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RESONANCE AND STRONG RESONANCE FOR SEMILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^N$

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Abstract. We prove the existence of weak solutions for the semilinear elliptic problem

$$-\Delta u = \lambda h u + a g(u) + f, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

where $\lambda \in \mathbb{R}$, $f \in L^{2N/(N+2)}$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous bounded function, and $h \in L^{N/2} \cap L^{\alpha}$, $\alpha > N/2$. We assume that $a \in L^{2N/(N+2)} \cap L^{\infty}$ in the case of resonance and that $a \in L^1 \cap L^{\infty}$ and $f \equiv 0$ for the case of strong resonance. We prove first that the Palais-Smale condition holds for the functional associated with the semilinear problem using the concentration-compactness lemma of Lions. Then we prove the existence of weak solutions by applying the saddle point theorem of Rabinowitz for the cases of non-resonance and resonance, and a linking theorem of Silva in the case of strong resonance. The main theorems in this paper constitute an extension to $\mathbb{R}^N$ of previous results in bounded domains by Ahmad, Lazer, and Paul [2], for the case of resonance, and by Silva [15] in the strong resonance case.

1. Introduction

Let $\mathcal{D}^{1,2}$ be the completion of $C_c^{\infty}(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \Big(\int |\nabla u|^2 \Big)^{1/2}.$$

It is known that $\mathcal{D}^{1,2}$ is a Hilbert space with inner product $\langle u, v\rangle = \int \nabla u \cdot \nabla v$. It is also known that $\mathcal{D}^{1,2}$ is embedded in $L^{2^*}(\mathbb{R}^N)$ (cf. [3]). In fact,

$$|u|_{L^{2^*}}^{2^*} \leqslant C^* \|u\|^2, \tag{1.1}$$

where $2^* = 2N/(N-2)$ and C^* is a constant depending on N.

In this paper we study the existence of solutions to the boundary–value problem

$$\begin{gathered} -\Delta u = \lambda h(x) u + a(x) g(u) + f(x), \quad x \in \mathbb{R}^N, \\ u \in \mathcal{D}^{1,2}, \end{gathered} \tag{1.2}$$

where $\lambda \in \mathbb{R}$, $f \in L^{2N/(N+2)}$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous, bounded function, $h \in L^{N/2} \cap L^{\alpha}$, for $\alpha > N/2$, and $a \in L^{\infty}$.

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Definition. For a bounded nonlinearity g, problem (1.2) is said to be at resonance if λ is an eigenvalue of the boundary-value problem

$$\begin{gathered} -\Delta u = \lambda h(x)u, \quad x \in \mathbb{R}^N, \\ u \in \mathcal{D}^{1,2}. \end{gathered} \tag{1.3}$$

If, in addition, $g(s) \to 0$ as $|s| \to \infty$, problem (1.2) is said to be strongly resonant. If λ is not an eigenvalue of (1.3), then (1.2) is said to be a non-resonance problem.

It is well known that, for $h \in L^{N/2}(\mathbb{R}^N) \cap L^{\alpha}(\mathbb{R}^N)$, $\alpha > N/2$, and $h > 0$ a.e., problem (1.3) possesses a sequence $\{\lambda_j\}$ of eigenvalues satisfying $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \dots$, with $\lambda_j \to \infty$ as $j \to \infty$, and the corresponding family of eigenfunctions, $\{\varphi_n\}$, forms a complete orthonormal system for $\mathcal{D}^{1,2}$. Furthermore, φ_1 can be chosen to be positive a.e. in $\mathbb{R}^N$.

The goal of this paper is to extend the solvability of a family of elliptic problems on bounded domains to the whole space $\mathbb{R}^N$, $N \geqslant 3$. In particular, we study the existence of weak solutions for problem (1.2) with $a \in L^{2N/(N+2)} \cap L^{\infty}$, for the case of resonance, and with $a \in L^1 \cap L^{\infty}$ and $f = 0$, for the case of strong resonance.

We prove the existence of weak solutions of (1.2) using variational methods; i.e., solutions of (1.2) are realized as critical points of the functional

$$J_\lambda(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{\lambda}{2}\int hu^2 - \int aG(u) - \int fu,$$

where $G(s) = \int_0^s g(t)dt$, $s \in \mathbb{R}$.

Our results are obtained using the saddle point theorem by Rabinowitz [14] and a linking theorem in [15], in conjunction with the concentration-compactness lemma of Lions [11]. The solvability of (1.2) in the resonance case can be obtained by imposing conditions on either g or $G(s)$. We prove the following existence results:

Theorem 1.1. *Let $g \in C(\mathbb{R}, \mathbb{R})$ be bounded and a be an element of $L^{2N/(N+2)} \cap L^{\infty}$. If $\lambda \in (\lambda_1, \lambda_2)$, where λ_1 and λ_2 are the first two eigenvalues of (1.3), then problem* (1.2) *has at least one solution for any $f \in L^{2N/(N+2)}$.*

Theorem 1.2. *Let $g \in C(\mathbb{R}, \mathbb{R})$ be bounded and $a \in L^{2N/(N+2)} \cap L^{\infty}$. If*

$$\lim_{|t| \to \infty} \Big\{ \int a(x)G(t\varphi_1) + t\int f(x)\varphi_1 \Big\} = +\infty, \tag{1.4}$$

then Problem (1.2) *with $\lambda = \lambda_1$ has a weak solution.*

Theorem 1.3. *Suppose that $g : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $\lim_{|s|\to\infty} g(s) = 0$, and that $a \in L^1 \cap L^{\infty}$. Let*

$$\Lambda := \liminf_{\|u\| \to \infty,\, u \in L_n} \int a(x)G(u)dx, \tag{1.5}$$

where $L_n := \operatorname{span}\{\varphi_i : \lambda_i = \lambda_n\}$. Then, if $\Lambda \in \mathbb{R}$ and

$$a(x)G(s) \leqslant a(x)|a|_{L^1}^{-1}\Lambda \quad \text{for every } s \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N, \tag{1.6}$$

problem (1.2) *with $\lambda = \lambda_n$ possesses at least one solution.*

The non-resonance result of Theorem 1.1 can be proved in the more general case in which λ lies between two consecutive eigenvalues $\lambda_k < \lambda_{k+1}$ of problem (1.3). Similarly, the resonance result of Theorem 1.2 also holds for higher eigenvalues $\lambda_k < \lambda_{k+1}$, $k > 1$. In this case the solvability condition (1.4) has to be modified appropriately. Problems at resonance have been of interest to researchers ever since the pioneering work of Landesman and Lazer [12] in 1970 for second order elliptic operators in bounded domains. The literature on resonance problems in bounded domains is quite vast; of particular interest to this paper are the works of Ahmad, Lazer and Paul [2] in 1976 and of Rabinowitz in 1978, in which critical point methods are applied. Theorem 1.2 is an extension to $\mathbb{R}^N$ of the Ahmad, Lazer and Paul result. There is also an extensive literature on strongly resonant problems in bounded domains. Theorem 1.3 is an extension to $\mathbb{R}^N$ of a result of Silva in [15].

Resonance problems on unbounded domains, and in particular in $\mathbb{R}^N$, have been studied recently by Costa and Tehrani [7] and by Jeanjean [10] for the operator $-\Delta + K$ for K positive, and by Stuart and Zhou [16] for radially symmetric solutions for asymptotically linear problems in $\mathbb{R}^N$. In all these references variational methods were used. Previously, Metzen [13] had used the method of approximated domains to obtain existence for non-resonant problems in unbounded domains, and Hetzer and Landesman [9] for resonant problems for a class of operators which includes the Schrödinger operator.

The main difficulty in proving Theorems 1.1, 1.2 and 1.3 arises in showing that some kind of compactness occurs, the so called Palais-Smale condition $(PS)_c$, when using the variational approach. Even in bounded domains, to prove that the $(PS)_c$ condition holds is a very delicate issue. As an example, in bounded domains $\Omega \subset \mathbb{R}^N$, it has been proved [17] that for certain functionals the $(PS)_c$ condition does not hold at the constant $c = (1/N)S^{N/2}$, where

$$S = \inf_{\phi \in H_0^1(\Omega),\, |\phi|_{L^{2^*}}=1} \int |\nabla \phi|^2.$$

The lack of compactness for problems in unbounded domains has been overcome by different approaches; for instance, approximation by bounded domains mentioned above, the use of Sobolev spaces of symmetric functions which possess compact embedding properties, or the use of weighted Sobolev spaces (see [6] and references therein).

From an heuristic point of view it seems that for each problem the $(PS)_c$ condition requires a specific and particular approach. In this paper we apply the concentration-compactness method of Lions [11], which basically consists of proving the existence of a set where compactness is available by using the restrictions imposed on the $(PS)_c$ sequences by the energy functional associated with the problem (1.2).

2. Variational Setting for Non-Resonance Problems

We study the existence of solutions for semilinear elliptic equations in $\mathbb{R}^N$ ($N \geqslant 3$) of the form

$$-\Delta u = \lambda h(x)u + a(x)g(u) + f(x),$$

where $\lambda \in \mathbb{R}$, $f \in L^{2N/(N+2)}$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, $|g(s)| \leqslant M$ for all $s \in \mathbb{R}$, $h \in L^{N/2} \cap L^{\alpha}$, $\alpha > N/2$ and $a \in L^{2N/(N+2)} \cap L^{\infty}$. In particular, we consider the boundary-value problem (1.2) which is is a non-linear perturbation of the linear eigenvalue problem (1.3).

It can be shown that if $h \in L^{N/2}(\mathbb{R}^N) \cap L^{\alpha}(\mathbb{R}^N)$, for $\alpha > \frac{N}{2}$, and $h > 0$ a.e., then Problem (1.3) has an increasing sequence of eigenvalues $0 < \lambda_1 \leqslant \lambda_2, \dots$ with $\lambda_j \to \infty$ as $j \to \infty$ and a corresponding sequence of eigenfunctions, $\{\varphi_j\}$, which forms a complete orthonormal system for $\mathcal{D}^{1,2}$. This is a consequence of the following result which is easily derived from [5, Lemma 2.1].

Lemma 2.1. *If $h \in L^{N/2}(\mathbf{R}^N) \cap L^\alpha(\mathbb{R}^N)$ for $\alpha > \frac{N}{2}$, then*

$$-\Delta w = hu \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^N)$$

has a weak solution in $\mathcal{D}^{1,2}$ for every $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. Moreover the operator $T_h : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathcal{D}^{1,2}(\mathbb{R}^N)$, defined by $T_h(u) = T(u) = w$, is compact.

Corollary 2.2. *Let $h \in L^{N/2} \cap L^\alpha$ for $\alpha > N/2$ and define $F : \mathcal{D}^{1,2} \to \mathbb{R}$ by $F(u) := \int hu^2$, then F is weakly continuous; that is, if $u_n \to u$ weakly in $\mathcal{D}^{1,2}$, then $F(u_n) \to F(u)$.*

Moreover, the condition $h \in L^\alpha$ for $\alpha > N/2$ can also be used to show, as a consequence of the weak Harnack inequality [8, Theorem 8.20] that $\varphi_1 > 0$ a.e. in $\mathbb{R}^N$, λ_1 is simple, and the zero-set of the eigenfunctions φ_j, $j \geqslant 1$, has Lebesgue measure zero. This last property is known as *unique continuation* [1].

Solutions of (1.2) happen to be critical points of the functional

$$J_\lambda(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{\lambda}{2}\int h(x)u^2 - \int a(x)G(u) - \int f(x)u \tag{2.1}$$

for $u \in \mathcal{D}^{1,2}$, where $J_\lambda \in \mathcal{C}^1(\mathcal{D}^{1,2}, \mathbb{R}^N)$ has Fréchet derivative

$$J'_\lambda(u)v = \int \nabla u \nabla v - \lambda \int h(x)uv - \int a(x)g(u)v - \int f(x)v$$

for all $u, v \in \mathcal{D}^{1,2}$. This is a straightforward consequence of the definition of Fréchet derivative and the conditions on a, f, g, and h.

We will use the following version of the concentration-compactness lemma of Lions [11].

Lemma 2.3 (Lions Concentration-Compactness Lemma)**.** *Let $(\rho_n)_{n\geqslant 1}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying: $\rho_n \geqslant 0$ in $\mathbb{R}^N$ and $\int \rho_n dx = \sigma$, where $\sigma > 0$ is fixed. Then there exists a subsequence $(\rho_{n_k})_{k\geqslant 1}$ satisfying one of the three possibilities:*

- (i) *(Compactness) There exists $y_k \in \mathbb{R}^N$ such that $\rho_{n_k}(\cdot + y_k)$ is tight; that is, for all $\varepsilon > 0$ there exists $R < \infty$ such that $\int_{y_k + B_R} \rho_{n_k}(x) \geqslant \sigma - \varepsilon$ for all k.*
- (ii) *(Vanishing) $\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{y+B_R} \rho_{n_k}(x)dx = 0$ for all $R < \infty$.*
- (iii) *(Dichotomy) There exists $\alpha \in (0, \sigma)$ such that, for all $\varepsilon > 0$, there exist $k_o \geqslant 1$, a sequence $\{y_n\} \subset \mathbb{R}^N$, a number $R > 0$ and a sequence $\{R_n\} \subset \mathbb{R}_+$, with $R < R_1$, $R_n < R_{n+1} \to +\infty$, such that, if we set $\rho_n^1 = \rho_n \chi_{\{|x-y_n|\leqslant R\}}$ and $\rho_n^2 = \rho_n \chi_{\{|x-y_n|\geqslant R_n\}}$, then*

$$\begin{gathered} \|\rho_{n_k} - (\rho_k^1 + \rho_k^2)\|_{L^1} \leqslant \varepsilon, \quad \Big|\int_{\mathbb{R}^N} \rho_k^1 dx - \alpha\Big| \leqslant \varepsilon, \\ \Big|\int_{\mathbb{R}^N} \rho_k^2 dx - (\sigma - \alpha)\Big| \leqslant \varepsilon, \quad \text{for all } k \geqslant k_o, \\ \operatorname{dist}(\operatorname{supp}\rho_k^1, \operatorname{supp}\rho_k^2) \xrightarrow{k} +\infty. \end{gathered} \tag{2.2}$$

Next, we establish a compactness statement that will be used throughout this paper. First, we recall the Palais-Smale condition.

Palais-Smale condition. Suppose that E is a real Banach space, and let $\mathcal{C}^1(E,\mathbb{R})$ denote the set of functionals whose Fréchet derivative is continuous on E. A functional $I \in \mathcal{C}^1(E,\mathbb{R})$ satisfies the Palais-Smale condition at the level $c \in \mathbb{R}$, denoted $(PS)_c$, if any sequence $(u_n) \subset E$ for which

(i) $I(u_n) \to c$ as $n \to \infty$, and
(ii) $\|I'(u_n)\| \to 0$ as $n \to \infty$,

possesses a convergent subsequence. If $I \in \mathcal{C}^1(E,\mathbb{R})$ satisfies $(PS)_c$ for every $c \in \mathbb{R}$, we say that I satisfies the (PS) condition. Any sequence (u_n) for which (i) and (ii) hold is called a $(PS)_c$ sequence for I.

Proposition 2.4. *Let $J_\lambda : \mathcal{D}^{1,2} \to \mathbb{R}$ be as defined by (2.1), where $\lambda \in \mathbb{R}$, g is a continuous function with $|g(s)| \leqslant M$ for all $s \in \mathbb{R}$, $f \in L^{2N/(N+2)}$, $h \in L^{N/2} \cap L^\alpha$ for $\alpha > N/2$, and $a \in L^{2N/(N+2)} \cap L^\infty$. Then, if every $(PS)_c$ sequence for J_λ is bounded, J_λ satisfies the $(PS)_c$ condition.*

Proof. Let (u_n) be a $(PS)_c$ sequence for J_λ. Thus, by assumption, (u_n) is bounded. Without loss of generality, we may assume that $\|u_n\|^2 = \int |\nabla u_n|^2 > 0$ for all n. Define

$$\rho_n := |\nabla u_n|^2 \quad \text{for all } n.$$

Thus (ρ_n) is a sequence in $L^1(\mathbb{R}^N)$ satisfying (passing to a subsequence if necessary) $\int \rho_n \to \tau > 0$ as $n \to \infty$. Defining

$$\rho_n' = \frac{\rho_n}{\int \rho_n} \quad \text{for all } n,$$

we have $\int \rho_n' = 1 > 0$ for all n. Hence, using (ρ_n') for (ρ_n), we may assume that (ρ_n) satisfies the hypotheses of the Lions Concentration-Compactness Lemma 2.3 with $\sigma = 1$.

(A) Claim: Vanishing does not hold. Let $B_R(y) = \{x \in \mathbb{R}^N : |x-y| < R\}$. Assume by contradiction that vanishing in Lemma 2.3 does hold. Then there exists $\{n_k\}_{k\geqslant 1}$ such that (u_{n_k}) converges weakly to 0 in $\mathcal{D}^{1,2}$. To see why this is the case, let ϕ be any function in $\mathcal{D}^{1,2}$. Then, given $\varepsilon > 0$, there exists $R' > 0$ such that

$$\Big(\int_{[B_{R'}(0)]^c} |\nabla\phi|^2\Big)^{1/2} < \frac{\varepsilon}{2\sup_k \|u_{n_k}\|}.$$

On the other hand, by the Cauchy-Schwartz inequality,

$$\Big|\int \nabla u_{n_k}\cdot\nabla\phi\Big| \leqslant \|u_{n_k}\|\Big(\int_{[B_{R'}(0)]^c} |\nabla\phi|^2\Big)^{1/2} + \|\phi\|\Big(\int_{B_{R'}(0)} |\nabla u_{n_k}|^2\Big)^{1/2}.$$

Moreover, since vanishing in Lemma 2.3 implies the existence of a subsequence (u_{n_k}) and k_0 such that

$$\int_{B_{R'}(0)} |\nabla u_{n_k}|^2 < \big(\frac{\varepsilon}{2\|\phi\|}\big)^2 \quad \text{if } k \geqslant k_0,$$

it follows that $\int \nabla u_{n_k}\cdot\nabla\phi < \varepsilon$ for all $k \geqslant k_o$. Since $\varphi \in \mathcal{D}^{1,2}$ was arbitrary, $u_{n_k} \to 0$ weakly in $\mathcal{D}^{1,2}$.

Now, using the assumption that $\|J_\lambda'(u_{n_k})\| \to 0$ as $k \to \infty$, we have

$$\int |\nabla u_{n_k}|^2 - \int a g(u_{n_k}) u_{n_k} = o(1) \quad \text{as } k \to \infty, \tag{2.3}$$

since the map $u \mapsto \int hu^2$ is weakly continuous by Corollary 2.2, and $u \mapsto \int fu$ is in $(\mathcal{D}^{1,2})^*$. On the other hand, since $a \in L^{2N/(N+2)}$, given $\varepsilon > 0$, there exists R_* such that

$$\Big(\int_{[B_{R_*}(0)]^c} |a|^{2N/(N+2)}\Big)^{\frac{N+2}{2N}} < \frac{\varepsilon}{2M \sup_k \|u_{n_k}\|}. \tag{2.4}$$

Moreover, vanishing implies that there exists k_1 such that for $k > k_1$,

$$\int_{B_{R_*}(0)} |\nabla u_{n_k}|^2 < \Big(\frac{\varepsilon}{2M|a|_{L^{\frac{2N}{N+2}}}}\Big)^{2\cdot 2^*}. \tag{2.5}$$

Thus, applying Hölder's inequality and the estimates in (2.4) and (2.5), we conclude

$$\Big|\int a(x)g(u_{n_k})u_{n_k}\Big| < \varepsilon \quad \text{for all } k \geqslant k_1.$$

Hence, since ε was arbitrary, it follows from (2.3) that $\lim_{k\to\infty} \int |\nabla u_{n_k}|^2 = 0$, which contradicts $\int |\nabla u_{n_k}|^2 = \sigma > 0$ for all k.

(B) Claim: Dichotomy does not hold. If dichotomy occurs, then there exists $\alpha \in (0, \sigma)$ such that, given $\varepsilon > 0$, we can chose $R > 0$ with

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_{\frac{R}{2}}(y)} |\nabla u_{n_k}|^2 > \alpha - \varepsilon.$$

Moreover, there exists $k_o \geqslant 1$ such that, for $k \geqslant k_o$,

$$\alpha - \varepsilon < \sup_{y\in\mathbb{R}^N} \int_{B_{\frac{R}{2}}(y)} |\nabla u_{n_k}|^2 < \alpha + \varepsilon;$$

thus, for each $k \geqslant k_o$, there exists $y_k \in \mathbb{R}^N$ such that

$$\alpha - \varepsilon < \int_{B_{\frac{R}{2}}(y_k)} |\nabla u_{n_k}|^2 < \alpha + \varepsilon. \tag{2.6}$$

Furthermore, from Property (2.2) in Lemma 2.3, there exists an increasing sequence (R_k), with $R_1 \geqslant R$ and $R_k \to \infty$ as $k \to \infty$, such that

$$\sigma - \alpha - \epsilon \leqslant \int_{\{B_{3R_k}(y_k)\}^c} |\nabla u_{n_k}|^2 \leqslant \sigma - \alpha + \epsilon \quad \text{for all } k \geqslant k_o. \tag{2.7}$$

Consequently,

$$\int_{\frac{R}{2}<|x-y_k|<3R_k} |\nabla u_{n_k}|^2 < 2\varepsilon \quad \text{for all } k \geqslant k_o. \tag{2.8}$$

Note that (2.8) implies

$$\int_{R<|x-y_k|\leqslant 2R_k} |u_{n_k}|^{2^*} \leqslant \theta(\varepsilon) \quad \text{for all } \ k \geqslant k_o, \tag{2.9}$$

where $\theta \to 0$ as $\varepsilon \to 0$. To see why (2.9) holds, take $\eta_k \in C_0^\infty(\mathbb{R}^N)$ such that $\eta_k(x) = 0$ if $|x| \leqslant R/2$ or $|x| \geqslant 3R_k$, and $\eta_k(x) = 1$ if $R \leqslant |x| \leqslant 2R_k$. By Sobolev's inequality (1.1) we have that

$$\begin{aligned}\Big(\int |\eta_k u_k|^{2^*}\Big)^{1/2^*} &\leqslant C\Big(\int |\nabla(\eta_k u_k)|^2\Big)^{1/2}\\ &\leqslant C\Big(\int |\nabla\eta_k|^2 u_k^2 + 2\int u_k\eta_k\nabla\eta_k\cdot\nabla u_k + \int \eta_k^2|\nabla u_k|^2\Big)^{1/2}\\ &\leqslant C\Big(C_1\int_\Omega |u_k|^2 + C_2\int_\Omega |u_k| + C_3\int_{\frac{R}{2}\leqslant|x-y_k|\leqslant 3R_k} |\nabla u_k|^2\Big)^{1/2}.\end{aligned}$$

Where we have written u_k for u_{n_k}, and $\Omega = \{x : \nabla\eta_k \neq 0\} = \{R/2 \leqslant |x-y_k| \leqslant R\} \cup \{2R_k \leqslant |x-y_k| \leqslant 3R_k\}$. Clearly, $\Omega \subset \{R/2 \leqslant |x-y_k| \leqslant 3R_k\}$. By (2.8) in conjunction with Sobolev's inequality, we also obtain that $\int_\Omega |u_k|^2 < C\varepsilon$ and $\int_\Omega |u_k| < C\varepsilon$. Consequently, it follows from (2.9) and the previous estimate that $\int_{R\leqslant|x-y_k|\leqslant 2R_k} |u_k|^{2^*} \leqslant \int |\eta_k u_k|^{2^*} \leqslant \theta(\varepsilon)$ as required.

Now take $\zeta \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leqslant \zeta \leqslant 1$ and

$$\zeta(x) = \begin{cases} 1 & \text{for } |x| \leqslant 1,\\ 0 & \text{for } |x| \geqslant 2,\end{cases}$$

and let $\phi(x) = 1-\zeta(x)$. Put $\zeta_k(x) = \zeta\left(\frac{x-y_k}{R}\right)$ and $\phi_k(x) = \phi\left(\frac{x-y_k}{R_k}\right)$ for all $x \in \mathbb{R}^N$, and define $u_k^\dagger(x) := (\zeta_k \cdot u_{n_k})(x)$ and $u_k^\ddagger(x) := (\phi_k \cdot u_{n_k})(x)$. Then for each k,

$$u_k^\dagger(x) = \begin{cases} u_{n_k} & \text{if } |x-y_k| \leqslant R,\\ 0 & \text{if } |x-y_k| \geqslant 2R,\end{cases} \tag{2.10}$$

and

$$u_k^\ddagger(x) = \begin{cases} 0 & \text{if } |x-y_k| \leqslant R_k,\\ u_{n_k} & \text{if } |x-y_k| \geqslant 2R_k.\end{cases} \tag{2.11}$$

We have two cases: (y_k) is bounded and (y_k) is unbounded.
(i) Assume (y_k) is bounded. Note that

$$\begin{aligned}\Big|\int hu_k^\ddagger(u_k^\ddagger - u_k)\Big| &\leqslant \int |h||u_k^\ddagger||u_k^\ddagger - u_k|\\ &\leqslant \int_{R_k\leqslant|x-y_k|\leqslant 2R_k} |h||\phi_k||\phi_k - 1||u_k|^2\\ &\leqslant \int_{R_k\leqslant|x-y_k|\leqslant 2R_k} |h||u_k|^2.\end{aligned}$$

Thus, by Hölder's inequality,

$$\begin{aligned}\Big|\int hu_k^\ddagger(u_k^\ddagger - u_k)\Big| &\leqslant C|h|_{\frac{N}{2}}\Big(\int_{R_k\leqslant|x-y_k|\leqslant 2R_k} |u_k|^{2^*}\Big)^{2/2^*}\\ &\leqslant C_1\theta(\epsilon)^{2/2^*} := \theta_1(\epsilon) \xrightarrow{\epsilon} 0 \quad \text{for } k \geqslant k_o,\end{aligned}$$

where the last inequality follows from (2.9). Thus $\int hu_k^\ddagger u_k = \int h|u_k^\ddagger|^2 + \theta_1(\epsilon)$ for $k \geqslant k_o$. We claim that $u_k^\ddagger \to 0$ weakly in $\mathcal{D}^{1,2}$. It will then follow, using Corollary (2.2), that

$$\int hu_k^\ddagger u_k \to 0 \quad \text{as } k \to \infty. \tag{2.12}$$

Take any $\phi \in \mathcal{D}^{1,2}$. For each $\varepsilon > 0$, there exists $R' > 0$ such that

$$\Big(\int_{[B_{R'}(0)]^c} |\nabla\phi|^2\Big)^{1/2} < \frac{\varepsilon}{\sup_k \|u_k\|}. \tag{2.13}$$

Since $\{y_k\}$ is bounded in $\mathbb{R}^N$, there exists y^* such that $y_k \to y^*$ (taking a subsequence if necessary). Choose $n_0 \geqslant k_o$ such that $B_{R'}(0) \subset B_{R_{n_0}}(y^*)$; this is possible since $R_k \to \infty$. Then, for $k > n_0$, by (2.11),

$$\begin{aligned}
\int \nabla u_k^\ddagger \cdot \nabla\phi &= \int_{[B_{R_{n_0}}(y^*)]^c} \nabla u_k^\ddagger \cdot \nabla\phi \\
&\leqslant \|u_k^\ddagger\| \Big(\int_{[B_{R_{n_0}}(y^*)]^c} |\nabla\phi|^2\Big)^{1/2} \\
&\leqslant \|u_k^\ddagger\| \Big(\int_{[B_{R'}(0)]^c} |\nabla\phi|^2\Big)^{1/2} < \varepsilon,
\end{aligned}$$

by the Cauchy-Schwarz inequality, the fact that $[B_{R_{n_0}}(y^*)]^c \subset [B_{R'}(0)]^c$ and (2.13). Therefore, since $\phi \in \mathcal{D}^{1,2}$ and $\varepsilon > 0$ were arbitrary, it follows that $u_k^\ddagger \to 0$ weakly in $\mathcal{D}^{1,2}$. Thus, $\int h u_k^\ddagger u_k \to 0$ as $k \to \infty$ as stated in (2.12).

On the other hand, since $a \in L^{2N/(N+2)}$, given $\varepsilon > 0$, there exists R'' such that

$$\Big(\int_{[B_{R''}(0)]^c} |a|^{2N/(N+2)}\Big)^{\frac{N+2}{2N}} < \frac{\varepsilon}{M \sup_k \|u_k\|}.$$

Thus, by (2.11) and similar arguments to those used above, there exist $n_0 \geqslant k_o$ such that $B_{R''}(0) \subset B_{R_{n_0}}(y^*)$ and

$$\begin{aligned}
\Big|\int a(x) g(u_k) u_k^\ddagger\Big| &\leqslant M \int a(x)|u_k^\ddagger| \\
&\leqslant M \int_{[B_{R_{n_0}}(y^*)]^c} a|u_k^\ddagger| \\
&\leqslant M\|u_k^\ddagger\| \Big(\int_{[B_{R_{n_0}}(y^*)]^c} |a|^{2N/(N+2)}\Big)^{(N+2)/(2N)} \\
&\leqslant \varepsilon, \quad \text{if } k > n_0.
\end{aligned}$$

Consequently,

$$\int a g(u_k) u_k^\ddagger = o(1) \quad \text{as } k \to \infty. \tag{2.14}$$

Now,

$$\begin{aligned}
&\Big|\int \nabla u_k \cdot \nabla u_k^\ddagger - \int |\nabla u_k^\ddagger|^2\Big| \\
&= \Big|\int_{R_k \leqslant |x-y_k| \leqslant 2R_k} \nabla(\phi_k u_k) \cdot \nabla u_k\Big| \\
&\leqslant \int_{R_k \leqslant |x-y_k| \leqslant 2R_k} |u_k||\nabla u_k \cdot \nabla\phi_k| + \int_{R_k \leqslant |x-y_k| \leqslant 2R_k} |\phi_k||\nabla u_k|^2 \\
&\leqslant C_1 \int_{R_k \leqslant |x-y_k| \leqslant 2R_k} |u_k| + C_2 \int_{R_k \leqslant |x-y_k| \leqslant 2R_k} |\nabla u_k|^2.
\end{aligned}$$

Hence, by (2.9) and a similar argument used to obtain (2.9) applied to the first integral, as well as the Sobolev embedding Theorem,

$$\int \nabla u_k \cdot \nabla u_k^{\ddagger} = \int |\nabla u_k^{\ddagger}|^2 + \theta_2(\varepsilon) \quad \text{for } k \geqslant k_o, \tag{2.15}$$

where $\theta_2 \to 0$ as $\varepsilon \to 0$. Since (u_k) is a $(PS)_c$ sequence, as $k \to \infty$,

$$\langle J'_\lambda(u_k), u_k^{\ddagger}\rangle = \int \nabla u_k \cdot \nabla u_k^{\ddagger} - \lambda \int h u_k u_k^{\ddagger} - \int a g(u_k) u_k^{\ddagger} - \int f u_k^{\ddagger} = o(1)\,.$$

It then follows from (2.12), (2.14) and (2.15) together with the fact that $\int f u_k^{\ddagger} \to 0$ as $k \to \infty$, that

$$\int |\nabla u_k^{\ddagger}|^2 = o(1) \quad \text{as } k \to \infty.$$

Therefore, from (2.7),

$$\sigma - \alpha - \varepsilon \leqslant \int_{|x-y_k| \geqslant 3R_k} |\nabla u_k^{\ddagger}|^2 \leqslant \int |\nabla u_k^{\ddagger}|^2 = o(1) \quad \text{as } k \to \infty,$$

with $\sigma - \alpha - \varepsilon > 0$ for ε small, which is a contradiction. Consequently, (y_k) cannot be bounded and dichotomy does not hold in this case.

(ii) Assume now that (y_k) is unbounded. For this case we use $u_k^{\dagger}$ to get a contradiction. First, we show that $u_k^{\dagger} \xrightarrow{k} 0$ weakly in $\mathcal{D}^{1,2}$. Let ϕ be any function in $\mathcal{D}^{1,2}$. Given $\varepsilon > 0$, there exists $R' > 0$ such that $\int_{[B_{R'}(0)]^c} \nabla\phi < \varepsilon / \sup_k \|u_k\|$. Since $\{y_k\}$ is not bounded, there exists n_0 such that $|y_{n_0}| > R' + 2R$, where R is as in (2.10). We then have that $B_{2R}(y_{n_0}) \subset [B_{R'}(0)]^c$, so in view of (2.10),

$$\begin{aligned}
\int \nabla u_k^{\dagger} \cdot \nabla\phi &= \int_{[B_{2R}(y_{n_0})]} \nabla u_k^{\dagger} \cdot \nabla\phi \\
&\leqslant \|u_k^{\dagger}\| \int_{[B_{2R}(y_{n_0})]} \nabla\phi \\
&\leqslant \|u_k^{\dagger}\| \int_{[B_{R'}(0)]^c} \nabla\phi \\
&\leqslant \varepsilon, \quad \text{if } k > n_0.
\end{aligned}$$

Since $\phi \in \mathcal{D}^{1,2}$ and $\varepsilon > 0$ are arbitrary, we conclude that $u_k^{\dagger} \xrightarrow{k} 0$ weakly in $\mathcal{D}^{1,2}$.

From the assumption that (u_m) is a bounded $(PS)_c$ sequence, we have

$$\int \nabla u_k \cdot \nabla u_k^{\dagger} - \lambda \int h u_k u_k^{\dagger} - \int a g(u_k) u_k^{\dagger} - \int f u_k^{\dagger} = o(1) \quad \text{as } k \to \infty. \tag{2.16}$$

Observe that

$$\begin{aligned}
\Big| \int h u_k^{\dagger}(u_k^{\dagger} - u_k) \Big| &\leqslant \int |h||u_k^{\dagger}||u_k^{\dagger} - u_k| \\
&\leqslant \int_{R \leqslant |x-y_k| \leqslant 2R} |h||\zeta_k||\zeta_k - 1||u_k|^2 \\
&\leqslant C|h|_{\frac{N}{2}} \Big(\int_{R \leqslant |x-y_k| \leqslant 2R} |u_k|^{2^*} \Big)^{2/2^*},
\end{aligned}$$

by Hölder's inequality. So, by (2.9), since $R < R_k$ for all k,

$$\Big| \int h u_k^{\dagger}(u_k^{\dagger} - u_k) \Big| \leqslant C_1 \theta(\epsilon)^{2/2^*} := \theta_3(\epsilon) \xrightarrow{\epsilon} 0 \quad \text{for } k \geqslant k_o.$$

Observe that $\int h|u^\dagger|^2 \to 0$ by Corollary 2.2, since $u_k^\dagger \xrightarrow{k} 0$ weakly in $\mathcal{D}^{1,2}$. This, in conjunction with the above estimate, implies

$$\int h u_k^\dagger u_k \xrightarrow{k} 0. \tag{2.17}$$

Furthermore,

$$\Big| \int a(x) g(u_k) u_k^\dagger \Big| \xrightarrow{k} 0. \tag{2.18}$$

Effectively, given $\varepsilon > 0$, since $a \in L^{2N/(N+2)}$, there exists $R''' > 0$ such that

$$\Big(\int_{[B_{R'''}(0)]^c} |a|^{2N/(N+2)} \Big)^{\frac{N+2}{2N}} < \frac{\varepsilon}{M \sup_k \|u_k\|}. \tag{2.19}$$

Since $\{y_k\}$ is unbounded, we take n_0 such that $|y_{n_0}| > R''' + 2R$. Then $B_{2R}(y_{n_0}) \subset [B_{R'''}(0)]^c$; thus, by (2.10) and (2.19),

$$\begin{aligned}
\Big| \int a g(u_k) u_k^\dagger \Big| &\leqslant M \int_{B_{2R}(y_{n_0})} |a||g(u_k) u_k^\dagger| \\
&\leqslant M \|u_k\| \Big(\int_{[B_{R'''}(0)]^c} |a|^{2N/(N+2)} \Big)^{(N+2)/(2N)} \\
&< \varepsilon \quad \text{for all } k,
\end{aligned}$$

from which (2.18) follows. Finally,

$$\int \nabla u_k \cdot \nabla u_k^\dagger = \int |\nabla u_k^\dagger|^2 + \theta_4(\varepsilon), \tag{2.20}$$

where $\theta_4 \to 0$ as $\varepsilon \to 0$. This follows from (2.9), the estimate

$$\Big| \int \nabla u_k \cdot \nabla u_k^\dagger - \int |\nabla u_k^\dagger|^2 \Big| \leqslant C_1 \int_{R \leqslant |x-y_k| \leqslant 2R} |u_k| + C_2 \int_{R \leqslant |x-y_k| \leqslant 2R} |\nabla u_k|^2,$$

and an argument similar to that used to obtain (2.9), by observing that $\{R \leqslant |x-y_k| \leqslant 2R\} \subset \{R \leqslant |x-y_k| \leqslant 2R_k\}$ and applying the Sobolev's embedding Theorem. Thus, using (2.17), (2.18) and (2.20) in (2.16), and recalling (2.6), we obtain

$$0 < \alpha - \varepsilon \leqslant \int_{B_{\frac{R}{2}}(y_k)} |\nabla u_k^\dagger|^2 \leqslant \int |\nabla u_k^\dagger|^2 = o(1) \quad \text{as } k \to \infty,$$

which is a contradiction.

Since vanishing and dichotomy in Lemma 2.3 do not hold, necessarily compactness holds; i.e., there exists $\{y_n\} \subset \mathbb{R}^N$ such that for all ε there exists $R > 0$ such that

$$\int_{B_R(y_n)} |\nabla u_n|^2 \geqslant \sigma - \varepsilon \quad \text{for all } n. \tag{2.21}$$

Now, it follows from (2.21) and the fact that $\int |\nabla u_n|^2 = \sigma$ for all n that

$$\int_{[B_R(y_n)]^c} |\nabla u_n|^2 < \varepsilon \quad \text{for all } n. \tag{2.22}$$

Claim: $\{y_n\}$ is bounded. If $\{y_n\}$ is not bounded, then $u_n \to 0$ weakly in $\mathcal{D}^{1,2}$ as $n \to \infty$. To see why this is the case, take $\phi \in \mathcal{D}^{1,2}$ and let $\varepsilon > 0$. There exists $R' > 0$ such that

$$\Big(\int_{[B_{R'}(0)]^c} |\nabla\phi|^2\Big)^{1/2} < \varepsilon/(2\sup_n \|u_n\|). \tag{2.23}$$

Since $\{y_n\}$ is not bounded, we may assume that $|y_n| \to \infty$ as $n \to \infty$, and so there exists n_0 such that $|y_{n_0}| > R' + R_0$, where we choose $R_o > 0$, whose existence is guaranteed by part (i) of Lemma 2.3 (see also (2.22)), such that

$$\Big(\int_{[B_{R_0}(y_{n_0})]^c} |\nabla u_n|^2\Big)^{1/2} < \varepsilon/(2\|\phi\|). \tag{2.24}$$

Also, $B_{R_0}(y_{n_0}) \subset [B_{R'}(0)]^c$. Thus,

$$\begin{aligned}\Big|\int \nabla u_n \cdot \nabla\phi\Big| &\leqslant \|u_n\|\Big(\int_{B_{R_0}(y_{n_0})} |\nabla\phi|^2\Big)^{1/2} + \|\phi\|\Big(\int_{[B_{R_0}(y_{n_0})]^c} |\nabla u_n|^2\Big)^{1/2}\\ &\leqslant \|u_n\|\Big(\int_{[B_{R'}(0)]^c} |\nabla\phi|^2\Big)^{1/2} + \|\phi\|\Big(\int_{[B_{R_0}(y_{n_0})]^c} |\nabla u_n|^2\Big)^{1/2},\end{aligned}$$

so that, by (2.23) and (2.24),

$$\Big|\int \nabla u_n \cdot \nabla\phi\Big| < \varepsilon \quad \text{for all } n > n_0.$$

Since $\phi \in \mathcal{D}^{1,2}$ was arbitrary, we conclude that $u_n \to 0$ weakly in $\mathcal{D}^{1,2}$ as stated. Consequently, using the assumption that (u_n) is a bounded $(PS)_c$ sequence, we obtain

$$\int |\nabla u_n|^2 - \int a g(u_n)u_n = o(1) \quad \text{as } n \to \infty, \tag{2.25}$$

since $u \to \int hu^2$ is weakly continuous by Corollary 2.2, and $u \mapsto \int fu$ is also weakly continuous. Moreover,

$$\int a g(u_n)u_n = \int_{B_R(y_n)} a g(u_n)u_n + \int_{[B_R(y_n)]^c} a g(u_n)u_n. \tag{2.26}$$

Since $\{y_n\}$ is not bounded, it follows that

$$\Big(\int_{B_R(y_n)} |a|^{2N/(N+2)}\Big)^{\frac{N+2}{2N}} \to 0 \quad \text{as } n \to \infty.$$

Therefore,

$$\Big|\int_{B_R(y_n)} a g(u_n)u_n\Big| \leqslant C\Big(\int_{B_R(y_n)} |a|^{2N/(N+2)}\Big) \to 0 \quad \text{as } n \to \infty. \tag{2.27}$$

In addition, by (1.1) and (2.22),

$$\int_{[B_R(y_n)]^c} |u_n|^{2N/(N-2)} < C^*\varepsilon$$

So, by Hölder's inequality,

$$\Big|\int_{[B_R(y_n)]^c} a g(u_n)u_n\Big| \leqslant C\varepsilon^{1/2^*}.$$

Thus, in view of (2.26) and (2.27), we obtain from (2.25) and (2.21) that

$$\sigma - \varepsilon \leqslant \int_{B_R(y_n)} |\nabla u_n|^2 \leqslant \int |\nabla u_n|^2 = C\varepsilon^{1/2^*} + o(1) \quad \text{as } n \to \infty,$$

which clearly leads to a contradiction as $\varepsilon \to 0$. Therefore, $\{y_n\}$ cannot be unbounded. Thus, $\{y_n\}$ is bounded in the compactness case of the concentration-compactness Lemma 2.3.

Since $\{y_n\} \subset \mathbb{R}^N$ is bounded, there exists $R^* > 0$ such that $B_R(y_n) \subset B_{R^*}(0)$ for all $n = 1, 2, 3, \dots$. We may also choose R^* large enough so that

$$\int_{B_{R^*}(0)^c} |a|^{\frac{2N}{N+2}} < \varepsilon^{\frac{2N}{N+2}}. \tag{2.28}$$

Put $\Omega = B_{R^*}(0)$ and note that Ω satisfies the hypotheses of the Rellich-Kondrachov Theorem [8, Theorem 7.26, p. 171].

Given that (u_n) is a bounded sequence, there exists a subsequence, (u_{n_k}), such that $u_{n_k} \to u$ weakly in $\mathcal{D}^{1,2}$ as $k \to \infty$. Moreover, given $1 \leqslant t < 2^*$, we may assume that $u_{n_k} \to u$ strongly in $L^t(\Omega)$ as $k \to \infty$, since Ω is bounded, passing to a subsequence if necessary, by the Rellich-Kondrachov Theorem.

Observe that, since $B_R(y_n) \subset \Omega$ for all $n = 1, 2, 3, \dots$, then $\Omega^c \subset B_R(y_n)^c$ for all n. It then follows from (2.22) that

$$\int_{\Omega^c} |\nabla u_n|^2 < \varepsilon \quad \text{for all } n. \tag{2.29}$$

We want to show that

$$\int |\nabla (u_{n_k} - u)|^2 \to 0 \quad \text{as } k \to \infty. \tag{2.30}$$

We have

$$\begin{aligned} \int |\nabla (u_{n_k} - u)|^2 &= \int \nabla (u_{n_k} - u) \cdot \nabla (u_{n_k} - u) \\ &= \int \nabla u_{n_k} \cdot \nabla (u_{n_k} - u) - \int \nabla u \cdot \nabla (u_{n_k} - u), \end{aligned}$$

where $\int \nabla u \cdot \nabla (u_{n_k} - u) \to 0$ as $k \to \infty$, by the definition of weak convergence in $\mathcal{D}^{1,2}$. Consequently, (2.30) will follow if we can prove that

$$\lim_{k \to \infty} \int \nabla u_{n_k} \cdot \nabla (u_{n_k} - u) = 0. \tag{2.31}$$

Now from the fact that (u_n) is a bounded $(PS)_c$ sequence it follows that

$$\Big| \int \nabla u_{n_k} \cdot \nabla (u - u_{n_k}) - \lambda \int h u_{n_k} (u - u_{n_k}) - \int a g(u_{n_k})(u - u_{n_k}) \Big| = o(1) \tag{2.32}$$

as $k \to \infty$, since $\int f(u - u_n) \to 0$ as $n \to \infty$.

We estimate the second integral on the left–hand side of (2.32) as follows:

$$\Big| \int h u_{n_k} (u - u_{n_k}) \Big| \leqslant \Big| \int_\Omega h u_{n_k} (u - u_{n_k}) \Big| + \Big| \int_{\Omega^c} h u_{n_k} (u - u_{n_k}) \Big|,$$

where, by Hölder's inequality,

$$\Big| \int_\Omega h u_{n_k} (u - u_{n_k}) \Big| \leqslant |h|_{L^\alpha} \Big(\int_\Omega |u_{n_k}|^{2N/(N-2)} \Big)^{(N-2)/(2N)} \Big(\int_\Omega |u - u_{n_k}|^s \Big)^{1/s},$$

where $\frac{1}{s} = \frac{1}{2} + \frac{2}{N} - \frac{1}{\alpha}$, so that $s < 2^*$. Hence, by the Rellich-Kondrachov Theorem, we may assume that $u_{n_k} \to u$ strongly in $L^s(\Omega)$. Consequently,

$$\int_\Omega h u_{n_k} (u - u_{n_k}) = o(1) \quad \text{as } k \to \infty. \tag{2.33}$$

On the other hand, by Hölder's inequality and the assumption that (u_n) is bounded,

$$\Big|\int_{\Omega^c} hu_{n_k}(u-u_{n_k})\Big| \leqslant C|h|_{L^{\frac{N}{2}}}\Big(\int_{\Omega^c}|u_{n_k}|^{2N/(N-2)}\Big)^{(N-2)/(2N)}.$$

Thus, by (1.1) and (2.29),

$$\Big|\int_{\Omega^c} hu_{n_k}(u-u_{n_k})\Big| \leqslant C\varepsilon \quad \text{for all } k.$$

Therefore, it follows from (2.33) that

$$\limsup_{k\to\infty}\Big|\int hu_{n_k}(u-u_{n_k})\Big| \leqslant C\varepsilon. \tag{2.34}$$

Similarly, for the third integral on the left-hand side of (2.32),

$$\Big|\int_{\Omega} ag(u_{n_k})(u-u_{n_k})\Big| \leqslant C\int_{\Omega}|u-u_{n_k}| \to 0 \quad \text{as } k\to\infty, \tag{2.35}$$

since $a\in L^\infty$ and $u_{n_k}\to u$ strongly in $L^1(\Omega)$. To estimate the integral over Ω^c use Hölder's inequality together with the assumptions that (u_n) and g are bounded to obtain

$$\Big|\int_{\Omega^c} ag(u_{n_k})(u-u_{n_k})\Big| \leqslant C\Big(\int_{\Omega^c}|a|^{\frac{2N}{N+2}}\Big)^{(N+2)/(2N)}.$$

It then follows from (2.28) that

$$\Big|\int_{\Omega^c} ag(u_{n_k})(u-u_{n_k})\Big| \leqslant C\varepsilon.$$

Consequently, by (2.35),

$$\limsup_{k\to\infty}\Big|\int ag(u_{n_k})(u-u_{n_k})\Big| \leqslant C\varepsilon. \tag{2.36}$$

Hence, since ε is arbitrary, (2.31) follows from (2.32), (2.34) and (2.36), and (2.31) in turn implies (2.30); that is,

$$\|u_{n_k}-u\|^2 = \int|\nabla(u_{n_k}-u)|^2 = o(1) \quad \text{as } k\to\infty.$$

i.e. $u_{n_k}\to u$ strongly in $\mathcal{D}^{1,2}$. □

Now, with J_λ satisfying the $(PS)_c$ condition, once we can show that every $(PS)_c$ sequence is bounded, we are able to prove some existence results for problem (1.2). Existence will be obtained as a consequence of the following saddle point theorem of Rabinowitz.

Theorem 2.5 (Saddle Point Theorem [14]). *Let $E = V\oplus X$, where E is a real Banach space and $V\neq\{0\}$ and is finite dimensional. Suppose $I\in\mathcal{C}^1(E,\mathbb{R})$ satisfies the (PS) condition,*

- (I_1) *there is a constant α and a bounded neighborhood D of 0 in V such that $I\big|_{\partial D}\leqslant\alpha$, and*
- (I_2) *there is a constant $\beta>\alpha$ such that $I|_X\geqslant\beta$.*

Then, I possesses a critical value $c\geqslant\beta$. Moreover c can be characterized as

$$c = \inf_{h\in\Gamma}\max_{u\in\overline{D}} I(h(u)),$$

where $\Gamma = \{h\in\mathcal{C}(\overline{D},E) : h = id \text{ on } \partial D\}$.

First, we consider the case when λ in the problem (1.2) is not an eigenvalue of the eigenvalue problem (1.3):

$$\begin{gathered}-\Delta u = \lambda h(x) u \quad \text{in } \mathbb{R}^N,\ h > 0 \ a.e., \\ u \in \mathcal{D}^{1,2}.\end{gathered}$$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose (u_n) is a $(PS)_c$ sequence. First, we show that (u_n) is bounded in $\mathcal{D}^{1,2}$. We argue by contradiction. Assume $(\int |\nabla u_n|^2)^{1/2} = t_n \to \infty$ and define $v_n = u_n/t_n$; then, $(\int |\nabla v_n|^2)^{1/2} = 1$ for all n. So we have, passing to a subsequence if necessary, that $v_n \rightharpoonup v$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, since (v_n) is bounded.

Now we claim that $v(x) \equiv 0$. As a consequence of the assumption $\|J'_\lambda(u_n)\| \to 0$ as $n \to \infty$, we have

$$\begin{aligned}\langle J'_\lambda(u_n), \phi\rangle &= \int \nabla u_n \cdot \nabla \phi dx - \lambda \int h(x) u_n \phi dx \\ &\quad - \int a(x) g(u_n) \phi dx - \int f(x)\phi = o(1)\|\phi\|\end{aligned}$$

as $n \to \infty$, for all $\phi \in \mathcal{D}^{1,2}$. Dividing the previous equation by $t_n = |\nabla u_n|_2$ we obtain

$$\int \nabla v_n \cdot \nabla \phi - \lambda \int h(x) v_n \phi - \int a(x) \frac{g(u_n)}{t_n} \phi - \int \frac{f(x)\phi}{t_n} = o(1) \tag{2.37}$$

as $n \to \infty$. Given that g is bounded and that $\phi \in \mathcal{D}^{1,2}$ implies $\phi \in L^{2N/(N-2)}$, we obtain from (2.37)

$$\int \nabla v_n \cdot \nabla \phi - \lambda \int h v_n \phi - \frac{C}{t_n} = o(1) \quad \text{as } n \to \infty, \tag{2.38}$$

by Hölder's inequality. Thus, letting $n \to \infty$ in (2.38),

$$\int \nabla v \cdot \nabla \phi - \lambda \int h v \phi = 0.$$

Since λ is not an eigenvalue of problem (1.3), we conclude that $v = 0$ a.e. in $\mathbb{R}^N$. Substituting $\phi = v_n$ in (2.38) we obtain

$$\int |\nabla v_n|^2 - \lambda \int h v_n^2 - \frac{C}{t_n} = o(1) \quad \text{as } n \to \infty, \tag{2.39}$$

where we have used the fact that $\int h v_n^2 \to 0$ and $\int f v_n \to 0$ as $n \to \infty$ since $v_n \to v = 0$ weakly in $\mathcal{D}^{1,2}$ (see Corollary 2.2). Moreover, given that $\int |\nabla v_n|^2 = \|v_n\|^2 = 1$ we obtain from (2.39) that

$$1 - \frac{C}{t_n} = o(1) \quad \text{as } n \to \infty,$$

which leads clearly to a contradiction as $t_n \to \infty$. Therefore, u_n is bounded in $\mathcal{D}^{1,2}$. Thus, every $(PS)_c$ sequence is bounded. Hence, by Proposition 2.4, J_λ satisfies the $(PS)_c$ condition.

Next we prove that J_λ satisfies the hypotheses of the Saddle Point Theorem 2.5. Let φ_1 be an eigenfunction corresponding to λ_1. Recall that $\|\varphi_1\| = 1$. Consider

$$\begin{aligned} J_\lambda(t\varphi_1) &= \frac{1}{2}t^2 \int |\nabla\varphi_1|^2 - \frac{\lambda}{2}t^2 \int h(x)\varphi_1^2 - \int a(x)G(t\varphi_1) - t\int f(x)\varphi_1 \\ &= \frac{1}{2}t^2\big(1 - \frac{\lambda}{\lambda_1}\big) - \int a(x)G(t\varphi_1) - t\int f(x)\varphi_1. \end{aligned}$$

Recall that $|G(s)| \leqslant C|s|$ for all $s \in \mathbb{R}$. Therefore,

$$\begin{aligned} J_\lambda(t\varphi_1) &\leqslant -\frac{1}{2}\big(\frac{\lambda}{\lambda_1} - 1\big)t^2 + C|t| \int a(x)|\varphi_1| \\ &\leqslant -\frac{1}{2}\big(\frac{\lambda}{\lambda_1} - 1\big)t^2 + C_1|t|\big(|a|_{L^{2N/(N+2)}} + |f|_{L^{2N/(N+2)}}\big)|\varphi_1|_{L^{2N/(N-2)}}. \end{aligned}$$

Let $V = \operatorname{span}\{\varphi_1\}$; it then follows from the last inequality that

$$\lim_{\|v\|\to\infty,\ v\in V} J_\lambda(v) = -\infty.$$

Finally, let $X = V^\perp = \{w \in \mathcal{D}^{1,2} : \langle w, \varphi_1\rangle = 0\}$. Then $\lambda_2 \int hw^2 \leqslant \int |\nabla w|^2$ for all $w \in X$ and

$$J_\lambda(w) \geqslant \frac{1}{2}\big(1 - \frac{\lambda}{\lambda_2}\big)\|w\|^2 - C_1\big(|a|_{L^{2N/(N+2)}} + |f|_{L^{2N/(N+2)}}\big)\|w\|$$

for any $w \in X$. Therefore, $J_\lambda(w) \to +\infty$ as $\|w\| \to \infty$ in X. Consequently, (I_1) and (I_2) in the Saddle Point Theorem 2.5 hold, and so J_λ has a critical point, which establishes Theorem 1.1. □

Remark. This argument can be extended to the case $\lambda_k < \lambda < \lambda_{k+1}$ where λ_k and λ_{k+1} are consecutive eigenvalues of problem (1.3).

3. A Resonance Problem

In this section we consider the problem

$$\begin{gathered} -\Delta u = \lambda_1 h(x)u + a(x)g(u) + f(x), \\ u \in \mathcal{D}^{1,2}, \end{gathered} \tag{3.1}$$

where λ_1 is the first eigenvalue of (1.3) over $\mathbb{R}^N$. We can solve problem (3.1) if we impose a condition similar to one used by Ahmad, Lazer and Paul in [2] on $G(u)$ and f; that is condition (1.4) in the statement of Theorem 1.2.

Proof of Theorem 1.2. We first show that J_{λ_1} satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$, and then we verify that J_{λ_1} satisfies the conditions of the saddle point theorem of Rabinowitz (cf. Theorem 2.5).

Let (u_m) be a $(PS)_c$ sequence for the functional J_{λ_1} defined in (2.1) for $\lambda = \lambda_1$. We claim that (u_m) is bounded.

Write $u_m = v_m + w_m$, where $v_n \in span\{\varphi_1\} = V$ and $w_m \in V^\perp = X$ for each $m \in \mathbb{N}$. First we show that (w_m) is bounded in $\mathcal{D}^{1,2}$. Since $\|J'_{\lambda_1}(u_m)\| \overset{m}{\to} 0$, there exists $m_0 \in \mathbb{N}$ such that if $m \geqslant m_0$, then

$$\Big|\int \nabla u_m \cdot \nabla v - \lambda_1 \int hu_m v - \int ag(u_m)v - \int fv\Big| \leqslant \|v\| \quad \text{for all } v \in \mathcal{D}^{1,2}.$$

In particular, if $v = w_m$, we have

$$\Big|\int |\nabla w_m|^2 - \lambda_1 \int h w_m^2\Big| \leqslant \|w_m\| + \Big|\int a g(u_m) w_m\Big| + \Big|\int f w_m\Big| \quad \text{for } m \geqslant m_o.$$

Given that $\lambda_2 \int h v^2 \leqslant \|v\|^2$ for all $v \in X$, we obtain

$$\big(1 - \frac{\lambda_1}{\lambda_2}\big)\|w_m\|^2 \leqslant \|w_m\| + C\big(|a|_{L^{2N/(N+2)}} + |f|_{L^{2N/(N+2)}}\big)\|w_m\| \quad \text{for } m \geqslant m_o,$$

from which it follows that (w_m) is bounded in $\mathcal{D}^{1,2}$.

Next we show that (v_m) is bounded. Observe that $J_{\lambda_1}(u_m) \xrightarrow{m} c$ implies that $J_{\lambda_1}(u_m)$ is bounded; say $|J_{\lambda_1}(u_m)| \leqslant C_1$ for all m, where

$$\begin{aligned}
J_{\lambda_1}(u_m) &= \frac{1}{2}\int |\nabla u_m|^2 - \frac{\lambda_1}{2}\int h u_m^2 - \int a G(u_m) \\
&= \frac{1}{2}\int |\nabla w_m|^2 - \frac{\lambda_1}{2}\int h w_m^2 \\
&\quad - \int a\left[G(v_m + w_m) - G(v_m)\right] - \int a G(v_m) - \int f v_m.
\end{aligned}$$

Note that $|G(v_m + w_m) - G(v_m)| \leqslant M|w_m|$. Hence,

$$\Big|\int a[G(v_m + w_m) - G(v_m)]\Big| \leqslant M \int a|w_m| \leqslant C_3\|w_m\|.$$

So we obtain

$$\begin{aligned}
\Big|\int a G(v_m) + \int f v_m\Big| &\leqslant |J(u_m)| + C_2\|w_m\|^2 + C_3\|w_m\| \\
&\leqslant C_1 + C_2\|w_m\|^2 + C_3\|w_m\|.
\end{aligned}$$

Given that (w_m) is bounded, we have

$$\Big|\int a(x) G(v_m) + \int f v_m\Big| \leqslant C \quad \text{for all } m.$$

Therefore, if (1.4) holds, then (v_m) is bounded in $\mathcal{D}^{1,2}$, otherwise $\int a G(v_m) + \int f v_m$ would approach infinity as $m \to \infty$, by (1.4). We therefore conclude that (u_m) is bounded, and so by Theorem 1.1 we have that J_{λ_1} satisfies the $(PS)_c$ condition.

To show that the other hypotheses of the Saddle Point Theorem 2.5 are satisfied, we proceed as in the proof of Theorem 1.1. If $u \in X$, we have $u = \sum_{j=2}^{\infty} a_j \varphi_j$, hence

$$\int |\nabla u|^2 - \lambda_1 \int h u^2 = \sum_{j=2}^{2} a_j^2\big(1 - \frac{\lambda_1}{\lambda_j}\big) \geqslant \big(1 - \frac{\lambda_1}{\lambda_2}\big)\|u\|^2.$$

Moreover, since $|g(s)| \leqslant M$ for all $s \in \mathbb{R}$, we have that, for all $u \in \mathcal{D}^{1,2}$,

$$\Big|\int a G(u)\Big| \leqslant M \int |a||u| \leqslant M|a|_{L^{2N/(N+2)}}|u|_{L^{2N/(N-2)}} \leqslant C\|u\|.$$

Therefore J_λ is bounded from below on X; i.e. (I_2) in the Saddle Point Theorem 2.5 holds. Finally, if $v \in V$, we have

$$J_{\lambda_1}(v) = -\int a G(v) - \int f v.$$

But $\int aG(v) + \int fv \to \infty$ as $\|v\| \to \infty$ by (1.4) and, therefore, (I_1) in the Saddle Point Theorem (2.5) also holds. Hence, J_{λ_1} has a critical point and the theorem follows. $\square$

Remark. The existence result in Theorem 1.2 can be extended to the problem (1.2) with $\lambda = \lambda_n$ for $n > 1$ by modifying condition (1.4) appropriately. In fact, suppose the eigenspace corresponding to λ_n is $E_{\lambda_n} = \operatorname{span}\{\varphi_{n_1}, \varphi_{n_2}, \dots, \varphi_{n_k}\}$, then (1.4) is replaced by

$$\lim_{t_1^2+\cdots+t_k^2\to\infty} \int a(x)G(t_1\varphi_{n_1} + \cdots + t_k\varphi_{n_k}) + \int f(t_1\varphi_{n_1} + \cdots + t_k\varphi_{n_k}) = \infty.$$

Remark. Suppose $\lim_{s\to\infty} g(s) = g_\infty$ and $\lim_{s\to-\infty} g(s) = g_{-\infty}$ exist. Then, if $g_\infty > 0$ and $g_{-\infty} < 0$, $G(s) = \int_0^s g(t)dt \to \infty$ as $|s| \to \infty$. Consequently, by L' Hôspital's rule, the Lebesgue dominated convergence theorem and the fact that $\varphi_1 > 0$ a.e. in $\mathbb{R}^N$ we have that

$$\lim_{|t|\to\infty} \frac{1}{t} \int a(x)G(t\varphi_1) = \lim_{|t|\to\infty} \int ag(t\varphi_1)\varphi_1 = \begin{cases} g_\infty \int a\varphi_1 & \text{as } t \to \infty, \\ g_{-\infty} \int a\varphi_1 & \text{as } t \to -\infty. \end{cases}$$

Thus, the condition (1.4) in the resonance Theorem 1.2 holds if

$$g_\infty \int a\varphi_1 + \int f\varphi_1 > 0 \quad \text{and} \quad g_{-\infty} \int a\varphi_1 + \int f\varphi_1 < 0,$$

or

$$g_{-\infty} \int a\varphi_1 < -\int f\varphi_1 < g_\infty \int a\varphi_1. \tag{3.2}$$

This is the original Landesman-Lazer condition in [12] for the case of resonance around the first eigenvalue.

It can be shown that if

$$g_{-\infty} < g(s) < g_{+\infty} \quad \text{for all } s \in \mathbb{R},$$

then (3.2) is necessary and sufficient for the solvability of (3.1). If $g_{-\infty} = g_{+\infty}$, then the Landesman-Lazer condition (3.2) cannot hold, and if $g_{-\infty}$ and $g_{+\infty}$ are both zero, then condition (1.4) might not hold in general. This corresponds to what is known as strong resonance, which will be treated in the next section for the case $f \equiv 0$.

4. A Strongly Resonant Problem

As an example of a strongly resonant problem (cf. [4]) we have

$$\begin{gathered} -\Delta u = \lambda_n h(x)u + a(x)g(u), \quad n \geqslant 1, \\ u \in \mathcal{D}^{1,2}, \end{gathered} \tag{4.1}$$

where $g(s) \to 0$ as $|s| \to \infty$ and $a \in L^1 \cap L^\infty$. In this section we prove Theorem (1.3) which states that, under conditions (1.5) and (1.6) on G, problem (4.1) has a weak solution. This will extend to $\mathbb{R}^N$ the results in [15] for bounded domains. We use the following extension of the saddle point theorem by Rabinowitz.

Theorem 4.1 (Linking Theorem [15])**.** *Let $E = X_1 \oplus X_2$ be a real Banach space with X_1 finite dimensional. Suppose $I \in \mathcal{C}^1(E, \mathbb{R})$ and satisfies:*

- (I_0) *There exists $\beta \in \mathbb{R}$ such that $I(u) \leqslant \beta$ for every $u \in X_1$.*
- (I_1) *There exists $\gamma \in \mathbb{R}$ such that $I(u) \geqslant \gamma$ for every $u \in X_2$.*

(I_2) *There exist* $r_1 > 0$ *and* $\alpha > \gamma$ *such that* $I(u) \geqslant \alpha$ *for every* $u \in X_2$ *with* $\|u\|_E \geqslant r_1$.

If I *satisfies the* $(PS)_c$ *condition for every* $c > \gamma$ *and every* $(PS)_c$ *sequence is bounded, then* I *possesses a critical value* $b \geqslant \gamma$.

Proof of Theorem 1.3. Observe that since $L^1 \cap L^\infty \subset L^q$ for any $q \in (1, \infty)$, Proposition 2.4 applies to the functional J_{λ_n} given in equation (2.1) with $f = 0$.

Define the subspaces $E_k := \operatorname{span}\{\varphi_1, \dots, \varphi_k\}$ and $L_k := \operatorname{span}\{\varphi_i : \lambda_i = \lambda_k\}$ for every $k \in \mathbb{N}$; also, set $E_0 = \{0\}$. We show first that $J_{\lambda_n}(u)$ satisfies the $(PS)_c$ condition for every $c \in (-\Lambda, \infty)$. By Proposition 2.4 it is enough to show that if $c \in (-\Lambda, \infty)$ and (u_m) is a $(PS)_c$ sequence, then (u_m) is bounded.

Let (u_m) be a $(PS)_c$ sequence for J_λ. Assume by contradiction that (u_m) is not bounded. Write $u_m = u_m^+ + u_m^0 + u_m^-$, where $u_m^+ \in (E_n)^\perp$, $u_m^0 \in L_n$, and $u_m^- \in E_{n-1}$. Since $\|J'_{\lambda_n}(u_m)\| \to 0$ as $m \to \infty$, it follows that there exists $m_o \in \mathbb{N}$ for which

$$\Big| \int \nabla u_m \nabla u_m^+ - \lambda_n \int h u_m u_m^+ - \int a g(u_m) u_m^+ \Big| \leqslant \|u_m^+\| \quad \text{for } m \geqslant m_o. \tag{4.2}$$

On the other hand, since g is bounded,

$$\Big| \int a g(u_m) u_m^+ \Big| \leqslant M \int |a||u_m^+| \leqslant M |a|_{L^{2N/(N+2)}} \Big(\int |u_m^+|^{\frac{N-2}{2N}} \Big)^{2N/(N-2)} \leqslant C \|u_m^+\|$$

for all m. Consequently, it follows from (4.2) that there exists $C_1 > 0$ such that

$$\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} \|u_m^+\|^2 \leqslant C_1 \|u_m^+\| \quad \text{for } \ m \geqslant m_o.$$

Therefore, (u_m^+) is bounded.

For (u_m^-), since $u_m^- \in E_{n-1}$ implies $\lambda_{n-1} \geqslant \frac{\int |\nabla u_m^-|^2}{\int h(u_m^-)^2}$, by similar calculations as for u_m^+, we obtain that there exists $C_2 > 0$ such that

$$\Big| \frac{\lambda_{n-1} - \lambda_n}{\lambda_{n-1}} \Big| \|u_m^-\|^2 \leqslant C_2 \|u_m^-\| \quad \text{for } m \geqslant m_o.$$

Consequently, (u_m^-) is also bounded. Moreover, we will show shortly that

$$\|u_m^\pm\| \to 0 \qquad \text{as } m \to \infty. \tag{4.3}$$

This will follow from the fact that $g(s) \to 0$ as $|s| \to \infty$. In fact, from

$$\Big| \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} \Big| \|u_m^+\|^2 - \Big| \int a g(u_m) u_m^+ \Big| \leqslant \Big| \langle J'_{\lambda_n}(u_m), u_m^+ \rangle \Big|$$

and Hölder's inequality, we obtain

$$\Big| \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} \Big| \|u_m^+\| \leqslant \|J'_{\lambda_n}(u_m)\| + C \Big(\int |a|^{\frac{2N}{N+2}} |g(u_m)|^{\frac{2N}{N+2}} \Big)^{(N+2)/(2N)}. \tag{4.4}$$

Since $\|J'_{\lambda_n}(u_m)\| \to 0$ as $m \to \infty$, condition (4.3), for u_m^+, follows from (4.4) once we show that

$$\lim_{m \to \infty} \int |a|^{\frac{2N}{N+2}} |g(u_m)|^{\frac{2N}{N+2}} = 0. \tag{4.5}$$

Define $v_m = u_m^0 / \|u_m^0\|$ for all m. Then, since L_n is finite dimensional, we may assume, passing to a subsequence if necessary, that there exists $v \in L_n$ such that $\|v\| = 1$ and

$$v_m(x) \to v(x) \quad \text{a.e. as } m \to \infty. \tag{4.6}$$

For a given $\varepsilon > 0$, find $R > 0$ such that

$$\int_{[B_R(0)]^c} |a|^{2N/(N+2)} < \frac{\varepsilon}{M^{2N/(N+2)}}. \tag{4.7}$$

Since $\|u_m^+\|$ and $\|u_m^-\|$ are bounded, we may assume, as a consequence of the Rellich-Kondrachov Theorem [8, Theorem 7.26], passing to subsequences if necessary, that there exist functions $w^\pm \in H^1(B_R(0))$ such that $u_m^\pm(x) \to w^\pm$ a.e. in $B_R(0)$ as $m \to \infty$ [3, p. 58]. It then follows from

$$u_m(x) = \|u_m^0\|\Big(\frac{u_m^-(x)}{\|u_m^0\|} + v_m(x) + \frac{u_m^+(x)}{\|u_m^0\|}\Big),$$

(4.6), and the unique continuation property of the eigenfunctions that

$$|u_m(x)| \to \infty \quad \text{a.e. in } B_R(0) \quad \text{as } m \to \infty,$$

since $\|u_m^0\| \to \infty$ as $m \to \infty$. Therefore, by the Lebesgue dominated convergence theorem and the fact that $g(s) \to 0$ as $s \to \infty$,

$$\lim_{m\to\infty} \int_{B_R(0)} |a|^{2N/(N+2)} |g(u_m)|^{2N/(N+2)} = 0.$$

Hence, in view of (4.7),

$$\limsup_{m\to\infty} \int |a|^{2N/(N+2)} |g(u_m)|^{2N/(N+2)} \leqslant \varepsilon,$$

from which (4.5) follows. Consequently, (4.3) is established for (u_m^+). Similar calculations lead to the analogous result for (u_m^-).

Now, from

$$G(u_m) - G(u_m^0) = \int_0^1 g(u_m^0 + t(u_m^+ + u_m^-))(u_m^+ + u_m^-)dt\,,$$

we obtain

$$\begin{aligned}
\Big|\int aG(u_m) - \int aG(u_m^0)\Big| &\leqslant \Big|\int \int_0^1 ag(u_m^0 + t(u_m^+ + u_m^-))(u_m^+ + u_m^-)\Big| \\
&\leqslant \int_0^1 \int |a||g(u_m^0 + t(u_m^+ + u_m^-))(u_m^+ + u_m^-)|dtdx \\
&\leqslant M|a|_{\frac{2N}{N+2}} \Big(\int |u_m^+ + u_m^-|^{\frac{N-2}{2N}} dx\Big)^{2N/(N-2)} \\
&\leqslant C\|u_m^+ + u_m^-\|.
\end{aligned}$$

It then follows from (4.3) and the above inequality that

$$\int aG(u_m) = \int aG(u_m^0) + o(1) \text{ as } m \to \infty.$$

Thus,

$$\liminf_{m\to\infty} \int aG(u_m) \geqslant \liminf_{m\to\infty} \int aG(u_m^0) \geqslant \liminf_{\substack{\|u\|\to\infty \\ u\in L_n}} \int aG(u) = \Lambda.$$

Hence,

$$\liminf_{m\to\infty} \int aG(u_m) \geqslant \Lambda. \tag{4.8}$$

On the other hand, by $J_{\lambda_n}(u_m) \to c$ and (4.3) we have

$$\begin{aligned} c &= \lim_{m\to\infty} J_{\lambda_n}(u_m) = \lim_{m\to\infty} \Big\{ \int |\nabla u_m|^2 - \lambda_n \int h u_m^2 - \int aG(u_m) \Big\} \\ &= \lim_{m\to\infty} \Big\{ \int |\nabla u_m^+| - \lambda_n \int h(u_m^+)^2 + \int |\nabla u_m^-|^2 - \lambda_n \int h|u_m^-|^2 - \int aG(u_m) \Big\} \\ &= -\lim_{m\to\infty} \int aG(u_m). \end{aligned}$$

By hypothesis, $c > -\Lambda$, thus $\lim_{m\to\infty} \int aG(u_m) < \Lambda$, which contradicts (4.8). Therefore, (u_m) must be bounded if $c \in (-\Lambda, \infty)$.

To show the existence of a weak solution we use the Linking Theorem, Theorem 4.1 (see [15]). Define $(E_n)^\perp := X_2$, then $\lambda_{n+1} \leqslant \frac{\int |\nabla u|^2}{\int hu^2}$ for all $u \in X_2$. So, given any $u \in X_2$, it follows from (1.6) that

$$\begin{aligned} J_{\lambda_n}(u) &= \frac{1}{2} \int |\nabla u|^2 - \frac{1}{2}\lambda_n \int hu^2 - \int aG(u) \\ &\geqslant \frac{1}{2}\big(1 - \frac{\lambda_n}{\lambda_{n+1}}\big)\|u\|^2 - \int \frac{a}{|a|_{L^1}}\Lambda \\ &\geqslant -\Lambda; \end{aligned}$$

i.e., $J_{\lambda_n}(u) \geqslant \gamma \in \mathbb{R}$ for all $u \in X_2$ with $\gamma := -\Lambda$. So condition (I_1) in Theorem 4.1 holds.

On the other hand, from

$$J_{\lambda_n}(u) \geqslant \frac{1}{2}\big(1 - \frac{\lambda_n}{\lambda_{n+1}}\big)\|u\|^2 + \gamma,$$

it follows that $J_{\lambda_n}(u) \to \infty$ as $\|u\| \to \infty$ (since $\lambda_{n+1} > \lambda_n$), and therefore (I_2) in Theorem 4.1 also holds.

Now, define $X_1 := E_n$. If $n > 1$, we may write $u = u_1 + u_0$ where $u_1 \in E_{n-1}$ and $u_0 \in L_n$. For $u \in E_{n-1}$ we know that $\lambda_{n-1} \geqslant \frac{\int |\nabla u|^2}{\int hu^2}$; thus, for $u \in X_1$,

$$\begin{aligned} J_{\lambda_n}(u) &= \frac{1}{2} \int |\nabla(u_0 + u_1)|^2 - \frac{\lambda_n}{2} \int h(u_0 + u_1)^2 - \int aG(u) \\ &= \frac{1}{2} \int |\nabla u_1|^2 - \frac{\lambda_n}{2} \int hu_1^2 - \int aG(u) \\ &\leqslant -\frac{1}{2}\big(\frac{\lambda_n}{\lambda_{n-1}} - 1\big)\|u_1\|^2 - \int (aG(u) - aG(u_0)) - \int aG(u_0). \end{aligned}$$

From $G(u) - G(u_0) = \int_0^1 g(u_0 + tu_1)u_1 dt$ we get that $|G(u) - G(u_0)| \leqslant M|u_1|$, so that the above inequality becomes

$$J_{\lambda_n}(u) \leqslant -\frac{1}{2}\big(\frac{\lambda_n}{\lambda_{n-1}} - 1\big)\|u_1\|^2 + C\|u_1\| - \int aG(u_0). \tag{4.9}$$

It then follows from (4.9) and the condition (1.5) that there exists a real constant β such that $J_{\lambda_n}(u) \leqslant \beta$ for all $u \in X_1$ which is condition (I_0) in Theorem 4.1.

For $n = 1$ we have

$$J_{\lambda_n}(u) = -\int aG(u) \quad \text{for } u \in X_1,$$

which, by (1.5), yields $\beta \in \mathbb{R}$ such that $J_{\lambda_n}(u) \leqslant \beta$ for all $u \in X_1$ i.e., (I_0) in Theorem 4.1 holds. Therefore, $J_{\lambda_n}(u)$ has a critical value $b \geqslant \gamma$ and the theorem is established. □

As a consequence of Theorem 1.3, we have the following statement.

Corollary 4.2. *Let $g : \mathbf{R} \to \mathbf{R}$ be a continuous function satisfying $\lim_{|s|\to\infty} g(s) = 0$. Suppose that $a > 0$ a.e.,*

$$G(s) \to \xi \in \mathbb{R} \quad \text{as } |s| \to \infty, \quad \text{and} \quad G(s) \leq \xi \quad \text{for every } s \in \mathbb{R}.$$

Then problem (4.1) *has a weak solution.*

This corollary follows from Theorem 1.3, the unique continuation property of the eigenfunctions, and the Lebesgue dominated convergence theorem.

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