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EXISTENCE OF PERIODIC ORBITS FOR VECTOR FIELDS VIA FULLER INDEX AND THE AVERAGING METHOD

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Abstract. We prove a generalization of a theorem proved by Seifert and Fuller concerning the existence of periodic orbits of vector fields via the averaging method. Also we show applications of these results to Kepler motion and to geodesic flows on spheres.

1. Introduction

Let X_0 be a smooth vector field on a closed manifold M; we will refer to it as the *unperturbed* vector field. A smooth homotopy $\varepsilon \mapsto X_\varepsilon$ will be called a *perturbation* of X_0. A tipical situation when all the orbits of X_0 are closed leads to a *fibration by circles* of M, generated by the S^1-action whose infinitesimal generator is X_0. Relevant examples are harmonic oscillators having the same frequency, the geodesic flow on spheres and the regularized Kepler motion. We are interested in the problem of existence of periodic orbits for perturbed vector fields. Concerning this problem two important results should be mentioned: the Seifert-Fuller Theorem [8, 3], and the Reeb-Moser Theorem [9, 7]. In this article we clarify the relationships between these two results. In particular, we will prove the former by proving a generalization of the latter: in doing so, we realize an approach to the use of perturbation theory to draw conclusions about the qualitative dynamics proposed by Anosov in [1, page 181].

In the next section we prove the Seifert-Fuller Theorem via the averaging method for one-frequency systems. It is based on a generalization of the Reeb-Moser Theorem, contained in our Theorem 2.3. In the last section, we give some examples and applications: They concern the regularized Kepler motion and an existence result of periodic orbits which can be considered as a multidimensional version of the Poincaré-Bendixson Theorem. In the first case we also prove the existence of periodic orbits for Hamiltonian perturbations in the negative energy case, which generalizes an analogous one given by Moser in [7] in the nondegenerate case.

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2. A proof of a Theorem by Seifert and Fuller via Averaging Method on a Manifold

In this section we give a new proof - and a slight generalization - of a well-known result by Seifert [8] and Fuller [3], [4] concerning the existence of closed orbits for vector fields arising from perturbations of a fibration by circles on a closed manifold. Our approach is strongly motivated by Anosov's comments in [1]. Let us quote Anosov's own words in [1], page 181:

"We digress slightly to discuss a possible approach to the proof of this theorem (of Seifert-Fuller, or of Seifert-Reeb as referred to by Anosov) using perturbation theory, the description of what we refer to as the Reeb-Moser Theorem about the existence of closed nondegenerate orbits corresponding to nondegenerate singular points of the averaged vector field follows. But we do not, in fact, exclude cases in which the equilibrium points [of the averaged vector field] are degenerate or even non-isolated. Such cases could, of course, be investigated by perturbation theory, but it is not clear *a priori* what result of such an investigation would be and whether it would be possible to handle all the cases which arise in a uniform way ... In summary, perturbation theory provides effective computation procedure in a specific situation, but is less effective than topological considerations in studying the qualitative behaviour in the general case."

Actually, our generalization of the Seifert-Fuller Theorem will show how Anosov's approach to the use of perturbation theory for the qualitative study of differential systems can be made effective even in the degenerate cases.

To state the Seifert-Fuller Theorem we need a short introduction to the Fuller index theory. We give a simplified version of it, well-suited for our goals, and refer to [3], [4] for a thoroughly discussion of the theory and related results.

We point out that every mathematical object in this article is assumed smooth: C^2-regularity would be enough. Let M be a n-dimensional closed (i.e. compact boundaryless) manifold and let

$$X : M \to TM$$

be a vector field. Consider an open set $\Omega \subseteq M$, bounded away from the set $sing(X)$ of the singular points of X. Let $0 < T_1 < T_2 < +\infty$. Then, the set $\Omega\times]T_1, T_2[\subseteq M \times \mathbb{R}^+$ is *admissible* for X. When we denote by ϕ^t the flow of X and

$$\Pi(X) = \{(q, t) \in M \times \mathbb{R}^+ : \phi^t(q) = q\},$$

then $\partial(\Omega\times]T_1, T_2[) \cap \Pi(X) = \emptyset$.

If $\varepsilon \mapsto X_\varepsilon$ is a smooth homotopy of vector fields, $X_\varepsilon : M \to TM$, $0 \le \varepsilon \le 1$, we say that $\Omega\times]T_1, T_2[$ is *admissible for the homotopy* if it is admissible for every X_ε. In [3] Fuller defines a rational-valued, additive, homotopy invariant index, the *Fuller index*, relative to a vector field X and to an admissible set $\Omega\times]T_1, T_2[$. We denote it by $i_F(X; \Omega\times]T_1, T_2[)$. The most important property we will use about this index states that if $i_F(X; \Omega\times]T_1, T_2[) \ne 0$, then there exists a nontrivial periodic orbit of X, having nonempty intersection with Ω and period in $]T_1, T_2[$ (not necessarily minimal). Now we state the result by Seifert and Fuller.

Theorem 2.1 (Seifert [8], Fuller [4])**.** *Let M be a compact boundaryless n-dimensional manifold, fibered by circles by the S^1-action having as infinitesimal generator the vector field*

$$X_0 : M \to TM$$

or, equivalently, let us suppose that all the orbits of X_0 have one and the same minimal period, say 2π. Actually, it is enough to suppose that the minimal periods of the closed orbits of X_0 are bounded from above in M, and then reparametrize X_0. In dimension $n > 3$ there exist examples [10] *of closed manifolds foliated by circles having unbounded minimal periods (equiv. lenghts). Then, the orbit space*

$$\widetilde{M} = M/S^1$$

is a closed $(n-1)$-manifold, and

$$i_F(X_0; M \times]\pi, 3\pi[) = \chi(\widetilde{M}),$$

where $\chi(\widetilde{M})$ is the Euler characteristic of $\widetilde{M}$. Therefore, if $\chi(\widetilde{M}) \neq 0$ and ε is sufficiently small, each vector field X_ε of a given smooth homotopy has at least one closed orbit.

Remark. In general, the Fuller index is a rational number, while, in the cases when the statement of Theorem 2.1 applies, it is always an integer. This is a consequence of the fact that in the situation considered in the above theorem only closed orbits with minimal period in $]\pi, 3\pi[$ are detected.

To present our proof - and the promised slight generalization - of the above theorem, we need to introduce the basic elements of averaging method on a manifold, mainly focusing on the one-frequency case as treated by Moser in [7]. We will refer to

$$X_0 : M \to TM$$

as the *unperturbed vector field*, and to the X_ε's of a smooth homotopy as the *perturbations* of X_0. We will use the notation

$$X_\varepsilon = X_0 + \varepsilon P + O(\varepsilon^2).$$

The *averaged vector field* of X_ε on M, $\underline{X}_\varepsilon : M \to TM$, is defined as

$$\underline{X}_\varepsilon = \frac{1}{2\pi} \int_0^{2\pi} (\phi_0^t)_* X_\varepsilon dt, \tag{2.1}$$

where $(\phi_0^t)_* X_\varepsilon = d\phi_0^{-t} X_\varepsilon \circ \phi_0^t$ and ϕ_0^t denotes the flow of X_0. The main property of $\underline{\mathrm{X}}_\varepsilon$ is that

$$(\phi_0^t)_* \underline{\mathrm{X}}_\varepsilon = \underline{\mathrm{X}}_\varepsilon$$

or equivalently that $[X_0, \underline{\mathrm{X}}_\varepsilon] \equiv 0$.

As a straightforward consequence we get that, if

$$p : M \to \widetilde{M} = M/S^1$$

(p_* denotes the Fréchet derivative of p) is the projection of the S^1-bundle, then

$$\overline{X}_\varepsilon = p_* \underline{\mathrm{X}}_\varepsilon$$

is a well-defined vector field

$$\overline{X}_\varepsilon : \widetilde{M} \to T\widetilde{M}.$$

We still call it *averaged vector field* on $\widetilde{M}$. Recalling that $X_\varepsilon = X_0 + \varepsilon P + O(\varepsilon^2)$, and using the above formula (2.1), we obtain

$$\underline{X}_\varepsilon = X_0 + \varepsilon \underline{P} + O(\varepsilon^2),$$

where $\underline{P}$ is the averaged vector field of P (on M), defined as

$$\underline{P} = \frac{1}{2\pi}\int_0^{2\pi} (\phi_0^t)_* P\,dt.$$

Therefore, $\overline{X}_\varepsilon = \varepsilon\overline{P} + O(\varepsilon^2)$, where $\overline{P} = p_*\underline{P}$ is the averaged vector field on $\widetilde{M}$.

In local trivializing coordinates of the bundle $p : M \to \widetilde{M} = M/S^1$, corresponding to straightening coordinates of X_0, by using the "action coordinate" I to parametrize $\widetilde{M}$ and the "angular coordinate" θ, with $\theta = \theta\, mod\, 2\pi$, to parametrize S^1, we obtain

$$X_0 : \begin{cases} \dot{I} = 0 \\ \dot{\theta} = 1, \end{cases}$$

$$X_\varepsilon : \begin{cases} \dot{I} = \varepsilon g(I,\theta,\varepsilon) \\ \dot{\theta} = 1 + \varepsilon f(I,\theta,\varepsilon), \end{cases}$$

$$\underline{X}_\varepsilon : \begin{cases} \dot{I} = \varepsilon G(I) + O(\varepsilon^2) \\ \dot{\theta} = 1 + \varepsilon f(I,\theta,\varepsilon), \end{cases}$$

$$\overline{X}_\varepsilon : \left\{ \dot{I} = \varepsilon G(I) + O(\varepsilon^2). \right.$$

It is easy to check that $G(I)$ is the expression of $\overline{P}$ in local trivializing coordinates, that is,

$$G(I) = \frac{1}{2\pi}\int_0^{2\pi} g(I,\theta,0)d\theta.$$

The geometric meaning of the vector field $\overline{P}$, or equivalently of $G(I)$, is given by the following argument, essentially due to Moser [7]. The use of local trivializing coordinates allows us to locally identify the bundle $p : M \to \widetilde{M}$ with the product $U \times S^1$, where U is open in $\widetilde{M}$. On the other hand, U can be viewed as an $(m-1)$-dimensional submanifold of M, represented in local coordinates as $\{(I,0), |I| < R\}$, with $R > 0$ small enough.

For a sufficiently small $\overline{\varepsilon} > 0$, consider a cross section Σ of X_ε, $|\varepsilon| < \overline{\varepsilon}$, that is an $(m-1)$-dimensional submanifold of M, contained in U, which is transverse in each of its points to X_ε. In addition, consider the one parameter family of Poincaré maps

$$F = F_\varepsilon : A \times (-\overline{\varepsilon}, \overline{\varepsilon}) \to \Sigma,$$

where A is an open subset of Σ. The existence of closed orbits of X_ε, for $|\varepsilon| < \overline{\varepsilon}$, with initial data $(I,0)$, $I \in A$, and minimal period close to 2π, is then reduced to the existence of $I = I(\varepsilon)$ such that $F(I(\varepsilon),\varepsilon) = I(\varepsilon)$. Since F is smooth, it can be expanded as

$$F(I,\varepsilon) = I + \varepsilon\frac{\partial}{\partial\varepsilon}F(I,0) + O(\varepsilon^2),$$

where the equality $F(I,0) = I$ follows from the 2π-periodicity of X_0. Observe that $O(\varepsilon^2)$ is uniform with respect to I, if $|I| < R$. The crucial point in the averaging method for one frequency systems is the following equality, which clarifies the geometric meaning of the averaged vector field $\overline{P}$:

$$\frac{1}{2\pi}\frac{\partial}{\partial\varepsilon}F(I,0) = \overline{P}(I)$$

or, equivalently,

$$\frac{1}{2\pi}\frac{\partial}{\partial\varepsilon}F(I,0) = G(I) = \frac{1}{2\pi}\int_0^{2\pi} g(I,\theta,0)d\theta.$$

This property is easily verified. In fact, let us represent the flow ϕ_ε^t of X_ε in local coordinates as follows:

$$\phi_\varepsilon^t(I,\theta) = (\xi(t,I,\theta,\varepsilon), \zeta(t,I,\theta,\varepsilon)).$$

From the expression of X_ε and from the obvious identity $\phi_0^t(I,\theta) = (I, t+\theta)$ we get

$$\xi(t,I,\theta,\varepsilon) = \varepsilon\int_0^t g(I+O(\varepsilon), \tau+\theta+O(\varepsilon), \varepsilon)d\tau.$$

The Poincaré map has the equivalent definition

$$F(I,\varepsilon) = \xi(t(I,\varepsilon), I, 0, \varepsilon),$$

where $t(I,\varepsilon)$ is the first return time map on Σ, and, obviously,

$$t(I,\varepsilon) = 2\pi + O(\varepsilon).$$

Let us recall that all the $O(\varepsilon)$'s are uniform with respect I, with $|I| < R$. Finally

$$F(I,\varepsilon) = \varepsilon\int_0^{2\pi} g(I,\theta,0)d\theta + O(\varepsilon^2)$$

and thus

$$\frac{\partial}{\partial\varepsilon}F(I,0) = 2\pi G(I).$$

We can now collect in the following theorem some propositions which will be fundamental for our extension of the Reeb-Moser Theorem and therefore for our averaging-oriented proof of the Seifert-Fuller Theorem. These propositions are well-known, apart perhaps the statement i) which is obvious; nevertheless, we give a complete proof of them because it is elementary, basic for the developments of our article and slightly simplified in our case.

Theorem 2.2 (Reeb [9], Moser [7], Hale [5], Fuller [4])**.**

(i) *Let $\{\varepsilon_n\}$ be a real sequence converging to 0 and, for each n, let γ_{ε_n} be a closed orbit of X_{ε_n}, with minimal period in $]\pi, 3\pi[$ and corresponding to initial data $(I(\varepsilon_n), 0)$. Assume also that $I(\varepsilon_n)$ tends to 0. Then $G(0) = 0$.*

(ii) *(Reeb [9], Moser [7]) If $G(0) = 0$ and the linear operator $\frac{\partial}{\partial I}G(0)$ is non-singular, i.e. the averaged vector field $\overline{P}$ on M has a nondegenerate zero in $I = 0$, then there exist $\overline{\varepsilon} > 0$ and a neighborhood U of $I = 0$ in $\widetilde{M}$, such that for every ε, $|\varepsilon| < \overline{\varepsilon}$, there exists at least one closed orbit γ_ε of X_ε, corresponding to the initial datum $(I,0)$, $I \in U$, and such that $\gamma_\varepsilon \mapsto \{I = 0\}$ (Hausdorff topology).*

(iii) *(Hale [5]) Assume that $0 \in U \subseteq \mathbb{R}^{n-1}$ is a hyperbolic singular point of $\overline{P}$. Then the closed orbit γ_ε is hyperbolic, hence isolated among the closed orbits of X_ε having periods in $]\pi, 3\pi[$.*

(iv) *(Fuller [4]) In the same assumptions of the previous statement there exists a small tubular neighborhood $\widetilde{\gamma}_\varepsilon$ of γ_ε in M such that*

$$i_F(X_\varepsilon; \widetilde{\gamma}_\varepsilon \times]\pi, 3\pi[) = \operatorname{sign}(-\varepsilon)^n i_{P-H}(\overline{P}; 0),$$

where $i_{P-H}(\overline{P}; 0)$ is the Poincaré-Hopf index of $\overline{P}$ at 0.

Proof. From the assumptions of statement i) and from the basic relationship between the Poincaré map and the averaged vector field we get

$$F(I(\varepsilon_n), \varepsilon_n) - I(\varepsilon_n) = \varepsilon_n(\frac{\partial}{\partial \varepsilon} F(I(\varepsilon_n), \varepsilon_n) + O(\varepsilon_n)) = 0$$

for a sequence of nonzero $\varepsilon_n \to 0$ and consequently for $I(\varepsilon_n) \to 0$. This is clearly impossible if

$$\overline{P}(0) = \frac{1}{2\pi} \frac{\partial}{\partial \varepsilon} F(0, 0) \neq 0,$$

then i) follows.

To prove the second statement we must prove the existence of $\varepsilon \mapsto I(\varepsilon)$, $I(0) = 0$, such that

$$F(I(\varepsilon), \varepsilon) - I(\varepsilon) = 0$$

or equivalently, just expanding the Poincaré map with respect to the parameter ε, we must prove the existence of nontrivial solutions of

$$\varepsilon(\frac{1}{2\pi} \frac{\partial}{\partial \varepsilon} F(I, \varepsilon) + O(\varepsilon)) = 0.$$

Of course, this is a straightforward consequence of the Implicit Function Theorem, of the basic equality

$$\frac{1}{2\pi} \frac{\partial}{\partial \varepsilon} F(I, 0) = G(I)$$

and of the hypothesis that $G(I)$ has a nondegenerate zero at 0.

The proofs of both the statements iii) and iv) follow from the following argument. Let $\lambda_j(I, \varepsilon)$, $\mu_j(I)$, $j = 1, \dots, \dim \widetilde{M}$ be respectively the eigenvalues of

$$\frac{\partial}{\partial I} F(I, \varepsilon)$$

and of

$$\frac{\partial}{\partial I} \overline{P}(I).$$

Let us remark that, even if $X_0 : M \to TM$ does not admit a global section, i.e. a one-codimensional closed submanifold, diffeomorphic to $\widetilde{M}$, which is everywhere transverse to X_0, the functions $\lambda_j(I, \varepsilon)$, $j = 1, \dots, n = \dim \widetilde{M}$ are well-defined, if the multiplicity of the eigenvalues is considered. In fact, we can choose an one-codimensional distribution $\mathcal{D}$ of small disks on M, everywhere transverse to X_0, and we can compute the relative local first return maps: the eigenvalues $\lambda_j(I, \varepsilon)$ turn out to be independent of $\mathcal{D}$. Moreover, let us observe that all the conclusions about the computations of the various indices are not affected by a small smooth homotopy of P still keeping $I = 0$ as a hyperbolic singular point of $\overline{P}$, having eigenvalues of the linearization at 0 which are all distincts: hence we will suppose this is the situation we are dealing with.

Then, again as a straightforward consequence of the basic equality

$$F(I(\varepsilon), \varepsilon) = I(\varepsilon) + 2\pi\varepsilon \overline{P}(I) + O(\varepsilon^2),$$

we get

$$\frac{\partial}{\partial I} F(I(\varepsilon), \varepsilon) = E + 2\pi \frac{\partial}{\partial I} G(I)\varepsilon + O(\varepsilon^2)$$

and so finally, using the fact that the $\mu_j(I)$'s are all distinct, we get the equality

$$\lambda_j(I, \varepsilon) = 1 + 2\pi\varepsilon\mu_j(I) + O(\varepsilon^2)$$

from which both statements iii) and iv) easily follow. In fact, from the definition of the Fuller index for hyperbolic periodic orbits, see [3], we have that

$$i_F(X_\varepsilon; \widetilde{\gamma}_\varepsilon \times]\pi, 3\pi[) = (-1)^\sigma$$

where σ is the number of eigenvalues of the monodromy operator $\frac{\partial}{\partial I}F(I(\varepsilon), \varepsilon)$ in $]1, +\infty[$. Therefore σ is equal, in the case $\varepsilon > 0$, to the number of the μ_j's which are real and greater than zero, or, from the fact that the system is real, to the number of the μ_j's having positive real parts: this conclude the proof in the case when ε is positive; the case of negative ε is analogous. $\square$

We can now state and prove the main result of this section: it is an extension of statement ii) of the above thorem (Reeb-Moser Theorem) to the degenerate case. Let us consider the one-parameter family of vector fields on M

$$X_\varepsilon = X_0 + \varepsilon P + O(\varepsilon^2).$$

We are going to show how the topological properties of the Frechet derivative of $\varepsilon \mapsto X_\varepsilon$ - namely the vector field P - determine the existence of closed orbits of X_ε, with ε sufficiently small. A generalization of this approach will be considered in the remark at the end of this section.

Let $\widetilde{A}$ be an open subset of $\widetilde{M}$, whose boundary is a boundaryless $(m-2)$-dimensional manifold. We recall that, in this case, the *index* of the averaged vector field $\overline{P} : \widetilde{M} \to T\widetilde{M}$ in $\widetilde{A}$ is well defined if $sing\overline{P} \cap \partial\widetilde{A} = \emptyset$, where $sing\overline{P}$ is the set of singular points of $\overline{P}$. We have the equality

$$\operatorname{ind}(\overline{P}, \widetilde{A}) = \deg(\frac{\overline{P}}{\|\overline{P}\|}, \partial\widetilde{A}),$$

where $\deg(\frac{\overline{P}}{\|\overline{P}\|}, \partial\widetilde{A})$ stands for the ordinary Brouwer degree.

Theorem 2.3. *Let $\widetilde{A}$ be an open subset of $\widetilde{M}$ with $\partial\widetilde{A}$ a compact boundaryless manifold. Suppose $sing\overline{P} \cap \partial\widetilde{A} = \emptyset$. Let $p : M \to \widetilde{M}$ be the bundle projection map and $A = p^{-1}(\widetilde{A}) \subseteq M$. Then, there exists $\overline{\varepsilon} > 0$ such that, for every ε, $|\varepsilon| < \overline{\varepsilon}$, the set $A \times]\pi, 3\pi[$ is admissible for X_ε and*

$$i_F(X_\varepsilon; A \times]\pi, 3\pi[) = \operatorname{sign}(-\varepsilon)^n \operatorname{ind}(\overline{P}, \widetilde{A}), \tag{2.2}$$

where $n = \dim \widetilde{M}$.

Before giving the proof of this theorem we deduce as a corollary Theorem 2.1

$$i_F(X_\varepsilon; M \times]\pi, 3\pi[) = \chi(\widetilde{M}).$$

Proof of Theorem 2.1. We just need to use the above theorem and the additive property of the index, together with the well-known equality of the global index of a vector field on a closed manifold and the Euler characteristic of the manifold itself. The presence of the factor $sign(-\varepsilon)^n$ in the formula (2.2) is obviously immaterial when the global situation is considered. This is clear if $\dim \widetilde{M}$ is even and this is a consequence of the fact that $\chi(\widetilde{M}) = 0$ in the case when $\dim \widetilde{M}$ is odd . $\square$

Proof of Theorem 2.3. We carry on the proof in the case $\widetilde{A}$ is completely contained in one local chart of $\widetilde{M}$ and referred to local coordinates I as $\widetilde{A} = B_R(0) = \{I \in \widetilde{M} : |I| < R\}$. The general case is completely analogous and can be reduced to the above situation by choosing local trivializing coordinates (I, θ) on the bundle, decomposing $\widetilde{A}$ in local charts and patching the various parts of it together, taking into account the additivity property of the index.

Let us observe that in our situation, we have

$$A = B_R(0) \times S^1.$$

Moreover, as we are working in local coordinates, we will refer to $\overline{P}$ as $G(I)$. The assumption that $G(I) \neq 0$, for $I \in \partial B_R(0)$, and statement i) of Theorem 2.2 imply that for $|\varepsilon| < \overline{\varepsilon}$, $\overline{\varepsilon}$ sufficiently small, X_ε has no closed orbits with periods in $]\pi, 3\pi[$ passing through points of ∂A. This proves that A is admissible for the X_ε's.

In the following part of the proof we suppose ε to be fixed and sufficiently small, according to the above specified request, and we compute $i_F(X_\varepsilon; A\times]\pi, 3\pi[)$. Let

$$\rho : M \to \mathbb{R}^+$$

be a smooth bump function, such that $\rho \equiv 0$ in $M - p^{-1}(B_R(0)) = M - A$, while $\rho \equiv 1$ in $p^{-1}(B_{\mu R}(0))$ where μ is sufficiently small in order that $G(I) \neq 0$ for $\mu R \leq |I| \leq R$. The bump function ρ allows to localize a smooth homotopy $\lambda \mapsto X_{\varepsilon,\lambda}$ in the local chart containing $B_R(0) \times S^1$. Therefore, we just need to define $X_{\varepsilon,\lambda}$ in local coordinates (I, θ). Let $n = \dim M$ and $V \in \mathbb{R}^{n-1} - \{0\}$, and let us define such (local) homotopy as

$$\lambda \mapsto X_{\varepsilon,\lambda}(I, \theta) = X_\varepsilon(I, \theta) + \lambda\rho(I, \theta)V.$$

Of course, $A\times]\pi, 3\pi[$ is still admissible for $X_{\varepsilon,\lambda}$, for sufficiently small λ. Let λ be one of such sufficiently small values: a straightforward application of Sard's Theorem implies that for almost any choice of V, the averaged vector field $\overline{X}_{\varepsilon,\lambda}$ has only hyperbolic singular points in $B_R(0)$. From the basic results of degree theory we have the following chain of equalities

$$\begin{aligned}
\operatorname{ind}(\overline{P}, B_R(0)) &= \deg(\frac{\overline{P}}{\|\overline{P}\|}, \partial B_R(0)) \\
&= \deg(\frac{\overline{X}_\varepsilon}{\|\overline{X}_\varepsilon\|}, \partial B_R(0)) \\
&= \deg(\frac{\overline{X}_{\varepsilon,\lambda}}{\|\overline{X}_{\varepsilon,\lambda}\|}, \partial B_R(0)) \\
&= \sum_{I_j \in \operatorname{sing}(\overline{X}_{\varepsilon,\lambda})} i_{P-H}(\overline{X}_{\varepsilon,\lambda}; I_j).
\end{aligned}$$

On the other hand, statement iv) of Theorem 2.2 and the homotopy invariance of the Fuller index give

$$\begin{aligned}
\sum_{I_j \in \operatorname{sing}(\overline{X}_{\varepsilon,\lambda})} i_{P-H}(\overline{X}_{\varepsilon,\lambda}; I_j) &= \operatorname{sign}(-\varepsilon)^n i_F(X_{\varepsilon,\lambda}; p^{-1}(B_R(0)\times]\pi, 3\pi[) \\
&= \operatorname{sign}(-\varepsilon)^n i_F(X_\varepsilon; p^{-1}B_R(0)\times]\pi, 3\pi[)
\end{aligned}$$

and finally

$$\operatorname{sign}(-\varepsilon)^n i_F(X_\varepsilon; p^{-1}B_R(0)\times]\pi, 3\pi[) = \operatorname{ind}(\overline{P}, B_R(0)).$$

□

Remark. It is easy to see that, as a consequence of Theorem 2.3, a closed orbit of X_ε with initial datum $(I,0)$, $I \in \widetilde{A}$, exists whenever $\operatorname{ind}(X_\varepsilon, p^{-1}(\widetilde{A})) \neq 0$.

Remark. One could try to use the Fuller index approach to investigate the existence of periodic orbits of minimal period greater than 2π. Actually, these trajectories do not exist. More precisely we can state the following property:

> for any given number $T > 2\pi$, there exists $\overline{\varepsilon}$ such that for every $0 < \varepsilon < \overline{\varepsilon}$ the vector field X_ε has no periodic orbits having minimal periods in $]2\pi, T[$.

The proof of this claim is an obvious consequence of the fact that the eigenvalues $\lambda_j(I,\varepsilon)$ defined in the proof of Theorem 2.2 verify $\lambda_j(i,\varepsilon) = 1 + O(\varepsilon)$.

Remark. Theorem 2.3 easily generalizes to the case when $\overline{P} \equiv 0$ on $\widetilde{M}$. Let

$$\begin{gathered} X_\varepsilon = X_0 + \varepsilon P + \dots + \varepsilon^k P^{(k)} + O(\varepsilon^{k+1})\,, \\ \underline{P}^{(k)} = \frac{1}{2\pi}\int_0^{2\pi} (\phi_0^t)_* P^{(k)} dt\,, \\ \overline{P}^{(k)} = p_* \underline{P}^{(k)}. \end{gathered}$$

Also suppose that $\overline{P} \equiv \dots \equiv \overline{P}^{(k-1)} \equiv 0$ while $\overline{P}^{(k)} \neq 0$. Then the same arguments leading to Theorem 2.3 give

$$i_F(X_\varepsilon; A\times]\pi, 3\pi[) = \operatorname{sign}(-\varepsilon)^{kn} \operatorname{ind}(\overline{P}^{(k)}, \widetilde{A}).$$

3. Examples and applications

This final part of the article contains some applications of the results contained in the previous section. Specifically, we give applications of the "degenerate version of the Reeb-Moser Theorem", namely of Theorem 2.3, as well as applications of the classical Seifert-Fuller Theorem. In our opinion they have some interest, originality and relationship with the present article. This section is divided in two subsections, labeled by a latin letter and a short title.

Hamiltonian degenerate perturbations of the Kepler motion. Let

$$H_0(p,q) = \frac{1}{2}|p|^2 - \frac{1}{|q|}$$

be a Kepler Hamiltonan, $q = (q_1, \dots, q_n)$, $p = (p_1, \dots, p_n)$. In the case $n = 2$ H_0 is the Hamiltonian of the Newtonian gravitational field describing a two-body system. Let

$$H_\varepsilon(p,q,\varepsilon) = H_0(p,q) + \varepsilon K(p,q,\varepsilon)$$

be a perturbed Hamiltonian, where $K(p,q,\varepsilon)$ is smooth and satisfies a smoothness assumption also as a function

$$K(|p|^2 q - (2p\cdot q)p, \frac{p}{|p|^2}\varepsilon)$$

near $|q| = 2$, $p = 0$, $\varepsilon = 0$. Such a smoothness condition could be verified following an analogous case presented in [7, Section 5, p. 628].

Under these conditions the Hamiltonian motion on a negative energy level, say on $\{H_\varepsilon(p,q,\varepsilon) = -\frac{1}{2}\}$, can be embedded, as a flow, after a reparametrization of the independent variable and a smooth change of coordinates, in a Hamiltonian ε-perturbation of the geodesic flow X_0 on S^n (with respect to the standard metric on the n-sphere). This is the so called *regularization of the perturbed Kepler motion*, see [7] for details. Let

$$X_\varepsilon : T_1S^n \to T(T_1S^n)$$

be the corresponding one-parameter family of vector fields on the unitary tangent bundle of the n-sphere realizing the perturbation of the geodesic vector field

$$X_0 : T_1S^n \to T(T_1S^n).$$

In [7] Moser proved, as a consequence of statement ii) in Theorem 2.2, the following theorem of existence of periodic orbits for the perturbed geodesic flow on spheres, or equivalently for the perturbed Kepler motion.

Theorem 3.1 ([7])**.** *Let $\overline{X}_\varepsilon$ be the averaged vector field with respect to the unperturbed geodesic flow X_0 on a sphere. If, for ε sufficiently small, $\overline{X}_\varepsilon$ has a nondegenerate singular point, then X_ε has a (nondegenerate) periodic orbit.*

Moreover, let H_ε be the Hamiltonian of $\overline{X}_\varepsilon$ and consider the regularization of the perturbed Kepler motion on negative energy manifolds. For every ε sufficiently small such that a nondegenerate singular point of the averaged vector field arising from the regularization exists, it has at least one closed orbit. Actually such "closed" orbit could be a collision orbit. We will not consider this question here.

Our Theorem 2.3 permits to drop the (particularly heavy in the Hamiltonian case) non-degeneracy assumption in Theorem 3.1.

Theorem 3.2. *The same conclusion as in the previous theorem, regarding the existence of closed orbits for the perturbation X_ε of the geodesic flow on spheres, holds if $\overline{X}_\varepsilon = \varepsilon\overline{P}+O(\varepsilon^2)$ has a degenerate zero with nonzero index or more generally if there exists a ball B_R in the orbit space of X_0 such that*

$$\operatorname{ind}(\overline{P}, \partial B_R) \neq 0.$$

Actually, if ε is sufficiently small, the perturbed geodesic vector field X_ε has always at least one closed geodesic. An analogous conclusion holds for the perturbed Kepler motion.

Proof. The first part of the theorem is a straightforward application of Theorem 2.3, while the second one is a consequence of the Seifert-Fuller Theorem. In both cases we just need to reduce the dynamical situation to a geometric model well-suited for application of the one-frequency averaging method. We will do that referring to the application of the Seifert-Fuller Theorem, the rest of the proof being completely analogous. The geodesic vector field $X_0 : T_1S^n \to T(T_1S^n)$ defines a fibration by circles

$$T_1S^n \to G_{2,n+1},$$

where $G_{2,n+1}$ is the Grassmannian manifold of oriented 2-planes in $\mathbb{R}^{n+1}$, obtained after identification of a great circle in S^n by the 2-plane through the origin containing it. We apply now the Seifert-Fuller Theorem

$$i_F(X_0; T_1S^n \times]\pi, 3\pi[) = \chi(G_{2,n+1}) \neq 0.$$

In fact a straightforward computation of the Betti numbers of $G_{2,n+1}$, based for instance on the cell structure of the Grassmann manifolds as exposed in [6], together with the definition of the Euler characteristic as the alternating sum of the Betti numbers, leads to

$$\chi(G_{2,n+1}) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even.} \end{cases}$$

□

Remark. It is probably worthwhile mentioning that the above result and the approach used for its proof are not unrelated with the deeply studied problem of existence of closed geodesics after perturbation of the standard metric on S^n. We stress the fact that in the above theorem we considered *arbitrary* perturbations, and not only perturbations arising from a perturbation of the standard metric on the sphere. For such particular perturbations not only existence but also multiplicity results are known, obtained through a variational approach (see [2]).

A Poincaré-Bendixson type existence theorem of periodic orbits. This second application of the ideas related to the Fuller index approach in the averaging method for one-frequency systems deals with a situation which is frequently present in mechanics. Let $J \subseteq \mathbb{R}$ be an interval and M be a closed manifold: the dynamic variable h parametrizing J is called the *energy* of the *unperturbed dynamical system*

$$X_0 : J \times M \to \mathbb{R} \times TM.$$

Such a vector field verifies:

(i) h is a first integral for X_0,
(ii) $X_{0|\{h=c\}} : M \to TM$ generates a fibration by circles (with c-depending minimal periods T_c)
$$p(c) : M \to \widetilde{M},$$
(iii) for $c_1, c_2 \in J$ the fibrations $p(c_j) : M \to \widetilde{M}$, $j = 1, 2$, are isomorphic,
(iv) the Fuller index of the fibration by circles satisfies
$$i_F(X_{0|\{h=c\}}; M\times]\frac{1}{2}T_c, \frac{3}{2}T_c[) \neq 0.$$

In the sequel we will always refer to the natural splitting of the tangent bundle $T(J \times M) = \mathbb{R} \times TM$ and the analogous $T(J \times \widetilde{M}) = \mathbb{R} \times T\widetilde{M}$ defined by the bundle map. Therefore, the averaged vector field

$$\overline{P} : J \times \widetilde{M} \to \mathbb{R} \times T\widetilde{M}$$

is canonically decomposed as $\overline{P} = (\overline{P}^h, \overline{P}^{\widetilde{M}})$. s

Example. *Harmonic oscillators with the same frequency.* Let $x \in \mathbb{R}^4$ and

$$\dot{x} = X_0(x) = Ax,$$

where $A = I_1 \oplus I_2$ and

$$I_1 = I_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Here $J = \mathbb{R}^+$, $M = S^3$, $\widetilde{M} = S^2$ and the fibration $p : S^3 \to S^2$, which is the same for every energy level, is the Hopf fibration, with $i_F(X_0; S^3\times]\pi, 2\pi[) = 2$. Of course, this situation generalizes in an obvious way to the case of n harmonic oscillators with rationally dependent frequencies.

Example. *Geodesic flow on spheres.* Here

$$X_0 : TS^n \to T(TS^n)$$

is the geodesic vector field, $n \geq 2$, with respect to the usual metric on S^n. The tangent bundle is fibered through the level manifold of the kinetic energy first integral as

$$TS^n = (S^n \times \{0\}) \cup (\cup_{h>0} T_h S^n).$$

The exceptional 0-fiber is diffeomorphic to S^n, while all the other fibers are diffeomorphic to $T_1 S^n$. Therefore every energy interval $J = (h_1, h_2) \subseteq \mathbb{R}^+$ defines a phase space for (the restriction of) X_0 such that

$$X_0 : \cup_{h_1<h<h_2}(\{h\} \times T_h S^n) \to \cup_{h_1<h<h_2}(\{h\} \times T(T_h S^n)),$$

where $p(h) : T_h S^n \to G_{2,n+1}$ are isomorphic bundles for $h_1 < h < h_2$. Finally, we recall that

$$i_F(X_{0|\{h=c\}}; T_h S^n \times]\frac{\pi}{c}, \frac{3\pi}{c}[) = \chi(G_{2,n+1}) \neq 0.$$

These examples justify our attention to the perturbations

$$X_\varepsilon : J \times M \to \mathbb{R} \times TM,$$

where $X_\varepsilon = X_0 + \varepsilon P + O(\varepsilon^2)$ and X_0 satisfies the above listed properties i)-iv). It is easy to see that in general X_ε has no closed orbits: in the next theorem we will give some relevant hypotheses implying the existence of periodic orbits.

Theorem 3.3. *Let h_1, $h_2 \in J$ be two energy levels, $h_1 < h_2$, such that*

$$\overline{P}^h(h_1, I)\overline{P}^h(h_2, I) < 0$$

for every $I \in \widetilde{M}$. Assume in addition that $\chi(\widetilde{M}) \neq 0$. Then, for every sufficiently small ε, there exists a closed orbit of X_ε having minimal period between $\frac{1}{2}T(h_1)$ and $\frac{3}{2}T(h_2)$ and corresponding to an initial datum $q \in J \times M$ such that $h_1 < h(q) < h_2$.

Proof. We prove the theorem in the case $\pi < T(h_1) < T(h_2) < 3\pi$. This situation is the general one, up to a reparametrization of the independent variable and of the energy h, not affecting our geometric conclusions concerning the existence of a periodic orbit. We will apply the Fuller index theory to the set

$$\Omega = (\,]h_1, h_2[\times M) \times]\pi, 3\pi[\,.$$

Let us remark that, as M is boundaryless,

$$\partial\Omega = \{h_1\} \times M \times]\pi, 3\pi[\cup \{h_2\} \times M \times]\pi, 3\pi[\cup]h_1, h_2[\times M \times \{\pi\} \cup]h_1, h_2[\times M \times \{3\pi\}.$$

The assumption

$$\overline{P}^h(h_1, I)\overline{P}^h(h_2, I) < 0$$

together with statement i) of Theorem 2.2 permits to conclude that Ω is admissible for the perturbation if ε is sufficiently small, as we will always suppose for the rest of the proof. Therefore, to conclude the proof we keep ε fixed and we prove that $i_F(X_\varepsilon; \Omega) \neq 0$. Let us define

$$Y :]h_1, h_2[\times M \to \mathbb{R} \times TM$$

through its components with respect to the canonical splitting as

$$Y^M(h,q) = X_{0|\{h\}\times M},$$

$$Y^h(h,q) = \begin{cases} h - \frac{h_1+h_2}{2} & \text{if } \overline{P}^h(h_1,I) < 0, \\ \frac{h_1+h_2}{2} - h & \text{if } \overline{P}^h(h_1,I) > 0. \end{cases}$$

It is clear that $\{(h,q) : h - \frac{h_1+h_2}{2} = 0\} \simeq M$ is an invariant manifold for Y, fibered by circles by the Y-action, while no other point in $]h_1,h_2[\times M$ can be the initial datum for a periodic orbit of Y. It is also easy to see that Ω is admissible for Y and that - by an obvious homotopic perturbation - we get

$$i_F(Y;\Omega) = \chi(\widetilde{M}) \neq 0.$$

Therefore, the statement will be proved if we construct a smooth homotopy between Y and X_ε still having Ω as an admissible set. Let

$$\lambda \mapsto (1-\lambda)X_\varepsilon + \lambda Y = Z_{\varepsilon,\lambda}$$

connecting X_ε to Y. To prove the admissibility of Ω for $\lambda \mapsto Z_{\varepsilon,\lambda}$, we observe that

$$\overline{Z}_{\varepsilon,\lambda} = \varepsilon(1-\lambda)\overline{P} + \varepsilon\lambda\overline{Y}$$

and that $\overline{Y}^{\widetilde{M}} = 0$, while $\overline{Y}^h = h - \frac{h_1+h_2}{2}$. Now, let us suppose that $\overline{P}^h(h_1,I) < 0$ and $\overline{P}^h(h_2,I) > 0$ for $I \in \widetilde{M}$, the opposite situation being analogous. Then

$$\overline{Z}^{\widetilde{M}}_{\varepsilon,\lambda} = \varepsilon(1-\lambda)\overline{P}^{\widetilde{M}},$$
$$\overline{Z}^h_{\varepsilon,\lambda} = \varepsilon(1-\lambda)\overline{P}^h + \lambda\overline{Y}^h$$

and therefore for $\varepsilon \neq 0$,

$$\overline{Z}^h_{\varepsilon,\lambda}(h_1,I)\overline{Z}^h_{\varepsilon,\lambda}(h_2,I) < 0\,.$$

Arguing as in Theorem 2.2, statement (i), the above inequality implies that no periodic orbit of $Z_{\varepsilon,\lambda}$ can intersect

$$\{h_1\} \times M\times]\pi,3\pi[\cup\{h_2\} \times M\times]\pi,3\pi[.$$

Therefore, to prove that $\partial\Omega$ is admissible for $\lambda \mapsto Z_{\varepsilon,\lambda}$ we must prove that no periodic orbit of $Z_{\varepsilon,\lambda}$ intersects $]h_1,h_2[\times M \times \{\pi\}\cup]h_1,h_2[\times M \times \{3\pi\}$ or, which is the same, that no periodic orbit of $Z_{\varepsilon,\lambda}$ in $]h_1,h_2[\times M$ has period π or 3π. Let $Z_{0,\lambda} = (1-\lambda)X_0 + \lambda Y$, $0 \le \lambda \le 1$. For every $\lambda \in [0,1]$ these vector fields has no periodic orbits of period π or 3π. More precisely, if

$$\phi^t_{0,\lambda}(h,I) = (\phi^t_{0,\lambda;h}(h,I), \phi^t_{0,\lambda;I}(h,I))$$

is the flow of $Z_{0,\lambda}$, then, as

$$Z^M_{0,\lambda}(h,I) = X_{0|\{h\}\times M},$$

there exists $\delta > 0$ such that, for every $(h,I) \in]h_1,h_2[\times M$,

$$d(\phi^{\pi}_{0,\lambda;I}(h,I),I) > 2\delta\,,$$
$$d(\phi^{3\pi}_{0,\lambda;I}(h,I),I) > 2\delta\,,$$

where $d(\cdot,\cdot)$ is a distance defined by a Riemann metric on M. Then, if

$$\phi^t_{\varepsilon,\lambda}(h,I) = (\phi^t_{\varepsilon,\lambda;h}(h,I), \phi^t_{\varepsilon,\lambda;I}(h,I))$$

is the flow of $Z_{\varepsilon,\lambda}$ and if ε is sufficiently small, from the continuous dependence of the solutions from parameters one has that for every $(h,I) \in [h_1,h_2] \times M$

$$d(\phi^{\pi}_{\varepsilon,\lambda;I}(h,I),I) > \delta\,,$$
$$d(\phi^{3\pi}_{\varepsilon,\lambda;I}(h,I),I) > 2\delta$$

and therefore the vector fields $Z_{\varepsilon,\lambda}$ have no periodic orbits with periods π or 3π in $\in [h_1,h_2] \times M$. This concludes the proof. □

Remark. An application of the above theorem to the geodesic flow on spheres (second example above) gives an existence result for closed geodesics.

In some sense, the above theorem is a theorem of Poincaré-Bendixson type, too. In fact, the ω-invariance of a region of the (multidimensional) phase space, together with topological hypotheses on the averaged vector field, *including that it always points either inward or outward in the 2 boundary components*, imply the existence of a closed orbit.

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References

[1] D. V. Anosov et al.; *Smooth Dynamical Systems*, Dynamical Systems I, Encyclopaedia of Math. Sci., vol. 1, D. V. Anosov and V. I. Arnold eds., Springer, Berlin, 1988.
[2] W. Ballmann, G. Thorbergsson, W. Ziller; *Existence of closed geodesics on positively curved manifolds*, J. Diff. Geom., **18** (1983), 221-252.
[3] F. B. Fuller; *An index of fixed point type for periodic orbits*, Amer. J. Math., **89** (1967), 133-148.
[4] F. B. Fuller; *The existence of periodic orbits*, Mimeographed Seminar Notes of The Institute de Recherche Mathématiques Avancée, Strasbourg, 1968.
[5] J. K. Hale; *Integral Manifolds of Perturbed Differential Systems* Ann. Math., **73**, No. 3 (1961), 496-531.
[6] J. Milnor, J. Stasheff; *Characteristic Classes*, Princeton University Press, Princeton, 1974.
[7] J. Moser; *Regularization of Kepler's Problem and the Averaging Method on a Manifold*, Comm. Pure Appl. Math., **23** (1970), 609-636.
[8] H. Seifert; *Closed integral curves in 3-space and isotopic two-dimensional deformations*, Proc. Am. Math. Soc. **1** (1950), 287-302.
[9] G. Reeb; *Sur certaines propriétés topologiques des systèmes dynamiques*, Acad. Roy. Sci. Lett. et Beau Arts de Belgique Cl. de Sci., Memoires in 8^0, Ser. 2, vol. **27** No. 29 (1952).
[10] D. Sullivan; *A counterexample to the periodic orbit conjecture*, Publ. I.H.E.S. **46** (1976), 5-14.

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