IMPULSIVE NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH VARIABLE TIMES

MOUFFAK BENCHOHRA & ABDELGHANI OUAHAB

ABSTRACT. In this paper, we study the existence of solutions for first and second order impulsive neutral functional differential inclusions with variable times. Our main tool is a fixed point theorem due to Martelli for condensing multivalued maps.

1. INTRODUCTION

This paper concerns the existence of solutions for initial-value problems for first and second order neutral functional differential inclusions with impulsive effects at variable times. In Section 3, we consider the first order initial-value problem (IVP for short)

$$\frac{d}{dt}[y(t) - g(t, y_t)] \in F(t, y_t), \quad \text{a. e. } t \in J = [0, T], \ t \neq \tau_k(y(t)), \ k = 1, \dots, m,$$
(1.1)

$$y(t^+) = I_k(y(t)), \quad t = \tau_k(y(t)), \ k = 1, \dots, m,$$
(1.2)

$$y(t) = \phi(t), \quad t \in [-r, 0],$$
(1.3)

where $F : J \times D \to 2^{\mathbb{R}^n}$ is a compact convex valued multivalued map, $g : J \times D \to \mathbb{R}^n$ is given function, $D = \{\psi : [-r, 0] \to \mathbb{R}^n; \psi$ is continuous everywhere except for a finite number of points $\bar{t}$ at which $\psi(\bar{t})$ and $\psi(\bar{t}^+)$ exist and $\psi(\bar{t}^-) = \psi(\bar{t})\}$, $\phi \in D$, $0 < r < \infty$, $\tau_k : \mathbb{R}^n \to \mathbb{R}$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \dots, m$ are given functions satisfying some assumptions that will be specified later.

For any function y defined on $[-r, T]$ and any $t \in J$ we denote by y_t the element of D defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t. In Section 4, we consider the second order IVP

$$\frac{d}{dt}[y'(t) - g(t, y_t)] \in F(t, y_t), \quad \text{a. e. } t \in J = [0, T], \ t \neq \tau_k(y(t)), \ k = 1, \dots, m,$$
(1.4)

$$y(t^+) = I_k(y(t)), \quad t = \tau_k(y(t)), \ k = 1, \dots, m,$$
(1.5)

$$y'(t^+) = \overline{I}_k(y(t)), \quad t = \tau_k(y(t)), \; k = 1, \ldots, m, \tag{1.6}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \; y'(0) = \eta, \tag{1.7}$$

where g, F, I_k, and ϕ are as in the problem (1.1)–(1.3), $\overline{I}_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $\eta \in \mathbb{R}^n$.

The theory of impulsive differential equations have become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory in recent years, especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs of Bainov and Simeonov [3], Lakshmikantham *et al* [18], and Samoilenko and Perestyuk [24], the papers of Benchohra *et al* [4]-[7] and the references therein. The theory of impulsive differential equations with variable time is relatively less developed due to the difficulties created by the state-dependent impulses. Recently, some interesting extensions to impulsive differential equations with variable times have been done by Bajo and Liz [2], Frigon and O'Regan [10, 11, 12], Kaul *et al* [15], Kaul and Liu [16], [17], Lakshmikantham *et al* [19], [20], Liu and Ballinger [22] and the references cited therein.

The main theorems of this paper extend the problem (1.1)-(1.3) considered by Benchohra *et al* [4, 5, 6] when the impulse times are constant. Our approach is based on the Martelli fixed point theorem [23].

2. PRELIMINARIES

In this section, we introduce notation, definitions, and preliminary facts from multivalued analysis which are used throughout this paper. Let (a, b) be an open interval. $AC^i((a, b), \mathbb{R}^n)$ is the space of i-times differentiable functions $y : (a, b) \to \mathbb{R}^n$, whose i^{th} derivative, $y^{(i)}$, is absolutely continuous.

Let $(X, \| \cdot \|)$ be a Banach space. A multi-valued map $G : X \to 2^X$ has convex (closed) values if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B)$ is bounded in X for each bounded set B of X, i.e. $\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$. G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multi-valued G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). An upper semicontinuous map $G : X \to 2^X$ is said to be condensing if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompacteness. For properties of the Kuratowski measure, we refer to the book of Banas and Goebel [1]. We remark that a completely semicontinuous multivalued map is the easiest example of a condensing map. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. In the following, $CC(\mathbb{R}^n)$ denotes the set of all nonempty compact, convex subsets of $\mathbb{R}^n$.

Definition 2.1. A multi-valued map $F : J \times D \to 2^{\mathbb{R}^n}$ is said to be L^1-Carathéodory if

   (i) $t \mapsto F(t, u)$ is measurable for each $y \in D$;
   (ii) $u \mapsto F(t, u)$ is upper semi-continuous for almost all $t \in J$;

(iii) For each $q > 0$, there exists $\phi_q \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, u)\| = \sup\{|v| : v \in F(t, y)\} \leq \phi_q(t)$$

for all $\|u\|_D \leq q$ and for almost all $t \in J$.

For more details on multi-valued maps we refer the reader to the books of Deimling [8], Gorniewicz [13], Hu and Papageorgiou [14] and Tolstonogov [25].

For a function y defined on $[-r, T]$ we define the set

$$S_{F,y} = \{v \in L^1(J, \mathbb{R}^n) : \ v(t) \in F(t, y_t) \text{ for a.e. } t \in J\},$$

which is known as the set of *selection functions*.

The following lemmas are crucial in the proof of our main theorem.

Lemma 2.2 ([21]). *Let I be a compact real interval and X be a Banach space. Let F be a multi-valued map satisfying the Carathéodory conditions with the set of L^1-selections S_F nonempty, and let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$. Then the operator*

$$\Gamma \circ S_F : C(I, X) \to CC(C(I, X)), \quad y \mapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I, X) \times C(I, X)$.

Lemma 2.3 ([23]). *Let $N : X \to CC(X)$ be an upper semicontinuous and condensing map. If the set*

$$\mathcal{M} := \{y \in X : y \in \lambda N(y) \text{ for some } 0 < \lambda < 1\}$$

is bounded, then N has a fixed point.

To define the solutions of problems (1.1)-(1.3) and (1.4)-(1.7), we shall consider the space

$$PC = \big\{y : [0, T] \to \mathbb{R}^n : \text{there exist } 0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$$

$$\text{such that } t_k = \tau_k(y(t_k)), \ y(t_k^-) \text{ and } y(t_k^+) \text{ exist with } y(t_k^-) = y(t_k),$$

$$k = 1, \ldots, m, \text{ and } y \in C([t_k, t_{k+1}], \mathbb{R}^n) \ k = 0, \ldots, m\big\}.$$

Set $\Omega := \{y : [-r, T] \to \mathbb{R}^n : y \in D \cap PC\}$. In what follows, we will assume that F is an L^1-Carathéodory function.

3. First Order Impulsive NFDIs

Let us start by defining what we mean by a solution of problem (1.1)–(1.3).

Definition 3.1. A function $y \in \Omega \cap \cup_{k=0}^m AC((t_k, t_{k+1}), \mathbb{R}^n)$ is said to be a solution of (1.1)–(1.3) if there exists $v(t) \in F(t, y_t)$ a.e. $t \in [0, T]$ such that $\frac{d}{dt}[y(t) - g(t, y_t)] = v(t)$ a.e. on J, $t \neq \tau_k(y(t))$, $k = 1, \ldots, m$, $y(t^+) = I_k(y(t))$, $t = \tau_k(y(t))$, $k = 1, \ldots, m$, $y(t) = \phi(t)$, $t \in [-r, 0]$.

We are now in a position to state and prove our existence result for the problem (1.1)-(1.3). For the study of this problem we first list the following hypotheses:

(H1) The functions $\tau_k \in C^1(\mathbb{R}^n, \mathbb{R})$ for $k = 1, \ldots, m$. Moreover,

$$0 < \tau_1(x) < \ldots < \tau_m(x) < \tau_{m+1}(x) = T \quad \text{for all } x \in \mathbb{R}^n.$$

(H2) There exist constants c_k, such that $|I_k(x)| \leq c_k$, $k = 1, \ldots, m$ for each $x \in \mathbb{R}^n$.

(H3) The function g is completely continuous and there exist constants $0 \leq d_1 < 1$, $d_2 \geq 0$ such that

$$|g(t, u)| \leq d_1 \|u\| + d_2, \quad t \in [0, T], \quad u \in D.$$

(H4) There exist a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$, and $p \in L^1([0, T], \mathbb{R}_+)$ such that

$$\|F(t, u)\| \leq p(t)\psi(\|u\|) \quad \text{a.e. } t \in [0, b], \text{ and each } u \in D$$

with $\int_1^\infty \frac{d\gamma}{\psi(\gamma)} = \infty$

(H5) For all $(t, x) \in [0, T] \times \mathbb{R}^n$ and for all $y_t \in D$ we have $\langle \tau_k'(x), v(t) \rangle \neq 1$ for $k = 1, \dots, m$, for all $v \in S_{F,y}$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$.

(H6) g is a nonnegative function.

(H7) τ_k is a non-increasing function and $I_k(x) \leq x$ for all $x \in R^n, k = 1, \dots, m$.

(H8) For all $x \in \mathbb{R}^n$, $\tau_k(x) < \tau_{k+1}(I_k(x))$ for $k = 1, \dots, m$.

Theorem 3.2. *Assume that hypotheses (H1)-(H8) hold. Then the initial-value problem (1.1)–(1.3) has at least one solution on $[-r, T]$.*

The proof of this theorem will be given in several steps.

Step 1: Consider the problem

$$\frac{d}{dt}[y(t) - g(t, y_t)] \in F(t, y_t), \quad \text{a.e. } t \in [0, T], \tag{3.1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0]. \tag{3.2}$$

Transform the problem into a fixed point problem. Consider the operator $\mathcal{N} : C([-r, T], \mathbb{R}^n) \to 2^{C([-r, T], \mathbb{R}^n)}$ defined as $\mathcal{N}(y) = \{h \in C([-r, T], \mathbb{R}^n)\}$ where for $v \in S_{F,y}$,

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0) \\ \phi(0) - g(0, \phi(0)) + g(t, y_t) + \int_0^t v(s)ds, & \text{if } t \in [0, T]. \end{cases}$$

Remark 3.3. We can easily show that the fixed points of N are solutions to (3.1)–(3.2).

We shall show that the operator N is completely continuous. Using (H3) it suffices to show that the operator $N_1 : C([-r, T], \mathbb{R}^n) \to C([-r, T], \mathbb{R}^n)$ defined as $N_1(y) = \{h \in C([-r, T], \mathbb{R}^n)\}$, where

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0); \\ \phi(0) + \int_0^t v(s)ds, & \text{if } t \in [0, T], \end{cases}$$

is completely continuous. The proof will be given in several Claims.

Claim 1: $N_1(y)$ *is convex for each* $y \in C([-r, T], \mathbb{R}^n)$. Indeed, if v_1, v_2 belong to $N_1(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$, we have

$$h_i(t) = \phi(0) + \int_0^t v_i(s)ds, \ i = 1, 2.$$

Let $0 \leq d \leq 1$. Then for each $t \in J$ we have

$$(dh_1 + (1 - d)h_2(t) = \phi(0) + \int_0^t [dv_1(s) + (1 - d)v_2(s)]ds$$

Since $S_{F,y}$ is convex (because F has convex values) then

$$dh_1 + (1 - d)h_2 \in N_1(y).$$

Claim 2: N_1 *maps bounded sets into bounded sets in* $C([-r, T], \mathbb{R}^n)$. Indeed, it is sufficient to show that for any $q > 0$ there exists a positive constant ℓ such that for each $y \in B_q = \{y \in C([-r, T], \mathbb{R}^n) : \|y\|_\infty \leq q\}$ we have $\|N_1(y)\| \leq \ell$. Let $y \in B_q$ and $h \in N_1(y)$ then there exists $v \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = \phi(0) + \int_0^t v(s)ds.$$

Thus,

$$|h(t)| \leq |\phi(0)| + \int_0^t |v(s)|ds \leq \|\phi\|_\infty + \|h_q\|_{L^1} := \ell.$$

Claim 3: N_1 *maps bounded sets into equicontinuous sets of* $C([-r, T], \mathbb{R}^n)$. Let $u_1, u_2 \in J$, $u_1 < u_2$ and B_q be a bounded set of $C(J, \mathbb{R}^N)$ as in Claim 2. Let $y \in B_q$ and $h \in N_1(y)$. Then there exists $v \in S_{F,y}$ such that for each $t \in J$, we have

$$h(t) = \phi(0) + \int_0^t v(s)ds$$

Then

$$|N_1(y(u_2)) - N_1(y(u_1))| \leq \int_{u_1}^{u_2} h_q(s)ds.$$

As $u_2 \to u_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Claims 2 and 3 and the Arzela-Ascoli theorem we can conclude that $\mathcal{N} : C(J, \mathbb{R}^N) \to 2^{C(J,\mathbb{R}^N)}$ is a completely continuous multi-valued operator, and therefore, a condensing map.

Claim 4: N_1 *has a closed graph*. Let $y_n \to y_*$, $h_n \in N_1(y_n)$, and $h_n \to h_*$. We shall prove that $h_* \in N(y_*)$. $h_n \in N_1(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that for each $t \in J$,

$$h_n(t) = \phi(0) + \int_0^t v_n(s)ds.$$

We must prove that there exists $h_* \in S_{F,y_*}$ such that for each $t \in J$,

$$h_*(t) = \phi(0) + \int_0^t v_*(s)ds.$$

Clearly,

$$\left\|\left(h_n - \phi(0)\right) - \left(h_* - \phi(0)\right)\right\|_\infty \to 0, \quad \text{as } n \to \infty.$$

Consider the linear continuous operator $\Gamma : L^1(J, \mathbb{R}^n) \to C(J, \mathbb{R}^n)$,

$$v \mapsto (\Gamma v)(t) = \int_0^t v(s)ds.$$

¿From Lemma 2.2, it follows that $\Gamma \circ S_F$ is a closed graph operator. Since $\left(h_n(t) - \phi(0)\right) \in \Gamma(S_{F,y_n})$, it follows from Lemma 2.2 that for some $v_* \in S_{F,v_*}$,

$$h_*(t) = \phi(0) + \int_0^t v_*(s)ds.$$

Claim 5: *The following set is bounded,*

$$\mathcal{E}(\mathcal{N}) := \{y \in C([-r, T], \mathbb{R}^n) : y \in \lambda\mathcal{N}(y), \text{ for some } 0 < \lambda < 1\}.$$

Let $y \in \mathcal{E}(\mathcal{N})$. Then there exists $v \in S_{F,y}$ such that $y \in \lambda \mathcal{N}(y)$, for some $0 < \lambda < 1$. Thus, for each $t \in [0, T]$,

$$y(t) = \lambda \Big(\phi(0) - g(0, \phi) + g(t, y_t) + \int_0^t v(s)\, ds \Big).$$

This implies, by (H2)–(H4), that for each $t \in J$ we have

$$|y(t)| \leq \|\phi\| + d_1 \|\phi\| + d_1 \|y_t\| + 2 d_2 + \int_0^t p(s)\psi(\|y_s\|) ds.$$

We consider the function

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$

$$\mu(t) \leq \|\phi\| + d_1 \|\phi\| + d_1 \mu(t) + 2 d_2 + \int_0^t p(s)\psi(\mu(s)) ds.$$

Thus

$$\mu(t) \leq \frac{1}{1 - d_1} \Big[\|\phi\| + d_1 \|\phi\| + 2 d_2 + \int_0^t p(s)\psi(\mu(s)) ds \Big], \quad t \in J.$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|$ and the previous inequality holds.

Let us take the right-hand side of the above inequality to be $v(t)$. Then

$$c = v(0) = \frac{1}{1 - d_1} (\|\phi\| + d_1 \|\phi\| + 2 d_2),$$

$$\mu(t) \leq v(t), \quad t \in J,$$

$$v'(t) = \frac{1}{1 - d_1} p(t)\psi(\mu(t)), \quad t \in J.$$

Using the nondecreasing character of ψ, we obtain

$$v'(t) \leq \frac{1}{1 - d_1} p(t)\psi(v(t)).$$

This implies that for each $t \in J$,

$$\int_{v(0)}^{v(t)} \frac{d\gamma}{\psi(\gamma)} \leq \frac{1}{1 - d_1} \int_0^T p(s) ds < \int_{v(0)}^{\infty} \frac{d\gamma}{\psi(\gamma)}.$$

This inequality implies that there exists a constant K such that $v(t) \leq K$, $t \in J$, and hence $\mu(t) \leq K$, $t \in J$. Since for every $t \in [0, T]$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_\infty \leq K' = \max\{\|\phi\|, K\},$$

where K' depends only T, d_1, d_2, and on the functions p, ϕ and ψ. This shows that $\mathcal{E}(\mathcal{N})$ is bounded.

Set $X := C([-r, T], \mathbb{R}^n)$. As a consequence of Lemma 2.3 we deduce that N has a fixed point which is a solution of (3.1)–(3.2). Denote this solution by y_1. Define the function

$$r_{k,1}(t) = \tau_k(y_1(t)) - t \quad \text{for } t \geq 0.$$

Hypothesis (H1) implies that $r_{k,1}(0) \neq 0$ for $k = 1, \dots, m$. If $r_{k,1}(t) \neq 0$ on $[0, T]$ for $k = 1, \dots, m$; i.e.,

$$t \neq \tau_k(y_1(t)) \quad \text{on } [0, T] \quad \text{for } k = 1, \dots, m,$$

then y_1 is a solution of the problem (1.1)-(1.3).

It remains to consider the case when $r_{1,1}(t) = 0$ for some $t \in [0, T]$. Now since $r_{1,1}(0) \neq 0$ and $r_{1,1}$ is continuous, there exists $t_1 > 0$ such that

$$r_{1,1}(t_1) = 0, \quad \text{and} \quad r_{1,1}(t) \neq 0 \quad \text{for all } t \in [0, t_1).$$

Thus by (H1) we have $r_{k,1}(t) \neq 0$ for all $t \in [0, t_1)$, and $k = 1, \ldots, m$.

Step 2: Consider now the problem

$$\frac{d}{dt}[y(t) - g(t, y_t)] \in F(t, y_t), \quad \text{a.e. } t \in [t_1, T], \tag{3.3}$$

$$y(t_1^+) = I_1(y_1(t_1)). \tag{3.4}$$

Transform this problem into a fixed point problem. Consider the operator $N_2 : C([t_1, T], \mathbb{R}^n) \to 2^{C([t_1,T],\mathbb{R}^n)}$ defined as

$$N_2(y) = \Big\{ h \in C([t_1, T], \mathbb{R}^n) : h(t) = I_1(y_1(t_1)) - g(t_1, y_{t_1}) + g(t, y_t) + \int_{t_1}^t v(s)ds \Big\},$$

where $v \in S_{F,y}$. As in Step 1 we can show that N_2 is completely continuous, and that the following set is bounded,

$$\mathcal{E}(N_2) := \{ y \in C([t_1, T], \mathbb{R}^n) : y \in \lambda N_2(y), \quad \text{for some } 0 < \lambda < 1 \}.$$

Set $X := C([t_1, T], \mathbb{R}^n)$. As a consequence of Martelli's theorem, we deduce that N_2 has a fixed point y which is a solution to problem (3.3)–(3.4). Denote this solution by y_2. Define

$$r_{k,2}(t) = \tau_k(y_2(t)) - t \quad \text{for } t \geq t_1.$$

If $r_{k,2}(t) \neq 0$ on $(t_1, T]$ for all $k = 1, \ldots, m$, then

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [0, t_1], \\ y_2(t), & \text{if } t \in (t_1, T], \end{cases}$$

is a solution of the problem (1.1)–(1.3). It remains to consider the case when $r_{2,2}(t) = 0$, for some $t \in (t_1, T]$. By (H8), we have

$$\begin{aligned} r_{2,2}(t_1^+) &= \tau_2(y_2(t_1^+)) - t_1 \\ &= \tau_2(I_1(y_1(t_1))) - t_1 \\ &> \tau_1(y_1(t_1)) - t_1 \\ &= r_{1,1}(t_1) = 0. \end{aligned}$$

Since $r_{2,2}$ is continuous, there exists $t_2 > t_1$ such that $r_{2,2}(t_2) = 0$ and $r_{2,2}(t) \neq 0$ for all $t \in (t_1, t_2)$. It is clear by (H1) that

$$r_{k,2}(t) \neq 0 \quad \text{for all } t \in (t_1, t_2), \ k = 2, \ldots, m.$$

Suppose now that there is $\bar{s} \in (t_1, t_2]$ such that $r_{1,2}(\bar{s}) = 0$. Consider the function $L_1(t) = \tau_1(y_2(t) - g(t, y_t)) - t$. ¿From (H6)-(H8) it follows that

$$\begin{aligned} L_1(\bar{s}) &= \tau_1(y_2(\bar{s}) - g(\bar{s}, y_{\bar{s}})) - \bar{s} \\ &\geq \tau_1(y_2(\bar{s})) - \bar{s} \\ &= r_{1,2}(\bar{s}) = 0. \end{aligned}$$

Thus the function L_1 attains a nonnegative maximum at some point $s_1 \in (t_1, T]$. Since

$$\frac{d}{dt}[y_2(t) - g(t, y_{2t})] \in F(t, y_{2t}), \quad \text{a.e. } t \in (t_1, T),$$

then there exists $v(\cdot) \in L^1((t_1, T))$ with $v(t) \in F(t, y_{2t})$, a.e. $t \in (t_1, T)$ such that

$$\frac{d}{dt}[y_2(t) - g(t, y_{2t})] = v(t),$$

then

$$L_1'(s_1) = \tau_1'(y_2(s_1) - g(s_1, y_{2s_1}))\frac{d}{dt}[y_2(s_1) - g(s_1, y_{s_1})] - 1 = 0.$$

Therefore,

$$\langle \tau_1'(y_2(s_1) - g(s_1, y_{2s_1})), v(s_1) \rangle = 1,$$

which is a contradiction by (H4).

Step 3: We continue this process and take into account that $y_m := y\big|_{[t_m, T]}$ is a solution to the problem

$$\frac{d}{dt}[y(t) - g(t, y_t)] \in F(t, y_t), \quad \text{a.e. } t \in (t_m, T), \tag{3.5}$$

$$y(t_m^+) = I_m(y_{m-1}(t_m)). \tag{3.6}$$

The solution y of the problem (1.1)-(1.3) is then defined by

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [-r, t_1], \\ y_2(t), & \text{if } t \in (t_1, t_2], \\ \dots \\ y_m(t), & \text{if } t \in (t_m, T]. \end{cases}$$

4. Second Order Impulsive NFDIs

In this section, we study the initial-value problem (1.4)–(1.7).

Definition 4.1. A function $y \in \Omega \cap \cup_{k=0}^m AC((t_k, t_{k+1}), \mathbb{R}^n)$ is said to be a solution of (1.4)–(1.7) if there exists $v(t) \in F(t, y_t)$ a.e. $t \in [0, T]$ such that $\frac{d}{dt}[y'(t) - g(t, y_t)] = v(t)$ a.e. on J, $t \neq \tau_k(y(t))$, $k = 1, \dots, m$, $y(t^+) = I_k(y(t))$, $t = \tau_k(y(t))$, $k = 1, \dots, m$, $y'(t^+) = \overline{I}_k(y(t))$, $t = \tau_k(y(t))$ $k = 1, \dots, m$, $y(t) = \phi(t)$, $t \in [-r, 0]$ and $y'(0) = \eta$.

For the next theorem we need the following assumptions:
- (A1) There exist positive constants $\overline{d}_k$ such that $|\overline{I}_k(x)| \leq \overline{d}_k$, $k = 1, \dots, m$ for each $x \in \mathbb{R}^n$
- (A2) There exists a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and $p \in L^1([0, T], \mathbb{R}_+)$ such that $\|F(t, u)\| \leq p(t)\psi(\|u\|)$ a.e. $t \in [0, T]$ and each $u \in D$ with

$$\int_1^\infty \frac{d\gamma}{\gamma + \psi(\gamma)} = \infty,$$

- (A3) For all $(t, \bar{s}, x) \in [0, T] \times [0, T] \times \mathbb{R}^n$ and all $y_t \in D$ we have, for all $v \in S_{F,y}$,

$$\langle \tau_k'(x), \overline{I}_k(y(\bar{s})) - g(\bar{s}, y_{\bar{s}}) + g(t, y_t) + \int_{\bar{s}}^t v(s)ds \rangle \neq 1 \quad \text{for } k = 1, \dots, m.$$

- (A4) For all $x \in \mathbb{R}^n$, $\tau_k(I_k(x)) \leq \tau_k(x) < \tau_{k+1}(I_k(x))$ for $k = 1, \dots, m$.

Theorem 4.2. *Assume that (H1)-(H3) and (A1)-(A4) are satisfied. Then the IVP (1.4)–(1.7) has at least one solution.*

The proof of this theorem will be given in several steps.
Step 1: Consider the problem

$$\frac{d}{dt}[y'(t) - g(t, y_t)] \in F(t, y_t), \quad \text{a.e. } t \in [0, T], \tag{4.1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \ y'(0) = \eta. \tag{4.2}$$

Transform the problem into a fixed point problem. Consider the operator $\overline{N}_1 :$ $C([-r, T], \mathbb{R}^n) \to 2^{C([-r,T],\mathbb{R}^n)}$ defined as $\overline{N}_1(y) = \{h \in C([-r, T], \mathbb{R}^n)\}$ where

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0]; \\ \phi(0) + [\eta - g(0, \phi(0))]t + \int_0^t g(s, y_s)ds + \int_0^t (t - s)v(s)ds & \text{if } t \in [0, T]. \end{cases}$$

As in Theorem 3.2, we can show that $\overline{N}_1$ is completely continuous. Now we prove only that the following set is bounded,

$$\mathcal{E}(\overline{N}_1) := \{y \in C([-r, T], \mathbb{R}^n) : y \in \lambda \overline{N}_1(y), \text{ for some } 0 < \lambda < 1\}.$$

Let $y \in \mathcal{E}(\overline{N}_1)$. Then there exists $v \in S_{F,y}$ such that $y \in \lambda \overline{N}_1(y)$ for some $0 < \lambda < 1$. Thus for each $t \in [0, T]$ we have

$$y(t) = \lambda \phi(0) + \lambda [\eta - g(0, \phi(0))]t + \lambda \int_0^t g(s, y_s)ds + \lambda \int_0^t (t - s)v(s)ds.$$

This implies, by (H2), (H3), (A1), and (A2), that for each $t \in [0, T]$ we have

$$|y(t)| \le \|\phi\| + T(|\eta| + \|\phi\|d_1 + d_2) + \int_0^t d_1\|y_s\|ds + Td_2 + \int_0^t (T - s)p(s)\psi(\|y_s\|)ds$$

$$\le \|\phi\| + T(|\eta| + \|\phi\|d_1 + 2d_2) + \int_0^t M(s)\|y_s\|ds + \int_0^t M(s)\psi(\|y_s\|)ds,$$

where $M(t) = \max\{d_1, (T - t)p(t)\}$. Consider the function

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le T.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality, for $t \in [0, T]$, we have

$$\mu(t) \le \|\phi\| + T(|\eta| + \|\phi\|d_1 + 2d_2) + \int_0^t M(s)\mu(s)ds + \int_0^t M(s)\psi(\mu(s))ds.$$

Let us denote the right-hand side of the above inequality to be $v(t)$. Then

$$v(0) = \|\phi\| + T(|\eta| + \|\phi\|d_1 + 2d_2),$$

$$v'(t) = M(t)\mu(t) + M(t)\psi(\mu(t)), \ t \in [0, T].$$

Using the nondecreasing character of ψ we obtain, for a.e. $t \in [0, T]$,

$$v'(t) \le M(t)v(t) + M(t)\psi(v(t)) = M(t)[v(t) + \psi(v(t))].$$

This implies that, for each $t \in [0, T]$,

$$\int_{v(0)}^{v(t)} \frac{d\gamma}{\gamma + \psi(\gamma)} \le \int_0^T M(s)ds < \int_{v(0)}^{\infty} \frac{d\gamma}{\gamma + \psi(\gamma)}.$$

This inequality implies that there exists a constant b^* such that $v(t) \le b^*, \ t \in [0, T]$, and hence $\mu(t) \le b^*, \ t \in [0, T]$. Since for every $t \in [0, T], \|y_t\| \le \mu(t)$, we have

$$\|y\|_\infty \le \max\{\|\phi\|, b^*\},$$

where b^* depends only T and on the functions p and ψ. This shows that $\mathcal{E}(\overline{N}_1)$ is bounded.

Set $X := C([-r, T], \mathbb{R}^n)$. As a consequence of Martelli's theorem, we deduce that $\overline{N}_1$ has a fixed point y which is a solution to (4.1)–(4.2). Denote this solution by y_1. Define the function

$$r_{k,1}(t) = \tau_k(y_1(t)) - t \quad \text{for} \quad t \geq 0.$$

Hypothesis (H1) implies that $r_{k,1}(0) \neq 0$ for $k = 1, \ldots, m$. If $r_{k,1}(t) \neq 0$ on $[0, T]$ for $k = 1, \ldots, m$; i.e.,

$$t \neq \tau_k(y_1(t)) \quad \text{on } [0, T] \quad \text{and for } k = 1, \ldots, m.$$

Then y_1 is a solution of the problem (1.1)-(1.3).

It remains to consider the case when $r_{1,1}(t) = 0$ for some $t \in [0, T]$. Now since $r_{1,1}(0) \neq 0$ and $r_{1,1}$ is continuous, there exists $t_1 > 0$ such that

$$r_{1,1}(t_1) = 0, \quad \text{and} \quad r_{1,1}(t) \neq 0 \quad \text{for all } t \in [0, t_1).$$

Thus, by (H1) we have $r_{k,1}(t) \neq 0$ for all $t \in [0, t_1)$ and $k = 1, \ldots, m$.

Step 2: Consider now the problem

$$\frac{d}{dt}[y'(t) - g(t, y_t)] \in F(t, y_t), \quad \text{a.e. } t \in [t_1, T], \tag{4.3}$$

$$y(t_1^+) = I_1(y_1(t_1)), \tag{4.4}$$

$$y'(t_1^+) = \overline{I}_1(y_1(t_1)). \tag{4.5}$$

Transform the problem into a fixed point problem. Consider the operator $\overline{N}_2 : C([t_1, T], \mathbb{R}^n) \to 2^{C([t_1, T], \mathbb{R}^n)}$ defined as $\overline{N}_2(y) = \{h \in C([t_1, T], \mathbb{R}^n)\}$ where

$$h(t) = I_1(y_1(t_1)) + (t - t_1)\overline{I}_1(y_1(t_1)) - (t - t_1)g(t_1, y_{t_1}) + \int_{t_1}^t g(s, y_s)ds + \int_{t_1}^t (t - s)v(s)ds,$$

with $v \in S_{F,y}$. As in Step 1 we can show that $\overline{N}_2$ is completely continuous, and that the following set is bounded,

$$\mathcal{E}(\overline{N}_2) := \{y \in C([t_1, T], \mathbb{R}^n) : y \in \lambda \overline{N}_2(y), \text{ for some } 0 < \lambda < 1\}.$$

Set $X := C([t_1, T], \mathbb{R}^n)$. As a consequence of Martelli's theorem, we deduce that N_2 has a fixed point y which is a solution to problem (4.3)–(4.5). Denote this solution by y_2. Define

$$r_{k,2}(t) = \tau_k(y_2(t)) - t \quad \text{for } t \geq t_1.$$

If $r_{k,2}(t) \neq 0$ on $(t_1, T]$ for all $k = 1, \ldots, m$, then

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [0, t_1], \\ y_2(t), & \text{if } t \in (t_1, T], \end{cases}$$

is a solution of the problem (1.4)–(1.7). It remains to consider the case when $r_{2,2}(t) = 0$, for some $t \in (t_1, T]$. By (A4) we have

$$\begin{aligned} r_{2,2}(t_1^+) &= \tau_2(y_2(t_1^+)) - t_1 \\ &= \tau_2(I_1(y_1(t_1)) - t_1 \\ &> \tau_1(y_1(t_1)) - t_1 \\ &= r_{1,1}(t_1) = 0. \end{aligned}$$

Since $r_{2,2}$ is continuous, there exists $t_2 > t_1$ such that $r_{2,2}(t_2) = 0$, and $r_{2,2}(t) \neq 0$ for all $t \in (t_1, t_2)$. It is clear by (H1) that

$$r_{k,2}(t) \neq 0 \quad \text{for all } t \in (t_1, t_2), \quad k = 2, \ldots, m.$$

Suppose now that there is $\bar{s} \in (t_1, t_2]$ such that $r_{1,2}(\bar{s}) = 0$. From (A4), it follows that

$$\begin{aligned}
r_{1,2}(t_1^+) &= \tau_1(y_2(t_1^+)) - t_1 \\
&= \tau_1(I_1(y_1(t_1))) - t_1 \\
&\leq \tau_1(y_1(t_1)) - t_1 \\
&= r_{1,1}(t_1) = 0.
\end{aligned}$$

Thus, the function $r_{1,2}$ attains a nonnegative maximum at some point $s_1 \in (t_1, T]$. Since

$$\frac{d}{dt}[y_2'(t) - g(t, y_{2t})] \in F(t, y_{2t}), \quad \text{a.e. } t \in (t_1, T),$$

there exist $v(\cdot) \in L^1((t_1, T))$ with $v(t) \in F(t, y_{2t})$, a.e. $t \in (t_1, T)$ such that

$$y_2'(t) = \overline{I}_1(y(t_1)) - g(t_1, y_{t_1}) + g(t, y_t) + \int_{t_1}^{t} v(s)ds \,.$$

Then

$$r_{1,2}'(s_1) = \tau_1'(y_2(s_1))\left(\overline{I}_1(y(t_1)) - g(t_1, y_{t_1}) + g(s_1, y_{s_1}) + \int_{t_1}^{s_1} v(s)ds\right) - 1 = 0.$$

Therefore,

$$\langle \tau_1'(y_2(s_1)), \overline{I}_1(y(t_1)) - g(t_1, y_{t_1}) + g(s_1, y_{s_1}) + \int_{t_1}^{s_1} v(s)ds \rangle = 1,$$

which is a contradiction by (A3).

Step 3: We continue this process by taking into account that $y_m := y\big|_{[t_m, T]}$ is a solution to the problem

$$\frac{d}{dt}[y'(t) - g(t, y_t)] \in F(t, y_t), \ a.e. \ t \in (t_m, T), \tag{4.6}$$

$$y(t_m^+) = I_m(y_{m-1}(t_m)), \tag{4.7}$$

$$y'(t_m^+) = \overline{I}_m(y_{m-1}(t_m)). \tag{4.8}$$

The solution y of the problem (1.4)-(1.7) is then defined by

$$y(t) = \begin{cases}
y_1(t), & \text{if } t \in [-r, t_1], \\
y_2(t), & \text{if } t \in (t_1, t_2], \\
\ldots \\
y_m(t), & \text{if } t \in (t_m, T]\,.
\end{cases}$$

References

[1] J. Banas and K. Goebel, *Measures of Noncompactnes in Banach Spaces*, Marcel-Dekker, New York, 1980.

[2] I. Bajo and E. Liz, *Periodic boundary value problem for first order differential equations with impulses at variable times*, J. Math. Anal. Appl. **204** (1996), 65-73.

[3] D. D. Bainov and P.S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Ltd., Chichister, 1989.

[4] M. Benchohra, J. Henderson and S. K. Ntouyas, *An existence result for first order impulsive functional differential equations in Banach spaces*, Comput. Math. Appl. **42** (10&11) (2001), 1303-1310.

[5] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive neutral functional differential equations in Banach spaces*, Appl. Anal. **80** (3) (2001), 353-365.

[6] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive neutral functional differential inclusions in Banach spaces*, Appl. Math. Lett. **15** (2002), 917-924.

[7] M. Benchohra, J. Henderson and S. K. Ntouyas, *Multivalued impulsive neutral functional differential inclusions in Banach spaces*, Tamkang J. Math. **33** (1) (2002), 77-38.

[8] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York, 1992.

[9] Y. Dong, *Periodic boundary value problems for functional differential equations with impulses*, J. Math. Anal. Appl. **210** (1997), 170-181.

[10] M. Frigon and D. O'Regan, *Impulsive differential equations with variable times*, Nonlinear Anal. **26** (1996), 1913-1922.

[11] M. Frigon and D. O'Regan, *First order impulsive initial and periodic problems with variable moments*, J. Math. Anal. Appl. **233** (1999), 730-739.

[12] M. Frigon and D. O'Regan, *Second order Sturm-Liouville BVP's with impulses at variable moments*, Dynam. Contin. Discrete Impuls. Systems **8** (2), (2001), 149-159.

[13] L. Gorniewicz, *Topological Fixed Point Theory of Multivalued Mappings, Mathematics and its Applications*, 495, Kluwer Academic Publishers, Dordrecht, 1999.

[14] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer, Dordrecht, Boston, London, 1997.

[15] S. K. Kaul, V. Lakshmikantham and S. Leela, *Extremal solutions, comparison principle and stability criteria for impulsive differential equations with variable times*, Nonlinear Anal. **22** (1994), 1263-1270.

[16] S. K. Kaul and X. Z. Liu, *Vector Lyapunov functions for impulsive differential systems with variable times*, Dynam. Contin. Discrete Impuls. Systems **6** (1999), 25-38.

[17] S. K. Kaul and X. Z. Liu, *Impulsive integro-differential equations with variable times*, Nonlinear Studies, **8** (2001), 21-32.

[18] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[19] V. Lakshmikantham, S. Leela and S.K. Kaul, *Comparison principle for impulsive differential equations with variable times and stability theory*, Nonlinear Anal. **22** (1994), 499-503.

[20] V. Lakshmikantham, N.S. Papageorgiou, and J. Vasundhara, *The method of upper and lower solutions and monotone technique for impulsive differential equations with variable moments*, Appl. Anal. **15** (1993), 41-58.

[21] A. Lasota and Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781-786.

[22] X. Liu and G. Ballinger, *Existence and continuability of solutions for differential equations with delays and state-dependent impulses*, Nonlinear Anal. **51** (2002), 633-647.

[23] M. Martelli, *A Rothe's type theorem for non-compact acyclic-valued map*, Boll. Un. Mat. Ital. **4** (3), (1975), 70-76.

[24] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations, World Scientific*, Singapore, 1995.

[25] A. A. Tolstonogov, *Differential Inclusions in a Banach Space*, Kluwer Academic Publishers, Dordrecht, 2000.

Laboratoire de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000 Sidi Bel Abbès, Algérie

E-mail address: benchohra@univ-sba.dz