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THREE SYMMETRIC POSITIVE SOLUTIONS FOR LIDSTONE PROBLEMS BY A GENERALIZATION OF THE LEGGETT-WILLIAMS THEOREM

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Abstract. We study the existence of solutions to the fourth order Lidstone boundary value problem

$$y^{(4)}(t) = f(y(t), -y''(t)),$$
$$y(0) = y''(0) = y''(1) = y(1) = 0\,.$$

By imposing growth conditions on f and using a generalization of the multiple fixed point theorem by Leggett and Williams, we show the existence of at least three symmetric positive solutions. We also prove analogous results for difference equations.

1. Introduction

First we are concerned with the existence of multiple solutions for the fourth order Lidstone boundary value problem (BVP)

$$y^{(4)}(t) = f(y(t), -y''(t)), \quad 0 \le t \le 1\,, \tag{1}$$
$$y(0) = y''(0) = y''(1) = y(1) = 0\,, \tag{2}$$

where $f : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is continuous. We will impose growth conditions on f which ensure the existence of at least three symmetric positive solutions of (1), (2).

There is much current attention focused on questions of positive solutions of boundary value problems for ordinary differential equations, as well as for for finite difference equations; see [2, 5, 6, 8, 10, 11, 12, 13, 17, 18, 19, 20] to name a few. Much of this interest is due to the applicability of certain fixed point theorems of Krasnosel'skii [15] or Leggett and Williams [16] to obtain positive solutions or multiple positive solutions which lie in a cone. The recent book by Agarwal, O'Regan, and Wong [1] gives a good overview for much of the work which has been done and the methods used.

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In [3], Avery imposed conditions on g which yield at least three positive solutions to the second order conjugate BVP

$$y''(t) + g(y(t)) = 0\,, \quad 0 \le t \le 1\,, \tag{3}$$

$$y(0) = y(1) = 0\,, \tag{4}$$

using the Leggett-Williams Fixed Point Theorem. Henderson and Thompson [14] improved these results by using the symmetry of the associated Green's function and then Avery and Henderson [5] established similar results by applying the Five Functionals Fixed Point Theorem [4] (which is a generalization of the Leggett-Williams Fixed Point Theorem) to obtain the existence of three positive solutions of certain BVPs. Davis, Eloe, and Henderson [8] imposed conditions on f to yield at least three positive solutions to the $2m$th order Lidstone BVPs by applying the Leggett- Williams Fixed Point Theorem. Note that [8] is the only work which has allowed f to depend on higher order derivatives of y. This paper is in the same spirit as [5] and [8] since we apply the Five Functionals Fixed Point Theorem [4] and also allow f to depend on y''. This derivative dependence generalizes [9] as well.

In Section 2, we provide some background results and state the Five Functionals Fixed Point Theorem. In Section 3, we impose growth conditions on f which allow us to apply this theorem in obtaining three symmetric positive solutions of (1), (2). In Section 4, we prove discrete analogs of the results in Section 3.

2. Some Background Definitions and Results

In this section, we provide some background material from the theory of cones in Banach spaces in order that this paper be self-contained. We also state the Five Functionals Fixed Point Theorem for cone preserving operators.

Definition 1. Let $\mathcal{E}$ be a Banach space over $\mathbb{R}$. A nonempty, closed set $\mathcal{P} \subset \mathcal{E}$ is a *cone* provided

(a) $\alpha\mathbf{u} + \beta\mathbf{v} \in \mathcal{P}$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{P}$ and all $\alpha, \beta \ge 0$, and
(b) $\mathbf{u}, -\mathbf{u} \in \mathcal{P}$ implies $\mathbf{u} = \mathbf{0}$.

Definition 2. A Banach space $\mathcal{E}$ is a *partially ordered Banach space* if there exists a partial ordering $\preceq$ on $\mathcal{E}$ satisfying

(a) $\mathbf{u} \preceq \mathbf{v}$, for $\mathbf{u}, \mathbf{v} \in \mathcal{E}$ implies $t\mathbf{u} \preceq t\mathbf{v}$, for all $t \ge 0$, and
(b) $\mathbf{u}_1 \preceq \mathbf{v}_1$ and $\mathbf{u}_2 \preceq \mathbf{v}_2$, for $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{E}$ imply $\mathbf{u}_1 + \mathbf{u}_2 \preceq \mathbf{v}_1 + \mathbf{v}_2$.

Let $\mathcal{P} \subset \mathcal{E}$ be a cone and define $\mathbf{u} \preceq \mathbf{v}$ if and only if $\mathbf{v} - \mathbf{u} \in \mathcal{P}$. Then $\preceq$ is a partial ordering on $\mathcal{E}$ and we will say that $\preceq$ is the partial ordering induced by $\mathcal{P}$. Moreover, $\mathcal{E}$ is a partially ordered Banach space with respect to $\preceq$.

We also state the following definitions for future reference.

Definition 3. The map α is a *nonnegative continuous concave functional on* $\mathcal{P}$ provided $\alpha : \mathcal{P} \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in \mathcal{P}$ and $0 \le t \le 1$. Similarly we say the map β is a *nonnegative continuous convex functional on* $\mathcal{P}$ provided $\beta : \mathcal{P} \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \le t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in \mathcal{P}$ and $0 \le t \le 1$.

Definition 4. An operator, A, is *completely continuous* if A is continuous and compact, i.e. A maps bounded sets into precompact sets.

Let γ, β, θ be nonnegative continuous convex functionals on $\mathcal{P}$ and α, ψ be nonnegative continuous concave functionals on $\mathcal{P}$. Then for nonnegative numbers h, a, b, d, and c, we define the following convex sets:

$$\begin{aligned} P(\gamma, c) &= \{x \in \mathcal{P} \mid \gamma(x) < c\}, \\ P(\gamma, \alpha, a, c) &= \{x \in \mathcal{P} \mid a \leq \alpha(x),\ \gamma(x) \leq c\}, \\ Q(\gamma, \beta, d, c) &= \{x \in \mathcal{P} \mid \beta(x) \leq d,\ \gamma(x) \leq c\}, \\ P(\gamma, \theta, \alpha, a, b, c) &= \{x \in \mathcal{P} \mid a \leq \alpha(x),\ \ \theta(x) \leq b,\ \ \gamma(x) \leq c\}, \\ Q(\gamma, \beta, \psi, h, d, c) &= \{x \in \mathcal{P} \mid h \leq \psi(x),\ \ \beta(x) \leq d,\ \ \gamma(x) \leq c\}. \end{aligned}$$

In obtaining multiple symmetric positive solutions of (1), (2) the following so-called Five Functionals Fixed Point Theorem will be fundamental.

Theorem 1. [4] *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{E}$. Suppose α and ψ be nonnegative continuous concave functionals on $\mathcal{P}$ and γ, β, and θ be nonnegative continuous convex functionals on $\mathcal{P}$ such that, for some positive numbers c and m,*

$$\alpha(x) \leq \beta(x) \text{ and } \|x\| \leq m\gamma(x) \text{ for all } x \in \overline{P(\gamma, c)}.$$

Suppose further that $A : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is completely continuous and there exist constants $h, d, a, b \geq 0$ with $0 < d < a$ such that each of following is satisfied:

(C1) $\{x \in P(\gamma, \theta, \alpha, a, b, c) \mid \alpha(x) > a\} \neq \emptyset$
and $\alpha(Ax) > a$ *for* $x \in P(\gamma, \theta, \alpha, a, b, c)$,
(C2) $\{x \in Q(\gamma, \beta, \psi, h, d, c) \mid \beta(x) < d\} \neq \emptyset$
and $\beta(Ax) < d$ *for* $x \in Q(\gamma, \beta, \psi, h, d, c)$,
(C3) $\alpha(Ax) > a$ *provided* $x \in P(\gamma, \alpha, a, c)$ *with* $\theta(Ax) > b$,
(C4) $\beta(Ax) < d$ *provided* $x \in Q(\gamma, \beta, d, c)$ *with* $\psi(Ax) < h$.

Then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$\beta(x_1) < d,\ \ a < \alpha(x_2),\ \text{ and } d < \beta(x_3) \text{ with } \alpha(x_3) < a.$$

3. Three Symmetric Positive Solutions

In this section, we will impose growth conditions on f which allow us to apply Theorem 1 in regard to obtaining three symmetric positive solutions of (1), (2). We will apply Theorem 1 in conjunction with a completely continuous operator whose kernel, $G(t, s)$, is the Green's function for $-v'' = 0$, satisfying (4). In particular,

$$G(t, s) = \begin{cases} t(1 - s), & 0 \leq t \leq s \leq 1, \\ s(1 - t), & 0 \leq s \leq t \leq 1. \end{cases}$$

We will make use of various properties of $G(t,s)$, namely

$$\int_0^1 G(t,s)\,ds = \frac{t(1-t)}{2}, \quad 0 \le t \le 1,$$

$$\int_0^{\frac{1}{r}} G(1/2,s)\,ds = \int_{1-\frac{1}{r}}^1 G(1/2,s)\,ds = \frac{1}{4r^2}, \quad 2 < r, \tag{5}$$

$$\int_{\frac{1}{r}}^{\frac{1}{2}} G(1/2,s),ds = \int_{\frac{1}{2}}^{1-\frac{1}{r}} G(1/2,s)\,ds = \frac{r^2-4}{16r^2}, \quad 2 < r, \tag{6}$$

$$\int_{t_1}^{t_2} G(t_1,s)\,ds + \int_{1-t_2}^{1-t_1} G(t_1,s)\,ds = t_1(t_2-t_1), \quad 0 < t_1 < t_2 < 1/2, \tag{7}$$

$$\max_{0\le r\le 1} \frac{G(1/2,r)}{G(t,r)} = \frac{1}{2t}, \quad 0 < t \le 1/2, \tag{8}$$

$$\min_{0\le r\le 1} \frac{G(t_1,r)}{G(t_2,r)} = \frac{t_1}{t_2}, \quad 0 < t_1 < t_2 \le 1/2. \tag{9}$$

Let $\mathcal{E} = C[0,1]$ be endowed with the maximum norm,

$$\|v\| = \max_{0\le t\le 1} |v(t)|,$$

and for $0 < t_3 < 1/2$ define the cone $\mathcal{P} \subset \mathcal{E}$ by

$$\mathcal{P} = \Big\{ v \in \mathcal{E} \mid v(t) = v(1-t),\ v(t) \ge 0,\ v(t) \text{ is concave for all } t \in [0,1], \\ \text{and } \min_{t\in[t_3,1-t_3]} |v(t)| \ge 2t_3\|v\| \Big\}.$$

Finally, we define the nonnegative continuous concave functionals α, ψ and the nonnegative continuous convex functionals β, θ, γ on $\mathcal{P}$ by

$$\begin{aligned}
\gamma(v) &= \max_{t\in[0,t_3]\cup[1-t_3,1]} v(t) = v(t_3),\\
\psi(v) &= \min_{t\in\left[\frac{1}{r},1-\frac{1}{r}\right]} v(t) = v(1/r),\\
\beta(v) &= \max_{t\in\left[\frac{1}{r},1-\frac{1}{r}\right]} v(t) = v(1/2),\\
\alpha(v) &= \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} v(t) = v(t_1),\\
\theta(v) &= \max_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} v(t) = v(t_2),
\end{aligned}$$

where t_1, t_2, and r are nonnegative numbers such that

$$0 < t_1 < t_2 < \frac{1}{2} \text{ and } \frac{1}{r} \le t_2.$$

We observe here that for each $v \in \mathcal{P}$,

$$\alpha(v) = v(t_1) \le v(1/2) = \beta(v), \tag{10}$$

$$\|v\| = v(1/2) \le \frac{v(t_3)}{2t_3} = \frac{1}{2t_3}\gamma(v), \tag{11}$$

and also that $y \in \mathcal{P}$ is a solution of (1), (2) if and only if there exists a $v \in \mathcal{P}$ such that

$$y(t) = \int_0^1 G(t,s)v(s)\,ds, \quad 0 \le t \le 1,$$

where v is of the form

$$v(t) = \int_0^1 G(t,s) f\left(\int_0^1 G(s,\tau)v(\tau)d\tau, v(s)\right) ds, \quad 0 \le t \le 1.$$

In light of these preliminaries, we are now ready to present the main result of this section.

Theorem 2. *Suppose there exist* $0 < a < b < \frac{t_2}{t_1} b \le c$ *such that* f *satisfies each of the following growth conditions:*

(G1) $f(z,w) < \left(\frac{8r^2}{r^2-4}\right)\left(a - \frac{c}{r^2 t_3(1-t_3)}\right)$ *for all* $(z,w) \in \left[\frac{a(r-2)}{r^3}, \frac{a}{8}\right] \times \left[\frac{2a}{r}, a\right]$,

(G2) $f(z,w) \ge \frac{b}{t_1(t_2-t_1)}$ *for* $(z,w) \in \left[bt_1(t_2-t_1), \frac{c(t_1^2+t_2(1-2t_2))}{4t_3} + \frac{bt_2(t_2^2-t_1^2)}{2t_1}\right] \times \left[b, \frac{t_2}{t_1}b\right]$,

(G3) $f(z,w) \le \frac{2c}{t_3(1-t_3)}$ *for* $(z,w) \in \left[0, \frac{c}{16t_3}\right] \times \left[0, \frac{c}{2t_3}\right]$.

Then the Lidstone BVP (1), (2) *has at least three symmetric positive solutions* y_1, y_2, y_3, *such that*

$$\max_{t \in [0,t_3] \cup [1-t_3,1]} -y_i''(t) \le c, \quad \textit{for } i = 1,2,3,$$

$$\min_{t \in [t_1,t_2] \cup [1-t_2,1-t_1]} -y_1''(t) > b,$$

$$\max_{t \in \left[\frac{1}{r}, 1-\frac{1}{r}\right]} -y_2''(t) < a,$$

and

$$\min_{t \in [t_1,t_2] \cup [1-t_2,1-t_1]} -y_3''(t) < b \quad \textit{with} \quad \max_{t \in \left[\frac{1}{r}, 1-\frac{1}{r}\right]} -y_3''(t) > a,$$

for some v_1, v_2, $v_3 \in \mathcal{P}$ *satisfying*

$$y_i(t) = \int_0^1 G(t,s)v_i(s)\,ds, \quad i = 1,2,3.$$

Proof. Define the completely continuous operator A by

$$Av(t) = \int_0^1 G(t,s) f(Bv(s), v(s))\,ds$$

where

$$Bv(s) = \int_0^1 G(s,\tau)v(\tau)\,d\tau.$$

We seek three fixed points $v_1, v_2, v_3 \in \mathcal{P}$ of A which satisfy the conclusion of the theorem. We note first that if $v \in \mathcal{P}$, then from the properties of $G(t,s)$, $Av(t) \ge 0$, $Bv(t) \ge 0$, $(Av)''(t) = -f(B(v(t), v(t)) \le 0$, $0 \le t \le 1$, $Av(t_3) \ge 2t_3 Av(1/2)$, and $Av(t) = Av(1-t)$, $0 \le t \le 1/2$. Consequently, $A : \mathcal{P} \to \mathcal{P}$.

Also, for all $v \in \mathcal{P}$, by (10) we have $\alpha(v) \le \beta(v)$ and by (11), $\|v\| \le \frac{1}{2t_3}\gamma(v)$. If $v \in \overline{P(\gamma, c)}$, then $\|v\| \le \frac{1}{2t_3}\gamma(v) \le \frac{c}{2t_3}$, which implies that, for $s \in [0,1]$,

$$v(s) \in \left[0, \frac{c}{2t_3}\right] \text{ and } Bv(s) \in \left[0, \frac{c}{16t_3}\right].$$

By condition (G3) we have

$$\begin{aligned}\gamma(Av) &= \max_{t\in[0,t_3]\cup[1-t_3,1]} \int_0^1 G(t,s)f(Bv(s),v(s))\,ds \\ &= \int_0^1 G\left(t_3,s\right)f(Bv(s),v(s))\,ds \\ &\le \left(\frac{2c}{t_3(1-t_3)}\right)\int_0^1 G\left(t_3,s\right)\,ds \\ &= c.\end{aligned}$$

Therefore, $A:\overline{P(\gamma,c)}\to\overline{P(\gamma,c)}$.

It is immediate that

$$\begin{gathered}\left\{v\in P\left(\gamma,\theta,\alpha,b,\frac{t_2}{t_1}b,c\right)\mid \alpha(v)>b\right\}\neq\emptyset, \\ \left\{v\in Q\left(\gamma,\beta,\psi,\frac{2a}{r},a,c\right)\mid \beta(v)<a\right\}\neq\emptyset,\end{gathered}$$

and thus the first parts of (C1) and (C2) are satisfied.

In order to show the second part of (C1) holds, let $v\in P(\gamma,\theta,\alpha,b,\frac{t_2}{t_1}b,c)$. For each $s\in[t_1,t_2]\cup[1-t_2,1-t_1]$, we have $b\le v(s)\le\frac{t_2}{t_1}b$. Hence

$$bt_1(t_2-t_1)\le Bv(s)\le\frac{c(t_1^2+t_2(1-2t_2))}{4t_3}+\frac{bt_2(t_2^2-t_1^2)}{2t_1},$$

and by condition (G2) and (7) we see

$$\begin{aligned}\alpha(Av) &= \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} \int_0^1 G(t,s)f(Bv(s),v(s))\,ds \\ &= \int_0^1 G\left(t_1,s\right)f(Bv(s),v(s))\,ds \\ &> \int_{t_1}^{t_2} G\left(t_1,s\right)f(Bv(s),v(s))\,ds+\int_{1-t_2}^{1-t_1} G\left(t_1,s\right)f(Bv(s),v(s))\,ds \\ &\ge \left(\frac{b}{t_1(t_2-t_1)}\right)\int_{t_1}^{t_2} G\left(t_1,s\right)\,ds+\left(\frac{b}{t_1(t_2-t_1)}\right)\int_{1-t_2}^{1-t_1} G\left(t_1,s\right)\,ds \\ &= \left(\frac{b}{t_1(t_2-t_1)}\right)\left(\frac{t_1[(1-t_1)^2-(1-t_2)^2]}{2}+\frac{t_1(t_2^2-t_1^2)}{2}\right) \\ &= b.\end{aligned}$$

To verify the second part of (C2), let $v\in Q(\gamma,\beta,\psi,\frac{2a}{r},a,c)$. This implies that for each $s\in\left[\frac{1}{r},1-\frac{1}{r}\right]$, we have

$$v(s)\in\left[\frac{2a}{r},a\right]\text{ and }Bv(s)\in\left[\frac{a(r-2)}{r^3},\frac{a}{8}\right].$$

Thus, by condition (G1) and the calculations in (5) and (6),

$$\begin{aligned}
\beta(Av) &= \max_{t\in\left[\frac{1}{r},1-\frac{1}{r}\right]} \int_0^1 G(t,s) f(Bv(s),v(s))\,ds \\
&= \int_0^1 G\left(1/2,s\right) f(Bv(s),v(s))\,ds \\
&= 2\int_0^{\frac{1}{r}} G\left(1/2,s\right) f(Bv(s),v(s))\,ds + 2\int_{\frac{1}{r}}^{\frac{1}{2}} G\left(1/2,s\right) f(Bv(s),v(s))\,ds \\
&< \frac{c}{r^2 t_3(1-t_3)} + \left(\frac{8r^2}{r^2-4}\right)\left(a - \frac{c}{r^2 t_3(1-t_3)}\right)\left(\frac{r^2-4}{8r^2}\right) \\
&= a.
\end{aligned}$$

To show (C3) holds, suppose $v \in P(\gamma,\alpha,b,c)$ with $\theta(Av) > \frac{t_2}{t_1} b$. Using (9), we get

$$\begin{aligned}
\alpha(Av) &= \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} \int_0^1 G(t,s) f(Bv(s),v(s))\,ds \\
&= \int_0^1 G\left(t_1,s\right) f(Bv(s),v(s))\,ds \\
&= \int_0^1 \frac{G\left(t_1,s\right)}{G\left(t_2,s\right)} G\left(t_2,s\right) f(Bv(s),v(s))\,ds \\
&\geq \frac{t_1}{t_2}\int_0^1 G\left(t_2,s\right) f(Bv(s),v(s))\,ds \\
&= \frac{t_1}{t_2}\theta(Ay) \\
&> b.
\end{aligned}$$

Finally, to show (C4), we take $v \in Q(\gamma,\beta,a,c)$ with $\psi(Av) < \frac{2a}{r}$. Using (8), we have

$$\begin{aligned}
\beta(Av) &= \max_{t\in\left[\frac{1}{r},1-\frac{1}{r}\right]} \int_0^1 G(t,s) f(Bv(s),v(s))\,ds \\
&= \int_0^1 G\left(1/2,s\right) f(Bv(s),v(s))\,ds \\
&= \int_0^1 \frac{G\left(1/2,s\right)}{G\left(1/r,s\right)} G\left(1/r,s\right) f(Bv(s),v(s))\,ds \\
&\leq \frac{r}{2}\int_0^1 G\left(1/r,s\right) f(Bv(s),v(s))\,ds \\
&= \frac{r}{2}\psi(Ay) \\
&< a.
\end{aligned}$$

Therefore the hypotheses of Theorem 1 are satisfied and there exist three positive solutions y_1, y_2, and y_3 for the Lidstone BVP (1), (2). Moreover, these solutions are of the form

$$y_i(t) = \int_0^1 G(t,s) v_i(s)\,ds, \quad i = 1,2,3,$$

for some $v_1, v_2, v_3 \in \mathcal{P}$. $\square$

Remark. We have chosen to perform the analysis when f is autonomous. However, if $f = f(t, w, z)$ and in addition, for each (w, z), $f(t, w, z)$ is symmetric about $t = \frac{1}{2}$, then an analogous theorem would be valid with respect to the same cone $\mathcal{P}$.

4. Discrete Analogs

Motivated by the early multiple solutions results for difference equations [1, 2, 6] and specifically papers involving difference equations satisfying Lidstone boundary conditions [7], we want to extend Theorem 2 to discrete problems. To this end, we will again impose growth conditions on f which allow us to apply Theorem 1 and obtain three symmetric positive solutions of the discrete fourth order Lidstone BVP

$$\Delta^4 y(t-2) = f(y(t), -\Delta^2 y(t-1)), \quad a+2 \le t \le b+2 \tag{12}$$
$$y(a) = \Delta^2 y(a) = 0 = \Delta^2 y(b+2) = y(b+4), \tag{13}$$

where $f : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is continuous. We will apply Theorem 1 in conjunction with a completely continuous operator whose kernel, $G_1(t, s)$, is the Green's function for

$$\begin{gathered} -\Delta^2 w(t-1) = 0, \\ w(a+1) = 0 = w(b+3). \end{gathered}$$

In particular,

$$G_1(t,s) = \begin{cases} \dfrac{(t-a-1)(b+3-s)}{b+2-a}, & a+1 \le t \le s \le b+3\,, \\ \dfrac{(s-a-1)(b+3-t)}{b+2-a}, & a+1 \le s \le t \le b+3\,. \end{cases}$$

We will write our solutions of (12), (13) in the form

$$y(t) = \sum_{s=a+1}^{b+1} G_0(t,s) v(s)$$

where $G_0(t, s)$ is the Green's function for

$$\begin{gathered} -\Delta^2 w(t-1) = 0, \\ w(a) = 0 = w(b+4), \end{gathered}$$

and v is a fixed point of a completely continuous operator with kernel $G_1(t, s)$. In particular,

$$G_0(t,s) = \begin{cases} \dfrac{(t-a)(b+4-s)}{b+4-a}, & a \le t \le s \le b+4, \\ \dfrac{(s-a)(b+4-t)}{b+4-a}, & a \le s \le t \le b+4. \end{cases}$$

We will make use of various properties of $G_0(t, s)$ and $G_1(t, s)$, namely

$$\sum_{s=a+1}^{b+3} G_0(t,s) = \frac{(t-a)(b+4-t)}{2}, \quad a \le t \le b+4,$$
$$\sum_{s=a+2}^{b+2} G_1(t,s) = \frac{(t-a-1)(b+3-t)}{2}, \quad a+1 \le t \le b+3,$$

$$\sum_{s=a+k_3}^{b+4-k_3} G(t_m, s) = C_1, \tag{14}$$

$$\sum_{s=a+2}^{a+k_3-1} G_1(t_m, s) + \sum_{s=b+5-k_3}^{b+3} G_1(t_m, s) = C_2, \tag{15}$$

$$\sum_{s=a+k_1}^{a+k_2} G_1(t_1, s) + \sum_{s=b+4-k_2}^{b+4-k_1} G_1(t_1, s) = C_3, \tag{16}$$

$$\sum_{s=a+k_3}^{b+4-k_3} G_0(t_3, s) = C_4,$$

$$\sum_{s=a+k_1}^{a+k_2} G_0(t_1, s) + \sum_{s=b+4-k_2}^{b+4-k_1} G_0(t_1, s) = C_5,$$

$$\sum_{s=a+1}^{a+k_1-1} G_0(t_m, s) + \sum_{s=a+k_2+1}^{b+3-k_2} G_0(t_m, s) + \sum_{s=b+5-k_1}^{b+3} G_0(t_m, s) = C_6,$$

$$\sum_{s=a+k_1}^{a+k_2} G_0(t_m, s) + \sum_{s=b+4-k_2}^{b+4-k_1} G_0(t_m, s) = C_7,$$

$$\max_{a+2\le r\le b+2} \frac{G_1(t_m, r)}{G_1(t, r)} = \frac{t_m - a - 1}{t - a - 1}, \quad a+2 \le t \le t_m, \tag{17}$$

$$\min_{a+2\le r\le b+2} \frac{G_1(t', r)}{G_1(t'', r)} = \frac{t' - a - 1}{t'' - a - 1}, \quad a+2 \le t' \le t'' \le t_m, \tag{18}$$

For completeness, we have calculated and included the values of the above constants.

$$
\begin{aligned}
C_1 &= \frac{(t_m - a - 1)(b + 3 - t_m) - (t_3 - a - 1)^{(2)}}{2},\\
C_2 &= \frac{(t_3 - a - 1)^{(2)}}{2},\\
C_3 &= \frac{(t_1-a)[(t_2-a)^{(2)} + (b+4-t_1)^{(2)} - (t_1-a-1)^{(2)} - (b+3-t_2)^{(2)}]}{2(b + 2 - a)},\\
C_4 &= \frac{(t_3 - a)[(b + 5 - t_3)^{(2)} - (t_3 - a)^{(2)}]}{2(b + 4 - a)},\\
C_5 &= \frac{(t_1-a)[(t_2+1-a)^{(2)} + (b+5-t_1)^{(2)} - (t_1-a)^{(2)} - (b+4-t_2)^{(2)}]}{2(b + 4 - a)},\\
C_6 &= \frac{(b + 4 - t_m)(t_m - a) + (t_1 - a)^{(2)} - (t_2 - a + 1)^{(2)}}{2},\\
C_7 &= \frac{(t_2 - a + 1)^{(2)} - (t_1 - a)^{(2)}}{2},
\end{aligned}
$$

Define

$$t_m = \left\lfloor \frac{b + 4 + a}{2} \right\rfloor$$

and

$$M = \left\lfloor \frac{b+4-a}{2} \right\rfloor .$$

Let $\mathcal{E}_1 = \{y \mid y : [a, b+4] \to \mathbb{R}\}$ be endowed with the maximum norm,

$$\|y\|_1 = \max_{a \le t \le b+4} |y(t)| ,$$

and for $2 \le k_0 \le M$, let $t_0 = a + k_0$ and define the cone $\mathcal{P}_1 \subset \mathcal{E}_1$ by

$$\mathcal{P}_1 = \left\{ \begin{array}{l} y \in \mathcal{E}_1 \text{ such that } y(a+k) = y(b+4-k) \text{ for all } k \in [0, M], \\ y(t) \ge 0, \Delta^2 y(t-1) \le 0 \text{ for all } a+1 \le t \le b+3, \\ \text{and } \displaystyle\min_{t \in [a+k_0, b+4-k_0]} |y(t)| \ge \left(\frac{t_0 - a}{t_m - a} \right) \|y\|_1 \end{array} \right\} .$$

Similarly let $\mathcal{E}_0 = \{y \mid y : [a+1, b+3] \to \mathbb{R}\}$ be endowed with the maximum norm,

$$\|v\|_0 = \max_{a+1 \le t \le b+3} |v(t)| ,$$

and define the cone $\mathcal{P}_0 \subset \mathcal{E}_0$ by

$$\mathcal{P}_0 = \left\{ \begin{array}{l} v \in \mathcal{E}_0 \text{ such that } v(a+1+k) = v(b+3-k) \text{ for all } k \in [0, M-1], \\ v(t) \ge 0, \Delta^2 v(t-1) \le 0 \text{ for all } a+2 \le t \le b+2, \\ \text{and } \displaystyle\min_{t \in [a+k_0, b+4-k_0]} |v(t)| \ge \left(\frac{t_0 - a - 1}{t_m - a - 1} \right) \|v\|_0 \end{array} \right\} .$$

Finally, we define the nonnegative continuous concave functionals α, ψ and the nonnegative continuous convex functionals β, θ, γ on $\mathcal{P}_0$ by

$$\begin{aligned}
\gamma(v) &= \max_{t \in [a+1, a+k_0] \cup [b+4-k_0, b+3]} v(t) = v(t_0), \\
\psi(v) &= \min_{t \in [a+k_3, b+4-k_3]} v(t) = v(t_3), \\
\beta(v) &= \max_{t \in [a+k_3, b+4-k_3]} v(t) = v(t_m), \\
\alpha(v) &= \min_{t \in [a+k_1, a+k_2] \cup [b+4-k_2, b+4-k_1]} v(t) = v(t_1), \\
\theta(v) &= \max_{t \in [a+k_1, a+k_2] \cup [b+4-k_2, b+4-k_1]} v(t) = v(t_2),
\end{aligned}$$

where $t_1 = a + k_1$, $t_2 + a + k_2$,and $t_3 = a + k_3$ are nonnegative numbers such that

$$k_1, k_2, k_3 \in [2, M] \text{ and } k_1 \le k_2.$$

We observe here that for each $v \in \mathcal{P}_0$,

$$\alpha(v) = v(t_1) \le v(t_m) = \beta(v), \tag{19}$$

$$\|v\|_0 = v(t_m) \le \left(\frac{t_m - a - 1}{t_0 - a - 1} \right) v(t_0) = \left(\frac{t_m - a - 1}{t_0 - a - 1} \right) \gamma(v), \tag{20}$$

and also that $y \in \mathcal{P}_1$ is a solution of (12), (13) if and only if there exists a $v \in \mathcal{P}_0$ such that

$$y(t) = \sum_{s=a+1}^{b+3} G_0(t, s) v(s), \quad a \le t \le b+4,$$

where v is of the form

$$v(t) = \sum_{s=a+2}^{b+2} G_1(t,s) f\left(\sum_{\tau=a+1}^{b+3} G_0(s,\tau)v(\tau), v(s)\right), \quad a+1 \le t \le b+3.$$

In light of these preliminaries, we are now ready to present the main result of this section.

Theorem 3. *Suppose there exist* $0 < a' < b' < \left(\frac{t_2-a-1}{t_1-a-1}\right) b' \le c'$ *such that* f *satisfies each of the following growth conditions:*

($\Gamma 1$) $f(z,w) < \frac{a'}{C_1} - \frac{2c'C_2}{(t_0-a-1)(b+3-t_0)C_1}$ *for all* (z,w) *in*

$$\left[\frac{a'(t_3-a-1)C_4}{(t_m-a-1)}, \frac{a'(t_m-a)(b+4-t_m)}{2}\right] \times \left[\frac{a'(t_3-a-1)}{t_m-a-1}, a'\right]$$

($\Gamma 2$) $f(z,w) < \frac{b'}{C_3}$ *for all* (z,w) *in*

$$\left[b'C_5, \frac{b'(t_2-a-1)C_7}{2(t_1-a-1)} + \frac{2c'C_6}{(t_0-a-1)(b+3-t_0)}\right] \times \left[b, \frac{t_2-a-1}{t_1-a-1} b\right]$$

($\Gamma 3$) $f(z,w) \le \frac{2c'}{(t_0-a-1)(b+3-t_0)}$ *for all* (z,w) *in*

$$\left[0, \frac{c'(t_m-a)^{(2)}(b+4-t_m)}{2(t_0-a-1)}\right] \times \left[0, \frac{c'(t_m-a-1)}{t_0-a-1}\right].$$

Then the Lidstone BVP (12), (13) *has at least three symmetric positive solutions* y_1, y_2, $y_3 \in \mathcal{P}_1$, *such that*

$$\max_{t\in[a+1,a+k_0]\cup[b+4-k_0,b+3]} -\Delta^2 y_i(t-1) \le c', \quad \text{for } i = 1,2,3,$$

$$\min_{t\in[a+k_1,a+k_2]\cup[b+4-k_2,b+4-k_1]} -\Delta^2 y_1(t-1) > b',$$

$$\max_{t\in[a+k_3,b+4-k_3]} -\Delta^2 y_2(t-1) < a',$$

and

$$\min_{t\in[a+k_1,a+k_2]\cup[b+4-k_2,b+4-k_1]} -\Delta^2 y_3(t-1) < b'$$

with

$$\max_{t\in[a+k_3,b+4-k_3]} -\Delta^2 y_3(t-1) > a',$$

for some v_1, v_2, $v_3 \in \mathcal{P}_0$ *satisfying*

$$y_i(t) = \sum_{s=a+2}^{b+2} G_0(t,s)v_i(s), \quad i = 1,2,3.$$

Proof. Define the completely continuous operator A by

$$Av(t) = \sum_{s=a+2}^{b+2} G_1(t,s) f(Bv(s), v(s)), \quad a+1 \le t \le b+3,$$

where

$$Bv(s) = \sum_{\tau=a+1}^{b+3} G_0(s,\tau)v(\tau).$$

We seek three fixed points $v_1, v_2, v_3 \in \mathcal{P}_0$ of A which satisfy the conclusion of the theorem. We note first that if $v \in \mathcal{P}_0$, then from the properties of $G_1(t,s)$ and $G_0(t,s)$, it follows that $Av(t) \geq 0$, $Bv(t) \geq 0$, and

$$\begin{aligned} \Delta^2(Av)(t-1) &= -f(Bv(t), v(t)) \leq 0, \quad && a+2 \leq t \leq b+2, \\ \Delta^2(Bv)(t-1) &= -v(t) \leq 0, && a+2 \leq t \leq b+2, \end{aligned}$$

Moreover, $Av(t_0) \geq \left(\frac{t_0-a-1}{t_m-a-1}\right) Av(t_m)$, and $Av(a+k) = Av(b+4-k)$ for $1 \leq k \leq M$. Consequently, $A : \mathcal{P}_0 \to \mathcal{P}_0$.

Also, for all $v \in \mathcal{P}_0$, by (19) we have $\alpha(v) \leq \beta(v)$ and by (20),

$$\|v\|_0 \leq \left(\frac{t_m-a-1}{t_0-a-1}\right)\gamma(v).$$

If $v \in \overline{P(\gamma, c')}$, then

$$\|v\|_0 \leq \left(\frac{t_m-a-1}{t_0-a-1}\right)\gamma(v) \leq \frac{c'(t_m-a-1)}{(t_0-a-1},$$

which implies that, for $s \in [a+1, b+3]$,

$$v(s) \in \left[0, \frac{c'(t_m-a-1)}{t_0-a-1}\right] \quad \text{and} \quad Bv(s) \in \left[0, \frac{c'(t_m-a)^{(2)}(b+4-t_m)}{2(t_0-a-1)}\right].$$

By condition $(\Gamma 3)$ we have

$$\begin{aligned} \gamma(Av) &= \max_{t \in [a+1, a+k_0] \cup [b+4-k_0, b+3]} \sum_{s=a+2}^{b+2} G_1(t,s) f(Bv(s), v(s)) \\ &= \sum_{s=a+2}^{b+2} G_1(t_0,s) f(Bv(s), v(s)) \\ &\leq \left(\frac{2c'}{t_0-a-1)(b+3-t_0)}\right) \sum_{s=a+2}^{b+2} G_1(t_0,s) \\ &= c'. \end{aligned}$$

Therefore, $A : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$.

It is immediate that

$$\left\{ v \in P(\gamma, \theta, \alpha, b', \frac{b'(t_2-a-1)}{t_1-a-1}, c') \ : \ \alpha(v) > b' \right\} \neq \emptyset,$$
$$\left\{ v \in Q(\gamma, \beta, \psi, \frac{a'(t_3-a-1)}{t_m-a-1}, a', c') \ : \ \beta(v) < a' \right\} \neq \emptyset,$$

and thus the first parts of (C1) and (C2) are satisfied.

In order to show the second part of (C1) holds, let $v \in P(\gamma, \theta, \alpha, b', \frac{b'(t_2-a-1)}{t_1-a-1}, c')$. For each $s \in [a+k_1, a+k_2] \cup [b+4-k_2, b+4-k_1]$, we have

$$b' \leq v(s) \leq \frac{b'(t_2-a-1)}{t_1-a-1}.$$

Hence

$$b'C_5 \leq Bv(s) \leq \frac{b'(t_2-a-1)C_7}{2(t_1-a-1)} + \frac{2c'C_6}{(t_0-a-1)(b+3-t_0)},$$

and by condition (Γ2) and (16) we see

$$
\begin{aligned}
\alpha(Av) &= \min_{t\in[a+k_1,a+k_2]\cup[b+4-k_2,b+4-k_1]} \sum_{s=a+2}^{b+2} G_1(t,s)f(Bv(s),v(s)) \\
&= \sum_{s=a+2}^{b+2} G_1(t_1,s)f(Bv(s),v(s)) \\
&\geq \sum_{s=a+k_1}^{a+k_2} G_1(t_1,s)f(Bv(s),v(s)) + \sum_{s=b+4-k_2}^{b+4-k_1} G_1(t_1,s)f(Bv(s),v(s)) \\
&> \left(\frac{b'}{C_3}\right)\left(\sum_{s=a+k_1}^{a+k_2} G_1(t_1,s) + \sum_{s=b+4-k_2}^{b+4-k_1} G_1(t_1,s)\right) \\
&= b'.
\end{aligned}
$$

To verify the second part of (C2), let $v \in Q(\gamma,\beta,\psi,\frac{a'(t_3-a-1)}{t_m-a-1},a',c')$. This implies that for each $s\in[a+k_3,b+4-k_3]$, we have

$$
\frac{a'(t_3-a-1)}{t_m-a-1} \leq v(s) \leq a'
$$

and

$$
\frac{a'(t_3-a-1)C_4}{(t_m-a-1)} \leq Bv(s) \leq \frac{a'(t_m-a)(b+4-t_m)}{2}.
$$

Thus, by condition (Γ1) and the calculations in (14) and (15),

$$
\begin{aligned}
\beta(Av) &= \max_{t\in[a+k_3,b+4-k_3]} \sum_{s=a+2}^{b+2} G_1(t,s)f(Bv(s),v(s)) \\
&= \sum_{s=a+2}^{b+2} G_1(t_m,s)f(Bv(s),v(s)) \\
&= \sum_{s=a+2}^{a+k_3-1} G_1(t_m,s)f(Bv(s),v(s)) + \sum_{s=a+k_3}^{b+4-k_3} G_1(t_m,s)f(Bv(s),v(s)) \\
&+ \sum_{s=b+5-k_3}^{b+2} G_1(t_m,s)f(Bv(s),v(s)) \\
&< \left(\frac{2c'}{(t_0-a-1)(b+3-t_0)}\right)C_2 + \left(\frac{a'}{C_1} - \frac{2c'C_2}{(t_0-a-1)(b+3-t_0)C_1}\right)C_1 \\
&= a'.
\end{aligned}
$$

To show (C3) holds, suppose $v \in P(\gamma, \alpha, b', c')$ with $\theta(Av) > \frac{b'(t_2-a-1)}{t_1-a-1}$. Using (18), we get

$$\begin{aligned}
\alpha(Av) &= \min_{t\in[a+k_1,a+k_2]\cup[b+4-k_2,b+4-k_1]} \sum_{s=a+2}^{b+2} G_1(t,s)f(Bv(s),v(s)) \\
&= \sum_{s=a+2}^{b+2} G_1(t_1,s)f(Bv(s),v(s)) \\
&= \sum_{s=a+2}^{b+2} \left(\frac{G_1(t_1,s)}{G_1(t_2,s)}\right) G_1(t_2,s)f(Bv(s),v(s)) \\
&\geq \frac{t_1-a-1}{t_2-a-1} \sum_{s=a+2}^{b+2} G_1(t_2,s)f(Bv(s),v(s)) \\
&= \frac{t_1-a-1}{t_2-a-1}\theta(Av) \\
&> b'.
\end{aligned}$$

Finally, to show (C4), we take $v \in Q(\gamma, \beta, a', c')$ with $\psi(Av) < \frac{a'(t_3-a-1)}{t_m-a-1}$. Using (17), we have

$$\begin{aligned}
\beta(Av) &= \max_{t\in[a+k_3,b+4-k_3]} \sum_{s=a+2}^{b+2} G_1(t,s)f(Bv(s),v(s)) \\
&= \sum_{s=a+2}^{b+2} G_1(t_m,s)f(Bv(s),v(s)) \\
&= \sum_{s=a+2}^{b+2} \left(\frac{G_1(t_m,s)}{G_1(t_3,s)}\right) G_1(t_3,s)f(Bv(s),v(s)) \\
&\leq \left(\frac{t_m-a-1}{t_3-a-1}\right) \sum_{s=a+2}^{b+2} G_1(t_3,s)f(Bv(s),v(s)) \\
&= \left(\frac{t_m-a-1}{t_3-a-1}\right) \psi(Av) \\
&< a'.
\end{aligned}$$

The hypotheses of Theorem 1 are satisfied and as a result there exist three positive solutions y_1, y_2, and $y_3 \in \mathcal{P}_1$ for the Lidstone BVP (12), (13). Moreover, these solutions are of the form

$$y_i(t) = \sum_{s=a+1}^{b+1} G_0(t,s)v_i(s), \quad i = 1, 2, 3,$$

for some $v_1, v_2, v_3 \in \mathcal{P}_0$. □

Remark. We have chosen to perform the analysis when f is autonomous. However, if $f = f(t, w, z)$ and in addition, for each (w, z), $f(t, w, z)$ is symmetric about $t = t_m$, then an analogous theorem would be valid with respect to the same cone $\mathcal{P}_0$.

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