Electronic Journal of Differential Equations, Vol. 2005(2005), No. 100, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

CONTINUITY OF ATTRACTORS FOR A REACTION-DIFFUSION PROBLEM WITH RESPECT TO VARIATIONS OF THE DOMAIN

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Abstract. We show that for a class of dissipative semilinear parabolic problems, the global attractor varies continuously (upper and lower semi-continuity) with respect to perturbations of the spatial domain.

1. Introduction

Let Ω be an open bounded region in $\mathbb{R}^n$ with smooth boundary, where we consider the semilinear parabolic problem

$$\begin{gathered} u_t(x,t) = \Delta u(x,t) + f(u(x,t)), \quad x \in \Omega, t > 0 \\ u(x,t) = 0, \quad x \in \partial\Omega, t > 0. \end{gathered} \tag{1.1}$$

It is well known that, under appropriate growth conditions for the nonlinearity f, problem (1.1) is locally well-posed in various functional spaces (see [10]). With some additional dissipative conditions, the associated (global) dynamical system has a global attractor; see, for example [6, 7, 15]. The dependence of the global attractor of (1.1) on the parameters present in the equation is also an important object of study (see for example [8, 9, 13]). An excellent review of the subject can be found in [17].

Our purpose here is to address the problem of continuity of attractors for (1.1) when the parameter involved is the domain where the problem is posed. That is, we assume that Ω is a small perturbation of a fixed smooth region Ω_0 and we want to discuss the changes of the attractor of (1.1) with respect to the region Ω. As we shall see, small perturbations of Ω_0 cause small perturbations of the attractors. We say that Ω is a $\mathcal{C}^k$ *small perturbation* of Ω_0 if there exists a $\mathcal{C}^k$ diffeomorphism $h : \Omega_0 \to \mathbb{R}^n$ such that $\Omega = h(\Omega_0)$ and $\|h - i_{\Omega_0}\|_{C^k}$ is small (cf. Section 2) and *closeness* of attractors means upper semicontinuity and/or lower semicontinuity. One of the difficulties here is that the functional spaces change as we change the region. Our first task is then to find a way to compare the attractors of problem

2000 *Mathematics Subject Classification.* 35B40, 35K20, 58D25.
Key words and phrases. Parabolic equations; attractors; lower semi-continuity; perturbation of the domain.

Submitted October 27, 2004. Published September 20, 2005.
L. A. F. de Oliveira was supported by CAPES/MECD 023/01, Brazil.
A. L. Pereira was supported by grant 2002/07464-8 from FAPESP-SP-Brazil.
M. C. Pereira was supported by CNPq - Conselho Nac. Des. Científico e Tecnológico - Brazil.

(1.1) in different regions. One possible approach is the one taken by Henry in [11] which we describe very briefly.

Given an open bounded region $\Omega \subset \mathbb{R}^n$, consider the set

$$\text{Diff}^{\,m}(\Omega) = \{h \in C^m(\Omega, \mathbb{R}^n); h \text{ is injective and } \frac{1}{|\det h'(x)|} \text{ is bounded in } \Omega\}$$

and consider the collection of regions

$$\{h(\Omega_0); h \in \text{Diff}^{\,m}(\Omega_0)\}.$$

We introduce a topology in this set by defining a sub-basis of the neighborhoods of a given set Ω by

$$\{h(\Omega); \|h - i_\Omega\|_{C^m(\Omega,\mathbb{R}^n)} < \varepsilon, \varepsilon > 0 \text{ sufficiently small}\}.$$

When $\|h - i_\Omega\|_{C^m(\Omega,\mathbb{R}^n)}$ is small, h is a C^m embedding of Ω in $\mathbb{R}^n$, that is, a C^m diffeomorphism to its image $h(\Omega)$. Michelleti [14] has shown this topology is metrizable, and the set of regions C^m-diffeomorphic to Ω may be considered a separable metric space, which we denote by $\mathcal{M}_m(\Omega)$, or simply $\mathcal{M}_m$. We say that a function F defined in the space $\mathcal{M}_m$ with values in a Banach space is C^m or analytic if $h \mapsto F(h(\Omega))$ is C^m or analytic as a map of Banach spaces (h near i_Ω in $C^m(\Omega, \mathbb{R}^n)$). In this sense, we may express problems of perturbation of the boundary of a boundary value problem as problems of differential calculus in Banach spaces.

If $h : \Omega \mapsto \mathbb{R}^n$ is a C^k embedding, we may consider the 'pull-back' of h

$$h^* : C^k(h(\Omega)) \to C^k(\Omega) \quad (0 \le k \le m)$$

defined by $h^*(\varphi) = \varphi \circ h$, which is an isomorphism with inverse ${h^{-1}}^*$. Other function spaces can be used instead of C^k, and we will actually be interested mainly in Sobolev spaces and fractional power spaces. If $F_{h(\Omega)} : C^m(h(\Omega)) \to C^0(h(\Omega))$ is a (generally nonlinear) differential operator in $\Omega_h = h(\Omega)$ we can consider $h^* F_{h(\Omega)} {h^*}^{-1}$, which is a differential operator in the fixed region Ω.

Now it is easily seen that $v(\cdot, t)$ satisfies (1.1) in Ω_h if and only if $u(\cdot, t) = h^* v(\cdot, t)$ (that is, $u(x, t) = v(h(x), t)$) satisfies

$$\begin{gathered} u_t(x,t) = h^* \Delta_{\Omega_h} {h^*}^{-1} u(x,t) + f(u(x,t)), \quad x \in \Omega_0, t > 0 \\ u = 0, \quad x \in \partial\Omega_0, \end{gathered} \tag{1.2}$$

where $h^* \Delta_{\Omega_h} {h^*}^{-1} : H^2 \cap H^1(\Omega_0) \to L^2(\Omega_0)$ is defined by

$$\Big[h^* \Delta_{\Omega_h} {h^*}^{-1} u\Big](x) = \Delta_{\Omega_h}(u \circ h^{-1})(h(x)).$$

In particular, if $\mathcal{A}_h$ is the global attractor of (1.1) in $H_0^1(\Omega_h)$, then $\tilde{\mathcal{A}}_h = \{v \circ h | v \in \mathcal{A}_h\}$ is the global attractor of (1.2) in $H_0^1(\Omega_0)$ and conversely. In this way we can consider the problem of continuity of the attractors as $h \to i_{\Omega_0}$ in a fixed phase space.

For simplicity, we work here in L^2 spaces, assuming that the nonlinearity f is globally Lipschitz and satisfies the standard dissipation condition

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} \le 0.$$

This is not such a stringent requirement as it may seem at first in the problem at hand. We may, as is done in [15] for example (see also [3]), pose the problem in L^p

spaces. Choosing p big enough, we can, without assuming any growth condition, prove existence of the attractors and find estimates on their size in L^∞. It turns out that this bound depends only on a linear problem that can be chosen independently of the parameter in the problem we treat. This in turn allows as to perform the standard trick of 'cutting' f outside a ball containing the attractors so as to have it (and as many of its derivatives as wished), globally Lipschitz without changing the attractors.

During the last stages of preparation of this work, we learned of similar results (now already published) obtained by Arrieta and Carvalho [5]. The authors used a different method based on the spectral converge which allows more irregular perturbations. On the other hand, we believe that our method is simpler and gives more detailed results for the regular case. For instance, hyperbolicity of equilibria can be proved in our context for 'generic' regular regions (see [11]). Also, we believe that our method can be used to prove similar results for other (including nonlinear) boundary conditions (see Remark at the end of section 4). The upper semicontinuity of attractors, in the case of 'thin domains', have been obtained previously in [16], also using convergence of the spectra.

This paper is organized as follows. In section 2 we prove a result on continuity of linear semigroups with respect to variation of the generator following the same lines of Theorem 1.3.2 in [10] and apply it to the Laplacian operator in varying domains. We also prove a result on continuity of the unstable manifolds of the equilibria in the appendix. In Section 3 we prove that the nonlinear semigroup $T_h(t)$ generated by (1.2) is continuous with respect to all its arguments. Since continuity with respect to t and initial conditions follows easily from our assumptions, we concentrate in proving the continuity with respect to h, the 'perturbation' of the domain. In Section 4 we prove the main results of the paper, namely, that the family $\{\mathcal{A}_h : \|h - i_\Omega\| < \varepsilon_0\}$ is upper and, assuming hyperbolicity of the equilibria, also lower semicontinuous at i_Ω.

2. Continuity of the linear semigroup with respect to parameters

An abstract result.

Lemma 2.1. *Suppose A is a sectorial operator with $\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda - a|}$ for all λ in the sector $S_{a,\phi_0} = \{\lambda \mid \phi_0 \le |arg(\lambda - a)| \le \pi, \lambda \ne a\}$, for some $a \in \mathbb{R}$ and $0 \le \phi_0 < \pi/2$. Suppose also that B is a linear operator with $D(B) \supset D(A)$ and $\|Bx - Ax\| \le \varepsilon\|Ax\| + K\|x\|$, for any $x \in D(A)$, where K and ε are positive constants with $\varepsilon \le \frac{1}{4(1+LM)}$, $\quad K \le \frac{\sqrt{5}}{20M}\frac{\sqrt{2}L-1}{L^2-1}$, for some $L > 1$.*

Then B is also sectorial. More precisely, if $b = \frac{L^2}{L^2-1}a - \frac{\sqrt{2}L}{L^2-1}|a|$, $\phi = \max\{\phi_0, \frac{\pi}{4}\}$ and $M' = 2M\sqrt{5}$ then

$$\|(\lambda - B)^{-1}\| \le \frac{M'}{|\lambda - b|},$$

in the sector $S_{b,\phi} = \{\lambda \mid \phi \le |arg(\lambda - b)| \le \pi, \lambda \ne b\}$.

Proof. Simple computations show that in the sector $S_{b,\phi}$, we have

$$\frac{|\lambda|}{|\lambda - a|} \le L, \tag{2.1}$$

$$|\lambda - a| \ge \Big(\frac{\sqrt{2}L - 1}{L^2 - 1}\Big)|a|, \tag{2.2}$$

$$\frac{|\lambda - b|}{|\lambda - a|} \le \sqrt{5}. \tag{2.3}$$

Therefore,

$$\begin{aligned}
\|A(\lambda - A)^{-1}\| &\le \|(A-\lambda)(\lambda - A)^{-1}\| + |\lambda|\|(\lambda - A)^{-1}\| \\
&= \|I\| + |\lambda|\|(\lambda - A)^{-1}\| \\
&\le 1 + |\lambda|\frac{M}{|\lambda - a|} \\
&\le 1 + LM \quad \text{by (2.1)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|(A-B)(\lambda - A)^{-1}\| &\le \varepsilon\|A(\lambda - A)^{-1}\| + K\|(\lambda - A)^{-1}\| \\
&\le \varepsilon(1+LM) + K\frac{M}{|\lambda - a|} \le \frac{1}{2}
\end{aligned}$$

by (2.2). Therefore, $I + (A-B)(\lambda - A)^{-1}$ is invertible with $\|[I + (A-B)(\lambda - A)^{-1}]^{-1}\| \le 2$. From this we obtain

$$\begin{aligned}
\|(\lambda - B)^{-1}\| &= \|(\lambda - A + A - B)^{-1}\| \\
&= \|\left[(I + (A-B)(\lambda - A)^{-1})(\lambda - A)\right]^{-1}\| \\
&= \|(\lambda - A)^{-1}(I + (A-B)(\lambda - A)^{-1})^{-1}\| \\
&\le \|(\lambda - A)^{-1}\|\|(I + (A-B)(\lambda - A)^{-1})^{-1}\| \\
&\le \frac{2M}{|\lambda - a|} \\
&= \frac{2M}{|\lambda - b|}\frac{|\lambda - b|}{|\lambda - a|} \\
&\le \frac{2M\sqrt{5}}{|\lambda - b|}
\end{aligned}$$

by (2.3) as claimed. □

Remark 2.2. Observe that b can be made arbitrarily close to a by taking L sufficiently large. In particular, if $a > 0$ then $b > 0$.

Theorem 2.3. *Suppose that A is as in Lemma 2.1, Λ a topological space and $\{A_\gamma\}_{\gamma \in \Lambda}$ is a family of operators in X with $A_{\gamma_0} = A$ satisfying the following conditions:*

1. *$D(A_\gamma) \supset D(A)$, for all $\gamma \in \Lambda$;*
2. *$\|A_\gamma x - Ax\| \le \epsilon(\gamma)\|Ax\| + K(\gamma)\|x\|$ for any $x \in D(A)$, where $K(\gamma)$ and $\epsilon(\gamma)$ are positive functions with $\lim_{\gamma \to \gamma_0} \epsilon(\gamma) = 0$ and $\lim_{\gamma \to \gamma_0} K(\gamma) = 0$.*

Then, there exists a neighborhood V of γ_0 such that A_γ is sectorial if $\gamma \in V$ and the family of (linear) semigroups e^{-tA_γ} satisfy

$$\|e^{-tA_\gamma} - e^{-tA}\| \le C(\gamma)e^{-bt}$$

$$\|A\left(e^{-tA_\gamma} - e^{-tA}\right)\| \le C(\gamma)\frac{1}{t}e^{-bt}$$

$$\|A^\alpha\left(e^{-tA_\gamma} - e^{-tA}\right)\| \le C(\gamma)\frac{1}{t^\alpha}e^{-bt}, \quad 0 < \alpha < 1$$

for $t > 0$, where b is as in Lemma 2.1 and $C(\gamma) \to 0$ as $\gamma \to \gamma_0$.

Proof. If γ is sufficiently close to γ_0, $\varepsilon(\gamma) \le \frac{1}{4(1+LM)}$ and $K(\gamma) \le \frac{\sqrt{5}}{20M}\frac{\sqrt{2}L-1}{L^2-1}$. To simplify the notation we suppress, from now on, the dependence of K and ε on γ. By Lemma 2.1, A_γ is sectorial and the estimate

$$\|(\lambda - A_\gamma)^{-1}\| \le \frac{M'}{|\lambda - b|}$$

holds in the sector $S_{b,\phi} = \{\lambda \mid \phi \le |arg(\lambda - b)| \le \pi, \lambda \ne a\}$ with $M' = 2\sqrt{5}M$; M, b and ϕ are independent of γ.

If Γ is a contour in $-S_{b,\phi}$ with $|\arg \lambda - b| \to \pi - \phi$ as $|\lambda| \to \infty$ then, for any x in X

$$e^{-A_\gamma t}x - e^{-At}x = \frac{1}{2\pi i}\int_\Gamma \left[(\lambda + A_\gamma)^{-1}x - (\lambda + A)^{-1}x\right] e^{\lambda t}\, d\lambda.$$

We estimate the integrand as follows. Firstly we have, for $\lambda \in S_{b,\phi}$

$$\begin{aligned}
\|(\lambda - A_\gamma)^{-1} - (\lambda - A)^{-1}\| &\le \|(\lambda - A_\gamma)^{-1}\left[I - (\lambda - A_\gamma)(\lambda - A)^{-1}\right]\| \\
&\le \|(\lambda - A_\gamma)^{-1}\left[I - (\lambda - A + A - A_\gamma)(\lambda - A)^{-1}\right]\| \\
&\le \|(\lambda - A_\gamma)^{-1}\left[(A - A_\gamma)(\lambda - A)^{-1}\right]\| \\
&\le \|(\lambda - A_\gamma)^{-1}\|\|(A - A_\gamma)\cdot(\lambda - A)^{-1})\|.
\end{aligned}$$

Proceeding as in the proof of Lemma 2.1, we obtain

$$\|(A - A_\gamma)\cdot(\lambda - A)^{-1})\| \le \varepsilon(1 + LM) + K\frac{M}{|\lambda - a|}.$$

It follows that

$$\|(\lambda - A_\gamma)^{-1} - (\lambda - A)^{-1}\| \le \frac{M'}{|\lambda - b|}\big(\varepsilon(1 + LM) + K\frac{M}{|\lambda - a|}\big).$$

Therefore,

$$\begin{aligned}
\|e^{-A_\gamma t} - e^{-At}\| &\le \frac{1}{2\pi}\int_\Gamma \|(\lambda + A_\gamma)^{-1} - (\lambda + A)^{-1}\|\|e^{\lambda t}\|\,d\lambda| \\
&\le \frac{M'}{2\pi}\left(\varepsilon(1 + LM) + \frac{MK(L^2-1)}{(\sqrt{2}L-1)|a|}\right) e^{-bt}\int_\Gamma \frac{|e^{(\lambda+b)t}|}{|\lambda + b|}\,|d\lambda| \\
&\le C_1(\gamma)e^{-bt}\int_\Gamma \frac{|e^{\mu}|}{|\mu|}|\,d\mu|,
\end{aligned}$$

where $C_1(\gamma) \to 0$ as $\gamma \to 0$, as claimed.

Now, we have

$$\|A\left((\lambda - A_\gamma)^{-1} - (\lambda - A)^{-1}\right)\| \le \|A(\lambda - A_\gamma)^{-1}\|\|(A - A_\gamma)\cdot(\lambda - A)^{-1}\|.$$

Proceeding as in Lemma 2.1

$$\begin{aligned}
\|A(\lambda - A_\gamma)^{-1}\| &\le \|(A - A_\gamma)(\lambda - A_\gamma)^{-1}\| + \|A_\gamma(\lambda - A_\gamma)^{-1}\| \\
&\le \varepsilon\|A(\lambda - A_\gamma)^{-1}\| + \frac{KM}{|\lambda - a|} + 1 + LM'
\end{aligned}$$

and

$$\|(A - A_\gamma)(\lambda - A)^{-1}\| \le \varepsilon\,(1 + LM) + \frac{KM}{|\lambda - a|}. \tag{2.4}$$

From (2.4) and (2.4), we obtain

$$\begin{aligned}
&\|A\left((\lambda - A_\gamma)^{-1} - (\lambda - A)^{-1}\right)\| \\
&\leq \frac{1}{1-\varepsilon}\Big(\frac{KM}{|\lambda - a|} + 1 + LM'\Big)\Big\{\varepsilon(1+LM) + \frac{KM}{|\lambda - a|}\Big\} = C_2(\gamma),
\end{aligned}$$

where $C_2(\gamma) \to 0$ as $\gamma \to 0$. Then we have

$$\begin{aligned}
\|A\left(e^{-A_\gamma t} - e^{-At}\right)\| &\leq \frac{1}{2\pi}\int_\Gamma \|A\left((\lambda + A_\gamma)^{-1} - (\lambda + A)^{-1}\right)\| |\, e^{\lambda t}|| \,d\lambda| \\
&\leq \frac{1}{2\pi} C_2(\gamma) e^{-bt}\int_\Gamma |e^{(\lambda+b)t}|\, |d\lambda| \\
&\leq \frac{1}{2\pi} C_2(\gamma)\frac{e^{-bt}}{t}\int_\Gamma \frac{|e^{\mu}|}{|\mu|} |\, d\mu|.
\end{aligned}$$

The above inequality follows immediately from [10, Theorem 1.4.4]. □

2.1. **Application to the Laplacian operator in varying domains.** Suppose Ω is a $\mathcal{C}^2$ domain in $\mathbb{R}^n$ and $h : \Omega \to \mathbb{R}^n$ is a $\mathcal{C}^2$ embedding, i.e., a $\mathcal{C}^2$ diffeomorphism to its image.

Let $\Delta_{h(\Omega)}$ represent the Laplacian operator in the region $h(\Omega)$. Then we can consider the differential operator $h^*\Delta_{h(\Omega)}(h^*)^{-1}$ defined in the fixed region Ω. More explicitly, if $u \in \mathcal{C}^2(h(\Omega))$ and $x \in \Omega$, then

$$\left[(h^*\Delta_{h(\Omega)}(h^*)^{-1})u\right](x) = \left[\Delta_{h(\Omega)}(u \circ h^{-1})\right](h(x)).$$

We need to express the coefficients of $h^*\Delta_{h(\Omega)}(h^*)^{-1}$ in terms of h. To this end, we write

$$h(x) = h(x_1, x_2, \dots, x_n) = (h_1(x), h_2(x) \dots, h_n(x)) = (y_1, y_2, \dots, y_n) = y.$$

Then, we have

$$\begin{aligned}
\big(h^*\frac{\partial}{\partial y_i} h^{*-1}(u)\big)(x) &= \frac{\partial}{\partial y_i}(u \circ h^{-1})(h(x)) \\
&= \sum_{j=1}^n \frac{\partial u}{\partial x_j}(h^{-1}(y))\frac{\partial h_j^{-1}(y)}{\partial y_i}(y) \\
&= \sum_{j=1}^n \big[\big(\frac{\partial h}{\partial x}\big)^{-1}\big]_{j,i}(x)\frac{\partial u}{\partial x_j}(x) \\
&= \sum_{j=1}^n b_{i,j}(x)\ \frac{\partial u}{\partial x_j}(x),
\end{aligned}$$

where $b_{ij}(x)$ is the i,j entry in the inverse-transpose of the Jacobian matrix $h_x = [\frac{\partial h_i}{\partial x_j}]_{i,j=1}^n$. Therefore,

$$\begin{aligned}
h^*\frac{\partial^2}{\partial y_i^2} h^{*-1}(u)(x) &= \sum_{k=1}^n b_{i,k}(x)\frac{\partial}{\partial x_k}\Big(\sum_{j=1}^n b_{i,j}\frac{\partial u}{\partial x_j}\Big)(x) \\
&= \sum_{k=1}^n b_{i,k}(x)\sum_{j=1}^n \big[\frac{\partial b_{i,j}}{\partial x_k}(x)\cdot\frac{\partial u}{\partial x_j}(x) + b_{i,j}(x)\frac{\partial^2 u}{\partial x_k \partial x_j}(x)\big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j,k=1}^{n} b_{i,k}(x) b_{i,j}(x) \frac{\partial^2 u}{\partial x_k \partial x_j}(x) + \sum_{j,k=1}^{n} b_{i,k}(x) \frac{\partial b_{i,j}}{\partial x_k}(x) \frac{\partial u}{\partial x_j}(x) \\
&= \big(\frac{\partial^2}{\partial x_i^2}(u) \big)(x) + L_i(u)(x),
\end{aligned}
$$

where

$$
\begin{aligned}
L_i(u)(x) = (b_{ii}(x) - 1) \frac{\partial^2 u}{\partial x_i} &+ \sum_{j,k=1}^{n} (1 - \delta_{i,j,k}) b_{i,k}(x) b_{i,j}(x) \big(\frac{\partial^2 u}{\partial x_k \partial x_j} \big)(x) \\
&+ \sum_{j,k=1}^{n} b_{i,k}(x) \big(\frac{\partial}{\partial x_k} b_{i,j} \big)(x) \frac{\partial u}{\partial x_j}(x)
\end{aligned}
$$

with $\delta_{i,j,k} = 1$ if $i = j = k$, and 0 otherwise. Thus

$$
\big(h^* \Delta_{h(\Omega)} {h^*}^{-1}(u) \big) = \Delta_\Omega + Lu
$$

with

$$
Lu = \sum_{i=1}^{n} L_i u.
$$

Since $b_{j,k} \to \delta_{j,k}$ in $\mathcal{C}^2(\bar{\Omega})$ as $h \to i_\Omega$ in $\mathcal{C}^2(\bar{\Omega}, \mathbb{R}^n)$, the coefficients of L go to 0 as $h \to i_\Omega$ in $\mathcal{C}^2(\bar{\Omega}, \mathbb{R}^n)$. From the results in [1], we obtain

$$
\|Lu\| \leq \varepsilon(h) \|\Delta_\Omega u\| + K(h) \|u\| \tag{2.5}
$$

with $\varepsilon(h)$ and $K(h)$ going to 0 as $h \to i_\Omega$ in $\mathcal{C}^2(\bar{\Omega}, \mathbb{R}^n)$. (When dealing with Dirichlet boundary conditions, we can take $K(h) = 0$.) Therefore, the estimates obtained in Theorem 2.3 hold for the linear semigroup generated by the operators $h^* \Delta_{h(\Omega)} {h^*}^{-1}(u)$ in $L^2(\Omega)$.

3. Continuity of the nonlinear semigroup

We work first in an abstract setting.

Lemma 3.1. *Suppose Y is a Banach space, Λ is an open set in Y, $\{-A_\lambda\}_{\lambda \in \Lambda}$ is a family of operators in a Banach space X satisfying the conditions of Theorem 2.3 at $\lambda = \lambda_0$, U is an open set in $\mathbb{R}^+ \times X^\alpha$, $0 \leq \alpha < 1$ and $f : U \times \Lambda \to X$ is Hölder continuous in t, continuos in λ at λ_0 uniformly for (t,x) in bounded subsets of U, $\|f(t,x,\lambda) - f(t,y,\lambda)\| \leq L\|x - y\|_\alpha$,$||f(t,x,\lambda_0)|| \leq R$ for (t,x), (t,y) in U and $\lambda \in \Lambda$. Suppose the solution $x(t, x_0, \lambda$ of the problem*

$$
\begin{gathered}
\frac{dx}{dt} = A_\lambda x + f(t,x,\lambda), \quad t > t_0 \\
x(t_0) = x_0.
\end{gathered} \tag{3.1}
$$

exist for x_0 in bounded subsets of X^α λ in a neighborhood of λ_0 and $t_0 \leq t \leq T$.

Then the function $\lambda \mapsto x(t, x_0, \lambda) \in X^\alpha$ is continuous at λ_0 uniformly for x_0 in bounded subsets of X^α and $t_0 \leq t \leq T$.

Proof. Let b be the exponential rate of decay of the semigroup generated by A_λ, λ in the neighborhood of λ_0, given by Theorem 2.3. We write $x_\lambda(t)$ and $x(t)$ for the solutions of (3.1) with parameter values λ and λ_0. If x_0 belongs to a bounded set B of X^α, we have, by the variation of constants formula

$$\begin{aligned} x_\lambda(t) &\leq \|e^{A_\lambda(t-t_0)}x_0\|_\alpha + \int_{t_0}^t \|e^{A_\lambda(t-s)}f(s,x_\lambda(s),\lambda)\|_\alpha\,ds \\ &\leq e^{-b(t-t_0)}\|x_0\|_\alpha + R\int_{t_0}^t e^{-b(t-s)}(t-s)\,ds \end{aligned}$$

so $x_\lambda(t)$ remains in a bounded set for $x_0 \in B$ and $t-t_0$ bounded. By hypothesis there exists then a a function $\theta(\lambda)$ such that $\theta(\lambda) \to 0$ as $\lambda \to \lambda_0$ and $\|f(s,x(s),\lambda)-f(s,x(s),\lambda_0)\|\theta(\lambda)$, for $x_0 \in B$. Using again the the variation of constants formula

$$\begin{aligned} \|x_\lambda(t)-x(t)\|_\alpha \leq \|[e^{A_\lambda(t-t_0)} - e^{A(t-t_0)}]x_0\|_\alpha + \int_{t_0}^t \|e^{A_\lambda(t-s)}[f(s,x_\lambda(s),\lambda) \\ -f(s,x(s),\lambda)]\|_\alpha\,ds + \int_{t_0}^t \|e^{A_\lambda(t-s)}[f(s,x(s),\lambda)-f(s,x(s),\lambda_0)]\|_\alpha\,ds \\ + \int_{t_0}^t \|[e^{A_\lambda(t-s)} - e^{A(t-s)}]f(s,(x(s),\lambda)\|_\alpha\,ds \\ \leq C(\lambda)e^{-b(t-t_0)}(t-t_0)^\alpha\|x_0\| + LM\int_{t_0}^t (t-s)^{-\alpha}e^{-b(t-s)}\|x_\lambda(s)-x(s)\|_\alpha\,ds \\ +\theta(\lambda)M\int_{t_0}^t (t-s)^{-\alpha}e^{-b(t-s)}\,ds + C(\lambda)R\int_{t_0}^t (t-s)^{-\alpha}e^{-b(t-s)}\,ds. \end{aligned}$$

where M is such $\|e^{A_\lambda(t-t_0)}\| \leq Me^{-b(t-t_0)}$.

For $0 < t < T$, we have $\int_{t_0}^t (t-s)^{-\alpha}e^{-b(t-s)}\,ds \leq \eta(t-t_0)^{-\alpha}$, where η is a constant. Therefore, it follows from Gronwall's inequality (see [10]), that $\|x_\lambda(t)-x(t)\|_\alpha \leq \tilde{C}(\lambda)K(t-t_0)^{-\alpha}(1+\|x_0\|_\alpha)$, where $K = K(\alpha,T,B$ does not depend on the initial condition. This proves the claim. $\square$

Lemma 3.2. *Suppose, in addition to the hypotheses of Lemma 3.1, that the derivative $\frac{\partial f}{\partial x}(t,x,\lambda)$ exists, is continuous and bounded for $0 \leq t \leq T$, λ in a neighborhood of λ_0, and x in the ball of radius N. Then, the map $\lambda \mapsto \frac{\partial x(t,x_0,\lambda)}{\partial x_0} \in X^\alpha$ is continuous at λ_0 uniformly for $x_0 \in B$ and $t_0 \leq t \leq T$.*

Proof. The local existence and continuity of the derivative is shown in [10] (theorem 3.4.4). In fact the derivative $v_\lambda(t) = \frac{\partial x(t,x_0,\lambda)}{\partial x_0} \cdot \Delta x_0$ is the solution of the initial (linear) value problem

$$\begin{gathered} \frac{dy}{dt} = A_\lambda y + f_x(t,x(t),\lambda)y, \quad t > t_0 \\ y(t_0) = \Delta x_0. \end{gathered} \tag{3.2}$$

To prove continuity in λ we again use the variation of constants formula as in Lemma 3.1. Due to the linearity in v, we obtain now $\|v_\lambda(s)-v(s)\|_\alpha \leq C(\lambda)K(t-t_0)^{-\alpha}(\|\Delta x_0\|_\alpha)$, where K is a constant depending only on the size of the ball. From this, the result follows readily. $\square$

We now apply the results to our context.

Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^2$ region, $X = L^2(\Omega)$ and $\alpha = 1/2$. Using the results of section 2 and [10], it follows that (1.1) generates a nonlinear $\mathcal{C}^1$ semigroup $T(t,h)x$

in $X^\alpha = H_0^1(\Omega)$, for h in a neighborhood $\mathcal{H}$ of the inclusion i_Ω in $\mathcal{C}^2$. We then have the following result

Corollary 3.3. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^1$ bounded function with bounded derivative, $B \subset H_0^1(\Omega)$ is a bounded open set and $T(t,\mathcal{H})B$ is a bounded set in X^α for $t \in \mathbb{R}^+$. Then, the map $h \in \mathcal{H} \mapsto T(t,h) \in \mathcal{C}^1(B, H_0^1(\Omega))$ is continuous with respect to h at $h = i_\Omega$ for t in compact subsets of $\mathbb{R}^+$.*

Proof. The result follows immediately from Theorem 2.3 and Lemmas 3.1 and 3.2, by taking $U = X^\alpha = H_0^1(\Omega)$. □

4. Existence and continuity of attractors

We first mention some definitions and results from [7] that will be used in the sequel. Suppose Λ is a metric space, X a complete metric space, and, for each $\lambda \in \Lambda$, $T(t,\lambda) : X \to X$ is a $\mathcal{C}^r$-semigroup with $T(t,\lambda)x$ continuous in t, λ, x. For any λ let $T_\lambda(t) = T(\lambda, t) : X \to X$. We say that $T_\lambda(t)$ is *asymptotically smooth* if for any closed, bounded and positively invariant set B, there exists a compact set $K_\lambda(B) \subset B$ that attracts B. The family of mappings $\{T_\lambda(t) : \lambda \in \Lambda\}$ is *collectively asymptotically smooth* if $\bigcup_{\lambda\in\Lambda} K_\lambda(B)$ is compact (for any bounded positively invariant set B).

Theorem 4.1. *Suppose Λ is a metric space, X a complete metric space and $T_\lambda(t)$ is a $\mathcal{C}^r$-gradient semigroup on X, $r \geq 1$, for each $\lambda \in \Lambda$. Denote by E_λ the set of equilibria of $T_\lambda(t)$, for each $\lambda \in \Lambda$.*

If the family of semigroups $\{T_\lambda(t) : \lambda \in \Lambda\}$ is collectively asymptotically smooth, continuous in λ and $\bigcup_{\lambda\in\Lambda} E_\lambda$ is bounded, then the global attractor A_λ of $T_\lambda(t)$ exists and A_λ is upper semicontinuous at $\lambda_0 \in \Lambda$.

Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^2$-region and $h : \Omega \to \mathbb{R}^n$ the family of $\mathcal{C}^2$ embeddings with $||h - i_\Omega||_{\mathcal{C}^2} < \varepsilon$. Consider the family of semigroups $T(h,t)$ generated by (1.2).

We know that (1.1) generates a gradient system in $H_0^1(\Omega_h)$ with Lyapunov functional

$$\tilde{V}_h(\psi) = \int_{\Omega_h} \big[\frac{1}{2}|\nabla\psi(y)|^2 - F(\psi(y))\big]\, dy$$

where $F(v) = \int_0^v f(s)\, ds$. We define V_h in $H_0^1(\Omega)$ by

$$V_h(\phi) = \tilde{V}_h(\phi \circ h^{-1}). \tag{4.1}$$

Since u is a solution of (1.2) if and only if $v = h^{*-1}u$ is a solution of (1.1), we immediately obtain

Lemma 4.2. *The system generated by (1.2) is a gradient system with Lyapunov functional given by (4.1).*

Let $\|u\|_{H^1(\Omega)}$ (resp. $\|u\|_{H^1(\Omega_h)}$) denote the H^1 norm in Ω (resp. Ω_h.)

Define a new norm $\|u\|^h_{H^1(\Omega)}$ in $H^1(\Omega)$ by

$$\|u\|^h_{H^1(\Omega)} = \|u \circ h^{-1}\|_{H^1(\Omega_h)}. \tag{4.2}$$

Lemma 4.3. *Suppose Ω is a $\mathcal{C}^2$ region and $h : \Omega \to \Omega_h$ is a $\mathcal{C}^2$-diffeomorphism. Then, we have*

(1) $\|u\|^h_{H^1(\Omega)}$ *and* $\|u\|_{H^1(\Omega)}$ *are equivalent norms in* $H^1(\Omega)$ *, that is, there are positive constants* $K_1(h)$ *and* $K_2(h)$ *such that* $K_1(h)\|u\|^h_{H^1(\Omega)} \leq \|u\|_{H^1(\Omega)} \leq K_2(h)\|u\|^h_{H^1(\Omega)}$, *for any* u *in*$H^1(\Omega)$. *Furthermore* $K_1(h), K_2(h) \to 1$ *as* $h \to i_\Omega$ *in the* $\mathcal{C}^2$ *norm.*
(2) $K_1(h)|V(u)| \leq |V_h(u)| \leq K_2(h)|V(u)|$, *for any* u *in*$H^1(\Omega)$. *Furthermore* $K_1(h), K_2(h) \to 1$ *as* $h \to i_\Omega$ *in the* $\mathcal{C}^2$ *norm.*

Proof. We prove item (1), the proof of item (2) is similar.

$$\begin{aligned}(\|u\|^h_{H^1(\Omega)})^2 &= \int_{\Omega_h} (u\circ h^{-1}(y))^2 + |\nabla_h(u\circ h^{-1})|^2\, d\, y \\ &= \int_\Omega (u(x))^2 + |\ {}^T(h_x)^{-1}\cdot \nabla u(x)|^2\, |Jh(x)|\, dx,\end{aligned}$$

where ${}^T(h_x)^{-1}$ is the inverse transpose of the Jacobian matrix $h_x = [\frac{\partial h_i}{\partial x_j}]^n_{i,j=1}$ and $Jh(x) = \det h_x$.

Now $Jh(x)$ is clearly bounded from above and below since it is a positive continuous function in $\overline{\Omega}$. The same is true for the norm of the operator $||h_x||$ in $L(\mathbb{R}^n)$. From this the equivalence of the norms follows immediately.

If $h(x) = i_\Omega + r(x)$ with $\|r(x)\|, \|r'(x)\| \leq 1$ for $x \in \Omega$, then

$$(\|u\|^h_{H^1(\Omega)})^2 = \int_\Omega \left((u(x))^2 + |\ {}^T(h_x)^{-1}\cdot\nabla u(x)|^2\right)\ |\ Jh(x)|\, dx. \tag{4.3}$$

We have $Jh(x) = \det(I + r'(x)) = e^{\lambda(x)}$, with $\lambda(x) = \ln(\det(I + r'(x))) = \sum_1^\infty \frac{(-1)^{m-1}}{m} H_m(x)$, where $H_m = \mathrm{trace}(r'(x)^m)$. Now $e^{\lambda(x)} = 1 + \sum_{k=1}^\infty \frac{\lambda^k}{k!}$, and

$$|\sum_{k=1}^\infty \frac{\lambda^k}{k!}| \leq |\lambda| \sum_{k=1}^\infty \frac{|\lambda^{k-1}|}{k!} \leq |\lambda| \sum_{k=1}^\infty \frac{|\lambda^{k-1}|}{(k-1)!} \leq |\lambda| e^{|\lambda|}.$$

Since $|H_m(x)| = |\,\mathrm{trace}\,(r'(x)^m)\,| \leq n\|r'(x)\|^m$, it follows that

$$|\lambda(x)| \leq n\sum_{k=1}^\infty \frac{\|r'(x)\|^k}{k} = n\ln(1 - \|r'(x)\|).$$

Therefore, we obtain

$$|Jh(x)| - 1 \leq -n\ln(1 - \|r'(x)\|)(1 - \|r'(x)\|)^n. \tag{4.4}$$

Furthermore,

$${}^T(I + r'(x))^{-1} = (I + {}^Tr'(x))^{-1} = I + \sum_{k=1}^\infty (-1)^k\, {}^Tr'(x)^k.$$

Thus

$${}^T(I + r'(x))^{-1}\cdot\nabla u(x) = \nabla u(x) + \sum_{k=1}^\infty (-1)^k\, {}^T\, r'(x))^k \cdot \nabla u(x)$$

and so

$$\begin{aligned}\|{}^T(I + r'(x))^{-1}\nabla u(x)\|^2 = \|\nabla u(x)\|^2 &+ 2\langle \sum_{k=1}^\infty (-1)^k\, {}^Tr'(x))^k \cdot \nabla u(x), \nabla u(x)\rangle \\ &+ \|\sum_{k=1}^\infty (-1)^k\, {}^Tr'(x))^k \cdot \nabla u(x)\|^2.\end{aligned}$$

Since

$$\|\sum_{k=1}^{\infty}(-1)^k\,{}^T r'(x))^k\cdot\nabla u(x)\|\le\sum_{k=1}^{\infty}\|r'(x))\|^k\cdot\|\nabla u(x)\|\le\frac{\|r'(x\|}{1-\|r'(x)\|}\|\nabla u(x)\|$$

we obtain

$$\begin{aligned}&\|^T(I+r'(x))^{-1}\cdot\nabla u(x)\|^2-\|\nabla u(x)\|^2\\&\le\frac{2\|r'(x)\|}{1-\|r'(x\|}\|\nabla u(x)\|^2+\big(\frac{\|r'(x)\|}{1-\|r'(x)\|}\big)^2\|\nabla u(x)\|^2.\end{aligned}\tag{4.5}$$

From (4.3), (4.4) and (4.5), we obtain (taking $\|r'\|=\sup_{x\in\Omega}\|r'(x)\|\le 1/2$)

$$\begin{aligned}&(\|u\|^h_{H^1(\Omega)})^2-(\|u\|_{H^1(\Omega)})^2\\&\le\|u\|^2_{H^1(\Omega)}\cdot(-n\ln(1-\|r'\|)(1-\|r'\|)^n)+\frac{2\|r'\|}{1-\|r'\|}\|\nabla u\|^2+(\frac{\|r'\|}{1-\|r'\|})^2\|\nabla u\|^2\\&\le\|u\|^2_{H^1(\Omega)}\big(n(1-\|r'\|)^{n-1}+\frac{2-\|r'\|}{(1-\|r'\|)^2}\big)\|r'\|\\&\le\|u\|^2_{H^1(\Omega)}(n+8)\|r'\|.\end{aligned}$$

From this we obtain

$$\big|\|u\|^h_{H^1(\Omega)}-\|u\|_{H^1(\Omega)}\big|\le\|u\|_{H^1(\Omega)}\big(\frac{\sqrt{n+8}}{2}\sqrt{\|r'\|}\big)$$

which proves the claim. □

Lemma 4.4. *The family E_h of equilibria of (1.2) is uniformly bounded for h in a neighborhood of i_Ω.*

Proof. The equilibria of (1.2) in Ω_h are the solutions of

$$\begin{gathered}\Delta u(x)+f(u(x))=0\quad\text{in }\Omega_h\\ u=0\quad\text{in }\partial\Omega_h.\end{gathered}$$

Multiplying by u and integrating, we get

$$\int_{\Omega_h}u\Delta u\,dx=-\int_{\Omega_h}f(u)u\,dx\,.$$

Therefore,

$$\int_{\Omega_h}|\nabla u|^2\,dx=\int_{\Omega_h}f(u)u\,dx.$$

Since $\limsup_{|u|\to\infty}\frac{f(u)}{u}\le 0$, there exist $\varepsilon>0$ and $M(\varepsilon)>0$ such that $f(u)u<\varepsilon u^2$ for $|u|>M$.

Let $\Omega_1=\{x\in\Omega_h\mid|u(x)|>M\}$ and $\Omega_2=\Omega_h\setminus\Omega_1$. We have

$$\int_{\Omega_h}|\nabla u|^2\,dx\le\varepsilon\int_{\Omega_1}|u|^2\,dx+\int_{\Omega_2}Mf(u)\,dx\le\varepsilon\int_{\Omega_h}|u|^2\,dx+M\|f\||\Omega_h|$$

where $\|f\|=\sup_{|s|\le M}|f(s)|$ and $|\Omega_h|$ is the measure of Ω_h.

Since $\int_{\Omega_h}|\nabla u|^2\,dx\ge\lambda_0\int_{\Omega_h}|u|^2\,dx$, where λ_0 is the first eigenvalue of the Laplacian with Dirichlet boundary conditions, we obtain

$$(1-\frac{\varepsilon}{\lambda_0})\int_{\Omega_h}|\nabla u|^2\,dx\le\|f\|M(\varepsilon)|\Omega_h|\,.$$

Now, in a neighborhood of a fixed region Ω_0, both λ_0 and $|\Omega|$ are continuous functions of h and therefore bounded. If $\lambda^* \geq \lambda_0(h)$, $|\Omega| \leq K$ and $\varepsilon \leq \frac{\lambda^*}{2}$, then

$$\int_{\Omega_h} |\nabla u|^2 \, dx \leq 2\|f\| M(\varepsilon) K$$

as claimed. □

We are now in a position to prove our main results.

Theorem 4.5. *The flow generated by (1.2) has a global compact attractor $\mathcal{A}_h$ for each h in a neighborhood of i_Ω in $\mathcal{C}^2(\Omega, \mathbb{R}^n)$. The family of attractors $\mathcal{A}_h$ is upper semicontinuous in $X^{\frac{1}{2}} = H_0^1(\Omega)$ at $h = i_\Omega$.*

Proof. It follows from Lemmas 4.3 and 4.4 that, for each h in a neighborhood of i_Ω, the semigroup generated by (1.2) has an attractor and they are uniformly bounded in $H_0^1(\Omega)$. From regularity properties of the flow (see [10], Theorem 3.3.6) they are also bounded in X^β for $1/2 < \beta < 1$, and, therefore, their union is a compact set in $H_0^1(\Omega) = X^{1/2}$. From this and Theorem 4.1 the result follows immediately. □

Now we prove the lower semicontinuity property near the inclusion for the semigroup generated by (1.2), under the additional assumption that the equilibria are all hyperbolic. We observe that this property holds generically in h as proved by Henry in [11]. Our proof is based on the following result of Hale and Raugel (see [9], or [7, Theorem 4.10.8]).

Theorem 4.6. *Let X be a Banach space and, for $0 \leq \varepsilon \leq \varepsilon_0$, let $T_\varepsilon(t), t \geq 0$, be a family of semigroups on X. Suppose the following hypotheses hold*

- (H1) *$T_0(t)$ is a $\mathcal{C}^1$-gradient system, asymptotically smooth and orbits of bounded sets are bounded.*
- (H2) *The set E_0 of equilibrium points of $T_0(t)$ is bounded in X.*
- (H3) *Each element of E_0 is hyperbolic.*
- (H4) *For $\varepsilon \neq 0$, $T_\varepsilon(t)$ is a $\mathcal{C}^1$-semigroup which is asymptotically smooth.*
- (H5) *If E_ε is the set of equilibrium points of $T_\varepsilon(t)$ and $E_0 = \{\phi_1, \phi_2, \dots, \phi_N\}$, then there exists a neighborhood W_0 of E_0 such that*
$$W_0 \cap E_\varepsilon = \{\phi_{1,\varepsilon}, \phi_{2,\varepsilon}, \dots, \phi_{N,\varepsilon}\},$$
where each $\phi_{j,\varepsilon}$, $1 \leq j \leq N$, is hyperbolic and $\phi_{j,\varepsilon} \to \phi_j$ as $\varepsilon \to 0$.
- (H6) *$\delta_X(W^u_{\rm loc}(\phi_j), W^u_{{\rm loc},\varepsilon}(\phi_{j,\varepsilon})) \to 0$ as $\varepsilon \to 0$.*
- (H7) *$T_\varepsilon(t)x$ is continuous in ε uniformly with respect to (t, x) in bounded sets of $\mathbb{R}^+ \times X$.*

Then the family of sets $\{A_\varepsilon, 0 \leq \varepsilon \leq \varepsilon_0\}$ is continuous in X at $\varepsilon = 0$.

Theorem 4.7. *The family of attractors $\mathcal{A}_h$ of (1.2) is continuous in $X^{\frac{1}{2}} = H_0^1(\Omega)$ at $h = i_\Omega$*

Proof. Hypotheses (H1), (H2), (H4) and (H7) have already been proved, and (H3) is one of our hypotheses. (H5) follows by the Implicit Function Theorem applied to the map

$$\begin{aligned} F : H^2 \cap H_0^1(\Omega) \times \mathcal{H} &\to L^2(\Omega) \\ (u, h) &\to h^*(\Delta + \lambda) {h^*}^{-1} u + f(u), \end{aligned}$$

where $\mathcal{H}$ is a neighborhood of i_Ω, observing that the equilibria are all in a common compact set.

Finally (H6) is a consequence of Lemma 3.3 and results of [18] or [4], as observed in [17]. We also offer a direct proof in the appendix.

From this (H6) follows and the result is, therefore, proved. $\square$

We proved continuity in $X^{1/2} = H_0^1(\Omega)$ but our attractors actually belong to more regular spaces (see [3] for a discussion of this point). Using the regularization properties of the semigroup one can easily prove continuity also in these spaces. To be more precise, denote by Δ_p the Laplacian operator in L^p with Dirichlet boundary conditions. It is well-known that Δ_p is a sectorial operator with domain $W^{2,p} \cap W_0^{1,p}$. If Ω is regular enough (see [10]), the fractional power spaces X_p^α satisfy the continuous embeddings:

$$\begin{gathered} X^\alpha \subset W^{k,p}(\Omega) \quad \text{when } k - \frac{n}{q} < 2\alpha - \frac{n}{p}, \quad q \geq p; \\ X^\alpha \subset C^\nu(\Omega) \quad \text{when } 0 < \nu < 2\alpha - \frac{n}{p} \end{gathered}$$

We then have the following continuity result.

Corollary 4.8. *The family of attractors $\mathcal{A}_h$ of (1.2) is continuous at $h = i_\Omega$ in the topology of the fractional power space X_p^α, for any $p > 0$, $0 < \alpha < 1$. In particular, it is continuous in the topology of $C^{1+\delta}$ for any $0 < \delta < 1$.*

Proof. Since the first eigenvalue of the Dirichlet problem for the Laplacian is bounded away from zero for Ω in a neighborhood of the reference region Ω_0, it follows from results in [3] or [15], that the family of attractors $\mathcal{A}_h$ of (1.2) is contained in a bounded set of L^∞ and, therefore, also in a bounded set $B_p \subset L^p$ for any $0 < p < \infty$. Since we have assumed that f is globally bounded, we also obtain $\|f(u)\|_p \leq M$, for $u \in B_p$ (M can be taken independently of p). If u_0 is an initial condition in B_p, we obtain from the variation of constants formula

$$\begin{aligned} \|u_h(t)\|_\alpha &\leq \|e^{A_h(t-t_0)}u_0\|_\alpha + \int_{t_0}^t \|e^{A_h(t-s)} f(s, u_h(s), h)\|_\alpha \, ds \\ &\leq C(h) e^{-b(t-t_0)}(t-t_0)^\alpha \|u_0\| + M \int_{t_0}^t (t-s)^{-\alpha} e^{-b(t-s)} \, ds \end{aligned}$$

where $\|\cdot\|_\alpha$ denotes the norm in the fractional power space associated with the function $h^* \Delta_{h(\Omega)} {h^*}^{-1}$ in L^p. Taking $(t - t_o) = 1$ and using the invariance of the attractors we obtain an uniform bound for the family $\mathcal{A}_h$ in X_p^α. Since X_p^α is compactly imbedded in $X_p^{\alpha'}$ if $\alpha > \alpha'$, the family $\mathcal{A}_h$ actually lies in a compact subset of X_p^α. We then can show upper semicontinuity of the family arguing by contradiction. Suppose the family $\mathcal{A}_h$ is not upper semicontinous at $h = i_\Omega$. Then we may find a sequence $x_n \in \mathcal{A}_{h_n}$, with $h_n \to i_\Omega$ as $n \to \infty$ such that $dist_\alpha(x_n, \mathcal{A}) \geq \varepsilon > 0$, where $dist_\alpha$ is the distance in X_p^α. By compacity, we can extract a subsequence, which we still denote by x_n for simplicity, converging in X_p^α to some point x. We must have, of course, $dist_\alpha(x, \mathcal{A}) \geq \varepsilon$. However, if $p \geq 2$, $\alpha \geq \frac{1}{2}$, such a sequence must also converge in $X_2^{\frac{1}{2}} = H_0^1$. Since upper semicontinuity in H_0^1 has already been established, we conclude that x must lie in $\mathcal{A}_0 = \mathcal{A}_{i_\Omega}$, which is a contradiction.

The lower semicontinuity can be obtained in a similar way. $\square$

Remark 4.9. We have considered here only the reaction-diffusion problem with Dirichlet boundary conditions. Our method also applies to other boundary conditions but the extension is not completely straightforward. The difficulty with, say, Neumann boundary conditions is that they change with the 'change of coordinates' h, so we do not have a fixed space of functions to work with. One way to circumvent this difficulty is to consider the problem in a weaker space, where the boundary conditions do not appear explicitly but are included in the operator. This has been done, for instance in [6] and [15], to treat nonlinear boundary conditions. One then obtain continuity only in a weaker norm but then the 'bootstrap' arguments of Corollary 4.8 can be used to obtain continuity in better spaces.

5. Appendix

In this section, we study a more general equation than (1.1). Given an open bounded region Ω in $\mathbb{R}^n$, a Banach space X_Ω composed by real valued functions in Ω and a sectorial operator A_Ω in X_Ω, we consider the equation

$$\begin{aligned} v_t + A_{h(\Omega)}v &= f^h(v) \quad \text{in } h(\Omega),\ t>0 \\ v(0) &= v_0 \end{aligned} \tag{5.1}$$

where $h \in \operatorname{Diff}^m(\Omega)$, $v \in X_{h(\Omega)}$ and $f^h \in C^k(X^\alpha_{h(\Omega)}, X_{h(\Omega)})$ for some $k \geq 1$ and some $0 \leq \alpha < 1$.

As in (1.2), we work with the problem

$$\begin{aligned} u_t + h^* A_{h(\Omega)} {h^*}^{-1} u &= f^{i_\Omega}(u) \quad \text{in } \Omega,\ t>0 \\ u(0) &= u_0 \end{aligned} \tag{5.2}$$

which is equivalent the problem (5.1). In fact, if h^* is an isomorphism of $X_{h(\Omega)}$ in X_Ω and $h^* f^h({h^*}^{-1}u) = f^{i_\Omega}(u)$ for all h we have that v is solution of (5.1) if and only if $u = h^* v \in X_\Omega$ is solution of (5.2). So, we always think that $h^* : X_{h(\Omega)} \to X_\Omega$ is an isomorphism with inverse ${h^*}^{-1} = {h^{-1}}^* : X_\Omega \to X_{h(\Omega)}$.

Observe that $\{h^* A_{h(\Omega)} {h^*}^{-1} u\}_{h \in \operatorname{Diff}^m(\Omega)}$ is a family of operators in X_Ω. We will assume in this section that this family satisfies the conditions of the Theorem 2.3. Our first result is

Proposition 5.1. *Given $h_0 \in \operatorname{Diff}^m(\Omega)$, assume that $f^{h_0} : X^1_{h_0(\Omega)} \to X_{h_0(\Omega)}$ and $X^1_{h_0(\Omega)} \times \operatorname{Diff}^m(\Omega) \to X_{h_0(\Omega)} : (u,h) \to h^* A_{h(\Omega)} {h^*}^{-1} u$ are C^1 and that e is a hyperbolic equilibrium of (5.2) with $h = h_0$. Then, there exist bounded neighborhoods $U_0 \subset X^1_{h_0(\Omega)}$ and $\mathcal{H}_0 \subset \operatorname{Diff}^m(\Omega)$ of e and h_0, respectively, such that given $h \in \mathcal{H}_0$ there exists an unique equilibrium $e(h)$ of (5.2) in U_0 with the same Morse index as the equilibrium e. Also, the map $\mathcal{H}_0 \to U_0 : h \to e(h)$ is $\mathcal{C}^1$.*

Proof. We may assume that $h_0 = i_\Omega$. Consider the mapping

$$F : X^1_\Omega \times \operatorname{Diff}^m(\Omega) \to X_\Omega : (u,h) \to h^* A_{h(\Omega)} {h^*}^{-1} u - f^{i_\Omega}(u).$$

Of course, F is $\mathcal{C}^1$ and $F(e, i_\Omega) = 0$. Since e is hyperbolic we have that $\frac{\partial F}{\partial u}(e, i_\Omega) = A_\Omega - f'(e)$ is an isomorphism and, by the Implicit Function Theorem, there exists a neighborhood $\mathcal{H}_0$ of i_Ω and a $\mathcal{C}^1$ map $h \to e(h)$ of $\mathcal{H}_0$ in X^1_Ω such that $e(i_\Omega) = e$ and, for all $h \in \mathcal{H}_0$, $F(e(h), h) = 0$. Observe that the Implicit Function Theorem also implies that $\frac{\partial F}{\partial u}(e(h), h)$ is an isomorphism for all $h \in \mathcal{H}_0$, that is, $e(h)$ is a hyperbolic equilibrium for all $h \in \mathcal{H}_0$. Moreover, by the hypotheses of A_Ω and f^{i_Ω}, there exist real positive continuous functions $\epsilon(h)$ and $\delta(h)$ defined in $\mathcal{H}_0$ such that for all $h \in \mathcal{H}_0$

$$\begin{aligned}&\|(A_\Omega - g'(e) - h^* A_{h(\Omega)} {h^*}^{-1} + g'(e(h)))u\| \\ &\le \|(A_\Omega - h^* A_{h(\Omega)} {h^*}^{-1})u\| + \|g'(e) - g'(e(h))\| \, \|u\| \\ &\le \epsilon(h)\|A_\Omega u\| + \delta(h)\|u\|,\end{aligned}$$

for all $u \in D(A_\Omega)$. So, it follows from [12, Theorems 2.14 IV and 3.16 IV] that the Morse index of $e(h)$ is constant in $\mathcal{H}_0$. □

Let $\mathcal{H}$ be a neighborhood of i_Ω in $\mathrm{Diff}^m(\Omega)$ such that $e(h)$ is a hyperbolic equilibrium of (5.2) for all $h \in \mathcal{H}$ with $e(h) \in U \subset X_\Omega$ continuous in h. Suppose that $Re\,\sigma(A_\Omega) > 0$ and the function $f : X^\alpha = D(A_\Omega^\alpha) \to X = X_\Omega$ is C^1 and satisfies

$$f(e(h) + z) = h^* A_{h(\Omega)} {h^*}^{-1} e(h) + f'(e(h))z + r(z, h), \tag{5.3}$$

for all $h \in \mathcal{H}$, with $r(0, h) = 0$, $\sup_{\|z\|_\alpha \le \varrho} \|r(z, h_0) - r(z, h)\| \le C_{h_0}(h)$, $C_{h_0}(h) \to 0$ when $h \to h_0$ in $\mathcal{H}$, $\|r(z_1, h) - r(z_2, h)\| \le k(\varrho)\|z_1 - z_2\|_\alpha$ for $\|z_1\|_\alpha \le \varrho$, $\|z_2\|_\alpha \le \varrho$, $k(\varrho) \to 0$ when $\varrho \to 0+$ and $k(\cdot)$ is nondecreasing.

Assume also that the family of operators $\{h^* A_{h(\Omega)} {h^*}^{-1}\}_{h\in\mathcal{H}}$ satisfies the hypotheses of Theorem 2.3 and the Banach spaces $D(h^* A_{h(\Omega)} {h^*}^{-1\alpha})$ are all equivalent for some $0 \le \alpha < 1$, that is, given $0 \le \alpha < 1$, there are positive constants m_α and M_α such that

$$m_\alpha \|h^* A_{h(\Omega)} {h^*}^{-1\alpha} u\| \le \|A_\Omega^\alpha u\| \le M_\alpha \|h^* A_{h(\Omega)} {h^*}^{-1\alpha} u\|,$$

for all $u \in D(A_\Omega^\alpha)$ and all $h \in \mathcal{H}$. Since $e(h)$ is a hyperbolic equilibrium of the equations (5.2), we have that $L(h) = h^* A_{h(\Omega)} {h^*}^{-1} - f'(e(h))$ is an isomorphism for all $h \in \mathcal{H}$. We decompose X in subspaces X_1 and X_2 corresponding to the spectral sets $\sigma_1 = \sigma(L(i_\Omega)) \cap \{Re\lambda < 0\}$ and $\sigma_2 = \sigma(L(i_\Omega)) \cap \{Re\lambda > 0\}$. Let E_1, E_2 be the projections onto X_1 and X_2, respectively. The hypotheses on A_Ω and f imply the existence of positive real continuous functions $\epsilon(h)$ and $\delta(h)$ defined in $\mathcal{H}$ such that for all $h \in \mathcal{H}$, $L(h)$ is a sectorial operator in X and for all $u \in D(A_\Omega)$,

$$\|(L(i_\Omega) - L(h))u\| \le \epsilon(h)\|A_\Omega u\| + \delta(h)\|u\|.$$

By Theorem 2.3 and by [10, Theorem 1.5.3], if $\epsilon(h)$ and $\delta(h)$ are sufficiently small in $\mathcal{H}$ the following estimates hold for positive constants M and b independent on h

$$\|A_\Omega^\alpha e^{-L(h)_1 t}\| \le Me^{bt}, \quad \|e^{-L(h)_1 t}\| \le Me^{bt}, \quad t \le 0; \tag{5.4}$$

$$\|A_\Omega^\alpha e^{-L(h)_2 t}\| \le Mt^{-\alpha}e^{-bt}, \quad \|A_\Omega^\alpha e^{-L(h)_2 t} E_2 A_\Omega^{-\alpha}\| \le Me^{-bt} \quad t \ge 0. \tag{5.5}$$

Theorem 5.2. *Under the above hypotheses, there exists $\varrho > 0$ such that, for any $h \in \mathcal{H}$,*
1. The stable local manifold of $e(h)$

$$W^s_{\rm loc}(e(h)) = \{e(h) + z_0 \in X^\alpha : \|E_2 z_0\|_\alpha \le \frac{\varrho}{2M}, \|z(t, t_0, z_0, h)\|_\alpha \le \varrho \text{ for } t \ge t_0\}$$

where $z(t, t_0, z_0, h)$ is the solution of the equation

$$z_t + L(h)z = r(z, h) \quad \text{for } t \ge t_0 \tag{5.6}$$

with initial value z_0. When $z_0 + e(h) \in W^s_{\rm loc}(e(h))$, $\|z(t, t_0, z_0, h)\|_\alpha \to 0$ as $t \to \infty$.
2. The unstable local manifold $e(h)$

$$W^u_{\rm loc}(e(h)) = \{e(h) + z_0 \in X^\alpha; \|E_1 z_0\|_\alpha \le \frac{\varrho}{2M}, \|z(t, t_0, z_0, h)\|_\alpha \le \varrho \text{ for } t \le t_0, \}$$

where $z(t, t_0, z_0, h)$ is the solution of the equation (5.6) in $(-\infty, t_0)$ with initial value z_0. When $z_0 + e(h) \in W^u_{\mathrm{loc}}(e(h))$, $\|z(t, t_0, z_0, h)\|_\alpha \to 0$ as $t \to -\infty$.
3. If h_0 and h are sufficiently close, the solution $z(t, t_0, z_0(h_0), h_0)$ and the solution $z(t, t_0, z_0(h), h)$ of (5.6) are close in X^α uniformly in $[t_0, \infty)$ (or $(-\infty, t_0]$).
4. If $\beta(O, Q) = \sup_{o \in O} \inf_{q \in Q} \|q - o\|_\alpha$ for O, $Q \subset X^\alpha$, then for all $h_0 \in \mathcal{H}$,

$$\beta\Big(W^s_{\mathrm{loc}}(e(h)), W^s_{\mathrm{loc}}(e(h_0))\Big), \quad \beta\Big(W^s_{\mathrm{loc}}(e(h_0)), W^s_{\mathrm{loc}}(e(h))\Big),$$
$$\beta\Big(W^u_{\mathrm{loc}}(e(h)), W^u_{\mathrm{loc}}(e(h_0))\Big), \quad \beta\Big(W^u_{\mathrm{loc}}(e(h_0)), W^u_{\mathrm{loc}}(e(h))\Big)$$

approach zero as $h \to h_0$ in $\mathcal{H}$.

Proof. The results 1 and 2 follow from [10, Theorem 5.2.1], the independence of M and b of the variable h in $\mathcal{H}$ and the equivalence of the spaces $D(h^* A_{h(\Omega)} {h^*}^{-1^\alpha})$.

To prove 3 is sufficient to show that the map defined in the proof of [10, Theorem 5.2.1] is a uniform contraction and is continuous in h. We prove 3 only in the interval $[t_0, \infty)$, the other case is analogous. Let

$$U_0 = \{a \in X_2 : \|a\|_\alpha \le \frac{\varrho}{2M}\}$$

and

$$\begin{aligned} Z_0 = \big\{ & z : [t_0, \infty) \to X^\alpha ; z \text{ is continuous}, \\ & \sup \|z(t)\|_\alpha \le \varrho, E_2 z(t_0) = a \text{ with } \|a\|_\alpha \le \frac{\varrho}{2M} \big\}. \end{aligned}$$

The contraction map in [10, Theorem 5.2.1] now depends on the parameter h and is given by $G : Z_0 \times U_0 \times \mathcal{H} \to Z_0$ defined by

$$\begin{aligned} G(z, a, h)(t) = e^{-L(h)_2(t - t_0)} a + \int_{t_0}^t e^{-L(h)_2(t-s)} E_2 r(z(s), h) ds \\ - \int_t^\infty e^{-L(h)_1(t-s)} E_1 r(z(s), h) ds. \end{aligned}$$

Since estimates (5.4) and (5.5) are uniform in $\mathcal{H}$, we can choose $\varrho > 0$ as in [10, Theorem 5.2.1] so small as to have

$$M k(\varrho) \{ \|E_2\| \int_0^\infty u^{-\alpha} e^{-bu} du + \|E_1\| \int_0^\infty e^{-bu} du \} < \frac{1}{2}.$$

Therefore, G is a contraction map uniformly in $\mathcal{H}$ and U_0. Now, we need to prove that G is continuous in $\mathcal{H}$. If h_0 and $h \in \mathcal{H}$, then

$$\begin{aligned} & \|G(z, a, h_0)(t) - G(z, a, h)(t)\|_\alpha \\ & \le \|\Big(e^{-L(h_0)_2(t-t_0)} - e^{-L(h)_2(t-t_0)}\Big) E_2 z(t_0)\|_\alpha \\ & \quad + \int_{t_0}^t \|\Big(e^{-L(h_0)_2(t-s)} - e^{-L(h)_2(t-s)}\Big) E_2 r(z(s), h_0)\|_\alpha ds \\ & \quad + \int_{t_0}^t \|e^{-L(h)_2(t-s)} E_2 \Big(r(z(s), h_0) - r(z(s), h)\Big)\|_\alpha ds \\ & \quad + \int_t^\infty \|\Big(e^{-L(h_0)_1(t-s)} - e^{-L(h)_1(t-s)}\Big) E_1 r(z(s), h_0)\|_\alpha ds \\ & \quad + \int_t^\infty \|e^{-L(h)_1(t-s)} E_1 \Big(r(z(s), h_0) - r(z(s), h)\Big)\|_\alpha ds \end{aligned} \tag{5.7}$$

and so

$$\begin{aligned}
&\|G(z,a,h_0)(t)-G(z,a,h)(t)\|_\alpha\\
&\le C_\alpha(h)e^{-bt}\|z(t_0)\|_\alpha + C_{\alpha,1}(h)\Big(\varrho k(\varrho)\|E_1\|\int_0^\infty e^{-bu}du\Big)\\
&\quad + C_{\alpha,2}(h)\Big(\varrho k(\varrho)\|E_2\|\int_0^\infty u^{-\alpha}e^{-bu}du\Big)\\
&\quad + \sup_{\|z\|_\alpha\le\varrho}\|r(z,h_0)-r(z,h)\|\Big(M\|E_2\|\int_0^\infty u^{-\alpha}e^{-bu}du + M\|E_1\|\int_0^\infty e^{-bu}du\Big)
\end{aligned}$$

where $C_\alpha(h)$, $C_{\alpha,1}(h)$, $C_{\alpha,2}(h)$ and $\sup_{\|z\|_\alpha\le\varrho}\|r(z,h_0)-r(z,h)\|$ approach 0 as $h\to h_0$ in $\mathcal{H}$. Therefore,

$$\sup_{t\in[t_0,\infty)}\|G(z,a,h_0)(t)-G(z,a,h)(t)\|_\alpha \le C(h), \tag{5.8}$$

with $C(h)\to 0$ as $h\to h_0$ in $\mathcal{H}$ and so G is continuous em h_0.

Now, we prove 4 only for $W^s_{\rm loc}(e(h))$. The other cases are similar. For each $h\in\mathcal{H}$ we have by [10, Theorem 5.2.1] that $W^s_{\rm loc}(e(h))$ is image of the Lipschitz map $\Phi_h: U_0\to X^\alpha$, defined by

$$\Phi_h(a) = a - \int_{t_0}^\infty e^{-L(h)_1(t_0-s)}E_1 r(z(s,t_0,a,h),h)ds,$$

where $z(t,t_0,a,h)$ is the solution of the equation (5.6) for $t>t_0$ with initial value $z(t_0,t_0,a,h)=\Phi_h(a)$. Since $\Phi_h(a)=G(z,a,h)(t_0)$, it follows from (5.7) that

$$\|\Phi_{h_0}(\cdot)-\Phi_h(\cdot)\|_\alpha\to 0 \quad \text{as } h\to h_0.$$

Since $W^s_{\rm loc}(e(h))$ is the image of the application Φ_h we have

$$\beta\Big(W^s_{\rm loc}(e(h)), W^s_{\rm loc}(e(h_0))\Big) = \sup_{a\in U_0}\inf_{b\in U_0}\|\Phi_h(a)-\Phi_{h_0}(b)\|_\alpha.$$

Then, since $\inf_{b\in U_0}\|\Phi_h(a)-\Phi_{h_0}(b)\|_\alpha \le \|\Phi_h(a)-\Phi_{h_0}(a)\|_\alpha$ for all $a\in U_0$, we have

$$\beta\Big(W^s_{\rm loc}(e(h)), W^s_{\rm loc}(e(h_0))\Big) \le \sup_{a\in U_0}\|\Phi_h(a)-\Phi_{h_0}(a)\|_\alpha \to 0,$$

when $h\to h_0$ in $\mathcal{H}$, and the proof is complete. □

Acknowledgments. The authors wish to thank the anonymous referee for his/her helpful comments; in particular the suggestions that lead to Corollary 4.8 and Remark 4.9. Part of this work was carried out while the second author was visiting the Instituto Superior Técnico de Lisboa, Portugal. He wishes to thank for the support and warm hospitality while being there.

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