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REGULARITY OF SOLUTIONS TO THE NAVIER-STOKES EQUATION

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Abstract. Recently, Beirão da Veiga [15] obtained regularity for the Navier-Stokes equation in $\mathbb{R}^3$ by imposing conditions on the vorticity rather than the velocity. In this article, we obtain regularity by imposing conditions on only two components of the vorticity vector.

1. Introduction

We are concerned with the initial value problem of the Navier-Stokes equation in $\mathbb{R}^3 \times (0,T)$,

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p + \nu\Delta\,,, \tag{1}$$

$$\operatorname{div} v = 0\,, \tag{2}$$

$$v(x,0) = v_0(x)\,, \tag{3}$$

where $v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$, $x \in \mathbb{R}^3$, and $t \in (0,T)$. For simplicity we assume that the external force is zero, but it is easy to extend our results to the nonzero-external-force case.

Recall that a weak solution to the Navier-Stokes equation, which is called the Leray-Hopf weak solution, is defined as a vector field $v \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$ satisfying $\operatorname{div} v = 0$ in the distributional sense, and

$$\int_0^T \int_{\mathbb{R}^3} [v \cdot \phi_t + (v\cdot\nabla)\phi \cdot v + v \cdot \Delta\phi] dx\, dt = 0$$

for all $\phi \in [C_0^\infty(\mathbb{R}^3 \times (0,T))]^3$ with $\operatorname{div}\phi = 0$.

For $v_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} v_0 = 0$, the existence of weak solutions was established by Leray[13] and Hopf in [11]. A weak solution of the Navier-Stokes equation that belongs to $L^\infty\Big(0,T;H^1(\mathbb{R}^3)\Big) \cap L^2\Big(0,T;H^2(\mathbb{R}^3)\Big)$ is called a strong solution. It is well-known that for $v_0 \in H^1(\mathbb{R}^3)$ with $\operatorname{div} v_0 = 0$ there exists a local unique strong solution $v \in C([0,T);H^1(\mathbb{R}^3))$, and that the maximal time of existence T_*

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depends on the initial data $\|v_0\|_{H^{1/2}(\mathbb{R}^3)}$. Moreover, the strong solution belongs to the class $C\big((0,T_*);C^\infty(\mathbb{R}^3)\big)$, i.e., the solution becomes regular in the space variable immediately after the initial moment. The global-in time continuation of the local strong solution is an outstanding open problem in mathematical fluid mechanics. Another notion of solution, useful for the study of the Navier-Stokes equation, is that of mild solution initiated by Fujita and Kato [9].

In this note we are concerned with obtaining a sufficient condition for the global continuation of strong solutions. In this direction there is a classical result due to Serrin[14], which states that if a weak solution belongs to $L^{\alpha,\gamma}_T$, then v becomes the strong solution in $(0,T]$. Here $L^{\alpha,\gamma}_T = L^{\alpha,\gamma}_{[0,T]} = L^\alpha(0,T;L^\gamma(\mathbb{R}^3)$ with $\frac{2}{\alpha}+\frac{3}{\gamma}<1$, and $\alpha<\infty$. Later, Fabes-Jones-Riviere [8] extended the above criterion to the case $\frac{2}{\alpha}+\frac{3}{\gamma}=1$. The problem of regularity and uniqueness for the marginal case $\alpha=\infty$, $\gamma=3$ in Serrin's condition has been extensively studied by many authors; see for example [3], [10], and [12]. Recently, Beirão da Veiga [15] obtained a sufficient condition for regularity using the vorticity rather than the velocity. His result says that if the vorticity $\omega=\operatorname{curl} v$ of a weak solution v belongs to the space $L^{\alpha,\gamma}_T$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 2$ with $1<\alpha<\infty$, then v becomes the strong solution on $(0,T]$. Here we prove that it is sufficient to control only two components of the vorticity vector, or the gradients of the two components of the velocity vector. See Theorem 1 below.

Given a velocity field v, the two-component vorticity field is denoted by $\tilde\omega=\omega_1e_1+\omega_2e_2$, where $e_1=(1,0,0)$, $e_2=(0,1,0)$.

Theorem 1. *Let $v_0\in L^2(\mathbb{R}^3)$ with $\operatorname{div} v_0=0$ and $\omega_0=\operatorname{curl} v_0\in L^2(\mathbb{R}^3)$. If a Leray-Hopf weak solution v, satisfies $\tilde\omega\in L^{\alpha,\gamma}_T$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 2$, $1<\alpha<\infty$ and $\frac{3}{2}<\gamma<\infty$, or if $\|\tilde\omega\|_{L^{\infty,\frac{3}{2}}_T}$ is sufficiently small, then v becomes the classical solution on $(0,T]$.*

Remark 1. As an immediate corollary of the above theorem, we find that if the classical solution of the 3-D Navier-Stokes equations blows up at time T, then $\|\tilde\omega\|_{L^{\alpha,\gamma}_T}=\infty$, where $\tilde\omega$ is any two component vector of ω, and (α,γ) is a pair of real numbers satisfying $\frac{2}{\alpha}+\frac{3}{\gamma}\le 2$, $1<\alpha<\infty$. In other words, at finite blow-up time at least two components of the vortices must simultaneously blow up.

A related result is studied by Beale-Kato-Majda [1]. They show that for 3-D Euler equations, the blow up of full gradients of the velocity field is controlled by only three components of the vorticity field.

Remark 2. As another immediate corollary we obtain the global regularity for the 2-D Navier-Stokes equations, since in this case $\tilde\omega(x,t)=0$, for all $(x,t)\in\mathbb{R}^3\times(0,T)$.

Our second theorem concerns the regularity criterion in terms of gradients of the two components of velocity.

Theorem 2. *Let $\tilde v=v_1e_1+v_2e_2$ be the first two components of a Leray-Hopf weak solution of the Navier-Stokes equation corresponding to $v_0\in H^1(\mathbb{R}^3)$ with $\operatorname{div} v_0=0$. Suppose that $D\tilde v\in L^{\alpha,\gamma}_T$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 1$, where $2\le\alpha\le\infty$, and $3\le\gamma\le\infty$, then v becomes a classical solution in $(0,T]$.*

2. Proof of Main Theorems

The key idea in booth proofs is a careful observation of the structure of the nonlinear terms of the vorticity equations for the Navier-Stokes system. The structure of the nonlinear term has been emphasized in the works by Constantin and Fefferman [4], [5], [6], and [7].

Below we use the notation

$$\|u\|_p = \left(\int_{\mathbb{R}^3} |u(x)|^p dx \right)^{1/p}, \quad 1 \le p < \infty\,.$$

We also use C for various constants in the estimates below.

Proof of Theorem 1. Taking the curl on (1), we obtain

$$\omega_t + (v\cdot\nabla)\omega = (\omega\cdot\nabla)v + \nu\Delta\omega. \tag{4}$$

Multiplying (4) by ω in $L^2(\mathbb{R}^3)$, and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_2^2 + \nu\|\nabla\omega(t)\|_2^2 = \int_{\mathbb{R}^3} (\omega\cdot\nabla)v\cdot\omega\, dx\,. \tag{5}$$

Using the Biot-Savart law, v is written in terms of ω:

$$v(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times\omega(y,t)}{|x-y|^3}\, dy\,. \tag{6}$$

Substituting this into the right hand side of (5), we have

$$\begin{aligned}
\int_{\mathbb{R}^3} (\omega\cdot\nabla)v\cdot\omega\, dx &= \frac{3}{4\pi}\iint \frac{y}{|y|}\cdot\omega(x,t)\left\{\frac{y}{|y|^4}\times\omega(x+y,t)\cdot\omega(x,t)\right\} dy\, dx \\
&\Big(\text{We decompose } \omega = \tilde\omega + \omega',\ \tilde\omega = \omega_1 e_1 + \omega_2 e_2,\ \omega' = \omega_3 e_3 \\
&\text{for the vorticities in } \{\cdot\}\Big) \\
&= \frac{3}{4\pi}\iint \frac{y}{|y|}\cdot\omega(x,t)\left\{\frac{y}{|y|^4}\times\tilde\omega(x+y,t)\cdot\omega'(x,t)\right\} dy\, dx \\
&+ \frac{3}{4\pi}\iint \frac{y}{|y|}\cdot\omega(x+y,t)\left\{\frac{y}{|y|^4}\times\tilde\omega(x+y,t)\cdot\tilde\omega(x,t)\right\} dy\, dx \\
&+ \frac{3}{4\pi}\iint \frac{y}{|y|}\cdot\omega(x,t)\left\{\frac{y}{|y|^4}\times\omega'(x+y,t)\cdot\tilde\omega(x,t)\right\} dy\, dx\,,
\end{aligned} \tag{7}$$

where all the integrations with respect to y are in the sense of principal value.

We have thus

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} (\omega\cdot\nabla)v\cdot\omega dx\right| &\le C\int_{\mathbb{R}^3} |\omega(x,t)||P(\tilde\omega)||\omega'(x,t)|dx \\
&+ C\int_{\mathbb{R}^3} |\omega(x,t)||P(\tilde\omega)||\tilde\omega(x,t)|dx \\
&+ C\int_{\mathbb{R}^3} |\omega(x,t)||P(\omega')||\tilde\omega(x,t)|dx \\
&\le C\int_{\mathbb{R}^3} |\omega|^2|P(\tilde\omega)|dx + C\int_{\mathbb{R}^3} |\omega||P(\omega')||\tilde\omega|dx. \\
&=: I_1 + I_2,
\end{aligned}$$

where $P(\cdot)$ denotes the singular integral operator defined by the integrals with respect to y in (7). We first consider the case $\frac{3}{2} < \gamma < \infty$.

We have the following estimates

$$\begin{aligned} I_1 &\le \|P(\tilde\omega)\|_\gamma \|\omega\|^2_{\frac{2\gamma}{\gamma-1}} \quad \text{(by Hölder's inequality)} \\ &\le C\|\tilde\omega\|_\gamma \|\omega\|_2^{\frac{2\gamma-3}{\gamma}} \|\nabla\omega\|_2^{\frac{3}{\gamma}} \\ &\le C\|\tilde\omega\|_\gamma^{\frac{2\gamma}{2\gamma-3}} \|\omega\|_2^2 + \frac{\nu}{4}\|\nabla\omega\|_2^2 \quad \text{(by Young's inequality)}, \end{aligned} \tag{8}$$

where we used the Calderon-Zygmund and the Gagliardo-Nirenberg inequalities in the second inequality. For the second term of the right hand side of (8) we have by the Höder inequality and the Calderon-Zygmund inequality,

$$\begin{aligned} I_2 &\le \|\tilde\omega\|_\gamma \|P(\omega')\|_{\frac{2\gamma}{\gamma-1}} \|\omega\|_{\frac{2\gamma}{\gamma-1}} \quad \text{(by Hölder's inequality)} \\ &\le C\|\tilde\omega\|_\gamma \|\omega'\|_{\frac{2\gamma}{\gamma-1}} \|\omega\|_{\frac{2\gamma}{\gamma-1}} \quad \text{(by the Calderon-Zygmund inequality)} \\ &\le C\|\tilde\omega\|_\gamma \|\omega\|^2_{\frac{2\gamma}{\gamma-1}} \\ &\le C\|\tilde\omega\|_\gamma^{\frac{2\gamma}{2\gamma-3}} \|\omega\|_2^2 + \frac{\nu}{4}\|\nabla\omega\|_2^2 \quad \text{(by the similar estimates to (8))}. \end{aligned} \tag{9}$$

Thus, combining (8)-(9) with (5), we obtain

$$\frac{d}{dt}\|\omega(t)\|_2^2 + \nu\|\nabla\omega(t)\|_2^2 \le C\|\tilde\omega(t)\|_\gamma^{\frac{2\gamma}{2\gamma-3}} \|\omega(t)\|_2^2.$$

Applying the standard Gronwall lemma, we have

$$\begin{aligned} \|\omega(t)\|_2^2 + \int_0^t \|\nabla\omega(\tau)\|_2^2 \exp\left(C\int_\tau^t \|\tilde\omega(s)\|_\gamma^{\frac{2\gamma}{2\gamma-3}} ds\right) d\tau \\ \le \|\omega_0\|_2^2 \exp\left(C\int_0^t \|\tilde\omega(s)\|_\gamma^{\frac{2\gamma}{2\gamma-3}} ds\right). \end{aligned}$$

Since $0 < \frac{2\gamma}{2\gamma-3} \le \alpha$, by the Hölder inequality we obtain

$$\begin{aligned} \sup_{0\le t\le T} \|\omega(t)\|_2^2 + \nu\int_0^T \|\nabla\omega(t)\|_2^2 dt &\le \|\omega_0\|_2^2 \exp\left(C\int_0^T \|\tilde\omega(t)\|_\gamma^{\frac{2\gamma}{2\gamma-3}} dt\right) \\ &\le \|\omega_0\|_2^2 \exp\left(C\|\tilde\omega\|_{L_T^{\alpha,\gamma}}^{\frac{2\gamma}{2\gamma-3}} T^{\frac{2\gamma}{2\gamma-3}(2-\frac{2}{\alpha}-\frac{3}{\gamma})}\right). \end{aligned}$$

Thus, in this case $\|\tilde\omega\|_{L_T^{\alpha,\gamma}} < \infty$ implies

$$\omega \in L^\infty\big(0,T : L^2(\mathbb{R}^3)\big) \cap L^2\big(0,T : H^1(\mathbb{R}^3)\big).$$

Using the regularity of the strong solution, we obtain the conclusion of the theorem for $1 < \alpha < \infty$, $\frac{3}{2} < \gamma < \infty$.

Next, we consider the case $\alpha=\infty, \gamma=3/2$. In this case, instead of (8) and (9), we estimate as follows:

$$\{1\} \le \|P(\tilde\omega)\|_{\frac{3}{2}}\|\omega\|_6^2 \le C\|\tilde\omega\|_{\frac{3}{2}}\|\nabla\omega\|_2^2, \tag{10}$$

and

$$\begin{aligned} I_2 &\le \|\tilde\omega\|_{\frac{3}{2}}\|P(\omega')\|_6\|\omega\|_6 \\ &\le C\|\tilde\omega\|_{\frac{3}{2}}\|\omega'\|_6\|\omega\|_6 \\ &\le C\|\tilde\omega\|_{\frac{3}{2}}\|\omega\|_6^2 \le C\|\tilde\omega\|_{\frac{3}{2}}\|\nabla\omega\|_2^2. \end{aligned} \tag{11}$$

Combining (10)-(11) with (5), integrating over $[0,T]$, we deduce

$$\sup_{0\le t\le T}\|\omega(t)\|_2^2+\nu\int_0^T\|\nabla\omega(t)\|_2^2dt \le C\|\tilde\omega\|_{L_T^{\infty,\frac{3}{2}}}\int_0^T\|\nabla\omega(t)\|_2^2dt\,.$$

Thus, if $C\|\tilde\omega\|_{L_T^{\infty,3/2}}<\frac{\nu}{2}$, then we have again

$$\omega\in L^\infty\big(0,T:L^2(\mathbb{R}^3)\big)\cap L^2\big(0,T:H^1(\mathbb{R}^3)\big),$$

which implies the regularity of v as previously. $\square$

Proof of Theorem 2. We set $\tilde v=(v_1,v_2,0)$. Then, taking the first two components of the vorticity equation (4), we obtain

$$\tilde\omega_t+(v\cdot\nabla)\tilde\omega=(\omega\cdot\nabla)\tilde v+\nu\Delta\tilde\omega\,.$$

Taking $L^2(\mathbb{R}^3)$ inner product (12) with $\tilde\omega$, we have, after integration by part,

$$\frac{1}{2}\frac{d}{dt}\|\tilde\omega(t)\|_2^2+\nu\|\nabla\tilde\omega(t)\|_2^2=\int_{\mathbb{R}^3}(\omega\cdot\nabla)\tilde v\cdot\tilde\omega\,dx\,. \tag{12}$$

We first consider the case $2\le\alpha<\infty$, $3<\gamma\le\infty$. Using the Hölder and the Gagliardo-Nirenberg inequalities, we estimate

$$\begin{aligned} \Big|\int_{\mathbb{R}^3}(\omega\cdot\nabla)\tilde v\cdot\tilde\omega\,dx\Big| &\le \|\omega\|_2\|\nabla\tilde v\|_\gamma\|\tilde\omega\|_{\frac{2\gamma}{\gamma-2}} \\ &\le C\|\omega\|_2\|\nabla\tilde v\|_\gamma\|\tilde\omega\|_2^{\frac{\gamma-3}{\gamma}}\|\nabla\tilde\omega\|_2^{\frac{3}{\gamma}} \\ &\le C\|\omega\|_2^2+C\|\nabla\tilde v\|_\gamma^{\frac{2\gamma}{\gamma-3}}\|\tilde\omega\|_2^2+\frac{\nu}{2}\|\nabla\tilde\omega\|_2^2, \end{aligned} \tag{13}$$

where the case $\gamma=\infty$ corresponds to the obvious limit $\gamma\to\infty$ for the norms of the estimates. The estimates (13), combined with (12), yield

$$\frac{d}{dt}\|\tilde\omega(t)\|_2^2+\nu\|\nabla\tilde\omega(t)\|_2^2\le C\|\omega(t)\|_2^2+C\|\nabla\tilde v(t)\|_\gamma^{\frac{2\gamma}{\gamma-3}}\|\tilde\omega(t)\|_2^2.$$

Using the Gronwall lemma similarly to the proof of Theorem 1, we obtain

$$\begin{aligned} &\sup_{0\le t\le T}\|\tilde\omega(t)\|_2^2+\nu\int_0^T\|\nabla\tilde\omega(t)\|_2^2dt \\ &\qquad\le\left(\|\omega_0\|_2^2+\int_0^T\|\omega(t)\|_2^2dt\right)\exp\left(C\int_0^T\|\nabla\tilde v(t)\|_\gamma^{\frac{2\gamma}{\gamma-3}}dt\right) \\ &\qquad\le\left(\|\omega_0\|_2^2+\int_0^T\|Dv(t)\|_2^2dt\right)\exp\left(C\|\nabla\tilde v\|_{L_T^{\alpha,\gamma}}^{\frac{2\gamma}{\gamma-3}}T^{\frac{2\gamma}{\gamma-3}(1-\frac{2}{\alpha}-\frac{3}{\gamma})}\right), \end{aligned}$$

where we used the fact $\frac{2\gamma}{\gamma-3} \le \alpha$ in the second inequality. Thus, if $\nabla\tilde{v} \in L_T^{\alpha,\gamma}$, then we have by the Sobolev embedding, $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$,

$$\tilde{\omega} \in L^\infty\big(0,T : L^2(\mathbb{R}^3)\big) \cap L^2\big(0,T : L^6(\mathbb{R}^3)\big).$$

Since $\alpha = 2$ and $\gamma = 6$ satisfy $\frac{2}{\alpha} + \frac{3}{\gamma} \le 2$, the conclusion of Theorem 2 for the case $2 \le \alpha < \infty$, $3 < \gamma \le \infty$ follows from Theorem 1.

Next, we consider the case $\alpha = \infty$, $\gamma = 3$. In this case we estimate

$$\begin{aligned} \Big| \int_{\mathbb{R}^3} (\omega \cdot \nabla)\tilde{v} \cdot \tilde{\omega}\, dx \Big| &\le \|\omega\|_2 \|\nabla\tilde{v}\|_3 \|\tilde{\omega}\|_6 \\ &\le C\|\omega\|_2 \|\nabla\tilde{v}\|_3 \|\nabla\tilde{\omega}\|_2 \\ &\le C\|\omega\|_2^2 \|\nabla\tilde{v}\|_3^2 + \frac{\nu}{2}\|\nabla\tilde{\omega}\|_2^2 . \end{aligned}$$

This, together with (12), provide us with

$$\sup_{0\le t\le T} \|\tilde{\omega}(t)\|_2^2 + \nu \int_0^T \|\nabla\tilde{\omega}(t)\|_2^2 dt \le C\|\nabla\tilde{v}\|^2_{L_T^{\infty,3}} \int_0^T \|Dv(t)\|_2^2 dt$$

after integrating over $[0,T]$. This inequality, in turn, implies that if $\|\nabla\tilde{v}\|_{L_T^{\infty,3}} < \infty$, then

$$\tilde{\omega} \in L^\infty\big(0,T : L^2(\mathbb{R}^3)\big) \cap L^2\big(0,T : L^6(\mathbb{R}^3)\big) .$$

In a similar manner to the previous case, we conclude the present proof. $\square$

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References

1. J. Beale, T. Kato and A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Comm. Math. Phys. **94** (1984), 61-66.
2. M. Cannone, *Ondelettes, Paraproduits et Navier-Stokes*, Diderot Editeur (1995).
3. M. Cannone and F. Planchon, *On the non stationary Navier-Stokes equations with an external force*, preprint.
4. P. Constantin, *Geometric statistics in turbulence*, SIAM Review **36** (1994), 73 - 98.
5. ———, *Geometric and Analytic studies in Turbulences, in Trends and Perspectives in Applied Math.*, L. Sirovich ed. Appl.Math. Sciences 100, Springer -Verlag (1994), 21 - 54.
6. P. Constantin and Ch. Fefferman, *Direction of vorticity and the problem of global regularity for the Navier-Stokes equations*, Indiana Univ. Math.J. **42** (1994), 775 - 789.
7. P. Constantin, Ch. Fefferman and A. Majda, *Geometric constraints on potentially singular solutions for the 3-D Euler equations*, Comm. P.D.E. **21(3&4)** (1996), 559 - 571.
8. E. Fabes, B. Jones and N. Riviere, *The initial value problem for the Navier-Stokes equations with data in L^p*, Arch. Rat. Mech, Anal. **45** (1972), 222 - 248.
9. H. Fujita and T. Kato, *On the Navier-Stokes initial value problem I*, Arch. Rat. Mech, Anal. **16** (1964), 269-315.
10. G. Furioli, P. G. Lemarié-Rieusset and E. Terraneo, *Sur l'unicité dans $L^3(\mathbb{R}^3)$ des solutions mild des équations de Navier-Stokes*, C. R. Acad. Sci. Paris, t. 325, Série I (1997), 1253-1256.
11. E. Hopf, *Über die anfang swetaufgabe für die hydrodynamischer grundgleichungan*, Math. Nach. **4** (1951), 213 - 231.
12. H. Kozono and H. Sohr, *Regularity criterion on weak solutions to the Navier-Stokes equations*, Adv. Diff. Eqns **2 (No 4)**, 535-554.

13. J. Leray, *Sur le mouvement d'um liquide visqieux emlissant l'space*, Acta Math. **63** (1934), 193 - 248.
14. J. Serrin, *On the interior regularity of weak solutions of the Navier-Stokes equations*, Arch. Rat. Mech. Anal. **9** (1962), 187 - 191.
15. H. Beirão da Veiga, *Concerning the regularity problem for the solutions of the Navier-Stokes equations*, C.R. Acad. Sci. Paris, t.321. Série **I** (1995), 405 - 408.

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