?), $\omega(z,E,D) = 0$ a.e. in D and hence $\omega^*(z,E,D) \equiv 0$ in D.

The proof that $(dd^c\omega^*)^N = 0$ in $D \setminus \overline{E}$ in case E is nonpluripolar goes along the same lines as the proof for V_K in the Claim. □

Using this result, one can show (exercise 9) that locally bounded psh functions put no mass on pluripolar sets.

Corollary 7.4. *If u is psh and locally bounded in D and $E \subset D$ is pluripolar, then*

$$\int_E (dd^cu)^N = 0.$$

In particular, if $u \in L^+(\mathbb{C}^N)$, then $(dd^cu)^N$ puts no mass on pluripolar sets.

This second statement can be thought of as a (very weak) multivariate version of Propsition 2.4.

Exercises.

1. Let $u \in C^2(D)$ where D is a domain in $\mathbb{C}^N$. Prove that u is pluriharmonic in D if and only if $dd^cu = 0$ in D.
2. Let $u \in C^\infty(D)$ where D is a domain in $\mathbb{C}^N$. Prove that if u is psh in D and harmonic considered as a function in $D \subset \mathbb{R}^{2N}$, then u is pluriharmonic in D.
3. Let $u(z)$ be shm in a domain $D \subset \mathbb{C}$. Show that $U(z,w) := u(z)$ is a maximal psh function in $D \times \mathbb{C} \subset \mathbb{C}^2$.
4. Let $D \subset \mathbb{R}^N$ be a domain. Show that $u : D \to \mathbb{R}$ is convex if and only if $U(z_1,...,z_N) := u(\text{Re}z_1,...,\text{Re}z_N) : D + i\mathbb{R}^N \subset \mathbb{C}^N \to \mathbb{R}$ is psh.
5. Let $L : (\mathbb{C}\setminus\{0\}) \times (\mathbb{C}\setminus\{0\}) \to \mathbb{R}^2$ be defined as
$$L(z_1,z_2) = (\log|z_1|, \log|z_2|).$$
Suppose $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is of class C^2 on D and let $u := f \circ L$.
   (a) Show that u is psh (where defined) if f is convex.
   (b) Find a formula for $(dd^cu)^2$ in terms of the *real* Hessian of f.
6. Verify that for the set
$$E := \{(z_1,z_2) \in \mathbb{C}^2 : \text{Im}(z_1 + z_2^2) = \text{Re}(z_1 + z_2 + z_2^2) = 0\}$$
any complex line $L := \{(z_1,z_2) : a_1z_1 + a_2z_2 = b\},\ a_1,a_2,b \in \mathbb{C}$ intersects E in at most four points.

7. Let $D \subset \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$ be holomorphic. Show that
$$G(f) := \{(z, f(z)) : z \in D\}$$
is pluripolar. (A deep result of Shcherbina states that for *continuous* f on D, f is holomorphic if and only if $G(f)$ is pluripolar).
8. Prove the claim that for a compact, circled set $E \subset \mathbb{C}^N$ and a polynomial $p_d = h_d + h_{d-1} + \cdots + h_0$ of degree d written as a sum of homogeneous polynomials, $||h_j||_E \leq ||p_d||_E$, $j = 0, \ldots, d$. (*Hint: Fix a point $b \in E$ at which $|h_j(b)| = ||h_j||_E$ and use Cauchy's estimates on $\lambda \mapsto p_d(\lambda b) = \sum_{j=0}^{d} \lambda^j h_j(b)$*).
9. Use Proposition 7.3 and (50) to prove Corollary 7.4.

8 Transfinite diameter and polynomial interpolation in $\mathbb{C}^N$.

We have seen that, as in $\mathbb{C}$, for a compact set $K \subset \mathbb{C}^N$, either $V_K^* \equiv +\infty$, in which case K is pluripolar, or else $V_K^* \in L^+(\mathbb{C}^N)$. In the latter case, the measure $\mu_K = \frac{1}{(2\pi)^N}(dd^c V_K^*)^N$ plays the role of the equilibrium measure. However, since the complex Monge-Ampère operator is nonlinear, there is no natural notion of energy of measures which μ_K minimizes. Nevertheless, there is an analogue of the notion of transfinite diameter, and this turns out to be a nonnegative set function on compact sets which is zero precisely on the pluripolar sets. We highlight the main points of the fundamental work of Zaharjuta [28]. We begin by considering a function Y from the set of multiindices $\alpha \in \mathbf{N}^N$ to the nonnegative real numbers satisfying:

$$Y(\alpha+\beta) \leq Y(\alpha)\cdot Y(\beta) \text{ for all } \alpha,\ \beta \in \mathbf{N}^N. \tag{51}$$

We call a function Y satisfying (51) *submultiplicative*; we have two main examples below. Let $e_1(z),...,e_j(z),...$ be a listing of the monomials $\{e_i(z) = z^{\alpha(i)} = z_1^{\alpha_1}\cdots z_N^{\alpha_N}\}$ in $\mathbb{C}^N$ indexed using a lexicographic ordering on the multiindices $\alpha = \alpha(i) = (\alpha_1,...,\alpha_N) \in \mathbf{N}^N$, but with $\deg e_i = |\alpha(i)|$ nondecreasing. We write $|\alpha| := \sum_{j=1}^N \alpha_j$.

We define the following integers:

1. $m_d^{(N)} = m_d :=$ the number of monomials $e_i(z)$ of degree at most d in N variables;
2. $h_d^{(N)} = h_d :=$ the number of monomials $e_i(z)$ of degree exactly d in N variables;
3. $l_d^{(N)} = l_d :=$ the sum of the degrees of the m_d monomials $e_i(z)$ of degree at most d in N variables.

We have the following relations:

$$m_d^{(N)} = \binom{N+d}{d};\ h_d^{(N)} = m_d^{(N)} - m_{d-1}^{(N)} = \binom{N-1+d}{d} \tag{52}$$

and

$$h_d^{(N+1)} = \binom{N+d}{d} = m_d^{(N)};\ l_d^{(N)} = N\binom{N+d}{N+1} = (\frac{N}{N+1})\cdot d m_d^{(N)}. \tag{53}$$

The elementary fact that the dimension of the space of homogeneous polynomials of degree d in $N+1$ variables equals the dimension of the space of polynomials of degree at most d in N variables will be useful. Finally, we let

$$r_d^{(N)} = r_d := d h_d^{(N)} = d(m_d^{(N)} - m_{d-1}^{(N)})$$

which is the sum of the degrees of the h_d monomials $e_i(z)$ of degree exactly d in N variables. We observe that

$$l_d^{(N)} = \sum_{k=1}^{d} r_k^{(N)} = \sum_{k=1}^{N} k h_k^{(N)}. \tag{54}$$

Let $K \subset \mathbb{C}^N$ be compact. Here are two natural constructions of families of Chebyshev-type constants associated to K:

1. *Chebyshev constants*: Define the class of polynomials
$$P_i = P(\alpha(i)) := \{e_i(z) + \sum_{j<i} c_j e_j(z)\};$$
and the Chebyshev constants
$$Y_1(\alpha) := \inf\{||p||_K : p \in P_i\}.$$
We write $t_{\alpha,K} := t_{\alpha(i),K}$ for a Chebyshev polynomial; i.e., $t_{\alpha,K} \in P(\alpha(i))$ and $||t_{\alpha,K}||_K = Y_1(\alpha)$.
2. *Homogeneous Chebyshev constants*: Define the class of homogeneous polynomials
$$P_i^{(H)} = P^{(H)}(\alpha(i)) := \{e_i(z) + \sum_{j<i,\ \deg(e_j)=\deg(e_i)} c_j e_j(z)\};$$
and the homogeneous Chebyshev constants
$$Y_2(\alpha) := \inf\{||p||_K : p \in P_i^{(H)}\}.$$
We write $t_{\alpha,K}^{(H)} := t_{\alpha(i),K}^{(H)}$ for a homogeneous Chebyshev polynomial; i.e., $t_{\alpha,K}^{(H)} \in P^{(H)}(\alpha(i))$ and $||t_{\alpha,K}^{(H)}||_K = Y_2(\alpha)$.

Let Σ denote the standard $(N-1)-$simplex in $\mathbb{R}^N$; i.e.,

$$\Sigma = \{\theta = (\theta_1,...,\theta_N) \in \mathbb{R}^N : \sum_{j=1}^N \theta_j = 1,\ \theta_j \geq 0,\ j = 1,...,N\},$$

and let

$$\Sigma^0 := \{\theta \in \Sigma :\ \theta_j > 0,\ j = 1,...,N\}.$$

Given a submultiplicative function $Y(\alpha)$, define, as with the above examples, a new function

$$\tau(\alpha) := Y(\alpha)^{1/|\alpha|}. \tag{55}$$

An examination of lemmas 1, 2, 3, 5, and 6 in the fundamental paper by Zaharjuta [28] shows that (51) is the only property of the numbers $Y(\alpha)$ needed to establish those lemmas. To summarize, we have the following results for $Y : \mathbf{N}^N \to \mathbb{R}^+$ satisfying (51) and the associated function $\tau(\alpha)$ in (55).

Lemma 8.1. *For all $\theta \in \Sigma^0$, the limit*

$$T(Y,\theta) := \lim_{\alpha/|\alpha| \to \theta} Y(\alpha)^{1/|\alpha|} = \lim_{\alpha/|\alpha| \to \theta} \tau(\alpha)$$

exists.

We call $T(Y,\theta)$ a *directional Chebyshev constant* in the direction θ.

Lemma 8.2. *The function $\theta \to T(Y,\theta)$ is log-convex on Σ^0 (and hence continuous).*

Lemma 8.3. *Given $b \in \partial\Sigma$,*

$$\liminf_{\theta \to b,\ \theta \in \Sigma^0} T(Y,\theta) = \liminf_{i \to \infty,\ \alpha(i)/|\alpha(i)| \to b} \tau(\alpha(i)).$$

Lemma 8.4. *Let $\theta(k) := \alpha(k)/|\alpha(k)|$ for $k = 1,2,\ldots$ and let Q be a compact subset of Σ^0. Then*

$$\limsup_{|\alpha| \to \infty} \{\log \tau(\alpha(k)) - \log T(Y(\theta(k))) : |\alpha(k)| = \alpha,\ \theta(k) \in Q\} = 0.$$

Lemma 8.5. *Define*

$$\tau(Y) := \exp\Big[\frac{1}{meas(\Sigma)} \int_{\Sigma} \log T(Y,\theta) d\theta\Big]$$

Then

$$\lim_{d \to \infty} \frac{1}{h_d} \sum_{|\alpha|=d} \log \tau(\alpha) = \log \tau(Y);$$

i.e., using (55),

$$\lim_{d \to \infty} \Big[\prod_{|\alpha|=d} Y(\alpha)\Big]^{1/dh_d} = \tau(Y).$$

One can incorporate all of the $Y(\alpha)'$s for $|\alpha| \le d$; this is the content of the next result.

Theorem 8.6. *We have*

$$\lim_{d \to \infty} \Big[\prod_{|\alpha| \le d} Y(\alpha)\Big]^{1/l_d} \textit{ exists and equals } \tau(Y).$$

Proof. Define the geometric means

$$\tau_d^0 := \Big(\prod_{|\alpha|=d} \tau(\alpha)\Big)^{1/h_d},\ d = 1,2,\ldots$$

The sequence

$$\log \tau_1^0, \log \tau_1^0, \ldots (r_1 \text{ times}), \ldots, \log \tau_d^0, \log \tau_d^0, \ldots (r_d \text{ times}), \ldots$$

converges to $\log \tau(Y)$ by the previous lemma; hence the arithmetic mean of the first $l_d = \sum_{k=1}^d r_k$ terms (see (54)) converges to $\log \tau(Y)$ as well. Exponentiating this arithmetic mean gives

$$\Big(\prod_{k=1}^d (\tau_k^0)^{r_k}\Big)^{1/l_d} = \Big(\prod_{k=1}^d \prod_{|\alpha|=k} \tau(\alpha)^k\Big)^{1/l_d} = \Big(\prod_{|\alpha| \le d} Y(\alpha)\Big)^{1/l_d} \tag{56}$$

and the result follows. ☐

Returning to our examples (1) and (2), example (1) was the original setting of Zaharjuta [28] which he utilized to prove the existence of the limit in the definition of the *transfinite diameter* of a compact set $K \subset \mathbb{C}^N$. For $\zeta_1, \ldots, \zeta_n \in \mathbb{C}^N$, let

$$VDM(\zeta_1, \ldots, \zeta_n) = \det[e_i(\zeta_j)]_{i,j=1,\ldots,n} \tag{57}$$

$$= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \ldots & e_1(\zeta_n) \\ \vdots & \vdots & \ddots & \vdots \\ e_n(\zeta_1) & e_n(\zeta_2) & \ldots & e_n(\zeta_n) \end{bmatrix}$$

be a generalized Vandermonde determinant, in analogy with the univariate case, and for a compact subset $K \subset \mathbb{C}^N$ let

$$V_n = V_n(K) := \max_{\zeta_1, \ldots, \zeta_n \in K} |VDM(\zeta_1, \ldots, \zeta_n)|.$$

Then

$$\delta(K) = \lim_{d \to \infty} V_{m_d}^{1/l_d} \tag{58}$$

is the *transfinite diameter* of K; Zaharjuta [28] showed that the limit exists by showing that one has

$$\delta(K) = \exp\Big[\frac{1}{\text{meas}(\Sigma)} \int_{\Sigma^0} \log \tau(K,\theta) d\theta\Big] \tag{59}$$

where $\tau(K,\theta)=T(Y_1,\theta)$ from (1); i.e., the right-hand-side of (59) is $\tau(Y_1)$. This follows from Theorem 8.6 for $Y=Y_1$ and the estimate

$$\Big(\prod_{k=1}^{d}(\tau_k^0)^{r_k}\Big)^{1/l_d}\leq V_{m_d}^{1/l_d}\leq (m_d!)^{1/l_d}\Big(\prod_{k=1}^{d}(\tau_k^0)^{r_k}\Big)^{1/l_d}$$

in [28] (compare the estimate (56)). A set of points $z_1,...,z_{m_d}\in K$ with

$$V_{m_d}=V_{m_d}(K)=|VDM(z_1,...,z_{m_d})|$$

is called a set of *Fekete points of order d for K*.

For a compact *circled* set $K\subset\mathbb{C}^N$; i.e., $z\in K$ if and only if $e^{i\phi}z\in K$, $\phi\in[0,2\pi]$, one need only consider homogeneous polynomials in the definition of the directional Chebyshev constants $\tau(K,\theta)$. In other words, in the notation of (1) and (2), $Y_1(\alpha)=Y_2(\alpha)$ for all α so that

$$T(Y_1,\theta)=T(Y_2,\theta) \text{ for circled sets } K.$$

This is because for such a set, if we write a polynomial p of degree d as $p=\sum_{j=0}^{d}H_j$ where H_j is a homogeneous polynomial of degree j, then, from the Cauchy integral formula, $||H_j||_K\leq||p||_K,\ j=0,...,d$ (see the Claim and exercise 8 in the previous section). Moreover, a slight modification of Zaharjuta's arguments proves the existence of the limit of appropriate roots of maximal *homogeneous* Vandermonde determinants; i.e., the homogeneous transfinite diameter $d^{(H)}(K)$ of a compact set. From the above remarks, it follows that

$$\text{for circled sets } K,\ \delta(K)=d^{(H)}(K). \tag{60}$$

We will use this in the next section. Since we will be using the homogeneous transfinite diameter, we amplify the discussion. We relabel the standard basis monomials $\{e_i^{(H,d)}(z)=z^{\alpha(i)}=z_1^{\alpha_1}\cdots z_N^{\alpha_N}\}$ where $|\alpha(i)|=d,\ i=1,...,h_d$, we define the $d-$homogeneous Vandermonde determinant

$$VDMH_d((\zeta_1,...,\zeta_{h_d}):=\det\big[e_i^{(H,d)}(\zeta_j)\big]_{i,j=1,...,h_d}. \tag{61}$$

Then

$$d^{(H)}(K)=\lim_{d\to\infty}\big[\max_{\zeta_1,...,\zeta_{h_d}\in K}|VDMH_d(\zeta_1,...,\zeta_{h_d})|\big]^{1/dh_d} \tag{62}$$

is the homogeneous transfinite diameter of K; the limit exists and equals

$$\exp\Big[\frac{1}{\text{meas}(\Sigma)}\int_{\Sigma^0}\log T(Y_2,\theta)d\theta\Big]$$

where $T(Y_2,\theta)$ comes from (2).

A useful fact is that

$$\delta(K)=\delta(\widehat{K}) \text{ and } d^{(H)}(K)=d^{(H)}(\widehat{K}) \tag{63}$$

for K compact where

$$\widehat{K}:=\{z\in\mathbb{C}^N:|p(z)|\leq||p||_K,\ \text{all polynomials } p\}$$

is the polynomial hull of K.

Clearly if a compact set K is contained in an algebraic subvariety of $\mathbb{C}^N$ then $\delta(K)=0$ (why?). It turns out that *for $K\subset\mathbb{C}^N$ compact, $\delta(K)=0$ if and only if K is pluripolar* [23]. If the compact set $K\subset\mathbb{C}^N$ is L-regular, then for each $R>1$ we define

$$D_R\equiv\{z:V_K(z)<\log R\}; \tag{64}$$

then we clearly have, from (48), the Bernstein-Walsh inequality

$$|p(z)|\leq||p||_K R^{\deg p},\qquad z\in D_R \tag{65}$$

for every polynomial p in $\mathbb{C}^N$.

Recall that a compact set $K\subset\mathbb{C}^N$ is called polynomially convex if K coincides with its polynomial hull

$$\widehat{K}\equiv\{z\in\mathbb{C}^N:|p(z)|\leq||p||_K,\ p \text{ polynomial}\}.$$

For example, every compact set $K\subset\mathbb{R}^N=\mathbb{R}^N+i0\subset\mathbb{C}^N$ is polynomially convex (why?). For $N=1$, exercise 1 of section 4 showed that $K\subset\mathbb{C}$ is polynomially convex if and only if $\mathbb{C}-K$ is connected. With the above definitions, Theorem 4.4 goes over *exactly* to several complex variables:

Theorem 8.7. *Let K be an L-regular, polynomially convex compact set in $\mathbb{C}^N$. Let $R>1$, and let D_R be defined by (64). Let f be continuous on K. Then*

$$\limsup_{n\to\infty}d_n(f,K)^{1/n}\leq 1/R$$

if and only if f is the restriction to K of a function holomorphic in D_R.

Here, recall that for $f \in C(K)$,

$$d_n = d_n(f,K) \equiv \inf\{||f - p_n||_K : p_n \in \mathcal{P}_n\}.$$

For the rest of this section, we use n instead of d to index the degree of polynomials to avoid notational issues with the distance "d_n". To prove "only if" we may repeat the proof after the statement of Theorem 23, since K satisfies the Bernstein-Walsh inequality (65). The "if" proof, although not hard, requires some deeper knowledge of several complex variables.

We can utilize Lagrange interpolation in this higher-dimensional setting. Choose m_n points $A_n = \{a_{n1},...,a_{nm_n}\} \subset K$ and form the Vandermonde determinant

$$V_n(A_n) \equiv \det[e_i(a_{nj})]_{i,j=1,...,m_n}.$$

If $V_n(A_n) \neq 0$, we can form the FLIP's

$$l_{nj}(x) \equiv \frac{V_n(a_{n1},...,x,...,a_{nm_n})}{V_n(A_n)}, \qquad j = 1,...,m_n. \tag{66}$$

In the one (complex) variable case, we get *cancellation* in this ratio so that the formulas for the FLIP's simplify. In general, we still have $l_{nj}(a_{ni}) = \delta_{ji}$ and $l_{nj} \in \mathcal{P}_n$ since l_{nj} is a linear combination of $e_1,..,e_{m_n}$. Note that for a set of Fekete points of order n, we have $||l_{nj}||_K = 1$ for $j = 1,...,m_n$ (why?). For f defined on K,

$$(L_nf)(x) \equiv \sum_{j=1}^{m_n} f(a_{nj})l_{nj}(x)$$

is the Lagrange interpolating polynomial (LIP) for f at the points A_n. We call

$$\Lambda_n \equiv \sup_{x\in K} \sum_{j=1}^{m_n} |l_{nj}(x)|$$

the n-th *Lebesgue constant* for K,A_n. As in section 4, this is the norm of the linear operator

$$\mathcal{L}_n : C(K) \to \mathcal{P}_n \subset C(K)$$

defined by $\mathcal{L}_n(f) := L_nf$ where we equip $C(K)$ with the supremum norm. For a set of Fekete points of order n, we have $\Lambda_n \leq m_n$. We say that K is *determining* for $\bigcup \mathcal{P}_n$ if whenever $h \in \bigcup \mathcal{P}_n$ satisfies $h = 0$ on K, it follows that $h \equiv 0$. For these sets we can find points A_n for each n with $V_n(A_n) \neq 0$. We have the following elementary result, similar to the proof in one variable that arrays satisfying (32) yield good polynomial approximants to holomorphic functions.

Theorem 8.8. *Let K be determining for $\bigcup \mathcal{P}_n$ and let $A_n \subset K$ satisfy $V_n(A_n) \neq 0$ for each n. Given f bounded on K, if* $\limsup \Lambda_n^{1/n} = 1$, *then* $\limsup ||f - L_nf||_K^{1/n} = \limsup d_n^{1/n}$.

Proof. Fix $\epsilon > 0$ and choose, for each n, a polynomial $p_n \in \mathcal{P}_n$ with $||f - p_n||_K^{1/n} \leq d_n^{1/n} + \epsilon$. Since $p_n \in \mathcal{P}_n$, we have $L_np_n = p_n$ and

$$\begin{aligned}||f - L_nf||_K &= ||f - p_n + L_np_n - L_nf||_K\\ &\leq ||f - p_n||_K + \Lambda_n||f - p_n||_K = (1 + \Lambda_n)||f - p_n||_K.\end{aligned}$$

Using the hypothesis $\limsup \Lambda_n^{1/n} = 1$, we obtain the conclusion. □

Immediately from Theorems 8.7 and 8.8 we have

Corollary 8.9. *Let K be an L-regular, polynomially convex compact set in $\mathbb{C}^N$ and let $\{A_n\} \subset K$ satisfy* $\limsup \Lambda_n^{1/n} = 1$. *Then for any f holomorphic on a neighborhood of K, $L_nf \to f$ uniformly on K.*

As in the univariate case, for $K \subset \mathbb{C}^N$ compact, $L-$regular and polynomially convex, we can consider the following four properties which an array $\{a_{nj}\}_{j=1,...,m_n;\ n=1,2,...} \subset K$ may or may not possess:

1. $\lim_{n\to\infty} \Lambda_n^{1/n} = 1$;
2. $\lim_{n\to\infty} |VDM(a_{n1},...,a_{nm_n})|^{\frac{1}{l_n}} = \delta(K)$;
3. $\lim_{n\to\infty} \frac{1}{m_n}\sum_{j=1}^{m_n} \delta_{a_{nj}} = \mu_K$ weak-*;
4. $L_nf \rightrightarrows f$ on K for each f holomorphic on a neighborhood of K.

Corollary 8.9 shows that (1) $\implies$ (4); the univariate proof that (1) $\implies$ (2) generalizes to the multivariate setting; and a recent deep result in [6], which we give as Corollary 10.6 in section 10, shows that (2) $\implies$ (3). The reference [10] includes counterexamples to most other implications. A major problem with Lagrange interpolation of holomorphic functions in $\mathbb{C}^N$, $N > 1$, is the lack of a Hermite remainder formula. Together with the fact that one needs to insure, for each n, that the points $a_{n1},...,a_{nm_n}$ one chooses satisfy $VDM(a_{n1},...,a_{nm_n}) \neq 0$ (unisolvence), one might seek other polynomial interpolation procedures.

A more promising type of interpolation procedure has been successfully applied to many approximation problems by Tom Bloom and his collaborators. A natural extension of Lagrange interpolation to $\mathbb{R}^N$, $N > 1$ was discovered by P. Kergin (a student of Bloom) in his thesis. Indeed, Kergin interpolation acting on ridge functions (a univariate function composed with a linear form) is Lagrange interpolation. The Kergin interpolation polynomials generalize to the case of C^m functions in $\mathbb{R}^N$ both the Lagrange interpolation polynomials and those of Hermite.

As brief motivation, given $f \in C^m([0,1])$, say, and given $m+1$ points $t_0 < \cdots < t_m \in [0,1]$, if one constructs the Lagrange interpolating polynomial $L_m f$ for f at these points, then there exist (at least) $m-1$ points between pairs of successive t_j at which f' and $(L_m f)'$ agree; then there exist (at least) $m-2$ points between triples of successive t_j at which f'' and $(L_m f)''$ agree, etc. Given a set $\mathbf{A} = [A_0, A_1, \ldots, A_m] \subset \mathbb{R}^N$ of $m+1$ points and f a function of class C^m on a neighborhood of the convex hull of these points, there exists a unique polynomial $\mathcal{K}_A(f) = \mathcal{K}_A(f)(x_1, \ldots, x_N)$ of total degree m such that $\mathcal{K}_A(f)(A_j) = f(A_j)$, $j = 0, 1, \ldots, m$, and such that for every integer r, $0 \le r \le m-1$, every subset J of $\{0, 1, \ldots, m\}$ with cardinality equal to $r+1$, and every homogeneous differential operator Q of order r with constant coefficients, there exists ξ belonging to the convex hull of the (A_j), $j \in J$, such that $Qf(\xi) = Q\mathcal{K}_A(f)(\xi)$. In [7], Bloom gives a proof of this result by using a formula due to Micchelli and Milman [24] which gives an explicit expression for $\mathcal{K}_A(f)$. If $f = u + iv$ is holomorphic in a convex region D in $\mathbb{C}^N$, and if $\mathbf{A} = [A_0, A_1, \ldots, A_m] \subset D \subset \mathbb{C}^N = \mathbb{R}^{2N}$, then we can construct $\mathcal{K}_A(u)$ and $\mathcal{K}_A(v)$. It turns out (cf., [17]) that $\mathcal{K}_A(u) + i\mathcal{K}_A(v)$ is a holomorphic polynomial.

An alternate description, which we give in the holomorphic setting, is as follows (cf., [12]). Let D be a $\mathbb{C}$-convex domain in $\mathbb{C}^N$, i.e., the intersection of D with any complex line is connected and simply connected. Note that in $\mathbb{R}^N$ this is the same condition as convexity if we replace "complex line" by "real line." For any set $\mathbf{A} = [A_0, \ldots, A_d]$ of (not necessarily distinct) $d+1$ points in D there exists a unique linear projector $\mathcal{K}_{\mathbf{A}} : \mathcal{O}(D) \to \mathcal{P}_d$ (recall that $\mathcal{O}(D)$ is the space of holomorphic functions on D and $\mathcal{P}_d$ is the space of polynomials of N complex variables of degree less than or equal to d) such that

1. $\mathcal{K}_{\mathbf{A}}(f)(A_j) = f(A_j)$ for $j = 0, \cdots, d$,
2. $\mathcal{K}_{\mathbf{A}}(g \circ \lambda) = \mathcal{K}_{\lambda(\mathbf{A})}(g) \circ \lambda$ for every affine map $\lambda : \mathbb{C}^N \to \mathbb{C}$ and $g \in \mathcal{O}(\lambda(D))$, where $\lambda(\mathbf{A}) = (\lambda(A_0), \ldots, \lambda(A_d))$,
3. $\mathcal{K}_{\mathbf{A}}$ is independent of the ordering of the points in $\mathbf{A}$, and
4. $\mathcal{K}_{\mathbf{B}} \circ \mathcal{K}_{\mathbf{A}} = \mathcal{K}_{\mathbf{B}}$ for every subsequence $\mathbf{B}$ of $\mathbf{A}$.

The operator $\mathcal{K}_{\mathbf{A}}$ is called the Kergin interpolating operator with respect to $\mathbf{A}$.

Set $\mathcal{K}_d := \mathcal{K}_{\mathbf{A}_d}$ with $\mathbf{A}_d = [A_{d0}, \ldots, A_{dd}]$ and A_{dj} in a compact subset K of $D \subset \mathbb{C}^N$ for every $j = 0, \ldots, d$ and $d = 1, 2, 3, \ldots$. *Under what conditions on the array $\{\mathbf{A}_d\}_{d=1,2,\ldots}$ is it true that $\mathcal{K}_d(f)$ converges to f uniformly on K as $d \to \infty$ for every function f holomorphic in some neighborhood of $\overline{D}$?* Bloom and Calvi [12] attacked this problem with the aid of an integral representation formula for the remainder $f - \mathcal{K}_d(f)$ proved by M. Andersson and M. Passare [2]. Their solution reads as follows. Assume that the measures $\mu_d = (d+1)^{-1}\sum_{j=0}^d \delta_{A_{dj}}$ converge weak-* as $d \to \infty$ to a measure μ. In one variable, the answer comes from potential theory: one considers the logarithmic potential

$$V_\mu(z) := \int_K \log|z - \zeta| d\mu(\zeta)$$

and the required condition is that

$$\{z \in \mathbb{C} : V_\mu(z) \le \sup_K V_\mu\} \subset D.$$

For $N > 1$, given a linear form $p : \mathbb{C}^N \to \mathbb{C}$, define $\mu^p = p_*\mu$ as the push-forward of μ to $\mathbb{C}$ via p, i.e., for $f \in C_0(\mathbb{C})$,

$$\mu^p(f) := \int_{\mathbb{C}} f \, d\mu^p = \mu(f \circ p) := \int_{\mathbb{C}^N} (f \circ p) d\mu.$$

Set

$$\Psi_\mu(p, z) := \mu^p(\log|z - \cdot|) = \int_{\mathbb{C}} \log|z - \zeta| d\mu^p(\zeta),$$

and let $M_\mu(p)$ be the maximum of $z \mapsto \Psi_\mu(p, z)$ on $p(K)$. *If D has C^2 boundary and $\{z \in \mathbb{C} : \Psi_\mu(p, z) \le M_\mu(p)\} \subset p(D)$ for every linear form p on $\mathbb{C}^N$, then $\mathcal{K}_d(f)$ converges to f uniformly on K as $d \to \infty$ for every function f holomorphic in some neighborhood of $\overline{D}$.*

We call an array $\{\mathbf{A}_d\}_{d=1,2,\ldots}$ *extremal* for K if $\mathcal{K}_d(f)$ converges to f uniformly on K for each f holomorphic in a neighborhood of K. Of course, $\mathcal{K}_d(f)$ should make sense; i.e., f should be defined, e.g., in the convex (or more generally, the $\mathbb{C}-$convex) hull of K. In the setting of compact, convex subsets K of $\mathbb{R}^N$, Bloom and Calvi proved the following striking result.

Theorem 8.10. *[13] Let $K \subset \mathbb{R}^N$, $N \ge 2$, be a compact, convex set with nonempty interior. Then K admits extremal arrays if and only if $N = 2$ and K is the region bounded by an ellipse.*

For the Andersson-Passare remainder formula one needs an integral formula with a *holomorphic* kernel; moreover, one with a kernel that is the composition of a univariate function with an affine function. Together with property (2) of the Kergin interpolating operator, this allows a reduction of the multivariate problem to a univariate setting. For an outline of these items, see [21].

Exercises.

1. Let $K = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \le 1, \ z_2 = 0\}$. What is $\delta(K)$? Give a proof of your answer.
2. Let $K = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 \le 1\}$. What is $\delta(K)$? Give a proof of your answer.

3. *Extra Credit.* Let $K = \{(z_1, z_2) \in \mathbb{C}^2 : (\text{Re}z_1)^2 + (\text{Re}z_2)^2 \leq 1,\ \text{Im}z_1 = \text{Im}z_2 = 0\}$. What is $\delta(K)$? Give a proof of your answer.
4. Let $K = \{(z_1, z_2) \in \mathbb{C}^2 : 0 \leq |z_1| \leq |z_2| \leq 1\}$. Find $\widehat{K}$.
5. Let $\{A_n\}$ be a Fekete array for K; i.e., for each $n = 1, 2, ...$, the points $A_n = \{a_{n1}, ..., a_{nm_n}\} \subset K$ form a set of Fekete points of order n for K. Prove that $\lim_{n\to\infty} \Lambda_n^{1/n} = 1$.
6. Verify (63) that for K compact, $\delta(K) = \delta(\widehat{K})$ and $d^{(H)}(K) = d^{(H)}(\widehat{K})$. (*Hint: Compare the supremum norms of the Chebyshev polynomials $t_{\alpha,K}$, $t_{\alpha,\widehat{K}}$ and those of the homogeneous Chebyshev polynomials $t^{(H)}_{\alpha,K}$, $t^{(H)}_{\alpha,\widehat{K}}$*).

9 Weighted pluripotential theory in $\mathbb{C}^N$, $N>1$, Bergman functions and L^2-theory.

As in the univariate case, in weighted pluripotential theory in $\mathbb{C}^N$ for $N>1$ one restricts to closed but possibly unbounded sets. Again for $K\subset\mathbb{C}^N$ closed we let $\mathcal{A}(K)$ denote the collection of lowersemicontinuous $Q:=-\log w$ where w is a nonnegative, usc function on K with $\{z\in K: w(z)>0\}$ nonpluripolar; if K is unbounded, we require

$$|z|w(z)\to 0 \text{ as } |z|\to\infty,\ z\in K. \tag{67}$$

We define the weighted extremal function or weighted pluricomplex Green function $V^*_{K,Q}(z):=\limsup_{\zeta\to z}V_{K,Q}(\zeta)$ where

$$V_{K,Q}(z):=\sup\{u(z): u\in L(\mathbb{C}^N),\ u\leq Q \text{ on } K\}.$$

We have $V^*_{K,Q}\in L^+(\mathbb{C}^N)$. In the unbounded case, we again remind the reader that property (67) is equivalent to

$$Q(z)-\log|z|\to\ +\infty \text{ as } |z|\to\infty \text{ through points in } K;$$

hence $V_{K,Q}$ is well-defined and equals $V_{K\cap\mathcal{B}_R,Q}$ for $R>0$ sufficiently large where $\mathcal{B}_R=\{z:|z|\leq R\}$ (Definition 2.1 and Lemma 2.2 of Appendix B in [26]). It is known that the support

$$S_w:=\mathrm{supp}(\mu_{K,Q})$$

of the *weighted extremal measure*

$$\mu_{K,Q}:=\frac{1}{(2\pi)^N}(dd^cV^*_{K,Q})^N$$

is compact (recall the definition of μ_K in (49)). The proof of (39), adjusted using the solution of the Dirichlet problem for the complex Monge-Ampère equation on a ball, shows that

$$S_w\subset S^*_w:=\{z\in K: V^*_{K,Q}(z)\geq Q(z)\}. \tag{68}$$

Moreover,

$$V^*_{K,Q}=Q \text{ q.e. on } S_w$$

(i.e., $V^*_{K,Q}=Q$ on $S_w\setminus F$ where F is pluripolar); and if $u\in L(\mathbb{C}^N)$ satisfies $u\leq Q$ q.e. on S_w then $u\leq V^*_{K,Q}$ on $\mathbb{C}^N$. Indeed,

$$V_{K,Q}(z)=\sup\{\frac{1}{deg(p)}\log|p(z)|: ||w^{deg(p)}p||_{S_w}\leq 1,\ p \text{ polynomial}\} \tag{69}$$

and

$$||w^{deg(p)}p||_{S_w}=||w^{deg(p)}p||_K.$$

Theorem 2.8 of Appendix B in [26] includes the slightly stronger statement that

$$V^*_{K,Q}(z)=\left[\sup\{\frac{1}{deg(p)}\log|p(z)|: ||w^{deg(p)}p||^*_K\leq 1,\ p \text{ polynomial}\}\right]^*$$

where

$$||w^{deg(p)}p||^*_K:=\inf\{||w^{deg(p)}p||_{K\setminus F}: F\subset K \text{ pluripolar}\}.$$

The unweighted case is when K is compact and $w\equiv 1$ ($Q\equiv 0$); we then write $V_K:=V_{K,0}$ to be consistent with the previous notation.

A natural definition of a weighted transfinite diameter uses weighted Vandermonde determinants. Let $K\subset\mathbb{C}^N$ be compact and let w be an admissible weight function on K. Given $\zeta_1,...,\zeta_{m_d}\in K$, let

$$W(\zeta_1,...,\zeta_{m_d}):=VDM(\zeta_1,...,\zeta_{m_d})w(\zeta_1)^d\cdots w(\zeta_{m_d})^d$$

$$=\det\begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \ldots & e_1(\zeta_{m_d})\\ \vdots & \vdots & \ddots & \vdots\\ e_{m_d}(\zeta_1) & e_{m_d}(\zeta_2) & \ldots & e_{m_d}(\zeta_{m_d})\end{bmatrix}\cdot w(\zeta_1)^d\cdots w(\zeta_{m_d})^d$$

be a *weighted Vandermonde determinant*. Define a *$d-$th order weighted Fekete set for K and w* to be a set of m_d points $\zeta_1,...,\zeta_{m_d}\in K$ with the property that

$$W_{m_d}=W_{m_d}(K):=|W(\zeta_1,...,\zeta_{m_d})|=\sup_{\xi_1,...,\xi_{m_d}\in K}|W(\xi_1,...,\xi_{m_d})|.$$

In analogy with the univariate notation, we also set

$$\delta^d_w(K):=W^{1/l_d}_{m_d}.$$

Define

$$\delta^w(K):=\limsup_{d\to\infty}W^{1/l_d}_{m_d}=\limsup_{d\to\infty}\delta^d_w(K). \tag{70}$$

We will show in Proposition 9.1 that $\lim_{d\to\infty}W^{1/l_d}_{m_d}$ (the weighted analogue of (58)) exists.

Proposition 9.1. *Let $K \subset \mathbb{C}^N$ be a compact set with an admissible weight function w. The limit*

$$\lim_{d\to\infty} \big[\max_{\lambda^{(i)}\in K} |VDM(\lambda^{(1)},...,\lambda^{(m_d^{(N)})})| \cdot w(\lambda^{(1)})^d \cdots w(\lambda^{(m_d^{(N)})})^d\big]^{1/l_d^{(N)}}$$

exists (and equals $\delta^w(K)$).

Proof. Following [9], we define the circled set

$$F = F(K,w) := \{(t,z) = (t,t\lambda) \in \mathbb{C}^{N+1} : \lambda \in K,\ |t| = w(\lambda)\}.$$

We first relate weighted Vandermonde determinants for K with homogeneous Vandermonde determinants for the compact set

$$F(D) := \{(t,z) = (t,t\lambda) \in \mathbb{C}^{N+1} : \lambda \in K,\ |t| \le w(\lambda)\}. \tag{71}$$

Note that $F \subset \overline{F} \subset F(D) \subset \widehat{\overline{F}}$ (cf., [9], (2.4)) where $\widehat{\overline{F}}$ is the polynomial hull of $\overline{F}$ (recall (63)); thus

$$d^{(H)}(\overline{F}) = d^{(H)}(F(D)). \tag{72}$$

To this end, for each positive integer d, choose

$$m_d^{(N)} = \binom{N+d}{d}$$

(recall (52)) points $\{(t_i, z^{(i)})\}_{i=1,...,m_d^{(N)}} = \{(t_i, t_i\lambda^{(i)})\}_{i=1,...,m_d^{(N)}}$ in $F(D)$ and form the $d-$homogeneous Vandermonde determinant

$$VDMH_d((t_1,z^{(1)}),...,(t_{m_d^{(N)}}, z^{(m_d^{(N)})})).$$

We extend the lexicographical order of the monomials in $\mathbb{C}^N$ to $\mathbb{C}^{N+1}$ by letting t precede any of $z_1,...,z_N$. Writing the standard basis monomials of degree d in $\mathbb{C}^{N+1}$ as

$$\{t^{d-j}e_k^{(H,d)}(z) : j = 0,...,d;\ k = 1,...,h_j\};$$

i.e., for each power $d-j$ of t, we multiply by the standard basis monomials of degree j in $\mathbb{C}^N$, and dropping the superscript (N) in $m_d^{(N)}$, we have the $d-$homogeneous Vandermonde matrix

$$\begin{bmatrix} t_1^d & t_2^d & \cdots & t_{m_d}^d \\ t_1^{d-1}e_2(z^{(1)}) & t_2^{d-1}e_2(z^{(2)}) & \cdots & t_{m_d}^{d-1}e_2(z^{(m_d)}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{m_d}(z^{(1)}) & e_{m_d}(z^{(2)}) & \cdots & e_{m_d}(z^{(m_d)}) \end{bmatrix}$$
$$= \begin{bmatrix} t_1^d & t_2^d & \cdots & t_{m_d}^d \\ t_1^{d-1}z_1^{(1)} & t_2^{d-1}z_1^{(2)} & \cdots & t_{m_d}^{d-1}z_1^{(m_d)} \\ \vdots & \vdots & \ddots & \vdots \\ (z_N^{(1)})^d & (z_N^{(2)})^d & \cdots & (z_N^{(m_d)})^d \end{bmatrix}.$$

Factoring t_i^d out of the $i-$th column, we obtain

$$VDMH_d((t_1,z^{(1)}),...,(t_{m_d},z^{(m_d)})) = t_1^d \cdots t_{m_d}^d \cdot VDM(\lambda^{(1)},...,\lambda^{(m_d)});$$

thus, writing $|A| := |\det A|$ for a square matrix A,

$$\begin{vmatrix} t_1^d & t_2^d & \cdots & t_{m_d}^d \\ t_1^{d-1}z_1^{(1)} & t_2^{d-1}z_1^{(2)} & \cdots & t_{m_d}^{d-1}z_1^{(m_d)} \\ \vdots & \vdots & \ddots & \vdots \\ (z_N^{(1)})^d & (z_N^{(2)})^d & \cdots & (z_N^{(m_d)})^d \end{vmatrix} \tag{73}$$
$$= |t_1|^d \cdots |t_{m_d}|^d \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1^{(1)} & \lambda_1^{(2)} & \cdots & \lambda_1^{(m_d)} \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_N^{(1)})^d & (\lambda_N^{(2)})^d & \cdots & (\lambda_N^{(m_d)})^d \end{vmatrix},$$

where $\lambda_k^{(j)} = z_k^{(j)}/t_j$ provided $t_j \neq 0$. By definition of $F(D)$, since $(t_i, z^{(i)}) = (t_i, t_i\lambda^{(i)}) \in F(D)$, we have $|t_i| \le w(\lambda^{(i)})$. Clearly the maximum of

$$|VDMH_d((t_1,z^{(1)}),...,(t_{m_d},z^{(m_d)}))|$$

over points in $F(D)$ will occur when all $|t_j| = w(\lambda^{(j)}) > 0$ (recall w is an admissible weight) so that from (73)

$$\max_{(t_i,z^{(i)})\in F(D)} |VDMH_d((t_1,z^{(1)}),\dots,(t_{m_d},z^{(m_d)}))| =$$

$$\max_{\lambda^{(i)}\in K} |VDM(\lambda^{(1)},\dots,\lambda^{(m_d)})|\cdot w(\lambda^{(1)})^d\cdots w(\lambda^{(m_d)})^d.$$

As mentioned in the discussion of (62) the limit

$$\lim_{d\to\infty}\Big[\max_{(t_i,z^{(i)})\in F(D)} |VDMH_d((t_1,z^{(1)}),\dots,(t_{m_d},z^{(m_d)}))|\Big]^{1/dh_d^{(N+1)}}$$

$$=: d^{(H)}(F(D))$$

exists; thus the limit

$$\lim_{d\to\infty}\Big[\max_{\lambda^{(i)}\in K} |VDM(\lambda^{(1)},\dots,\lambda^{(m_d)})|\cdot w(\lambda^{(1)})^d\cdots w(\lambda^{(m_d)})^d\Big]^{1/l_d^{(N)}} := \delta^w(K)$$

exists. □

Corollary 9.2. *For $K\subset\mathbb{C}^N$ a nonpluripolar compact set with an admissible weight function w and*

$$F = F(K,w) := \{(t,z) = (t,t\lambda)\in\mathbb{C}^{N+1} : \lambda\in K,\ |t| = w(\lambda)\},$$

$$\delta^w(K) = d^{(H)}(\overline{F})^{\frac{N+1}{N}} = \delta(\overline{F})^{\frac{N+1}{N}}. \tag{74}$$

Proof. The first equality follows from the proof of Proposition 9.1 using (72) and the relation

$$l_d^{(N)} = (\frac{N}{N+1})\cdot dh_d^{(N+1)}$$

(see (53)). The second equality is (60). □

Given a compact set $K\subset\mathbb{C}^N$ and a measure ν on K, we say that (K,ν) satisfies the Bernstein-Markov inequality if, as in the univariate case, there is a strong comparability between L^2 and L^∞ norms of holomorphic polynomials on K. Precisely, for all $p_d\in\mathcal{P}_d$,

$$||p_d||_K \le M_d||p_d||_{L^2(\nu)} \text{ with } \limsup_{d\to\infty} M_d^{1/d} = 1;$$

equivalently, given $\epsilon > 0$, there exists a constant $\tilde{M} = \tilde{M}(\epsilon)$ such that

$$||p_d||_K \le \tilde{M}(1+\epsilon)^d||p_d||_{L^2(\nu)}.$$

If K is $L-$regular, (K,μ_K) satisfies the Bernstein-Markov inequality where μ_K is the extremal measure $\frac{1}{(2\pi)^N}(dd^cV_K)^N$ from (49). One can even find a Bernstein-Markov measure ν which is rather "sparse" in the sense that there exists a countable subset $K'\subset K$ with $\nu(K') = \nu(K)$. The next result shows that *any* compact set admits a Bernstein-Markov measure; indeed, the construction below provides a "sparse" example.

Proposition 9.3. *Let $K\subset\mathbb{C}^N$ be an arbitrary compact set. Then there exists a measure $\nu\in\mathcal{M}(K)$ such that (K,ν) satisfies a Bernstein-Markov property.*

Proof. To construct ν, we first observe that if K is a finite set, any measure ν which puts positive mass at each point of K will work. If K has infinitely many points, for each $k = 1,2,\dots$ let m_k =dim$\mathcal{P}_k(K)$, the holomorphic polynomials on $\mathbb{C}^N$ restricted to K. Then $\lim_{k\to\infty} m_k = \infty$ and $m_k \le \binom{N+k}{k} = 0(N^k)$. For each k, let

$$\mu_k := \frac{1}{m_k}\sum_{j=1}^{m_k}\delta(z_j^{(k)})$$

where $\{z_j^{(k)}\}_{j=1,\dots,m_k}$ is a set of Fekete points of order k for K relative to the vector space $\mathcal{P}_k(K)$; i.e., if $\{e_1,\dots,e_{m_k}\}$ is any basis for $\mathcal{P}_k(K)$,

$$\left|\det[e_i(z_j^{(k)})]_{i,j=1,\dots,m_k}\right| = \max_{q_1,\dots,q_{m_k}\in K}\left|\det[e_i(q_j)]_{i,j=1,\dots,m_k}\right|. \tag{75}$$

Define

$$\nu := c\sum_{k=3}^{\infty}\frac{1}{k(\log k)^2}\mu_k$$

where $c > 0$ is chosen so that $\nu\in\mathcal{M}(K)$. If $p\in\mathcal{P}_k(K)$, we have

$$p(z) = \sum_{j=1}^{m_k} p(z_j^{(k)})l_j^{(k)}(z)$$

where $l_j^{(k)} \in \mathcal{P}_k(K)$ with $l_j^{(k)}(z_k^{(k)}) = \delta_{jk}$. We have $||l_j^{(k)}||_K = 1$ from (75) and hence

$$||p||_K \leq \sum_{j=1}^{m_k} |p(z_j^{(k)})|.$$

On the other hand,

$$||p||_{L^2(d\nu)} \geq ||p||_{L^1(d\nu)} \geq \frac{c}{k(\log k)^2}\int_K |p|d\mu_k$$

$$= \frac{c}{km_k(\log k)^2}\sum_{j=1}^{m_k} |p(z_j^{(k)})|.$$

Thus we have

$$||p||_K \leq \frac{km_k(\log k)^2}{c}||p||_{L^2(d\nu)}.$$

□

We return to the setting of Theorem 8.7, i.e., K is a polynomially convex L-regular compact set in $\mathbb{C}^N$. Given a measure ν such that (K,ν) satisfies a Bernstein-Markov property, we show that best $L^2(\nu)$-approximating polynomials to certain functions $f \in C(K)$ – which are in principle easy to calculate – have asymptotic behavior similar to best supremum norm polynomial approximants. It will be convenient to let n denote the degree of a polynomial $p_n \in \mathcal{P}_n$ since we recall the notation

$$d_n = d_n(f,K) = \inf\{||f - p_n||_K : p_n \in \mathcal{P}_n\}.$$

Proposition 9.4. *Let K be a polynomially convex L-regular compact set in $\mathbb{C}^N$ and let ν be a measure supported on K such that (K,ν) satisfies the Bernstein-Markov property. If $f \in C(K)$ satisfies*

$$\limsup_{n\to\infty} d_n(f,K)^{1/n} = \rho < 1,$$

and if $\{p_n\}$ is a sequence of best $L^2(\nu)$-approximants to f, then

$$\limsup_{n\to\infty} ||f - p_n||_K^{1/n} = \rho.$$

Proof. Note the hypothesis implies that f extends to be holomorphic on a neighborhood of K by Theorem 8.7. For simplicity we take $\nu(K) = 1$. The proof follows trivially from the fact that if $\rho < r < 1$ and $\{q_n\}$ are best sup-norm approximating polynomials, so that $||f - q_n||_K \leq Mr^n$ for some M (independent of n), then

$$||f - p_n||_{L^2(\nu)} \leq ||q_n - f||_{L^2(\nu)} \leq ||q_n - f||_K \leq Mr^n.$$

Thus we have $||p_n - p_{n-1}||_{L^2(\nu)} \leq Mr^n(1 + 1/r)$ which shows that $p_0 + \sum_{n=1}^{\infty}(p_n - p_{n-1})$ converges to f in $L^2(\nu)$ and pointwise ν-a.e. to f on K. By the Bernstein-Markov property, for each $\epsilon < 1/r - 1$ there exists $\tilde{M} > 0$ with

$$||p_n - p_{n-1}||_K \leq \tilde{M}(1+\epsilon)^n||p_n - p_{n-1}||_{L^2(\nu)} \leq \tilde{M}[(1+\epsilon)r]^n M(1 + 1/r)$$

showing that $p_0 + \sum_{n=1}^{\infty}(p_n - p_{n-1})$ converges uniformly to a continuous function g on K (holomorphic on the interior of K). Since f and g are continuous and $g = f$ ν-a.e. on K, $g = f$ on K. Then

$$||f - p_n||_K = ||\sum_{k=n+1}^{\infty}(p_k - p_{k-1})||_K \leq \tilde{M}[(1+\epsilon)r]^{n+1}M\frac{(1+1/r)}{[1-(1+\epsilon)r]}$$

showing that $\limsup_{n\to\infty} ||p_n - f||_K^{1/n} \leq (1+\epsilon)r$. □

We recall briefly the basic theory of reproducing kernels on a Hilbert space in the context of the Hilbert space H_n consisting of elements in $\mathcal{P}_n$ equipped with the L^2-norm associated to a (probability) measure ν with compact support K. We presume that the measure is "thick" enough so that $||p||_{L^2(\nu)}^2 := \int_K |p|^2 d\nu = 0$ for $p \in \mathcal{P}_n$ implies $p \equiv 0$. Then for each $z \in K$, the linear functional of point evaluation $z \to p(z)$ is continuous as a map from H_n to $\mathbb{C}$ (why?). Thus, by the Riesz representation theorem, this functional is given by taking an inner product (in the norm of H_n) with a fixed element $Q_z \in \mathcal{P}_n$; i.e.,

$$p(z) = \int_K p\overline{Q}_z d\nu \text{ for } p \in \mathcal{P}_n.$$

Define $K_n^\nu(z,w) := \overline{Q}_z(w)$. One can check that if $\{q_j^{(n)}\}_{j=1,\dots,m_n}$ is an orthonormal basis for $\mathcal{P}_n$ with respect to $L^2(\nu)$, then

$$K_n^\nu(z,w) = \sum_{j=1}^{m_n} q_j^{(n)}(z)\overline{q}_j^{(n)}(w)$$

(note here that $m_n = \binom{N+n}{n}$). Indeed, observing that for any $p \in \mathcal{P}_n$ we have

$$p(z) = \sum_{j=1}^{m_n} \Big(\int_K p(w)\overline{q}_j^{(n)}(w) d\nu(w) \Big) q_j^{(n)}(z),$$

we see that

$$\int_K p(w)\Big(\sum_{j=1}^{m_n} q_j^{(n)}(z)\overline{q}_j^{(n)}(w)\Big) d\nu(w) =$$

$$\sum_{j=1}^{m_n} \Big(q_j^{(n)}(z) \int_K p(w)\overline{q}_j^{(n)}(w) d\nu(w)\Big) = p(z),$$

verifying that $\overline{Q}_z(w) = \sum_{j=1}^{m_n} q_j^{(n)}(z)\overline{q}_j^{(n)}(w)$. Restricting this reproducing kernel to the diagonal $\{z = w\}$, we call

$$B_n^\nu(z) := K_n^\nu(z,z) = \sum_{j=1}^{m_n} |q_j^{(n)}(z)|^2$$

the $n-th$ *Bergman function* of K, ν. It is known if (K,ν) satisfies the Bernstein-Markov inequality that

$$\lim_{n\to\infty} \frac{1}{2n} \log B_n^\nu(z) = V_K(z) \tag{76}$$

locally uniformly on $\mathbb{C}^N$ (cf., [16]).

As an easy example, take $K = \{z \in \mathbb{C} : |z| \le 1\}$, the closed unit disk in $\mathbb{C}$, and take $\nu = \frac{1}{2\pi} d\theta = \mu_K$. It is easy to see that the monomials $1, z, ..., z^n$ give an orthonormal basis for $\mathcal{P}_n$ in $L^2(\nu)$, and thus

$$B_n^\nu(z) = \sum_{j=0}^{n} |z|^{2j} = \frac{|z|^{2n+2} - 1}{|z|^2 - 1}.$$

Clearly, then, $\lim_{n\to\infty} \frac{1}{2n} \log B_n^\nu(z) = \log^+ |z|$ locally uniformly (exercise).

What happens in the weighted situation? For $K \subset \mathbb{C}^N$ compact, $w = e^{-Q}$ an admissible weight function on K, and ν a measure on K, we say that the triple (K,ν,Q) satisfies a weighted Bernstein-Markov property if there is a strong comparability between L^2 and L^∞ norms of *weighted* polynomials on K; precisely, for all $p_n \in \mathcal{P}_n$, writing $||w^n p_n||_K := \sup_{z\in K} |w(z)^n p_n(z)|$ and $||w^n p_n||^2_{L^2(\nu)} := \int_K |p_n(z)|^2 |w(z)|^{2n} d\nu(z)$,

$$||w^n p_n||_K \le M_n ||w^n p_n||_{L^2(\nu)} \text{ with } \limsup_{n\to\infty} M_n^{1/n} = 1.$$

If K is locally regular and w is continuous, taking $\nu = (dd^c V_{K,Q})^N$ we have (K,ν,Q) satisfies a weighted Bernstein-Markov property (cf., [9]). Now if (K,ν,Q) satisfies a weighted Bernstein-Markov property we have that

$$\lim_{n\to\infty} \frac{1}{2n} \log K_n^{\nu,w}(z,z) = V_{K,Q}(z) \tag{77}$$

locally uniformly on $\mathbb{C}^N$ where

$$K_n^{\nu,w}(z,\zeta) := \sum_{j=1}^{m_n} q_j^{(n)}(z)\overline{q_j^{(n)}(\zeta)}.$$

and

$$B_n^{\nu,w}(z) := K_n^{\nu,w}(z,z) w(z)^{2n} := \sum_{j=1}^{m_n} |q_j^{(n)}(z)|^2 w(z)^{2n} \tag{78}$$

is the $n-th$ *Bergman function* of K, w, ν (cf., [8]). Here, $\{q_j^{(n)}\}_{j=1,...,m_n}$ is an orthonormal basis for $\mathcal{P}_n$ with respect to the weighted L^2-norm $p \to ||w^n p_n||_{L^2(\nu)}$. A sketch of the proof of (77) and/or (76) runs as follows: first, one shows that if

$$\Phi_{K,Q,n}(z) := \sup\{|p(z)| : ||w^{\deg p} p||_K \le 1, \ p \in \mathcal{P}_n\},$$

then

$$\frac{1}{n} \log \Phi_{K,Q,n} \to V_{K,Q}$$

locally uniformly on $\mathbb{C}^N$ (see Corollary (4.3) and exercise 4 of section 5 for univariate versions). Next, one verifies the inequality

$$\frac{[\Phi_{K,Q,n}(z)]^2}{m_n} \le K_n^{\nu,w}(z,z) \le m_n \cdot M_n^2 [\Phi_{K,Q,n}(z)]^2.$$

The left-hand inequality follows simply from the reproducing property of the kernel function $K_n^{\nu,w}(z,\zeta)$; i.e., for any $p \in \mathcal{P}_n$,

$$p(z) = \int_K K_n^{\nu,w}(z,\zeta)p(\zeta)w(\zeta)^{2n}d\nu(\zeta),$$

and the Cauchy-Schwartz inequality; it is the right-side inequality which utilizes the weighted Bernstein-Markov property. Indeed, for an element $q_j^{(n)} \in \mathcal{P}_n$ in the orthonormal basis,

$$||w^n q_j^{(n)}||_K \le M_n \text{ and } \frac{|q_j^{(n)}(z)|}{||w^n q_j^{(n)}||_K} \le \Phi_{K,Q,n}(z)$$

imply

$$|q_j^{(n)}(z)| \le M_n \Phi_{K,Q,n}(z)$$

so that

$$K_n^{\nu,w}(z,z) = \sum_{j=1}^{m_n} |q_j^{(n)}(z)|^2 \le m_n \cdot M_n^2 [\Phi_{K,Q,n}(z)]^2.$$

These results were proved in the unweighted case, i.e., (76), by Bloom and Shiffman [16] and in the general (weighted) case, i.e., (77), by Bloom [8].

From the local uniform convergence in (77) follows the weak-* convergence of the Monge-Ampère measures

$$[dd^c \frac{1}{2n} \log K_n^{\nu,w}(z,z)]^N \to (dd^c V_{K,Q}^*)^N \text{ weak-}*.$$

One of the main results in the next section is a much stronger version of "Bergman asymptotics" to be proved in Corollary 10.5: if (K,ν,w) satisfies a weighted Bernstein-Markov inequality, then

$$\frac{1}{m_n} B_n^{\nu,w} d\nu \to \mu_{K,Q} := \frac{1}{(2\pi)^N}(dd^c V_{K,Q}^*)^N \text{ weak-}*.$$

This was proved in the one variable case ($N = 1$) in [14].

We continue with a multivariate version of Theorem 5.3, the relation of the weighted Bernstein-Markov property and weighted transfinite diameter. Here, we use the notion

$$G_n^{\nu,w} := \left[\int_K \overline{e_i(z)} e_j(z) w(z)^{2n} d\nu\right] \in \mathbb{C}^{m_n \times m_n} \tag{79}$$

for the Gram matrix of the standard basis monomials $e_i \in \mathcal{P}_n$ with respect to the measure ν and weight w. Recall that

$$l_n = \sum_{j=1}^{m_n} deg(e_j) = \frac{Nnm_n}{N+1}.$$

Thus, in the formulas below, $\frac{N+1}{2Nnm_n}$ is simply $\frac{1}{2l_n}$.

Proposition 9.5. *Let $K \subset \mathbb{C}^N$ be a compact set and let w be an admissible weight function on K. If ν is a measure on K with (K,ν,Q) satisfying a weighted Bernstein-Markov property, then*

$$\lim_{n\to\infty} \frac{N+1}{2Nnm_n} \cdot \log \det G_n^{\nu,w} = \log \delta^w(K).$$

Proof. Note first that $\det G_n^{\nu,w} = \prod_{j=1}^{m_n} ||r_j||^2_{L^2(w^{2n}\nu)}$ where $\{r_1,...,r_{m_n}\}$ are an orthogonal basis of $\mathcal{P}_n$ obtained by applying Gram-Schmidt to the standard basis monomials of $\mathcal{P}_n$. Defining, analogous to (20),

$$Z_n := Z_n(K,w,\nu)$$

$$:= \int_K \cdots \int_K |VDM(z_1,...,z_{m_n})|^2 w(z_1)^{2n} \cdots w(z_{m_n})^{2n} d\nu(z_1)\cdots d\nu(z_{m_n})$$

we show that

$$\lim_{n\to\infty} Z_n^{\frac{N+1}{2Nnm_n}} = \delta^w(K).$$

To see this, clearly

$$Z_n \le \delta_n^w(K)^{\frac{2Nnm_n}{N+1}} \nu(K)^{m_n}. \tag{80}$$

On the other hand, taking points $x_1,...,x_{m_n}$ achieving the maximum in $\delta_n^w(K)$, we have, upon applying the weighted Bernstein-Markov property to the weighted polynomial

$$z_1 \to VDM(z_1,x_2...,x_{m_n})w(z_1)^n \cdots w(x_{m_n})^n,$$

$$\delta_n^w(K)^{\frac{2Nnm_n}{N+1}} = |VDM(x_1,...,x_{m_n})|^2 w(x_1)^{2n}\cdots w(x_{m_n})^{2n}$$
$$\le M_n^2 \int_K \cdots \int_K |VDM(z_1,x_2...,x_{m_n})|^2 w(z_1)^{2n}\cdots w(x_{m_n})^{2n} d\nu(z_1).$$

Repeating this argument in each variable we obtain

$$\delta_n^w(K)^{\frac{2Nnm_n}{N+1}} \le M_n^{2m_n} Z_n. \tag{81}$$

Note that (80) and (81) give

$$Z_n \le \delta_n^w(K)^{\frac{2Nnm_n}{N+1}} \nu(K)^N \le \nu(K)^N M_n^{2m_n} Z_n.$$

Since $[\nu(K)^N M_n^{2m_n}]^{\frac{N+1}{2Nnm_n}} \to 1$, using (70)

$$\lim_{n\to\infty} Z_n^{\frac{N+1}{2Nnm_n}}$$

exists and equals

$$\lim_{n\to\infty} \delta_n^w(K)^{\frac{N+1}{Nnm_n}}.$$

Using elementary row operations in $|VDM(z_1,...,z_{m_n})|^2$ in the integrand of Z_n, we can replace the monomials $\{e_j\}$ by the orthogonal basis $\{r_1,...,r_{m_n}\}$ and obtain

$$Z_n = m_n! \prod_{j=1}^{m_n} ||r_j||^2_{L^2(w^{2n}\nu)}.$$

Putting everything together gives the result. Note that

$$Z_n = m_n! \cdot \det(G_n^{\nu,w})$$

(see (83) below). □

Definition 9.1. If a probability measure μ has the property that

$$\det(G_n^{\mu',w}) \le \det(G_n^{\mu,w}) \tag{82}$$

for all other probability measures μ' on K then μ is said to be an *optimal measure* of degree n for K and w.

Note we have fixed the usual monomial basis to compute our Gram matrices but it is an easy exercise to show that the notion of optimal measure is independent of the basis we choose. We continue with some algebraic preliminaries relating Gram determinants, Bergman functions, and generalized Vandermonde determinants, whose proofs we leave as exercises.

Lemma 9.6. *Suppose that $\mu \in \mathcal{M}(K)$ and that w is an admissible weight. Then*

$$\det(G_n^{\mu,w}) = \frac{1}{m_n!}\int_{K^{m_n}} |VDM(z_1,\cdots,z_{m_n})|^2 \tag{83}$$
$$\cdot w(z_1)^{2n}\cdots w(z_{m_n})^{2n} d\mu(z_1)\cdots d\mu(z_{m_n}) = \frac{Z_n}{m_n!}$$

and

$$B_n^{\mu,w}(z) = \frac{m_n}{Z_n}\int_{K^{m_n-1}} |VDM(z,z_2,\cdots,z_{m_n})|^2 \tag{84}$$
$$\cdot w(z)^{2n} w(z_2)^{2n}\cdots w(z_{m_n})^{2n} d\mu(z_2)\cdots d\mu(z_{m_n}).$$

A similar argument to the proof of Proposition 9.5 shows that the Gram determinants associated to a sequence of weighted optimal measures also converges to $\delta^w(K)$ (exercise 7). In this proposition, we again compute the Gram determinant with respect to the standard basis monomials.

Proposition 9.7. *Let K be compact and w an admissible weight function. For $n = 1,2,...$, let μ_n be an optimal measure of order n for K and w. Then*

$$\lim_{n\to\infty} \det(G_n^{\mu_n,w})^{\frac{N+1}{2Nnm_n}} = \delta^w(K).$$

The connection between (weighted) optimal measures and (weighted) Bergman functions is the following.

Proposition 9.8. *Let w be an admissible weight on K. A probability measure μ is an optimal measure of degree n for K and w if and only if*

$$\max_{z\in K} B_n^{\mu,w}(z) = m_n. \tag{85}$$

For the proof of Proposition 9.8, cf., [18]. As a corollary, we obtain the following key property of optimal measures.

Lemma 9.9. *Suppose that μ is an optimal measure of degree n for K and w Then*

$$B_n^{\mu,w}(z) = m_n, \quad a.e.\ \mu.$$

Proof. On the one hand, by Proposition 9.8

$$\max_{z\in K} B_n^{\mu,w}(z) = m_n,$$

while on the other hand, by orthonormality of the $q_j^{(n)}$ in (78) (with $\nu = \mu$)

$$\int_K B_n^{\mu,w}\, d\mu = \int_K \sum_{j=1}^{m_n} |q_j^{(n)}(z)|^2 w(z)^{2n}\, d\mu(z) = m_n,$$

and the result follows. □

Exercises.

1. Give a proof of (68) analogous to the univariate proof of (39) using the solution to the Dirichlet problem for the complex Monge-Ampère operator in a ball.
2. Suppose K is the closed unit ball and Q is continuous on K and *plurisuperharmonic* on the interior of K (i.e., $-Q$ is psh). What can you say about S_w?
3. Suppose K is the closed unit ball and Q is continuous on K and is a maximal psh function on the interior of K. What can you say about S_w?
4. Find $V_{K,Q}$ for K the closed unit ball and $Q(z) = -|z|^2$.
5. Verify that the dimension of the space of homogeneous polynomials of degree d in $\mathbb{C}^{N+1}$ equals the dimension of the space of polynomials of degree at most d in $\mathbb{C}^N$.
6. Verify equations (83) and (84) of Lemma 9.6.
7. Prove Proposition 9.7. (*Hint: Use the fact that* $\det(G_n^{\nu_n,w}) \leq \det(G_n^{\mu_n,w})$ *where* $\nu_n = \frac{1}{m_n}\sum_{k=1}^{m_n} \delta_{x_k}$ *and* $x_1,\dots,x_{m_n}$ *are points in* K *achieving the maximum in* $\delta_n^w(K)$.)

10 Recent results in pluripotential theory.

In this final section, we outline proofs of the strong Bergman asymptotic result mentioned in the previous section as well as the analogue of Proposition 5.2 for asymptotic weighted Fekete arrays in $\mathbb{C}^N$. These results are based on work of R. Berman and S. Boucksom. As the reader will see, the weighted theory is essential even if one only wants these results in the unweighted case.

Given a compact set $K \subset \mathbb{C}$, a discretization of the logarithmic energy minimization problem $\inf_{\mu\in\mathcal{M}(K)} I(\mu)$ led to the notion of transfinite diameter $\delta(K)$. In the nonpolar case, the energy-minimizing measure is given by $\mu_K = \frac{1}{2\pi}\Delta V_K^*$. Thus, in a sense, the notion of logarithmic energy relates $\delta(K)$ with V_K^*. How can we relate these two notions in $\mathbb{C}^N$, $N > 1$ without a notion of energy of a measure?

Proposition 10.11 below provides part of the answer; but Theorem 10.3 is the key. We begin by defining a special functional on the class $L^+(\mathbb{C}^N)$. The strictly psh function $u_0(z) := \frac{1}{2}\log(1+|z|^2)$ belongs to this class. For $u \in L^+(\mathbb{C}^N)$ we define

$$E(u) := \frac{1}{N+1}\int_{\mathbb{C}^N}\sum_{j=0}^{N}(u-u_0)(dd^cu)^j\wedge(dd^cu_0)^{N-j}. \tag{86}$$

The functional E is a primitive for the complex Monge-Ampère operator in a sense that will be made precise in Proposition 10.2. In the univariate case; i.e., $N=1$,

$$E(u) = \frac{1}{2}\int_{\mathbb{C}}(u-u_0)dd^c(u+u_0). \tag{87}$$

Next, for $Q \in \mathcal{A}(K)$, define

$$P(Q) = P_K(Q) := V_{K,Q}^*.$$

We record some straightforward properties of this operator P.

Proposition 10.1. *The operator $P : \mathcal{A}(K) \to L^+(\mathbb{C}^N)$ is increasing and concave: for $0 \le t \le 1$ and $Q_1, Q_2 \in \mathcal{A}(K)$,*

$$P(Q_1) \le P(Q_2) \text{ if } Q_1 \le Q_2 \text{ and}$$

$$P(tQ_1 + (1-t)Q_2) \ge tP(Q_1) + (1-t)P(Q_2).$$

In addition, P is Lipschitz: for $t\in\mathbb{R}$, $Q_1 \in \mathcal{A}(K)$ and $Q_2 \in C(K)$,

$$|P(Q_1+tQ_2) - P(Q_1)| \le C|t|,\ C = C(Q_1,Q_2). \tag{88}$$

The composition of the E and P operators is Gateaux differentiable; this non-obvious result (Theorem 10.3) was proved by Berman and Boucksom in [5] and is the key to many recent results in (weighted) pluripotential theory.

Proposition 10.2. *The functional E is increasing and concave; i.e., for $u, v \in L^+(\mathbb{C}^N)$ the function $f(t) := E((1-t)u+tv)$ is twice differentiable for $0 \le t \le 1$ with $f'(t) \ge 0$ and $f''(t) \le 0$. We have*

$$f'(0) := \lim_{t\downarrow 0^+}\frac{f(t)-f(0)}{t} = \int_{\mathbb{C}^N}(v-u)(dd^cu)^N. \tag{89}$$

Proof. We will verify (89) and leave the rest to the reader. To understand the idea, we consider first the univariate case, $N=1$. It suffices to show that

$$f(t)-f(0) = t\int_{\mathbb{C}}(v-u)(dd^cu) + 0(t^2).$$

From the definition in (87), setting $w := v-u$,

$$\begin{aligned}
2[f(t)-f(0)] &= 2\big(E(u+t(v-u)-E(u)\big) = 2\big(E(u+tw)-E(u)\big)\\
&= \int_{\mathbb{C}}(u+tw-u_0)[dd^c(u+tw+u_0)] - \int_{\mathbb{C}}(u-u_0)[dd^c(u+u_0)]\\
&= t\Big(\int_{\mathbb{C}}(u-u_0)dd^cw + \int_{\mathbb{C}}wdd^c(u+u_0)\Big) + 0(t^2)\\
&= 2t\int_{\mathbb{C}}wdd^cu + 0(t^2)
\end{aligned}$$

which gives the result.

For the multivariate case, we begin with the observation that if again we set $w := v-u$, then

$$\begin{aligned}
&\sum_{j=0}^{N}[dd^c(u+tw)]^j\wedge(dd^cu_0)^{N-j} - \sum_{j=0}^{N}(dd^cu)^j\wedge(dd^cu_0)^{N-j}\\
&= t\sum_{j=0}^{N}j[dd^cw\wedge(dd^cu)^{j-1}\wedge(dd^cu_0)^{N-j} + 0(t^2).
\end{aligned}$$

Then (all integrals are over $\mathbb{C}^N$)

$$
\begin{aligned}
(N+1)\big(E(u+t(v-u)-E(u)\big) &= (N+1)\big(E(u+tw)-E(u)\big)\\
&= \int [u+tw-u_0]\sum_{j=0}^{N}[dd^c(u+tw)]^j\wedge(dd^cu_0)^{N-j}\\
&\quad -\int (u-u_0)\sum_{j=0}^{N}(dd^cu)^j\wedge(dd^cu_0)^{N-j}\\
&= t\int (u-u_0)\sum_{j=0}^{N} j[dd^cw\wedge(dd^cu)^{j-1}\wedge(dd^cu_0)^{N-j}+0(t^2)\\
&\quad +\int tw\sum_{j=0}^{N}[dd^c(u+tw)]^j\wedge(dd^cu_0)^{N-j}\\
&= t\Big[\int (u-u_0)\sum_{j=0}^{N} j[dd^cw\wedge(dd^cu)^{j-1}\wedge(dd^cu_0)^{N-j}]\\
&\quad +\int w\sum_{j=0}^{N}(dd^cu)^j\wedge(dd^cu_0)^{N-j}\Big]+0(t^2)\\
&= t\Big[\int w\sum_{j=0}^{N} j[dd^c(u-u_0)\wedge(dd^cu)^{j-1}\wedge(dd^cu_0)^{N-j}]\\
&\quad +\int w\sum_{j=0}^{N}(dd^cu)^j\wedge(dd^cu_0)^{N-j}\Big]+0(t^2).
\end{aligned}
$$

In the last step we have used an "integration by parts" formula involving *differences* of functions in $L^+(\mathbb{C}^N)$; to wit: for $A,B,C,D\in L^+(\mathbb{C}^N)$ and $u_1,...,u_{N-1}\in L^+(\mathbb{C}^N)$ (so that $T:=dd^cu_1\wedge\cdots\wedge dd^cu_{N-1}$ is a positive closed $(N-1,N-1)$ current), we have

$$
\begin{aligned}
&\int_{\mathbb{C}^N}(A-B)(dd^cC-dd^cD)\wedge dd^cu_1\wedge\cdots\wedge dd^cu_{N-1}\\
&=\int_{\mathbb{C}^N}(C-D)(dd^cA-dd^cB)\wedge dd^cu_1\wedge\cdots\wedge dd^cu_{N-1}.
\end{aligned}
$$

Now check that

$$
\begin{aligned}
\sum_{j=0}^{N} jdd^c(u-u_0)\wedge(dd^cu)^{j-1}\wedge(dd^cu_0)^{N-j}+\sum_{j=0}^{N}(dd^cu)^j\wedge(dd^cu_0)^{N-j}\\
=(N+1)(dd^cu)^N
\end{aligned}
$$

(try the case $N=2$!) and the result follows. $\square$

Theorem 10.3. **[Berman-Boucksom]** *The functional defined for a nonpluripolar compact set $K\subset\mathbb{C}^N$ as the composition $E\circ P$ is Gateaux differentiable; i.e., for $Q\in\mathcal{A}(K)$, $F(t):=(E\circ P)(Q+tv)$ is differentiable for all $v\in C(K)$ and $t\in\mathbb{R}$. Furthermore,*

$$F'(0)=\int_K v(dd^cP(Q))^N. \tag{90}$$

The proof of Theorem 10.3 utilizes a global version of the comparison principle from section 7: for $u,v\in L^+(\mathbb{C}^N)$,

$$\int_{\{u<v\}}(dd^cv)^N\le\int_{\{u<v\}}(dd^cu)^N, \tag{91}$$

as well as the properties of the E and P operators in Proposition 10.1. The proof of (91) is outlined in exercise 2.

Theorem 10.3 is one ingredient used to obtain the following general result.

Proposition 10.4. *Let $K\subset\mathbb{C}^N$ be compact with admissible weight w. Let $\{\mu_n\}$ be a sequence of probability measures on K with the property that*

$$\lim_{n\to\infty}\frac{N+1}{2Nnm_n}\cdot\log\det G_n^{\mu_n,w}=\log\delta^w(K). \tag{92}$$

Then

$$\frac{1}{m_n}B_n^{\mu_n,w}d\mu_n\to\mu_{K,Q}=\frac{1}{(2\pi)^N}(dd^cV_{K,Q}^*)^N\ \textit{weak-}*. \tag{93}$$

In particular, from Proposition 10.4 and Proposition 9.5 we have a general strong Bergman asymptotic result.

Corollary 10.5. **[Strong Bergman Asymptotics]** *If (K,μ,w) satisfies a weighted Bernstein-Markov inequality, then*

$$\frac{1}{m_n}B_n^{\mu,w}d\mu \to \mu_{K,Q} \text{ weak-}*.$$

Another consequence of Proposition 10.4 is the analogue of Proposition 5.2 on asymptotic weighted Fekete arrays in $\mathbb{C}^N$.

Corollary 10.6. **[Asymptotic Weighted Fekete Points]** *Let $K\subset\mathbb{C}^N$ be compact with admissible weight w. For each n, take points $x_1^{(n)},x_2^{(n)},\cdots,x_{m_n}^{(n)}\in K$ for which*

$$\lim_{n\to\infty}\left[|VDM(x_1^{(n)},\cdots,x_{m_n}^{(n)})|w(x_1^{(n)})^n w(x_2^{(n)})^n\cdots w(x_{m_n}^{(n)})^n\right]^{\frac{(N+1)}{Nnm_n}}$$

$$=\delta^w(K) \tag{94}$$

(asymptotically weighted Fekete points) and let $\mu_n:=\frac{1}{m_n}\sum_{j=1}^{m_n}\delta_{x_j^{(n)}}$. Then $\mu_n\to\mu_{K,Q}$ weak $-*$.

Proof. Note that the hypothesis (94) is equivalent to (92) by observing (83) with $\mu=\mu_n$. By direct calculation, we have $B_n^{\mu_n,w}(x_j^{(n)})=m_n$ for $j=1,\dots,m_n$ and hence a.e. μ_n on K. Indeed, this property holds for *any* discrete, equally weighted measure $\mu_n:=\frac{1}{m_n}\sum_{j=1}^{m_n}\delta_{x_j^{(n)}}$ with

$$|VDM(x_1^{(n)},\cdots,x_{m_n}^{(n)})|w(x_1^{(n)})^n w(x_2^{(n)})^n\cdots w(x_{m_n}^{(n)})^n\neq 0$$

(exercise 3). The result follows immediately from Proposition 10.4, specifically, equation (93). □

Finally, using Lemma 9.9 and Proposition 9.7 in conjuction with Proposition 10.4, we conclude that a sequence of weighted optimal measures converges to $\mu_{K,Q}$.

Corollary 10.7. **[Weighted Optimal Measures]** *Let $K\subset\mathbb{C}^N$ be compact with admissible weight w. For each n, let μ_n be an optimal measure of degree n for K and w. Then $\mu_n\to\mu_{K,Q}$ weak $-*$.*

We proceed with an outline of the steps utilized to prove Proposition 10.4. Let w be an admissible weight function on K and fix $u\in C(K)$. Following the ideas in [1], [2], [3], [4], [5] we consider the perturbed weight $w_t(z):=w(z)\exp(-tu(z))$, $t\in\mathbb{R}$. For the moment, we let $\{\mu_n\}$ be any sequence of measures in $\mathcal{M}(K)$. We set

$$f_n(t):=-\frac{1}{2l_n}\log\det(G_n^{\mu_n,w_t}). \tag{95}$$

We have the following (see Lemma 6.4 in [5]).

Lemma 10.8. *We have*

$$f_n'(t)=\frac{N+1}{Nm_n}\int_K u(z)B_n^{\mu_n,w_t}(z)d\mu_n.$$

In particular,

$$f_n'(0)=\frac{N+1}{Nm_n}\int_K u(z)B_n^{\mu_n,w}(z)d\mu_n$$

and if $B_n^{\mu_n,w}=m_n$ a.e. μ_n,

$$f_n'(0)=\frac{N+1}{N}\int_K u(z)d\mu_n. \tag{96}$$

Before we give the proof, an illustrative example can be given if $\mu_n:=\frac{1}{m_n}\sum_{j=1}^{m_n}\delta_{x_j}$. Then $B_n^{\mu_n,w}(x_j)=m_n$ for $j=1,\dots,m_n$ (see exercise 2) so

$$\log\det(G_n^{\mu_n,w_t})$$
$$=\log\left(|W(x_1,\dots,x_{m_n})|^2e^{-2ntu(x_1)}\cdots e^{-2ntu(x_{m_n})}\right)$$

implies

$$\frac{d}{dt}\log\det(G_n^{\mu_n,w_t})|_{t=0}=\frac{d}{dt}\left(-2tn\sum_{j=1}^{N}u(x_j)\right)|_{t=0}$$

$$=-2n\sum_{j=1}^{m_n}u(x_j)=-2nm_n\int_K u(z)\frac{1}{m_n}B_n^{\mu_n,w}(z)d\mu_n.$$

Proof. The proof we offer here is based on the integral formulas of Lemma 9.6.

By (83) we may write

$$f_n(t) = -\frac{1}{2l_n}\log(F_n) + \frac{1}{2l_n}\log(m_n!)$$

where $l_n = (\frac{N}{N+1})nm_n$ and

$$F_n(t) := \int_{K^{m_n}} V\exp(-tU)d\mu$$

and

$$V := V(z_1, z_2, \cdots, z_{m_n}) = |VDM(z_1, \cdots, z_{m_n})|^2 w(z_1)^{2n} \cdots w(z_{m_n})^{2n},$$
$$U := U(z_1, z_2, \cdots, z_{m_n}) = 2n(u(z_1) + \cdots + u(z_{m_n})),$$
$$d\mu := d\mu_n(z_1)d\mu_n(z_2)\cdots d\mu_n(z_{m_n}).$$

Further, by (84) for $w = w_t$ and $\mu = \mu_n$, we have

$$B_n^{\mu_n,w_t}(z)$$
$$= \frac{m_n}{Z_n}\int_{K^{m_n-1}} V(z, z_2, z_3, \cdots, z_{m_n})\exp(-tU)d\mu_n(z_2)\cdots d\mu_n(z_{m_n})$$

where

$$Z_n = Z_n(t) := m_n!\det(G_n^{\mu_n,w_t}) = \int_{K^{m_n}} V\ \exp(-tU)d\mu.$$

Note that $Z_n(t) = F_n(t)$. Now

$$f_n'(t) = -\frac{1}{2l_n}\frac{F_n'(t)}{F_n(t)}$$

and we may compute

$$F_n'(t) = \int_{K^{m_n}} V(-U)\exp(-tU)d\mu_n(z_1)\cdots d\mu_n(z_{m_n})$$
$$= -2n\int_{K^{m_n}} (u(z_1) + \cdots + u(z_{m_n}))V\exp(-tU)d\mu_n(z_1)\cdots d\mu_n(z_{m_n}).$$

Notice that the integrand is symmetric in the variables and hence we may “de-symmetrize” to obtain

$$F_n'(t)$$
$$= -2nm_n\int_{K^{m_n}} u(z_1)V(z_1, \cdots, z_{m_n})\exp(-tU)d\mu_n(z_1)\cdots d\mu_n(z_{m_n})$$

so that, integrating in all but the z_1 variable, we obtain

$$F_n'(t) = -2nm_n\int_K u(z)B_n^{\mu_n,w_t}(z)\frac{Z_n}{n}d\mu_n(z).$$

Thus, using the fact that $Z_n(t) = F_n(t)$, we obtain

$$f_n'(t) = \frac{N+1}{Nm_n}\int_K u(z)B_n^{\mu_n,w_t}(z)d\mu_n(z)$$

as claimed. In particular,

$$f_n'(0) = \frac{N+1}{Nm_n}\int_K u(z)B_n^{\mu_n,w}(z)d\mu_n$$

and if $B_n^{\mu_n,w} = m_n$ a.e. μ_n, we recover (96):

$$f_n'(0) = \frac{N+1}{N}\int_K u(z)d\mu_n.$$

□

The next result was proved in a different way in [6], Lemma 2.2, and also in [11], Lemma 3.6.

Lemma 10.9. *The functions $f_n(t)$ are concave.*

Proof. We show that $f_n''(t) \le 0$. With the notation used in the proof of Lemma 10.8,

$$f_n''(t) = \frac{1}{2l_n}\frac{(F_n'(t))^2 - F_n''(t)}{F_n^2(t)}$$

and

$$\begin{aligned} F_n'(t) &= -\frac{1}{m_n!}\int_{K^{m_n}} UV\exp(-tU)d\mu, \\ F_n''(t) &= \frac{1}{m_n!}\int_{K^{m_n}} U^2 V\exp(-tU)d\mu. \end{aligned}$$

We must show that $(F_n'(t))^2 - F_n''(t) \ge 0$. Now, for a fixed t, we may mulitply V by a constant so that

$$\int_{K^{m_n}} V\exp(-tU)d\mu = 1.$$

Let $d\gamma := V\exp(-tU)d\mu$. Then by the above formulas for F_n' and F_n'', we must show that

$$\int_{K^{m_n}} U^2 d\gamma \ge \left(\int_{K^{m_n}} U\,d\gamma\right)^2,$$

but this is a simple consequence of the Cauchy-Schwarz inequality. □

The following "calculus lemma" is essential for the proof of Proposition 10.4.

Lemma 10.10. (Berman and Boucksom [5]) *Let $f_n(t)$ be a sequence of concave functions on $\mathbb{R}$ and $g(t)$ a function on $\mathbb{R}$. Suppose that*

$$\liminf_{n\to\infty} f_n(t) \ge g(t), \quad \forall t \in \mathbb{R}$$

and that

$$\lim_{n\to\infty} f_n(0) = g(0).$$

Suppose further that the f_n and g are differentiable at $t = 0$. Then

$$\lim_{n\to\infty} f_n'(0) = g'(0).$$

Here we really need differentiability at $t = 0$; one-sided differentiability is not sufficient. The last key ingredient we need for Proposition 10.4 is an amazing relationship between the weighted transfinite diameter $\delta^w(K)$ and $E(V_{K,Q}^*)$. Indeed, the proof of this result uses Theorem 10.3 and is more difficult but almost equivalent to Proposition 10.4.

Proposition 10.11. *For $K \subset \mathbb{C}^N$ compact and w an admissible weight on K, we have*

$$-\log\delta^w(K) = E(V_{K,Q}^*) - E(V_T). \tag{97}$$

With these preliminaries, we now prove Proposition 10.4.

Proof. Recall we are assuming the measures $\{\mu_n\}$ satisfy (92):

$$\lim_{n\to\infty}\frac{N+1}{2Nnm_n}\cdot\log\det G_n^{\mu_n,w} = \log\delta^w(K)$$

and we want to show (93):

$$\frac{1}{m_n}B_n^{\mu_n,w}d\mu_n \to \mu_{K,Q} = \frac{1}{(2\pi)^N}(dd^cV_{K,Q}^*)^N \text{ weak-}*.$$

For $u \in C(K)$ we again set $w_t(z) := w(z)\exp(-tu(z))$ which corresponds to $Q_t := Q + tu$ and $f_n(t)$ as in (95). From (92), for $t = 0$, $w_0 = w$ we have

$$\lim_{n\to\infty} f_n(0) = -\log(\delta^w(K)).$$

From (97) and Theorem 10.3, setting $g(t) = -\log(\delta^{w_t}(K))$,

$$g'(0) = \frac{N+1}{N(2\pi)^N}\int_K u(z)(dd^cV_{K,Q}^*)^N. \tag{98}$$

Now note that for each fixed t, the measure μ_n is a candidate for the optimal measure for K and w_t. If follows from Definition 9.1 that

$$\det(G_n^{\mu_n,w_t}) \le \det(G_n^{\mu_n^t,w_t})$$

where we denote an optimal measure for K and w_t by μ_n^t. Hence (see (95))

$$f_n(t) \geq -\frac{1}{2m_n}\log(\det(G_n^{\mu_n^t,w_t}))$$

and consequently from Proposition 9.7 we have

$$\liminf_{n\to\infty} f_n(t) \geq -\log(\delta^{w_t}(K)) = g(t). \tag{99}$$

It now follows from Lemma 10.10 that

$$\lim_{n\to\infty} f_n'(0) = g'(0).$$

In other words, by Lemma 10.8,

$$\begin{aligned}\lim_{n\to\infty}\frac{N+1}{N}\int_K u(z)d\mu_n &= \frac{N+1}{N(2\pi)^N}\int_K u(z)(dd^cV_{K,Q}^*)^N \\ &= \frac{N+1}{N}\int_K u(z)d\mu_{K,Q},\end{aligned}$$

and hence $\mu_n \to \mu_{K,Q}$ weak$-*$. □

The reader can consult [22] for a self-contained discussion of the results in this section.

Exercises.

1. Prove that the operator $P : \mathcal{A}(K) \to L^+(\mathbb{C}^N)$ is increasing and concave: for $0 \leq t \leq 1$ and $Q_1, Q_2 \in \mathcal{A}(K)$,
$$P(Q_1) \leq P(Q_2) \text{ if } Q_1 \leq Q_2 \text{ and}$$
$$P(tQ_1 + (1-t)Q_2) \geq tP(Q_1) + (1-t)P(Q_2).$$
2. Prove (91) using the following outline:
   (a) We can assume $u \geq 0$ (why?). For $\epsilon > 0$, apply (46) to $(1+\epsilon)u$ and v on the bounded set $\{(1+\epsilon)u < v\}$.
   (b) Show that $\bigcup_{j=1}^{\infty}\{(1+1/j)u < v\} = \{u < v\}$.
   (c) Apply (a) with $\epsilon = 1/j$ and conclude using (b) and monotone convergence.
3. Verify that $B_n^{\mu_n,w}(x_j^{(n)}) = m_n$ for $j = 1, \ldots, m_n$ for any discrete, equally weighted measure $\mu_n := \frac{1}{m_n}\sum_{j=1}^{m_n}\delta_{x_j^{(n)}}$ with
$$|VDM(x_1^{(n)}, \cdots, x_{m_n}^{(n)})| w(x_1^{(n)})^n w(x_2^{(n)})^n \cdots w(x_{m_n}^{(n)})^n \neq 0.$$
(*Hint: Show that the orthonormal polynomials are given by* $q_j^{(n)}(z) = \frac{\sqrt{m_n}\, l_{nj}(z)}{w(x_j^{(n)})^{2n}}$ *where* l_{nj} *is the FLIP associated to* $x_j^{(n)}$ *(recall (66)).*
4. Prove Lemma 10.10.

11 Appendix A: Differential forms and currents in $\mathbb{C}^N$.

We introduce some standard material on differential forms and currents. We may identify $\mathbb{C}^N$ with $\mathbb{R}^{2N}$ via the mapping

$$(z_1,\dots,z_N)=(x_1+iy_1,\dots,x_N+iy_N)\mapsto(x_1,y_1,\dots,x_N,y_N).$$

We have for $k=1,\dots,N$, the complex differentials

$$dz_k=dx_k+idy_k,\quad d\overline{z}_k=dx_k-idy_k.$$

We also recall the following notation—for a multi-index $I=(i_1,\dots,i_p)$ we write

$$|I|=p \text{ (the multi-index length)},$$

$$dz^I=dz_{i_1}\wedge\cdots\wedge dz_{i_p},\ \ d\overline{z}^I=d\overline{z}_{i_1}\wedge\cdots\wedge d\overline{z}_{i_p}.$$

The standard volume form in $\mathbb{C}^N\sim\mathbb{R}^{2N}$ is defined by

$$dV_{2N}:=\left(\frac{i}{2}\right)^N dz_1\wedge d\overline{z}_1\wedge\cdots\wedge dz_N\wedge d\overline{z}_N=dx_1\wedge dy_1\wedge\cdots\wedge dx_N\wedge dy_N.$$

Let D be a domain in $\mathbb{C}^N$ and k a nonnegative integer, $k\le 2N$. A complex differential k-form on D can be written as

$$\omega=\Sigma'_{|I|+|J|=k}\omega_{IJ}\,dz^I\wedge d\overline{z}^J$$

for some coefficient functions $\omega_{IJ}\in C^\infty(D,\mathbb{C}):=C^\infty(D)$. Here the 'prime' ($'$) indicates that we sum over increasing multi-indices only: if $I=(i_1,\dots,i_p)$, then $1\le i_1<i_2<\cdots<i_p\le N$. The *norm of* ω is given pointwise by

$$|\omega|=\left(\Sigma'_{|I|+|J|=k}|\omega_{IJ}|^2\right)^{\frac{1}{2}}.$$

It measures at each point of D the Euclidean norm of the k-form with respect to the orthonormal basis $\{dz^I\wedge d\overline{z}^J\}_{|I|+|J|=k}$.

We write $\bigwedge^k(D,\mathbb{C})$ to denote the complex vector space of (smooth) k-forms on D. The 0-forms, by convention, are the functions in $C^\infty(D,\mathbb{C})$. The space $\bigwedge^k(D,\mathbb{C})$ has some important subspaces. Given nonnegative integers p,q with $p+q=k$, we define $\bigwedge^{p,q}(D,\mathbb{C})$, the *forms of bidegree (p,q)*, as the set of all k-forms ω that can be written as

$$\omega=\Sigma'_{|I|=p,|J|=q}\omega_{IJ}\,dz^I\wedge d\overline{z}^J.$$

In pluripotential theory we often consider only the spaces $\bigwedge^{p,p}(D,\mathbb{C})$, where $0\le p\le N$ is a nonnegative integer. Note that $\bigwedge^{2N}(D,\mathbb{C})=\bigwedge^{N,N}(D,\mathbb{C})$. For such differential forms, we will define the notion of *positivity*.

Definition 11.1. An (N,N)-form ω on D is called *positive* if $\omega=\tau dV_{2N}$ for some function $\tau:D\to[0,\infty)$.

A (p,p)-form α is called *elementary strongly positive* if there are linearly independent complex linear mappings $\eta_j:\mathbb{C}^N\to\mathbb{C}$, $j=1,\dots,p$ such that

$$\alpha=\frac{i}{2}d\eta_1\wedge d\overline{\eta}_1\wedge\cdots\wedge\frac{i}{2}d\eta_p\wedge d\overline{\eta}_p.$$

A form ω is called *strongly positive* if $\omega=\sum\lambda_j\omega_j$ for m non-negative numbers $\lambda_1,\dots,\lambda_m$ and elementary strongly positive forms $\omega_1,\dots,\omega_m$, where m is a positive integer.

A (p,p)-form ω is called *positive* if for any strongly positive $(N-p,N-p)$-form η, the (N,N)-form $\omega\wedge\eta$ is positive.

As an example, the standard Kähler form in $\mathbb{C}^N$ is defined by $\beta:=\frac{i}{2}\sum_{j=1}^N dz_j\wedge d\overline{z}_j$. For a positive integer $p\le N$, $\beta^p=\beta\wedge\cdots\wedge\beta$ (p times) is a positive (p,p)-form. In particular, $\beta^N=N!dV_{2N}$.

We denote by $\mathcal{D}^k(D,\mathbb{C})$ the subspace of $\bigwedge^k(D,\mathbb{C})$ made up of those forms whose coefficients are in $C_0^\infty(D,\mathbb{C}):=C_0^\infty(D)$. They are called the *test forms of degree* k. Note that $\mathcal{D}^0(D,\mathbb{C})=C_0^\infty(D,\mathbb{C})$, the usual *test functions* of distribution theory. The *test forms of bidegree* (p,q), $\mathcal{D}^{p,q}(D,\mathbb{C})$, are defined similarly.

We equip $\mathcal{D}^0(D,\mathbb{C})$ with the topology characterized by the following convergence property: given test functions $\{\phi_j\}_{j=1}^\infty,\phi$ then $\phi_j\to\phi$ if there exists a compact set $K\subset D$ such that

1. $supp(\phi_j),supp(\phi)\subset K$
2. The functions ϕ_j converge uniformly to ϕ on K, and the derivatives (of all orders) of ϕ_j converge uniformly to the corresponding derivatives of ϕ.

The topology on $\mathcal{D}^k(D,\mathbb{C})$ is characterized by the property that given forms $\{\omega_j\},\omega$ in $\mathcal{D}^k(D,\mathbb{C})$, then $\omega_j\to\omega$ if and only if each coefficient of ω_j converges to the corresponding coefficient of ω in the above sense.

Definition 11.2. A *current* T *of degree* k is a linear functional on $\mathcal{D}^{2N-k}(D,\mathbb{C})$, i.e., an element of the dual space $\left(\mathcal{D}^{2N-k}(D,\mathbb{C})\right)'$. We will use the *dual pairing* notation $<T,\phi>$ to indicate the action of a current T on a test form ϕ.

We furnish the space of currents $\left(\mathcal{D}^{2N-k}(D,\mathbb{C})\right)'$ with the weak* topology, which is characterized by the property that given currents $\{T_j\}, T$ then $T_j \to T$ if and only if $< T_j, \phi > \to < T, \phi >$ for all $\phi \in \mathcal{D}^{2N-k}(D,\mathbb{C})$.

If T is a k-current and ψ is a smooth m-form with $k+m \le 2N$, then we define the $(k+m)$-current $T \wedge \psi$ by the formula

$$< T \wedge \psi\,,\, \phi > := < T\,,\, \psi \wedge \phi >, \quad \phi \in \mathcal{D}^{N-k-m}(D,\mathbb{C}). \tag{100}$$

Remark 4. One can extend the definition of differential k-forms to a larger class by allowing the forms to have *distribution coefficients*. Denoting the set of such forms by $\mathcal{D}'^k(D,\mathbb{C})$, it turns out that $\mathcal{D}'^k(D,\mathbb{C}) = \left(\mathcal{D}^{2N-k}(D,\mathbb{C})\right)'$. Similarly, we can also define $\mathcal{D}'^{p,q}(D,\mathbb{C})$ to be the (p,q)-forms with distribution coefficients. Then we also have $\mathcal{D}'^{p,q}(D,\mathbb{C}) = \left(\mathcal{D}^{N-p,N-q}(D,\mathbb{C})\right)'$, the currents of bidegree (p,q).

A distribution T, considered as a 0-current, acts on a test $2N$-form $\phi = \phi_{2N}\, dV_{2N}$ by the formula

$$< T, \phi > := (T, \phi_{2N}), \tag{101}$$

where the pairing $(\cdot,\cdot)$ on the right-hand side of 101 is the usual pairing of a distribution with a test function. If T is a k-current in $\mathbb{C}^N$ that can be written as $T = T_0\omega$ where T_0 is a distribution and ω is a k-form, then by (100) and (101), T acts on a test form ϕ of degree $2N-k$ as follows:

$$< T, \phi > = < T_0\,,\, \omega \wedge \phi > = (T_0\,,\, [\omega \wedge \phi]_{2N}).$$

In the above equation we use the subscript $2N$ to denote the coefficient of dV_{2N} in a $2N$-form on $\mathbb{C}^N$.

We will generalize the notion of positivity in Definition 11.1 to currents; first, we recall the notion of a positive distribution.

Definition 11.3. A *positive distribution* is a distribution S such that for any test function ϕ with range in $[0,\infty)$, we have $(S,\phi) \in [0,\infty)$.

Definition 11.4. For $k \le N$, a (k,k)-current T is called *positive* if for every strongly positive $(N-k, N-k)$-form ω, $T \wedge \omega = \tau\, dV_{2N}$ for some positive distribution τ.

Remark 5. A positive distribution can be extended to a linear functional on $C_0(D,\mathbb{C})$. The Riesz representation theorem says that for any continuous linear functional A on $C_0(D)$, there exists a unique measure μ such that $(A,\phi) = \int_D \phi\, d\mu$ for any $\phi \in C_0(D)$. The measure μ thus obtained is called a *Radon measure*. We may therefore identify positive distributions with Radon measures. If T is a current which can be written in the form $T = \mu\omega$, where ω is a k-form and μ is a Radon measure, then the action of T on a test $(2N-k)$-form ϕ is given by

$$< T\,,\, \phi > = < \mu\,,\, \omega \wedge \phi > = \int [\omega \wedge \phi]_{2N}\, d\mu.$$

12 Appendix B: Exercises on distributions.

1. If $g \in L^1_{loc}(\mathbb{R})$, we define the distribution $\mathcal{L}_g$ via

$$\mathcal{L}_g(\phi) = \int_{\mathbb{R}} \phi(x)g(x)dx$$

for $\phi \in C_0^\infty(\mathbb{R})$.

   (a) Show that if $\{g_n\} \subset L^1_{loc}(\mathbb{R})$ and $g_n \to g$ in $L^1_{loc}(\mathbb{R})$, then $\mathcal{L}_{g_n} \to \mathcal{L}_g$ as distributions.
   (b) Verify that if $g \in C^1(\mathbb{R})$ then $\mathcal{L}'_g = \mathcal{L}_{g'}$.

2. If $g_1, g_2 \in L^1_{loc}(\mathbb{R})$ and $g_1 = g_2$ a.e., then clearly $\mathcal{L}_{g_1} = \mathcal{L}_{g_2}$ as distributions. Prove the converse: let $g_1, g_2 \in L^1_{loc}(\mathbb{R})$; suppose that

$$\mathcal{L}_{g_1}(\phi) = \mathcal{L}_{g_2}(\phi) \text{ for all } \phi \in C_0^\infty(\mathbb{R});$$

and show that $g_1 = g_2$ a.e. (*Hint: Clearly* $g_1 * \chi_{1/j} = g_2 * \chi_{1/j}$ *for all* $j = 1, 2, \ldots$ *where* $\chi(x) = \chi(|x|) \geq 0$ *with* $\chi \in C_0^\infty(\mathbb{R})$ *and* $\int_{\mathbb{R}} \chi(x)dx = 1$. *Thus it suffices to show that* $g_i * \chi_{1/j} \to g_i$, $i = 1, 2$ *in* $L^1_{loc}(\mathbb{R})$ *as* $j \to \infty$).

3. Let $f(x) = |x|$.

   (a) Show that if ϕ is a C^1-function (ϕ is differentiable and ϕ' is continuous) which is identically zero outside of an interval; e.g., $\phi(x) = 0$ if $|x| > M$ for some M, then

$$\int_{\mathbb{R}} \phi'(x)f(x)dx = -\int_{\mathbb{R}} \phi(x)f'(x)dx.$$

   (b) Show that if ϕ is a C^2-function (ϕ'' is continuous) which is identically zero outside of an interval; e.g., $\phi(x) = 0$ if $|x| > M$ for some M, then

$$\int_{\mathbb{R}} \phi''(x)f(x)dx = 2\phi(0).$$

   This shows in particular that as distributions,

$$\mathcal{L}''_{|x|} = 2\delta_0(x)$$

   where $\delta_0(x)$ is the delta function at 0; i.e., the distribution whose action on a test function $\phi(x)$ gives $\phi(0)$.

4. We defined the derivative $\mathcal{L}'$ of a distribution $\mathcal{L}$ by $\mathcal{L}'(f) := -\mathcal{L}(f')$ and the product of a distribution $\mathcal{L}$ and a smooth function g by

$$(g \cdot \mathcal{L})(f) := \mathcal{L}(gf).$$

   (a) Using this definition, find the distribution $x \cdot \delta_0(x)$; i.e., describe its action on a test function $f(x)$.
   (b) Using this definition, and the definition of distributional deriviative, find the distribution $x \cdot \delta'_0(x)$; i.e., describe its action on a test function $f(x)$.
   (c) Using this definition, and the definition of distributional deriviative, find the distribution $x^2 \cdot \delta''_0(x)$; i.e., describe its action on a test function $f(x)$.

5. We recall again the derivative $\mathcal{L}'$ of a distribution $\mathcal{L}$ in one variable is defined by $\mathcal{L}'(f) := -\mathcal{L}(f')$.

   (a) Suppose g is piecewise smooth on $\mathbb{R}$, differentiable on $\mathbb{R} \setminus \{0\}$, and has a (possible) jump discontinuity at 0; i.e., $g(0+) := \lim_{x\to 0^+} g(x)$ and $g(0-) := \lim_{x\to 0^-} g(x)$ exist but (perhaps) are different. Find the distribution $\mathcal{L}'_g$; i.e., describe its action on a test function $f(x)$.
   (b) Let $g(x)$ be the Heaviside function $H(x)$; i.e., $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x > 0$. What does your answer to (a) give for the action of $\mathcal{L}'_H$ on a test function $f(x)$?
   (c) Compare the distributions $\mathcal{L}'_{g_1}$ and $\mathcal{L}'_{g_2}$ where

$$g_1(x) = 0 \text{ for } x \leq 0 \text{ and } g_1(x) = x^2 \text{ for } x \geq 0 \text{ and}$$

$$g_2(x) = -1 \text{ for } x < 0 \text{ and } g_2(x) = x^2 \text{ for } x \geq 0;$$

   i.e., describe each one's action on a test function $f(x)$.

6. Suppose $\mathcal{L}$ is a distribution with $\mathcal{L}' = 0$, i.e., $\mathcal{L}'(f) = 0$ for all $f \in C_0^\infty(\mathbb{R})$. What can you conclude about $\mathcal{L}$?

7. Let $g(x, y) = H(x)H(y) : \mathbb{R}^2 \to \mathbb{R}$ where H is the (univariate) Heaviside function; i.e., $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x > 0$. Then g determines a distribution $\mathcal{L}_g$ on $C_0^\infty(\mathbb{R}^2)$ by

$$\mathcal{L}_g(f) := \int\int f(x, y)g(x, y)dA(x, y).$$

Determine the distribution $\Delta\mathcal{L}_g$; i.e., describe its action on a test function $f(x, y)$.

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