Electronic Journal of Differential Equations, Vol. 2007(2007), No. 177, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF POSITIVE SOLUTIONS FOR $p(x)$-LAPLACIAN PROBLEMS

GHASEM A. AFROUZI, HORIEH GHORBANI

Abstract. We consider the system of differential equations

$$\begin{gathered}-\Delta_{p(x)}u = \lambda[g(x)a(u) + f(v)] \quad \text{in } \Omega\\ -\Delta_{q(x)}v = \lambda[g(x)b(v) + h(u)] \quad \text{in } \Omega\\ u = v = 0 \quad \text{on } \partial\Omega\end{gathered}$$

where $p(x) \in C^1(\mathbb{R}^N)$ is a radial symmetric function such that $\sup |\nabla p(x)| < \infty$, $1 < \inf p(x) \leq \sup p(x) < \infty$, and where $-\Delta_{p(x)}u = -\operatorname{div} |\nabla u|^{p(x)-2}\nabla u$ which is called the $p(x)$-Laplacian. We discuss the existence of positive solution via sub-super-solutions without assuming sign conditions on $f(0), h(0)$.

1. Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions has been a new and interesting topic. Many results have been obtained on this kind of problems; see for example [3, 4, 5, 6, 7, 8, 13]. In [5, 6] Fan and Zhao give the regularity of weak solutions for differential equations with nonstandard $p(x)$-growth conditions. Zhang [11] investigated the existence of positive solutions of the system

$$\begin{gathered}-\Delta_{p(x)}u = f(v) \quad \text{in } \Omega\\ -\Delta_{p(x)}v = g(u) \quad \text{in } \Omega\\ u = v = 0 \quad \text{on } \partial\Omega\end{gathered} \tag{1.1}$$

where $p(x) \in C^1(\mathbb{R}^N)$ is a function, $\Omega \subset \mathbb{R}^N$ is a bounded domain. The operator $-\Delta_{p(x)}u = -\operatorname{div} |\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. Especially, if $p(x)$ is a constant p, System (1.1) is the well-known p-Laplacian system. There are many papers on the existence of solutions for p-Laplacian elliptic systems, for example [1, 3, 4, 5, 6, 7, 8, 9].

2000 *Mathematics Subject Classification.* 35J60, 35B30, 35B40.
Key words and phrases. Positive radial solutions; $p(x)$-Laplacian problems; boundary value problems.

Submitted July 18, 2007. Published December 17, 2007.

In [9] the authors consider the existence of positive weak solutions for the p-Laplacian problem

$$\begin{gathered}
-\Delta_p u = f(v) \quad \text{in } \Omega \\
-\Delta_p v = g(u) \quad \text{in } \Omega \\
u = v = 0 \quad \text{on } \partial\Omega\,.
\end{gathered} \tag{1.2}$$

There the first eigenfunctions is used for constructing the subsolution of p-Laplacian problems. Under the condition $\lim_{u\to+\infty} f(M(g(u))^{1/(p-1)}/u^{p-1} = 0$, for all $M > 0$, the authors show the existence of positive solutions for problem (1.2).

In this paper, at first, we consider the existence of positive solutions of the system

$$\begin{gathered}
-\Delta_{p(x)} u = F(x,u,v) \quad \text{in } \Omega \\
-\Delta_{p(x)} v = G(x,u,v) \quad \text{in } \Omega \\
u = v = 0 \quad \text{on } \partial\Omega
\end{gathered} \tag{1.3}$$

where $p(x) \in C^1(\mathbb{R}^N)$ is a function, $F(x,u,v) = [g(x)a(u) + f(v)]$, $G(x,u,v) = [g(x)b(v) + h(u)]$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain. Then we consider the system

$$\begin{gathered}
-\Delta_{p(x)} u = \lambda F(x,u,v) \quad \text{in } \Omega \\
-\Delta_{p(x)} v = \lambda G(x,u,v) \quad \text{in } \Omega \\
u = v = 0 \quad \text{on } \partial\Omega
\end{gathered} \tag{1.4}$$

where $p(x) \in C^1(\mathbb{R}^N)$ is a function, $F(x,u,v) = [g(x)a(u) + f(v)]$, $G(x,u,v) = [g(x)b(v) + h(u)]$, λ is a positive parameter and $\Omega \subset \mathbb{R}^N$ is a bounded domain.

To study $p(x)$-Laplacian problems, we need some theory on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [4]). If $\Omega \subset \mathbb{R}^N$ is an open domain, write

$$C_+(\Omega) = \{h : h \in C(\Omega), h(x) > 1 \text{ for } x \in \Omega\}$$

$h^+ = \sup_{x\in\Omega} h(x)$, $h^- = \inf_{x\in\Omega} h(x)$, for any $h \in C(\Omega)$, $L^{p(x)}(\Omega) = \{u|u$ is a measurable real-valued function, $\int_\Omega |u|^{p(x)}dx < \infty\}$.

Throughout the paper, we will assume that $p \in C_+(\Omega)$ and $1 < \inf_{x\in\mathbb{R}^N} p(x) \le \sup_{x\in\mathbb{R}^N} p(x) < N$. We introduce the norm on $L^{p(x)}(\Omega)$by

$$|u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \le 1\},$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space, we call it generalized Lebesgue space. The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, reflexive and uniform convex Banach space (see [4, Theorem 1.10, 1.14]).

The space $W^{1,p(x)}(\Omega)$ is defined by $W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$, and it is equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniform convex Banach space (see [4, Theorem 2.1]). We define

$$(L(u), v) = \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u \nabla v dx, \quad \forall u, v \in W^{1,p(x)}(\Omega),$$

then $L : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))^*$ is a continuous, bounded and is a strictly monotone operator, and it is a homeomorphism [7, Theorem 3.11].

Functions u, v in $W_0^{1,p(x)}(\Omega)$, is called a weak solution of (1.4); it satisfies

$$\begin{gathered}
\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \xi dx = \int_\Omega \lambda F(x,u,v)\xi dx, \quad \forall \xi \in W_0^{1,p(x)}(\Omega),\\
\int_\Omega |\nabla v|^{q(x)-2}\nabla v \nabla \xi dx = \int_\Omega \lambda G(x,u,v)\xi dx, \quad \forall \xi \in W_0^{1,p(x)}(\Omega).
\end{gathered}$$

We make the following assumptions

(H1) $p(x) \in C^1(\mathbb{R}^N)$ is a radial symmetric and $\sup |\nabla p(x)| < \infty$

(H2) $\Omega = B(0,R) = \{x||x| < R\}$ is a ball, where $R > 0$ is a sufficiently large constant.

(H3) $a, b \in C^1([0,\infty))$ are nonnegative, nondecreasing functions such that

$$\lim_{u\to+\infty} \frac{a(u)}{u^{P^- -1}} = 0, \quad \lim_{u\to+\infty} \frac{b(u)}{u^{P^- -1}} = 0\,.$$

(H4) $f, h \in C^1([0,\infty))$ are nondecreasing functions, $\lim_{u\to+\infty} f(u) = +\infty$, $\lim_{u\to+\infty} h(u) = +\infty$, and

$$\lim_{u\to+\infty} \frac{f(M(h(u))^{\frac{1}{p^- -1}})}{u^{p^- -1}} = 0, \quad \forall M > 0\,.$$

(H5) $g : [0,+\infty) \to (0,\infty)$ is a continuous function such that $L_1 = \min_{x\in\bar\Omega} g(x)$, and $L_2 = \max_{x\in\bar\Omega} g(x)$.

We shall establish the following result.

Theorem 1.1. *If* (H1)–(H5) *hold, then* (1.3) *has a positive solution.*

Proof. We establish this theorem by constructing a positive subsolution (ϕ_1, ϕ_2) and supersolution (z_1, z_2) of (1.3), such that $\phi_1 \le z_1$ and $\phi_2 \le z_2$. That is (ϕ_1, ϕ_2) and (z_1, z_2) satisfy

$$\begin{gathered}
\int_\Omega |\nabla \phi_1|^{p(x)-2}\nabla \phi_1 \cdot \nabla \xi dx \le \int_\Omega g(x)a(\phi_1)\xi dx + \int_\Omega f(\phi_2)\xi dx,\\
\int_\Omega |\nabla \phi_2|^{p(x)-2}\nabla \phi_1 \cdot \nabla \xi dx \le \int_\Omega g(x)b(\phi_2)\xi dx + \int_\Omega h(\phi_1)\xi dx,\\
\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla \xi dx \ge \int_\Omega g(x)a(z_1)\xi dx + \int_\Omega f(z_2)\xi dx,\\
\int_\Omega |\nabla z_2|^{p(x)-2}\nabla z_2 \cdot \nabla \xi dx \ge \int_\Omega g(x)b(z_2)\xi dx + \int_\Omega h(z_1)\xi dx,
\end{gathered}$$

for all $\xi \in W_0^{1,p(x)}(\Omega)$ with $\xi \ge 0$. Then (1.3) has a positive solution.

Step 1. We construct a subsolution of (1.3). Denote

$$\begin{gathered}
\alpha = \frac{\inf p(x) - 1}{4(\sup |\nabla p(x)| + 1)}, \quad R_0 = \frac{R-\alpha}{2},\\
b = \min\{a(0)L_1 + f(0), b(0)L_1 + h(0), -1\},
\end{gathered}$$

and let

$$\phi(r)=\begin{cases} e^{-k(r-R)}-1, & 2R_0<r\le R,\\ e^{\alpha k}-1+\int_r^{2R_0}(ke^{\alpha k})^{\frac{p(2R_0)-1}{p(r)-1}} & \\ \times[\frac{(2R_0)^{N-1}}{r^{N-1}}\sin(\varepsilon(r-2R_0)+\frac{\pi}{2})(L_1+1)]^{\frac{1}{p(r)-1}}\,dr, & 2R_0-\frac{\pi}{2\varepsilon}<r\le 2R_0,\\ e^{\alpha k}-1+\int_{2R_0-\frac{\pi}{2\varepsilon}}^{2R_0}(ke^{\alpha k})^{\frac{p(2R_0)-1}{p(r)-1}} & \\ \times[\frac{(2R_0)^{N-1}}{r^{N-1}}\sin(\varepsilon_0(r-2R_0)+\frac{\pi}{2})(L_1+1)]^{\frac{1}{p(r)-1}}\,dr, & r\le 2R_0-\frac{\pi}{2\varepsilon}, \end{cases}$$

where R_0 is sufficiently large, ε is a small positive constant which satisfies $R_0 \le 2R_0-\frac{\pi}{2\varepsilon}$,

In the following, we will prove that (ϕ,ϕ) is a subsolution of (1.3). Since

$$\phi'(r)=\begin{cases} e^{-k(r-R)}-1, & 2R_0<r\le R,\\ -(ke^{\alpha k})^{\frac{p(2R_0)-1}{p(r)-1}} & \\ \times[\frac{(2R_0)^{N-1}}{r^{N-1}}\sin(\varepsilon(r-2R_0)+\frac{\pi}{2})(L_1+1)]^{\frac{1}{p(r)-1}}\,dr, & 2R_0-\frac{\pi}{2\varepsilon}<r\le 2R_0,\\ 0, & 0\le r\le 2R_0-\frac{\pi}{2\varepsilon}, \end{cases}$$

it is easy to see that $\phi\ge 0$ is decreasing and $\phi\in C^1([0,R]),\phi(x)=\phi(|x|)\in C^1(\bar\Omega)$. Let $r=|x|$. By computation,

$$-\Delta_{p(x)}\phi=-\operatorname{div}|\nabla\phi(x)|^{p(x)-2}\nabla\phi(x))=-(r^{N-1}|\phi'(r)|^{p(r)-2}\phi'(r))'/r^{N-1}.$$

Then

$$-\Delta_{p(x)}\phi=\begin{cases} (ke^{-k(r-R)})^{p(r)-1}\big[-k(p(r)-1)+p'(r)\ln k & \\ -kp'(r)(r-R)+\frac{N-1}{r}\big], & 2R_0<r\le R,\\ \varepsilon(\frac{2R_0}{r})^{N-1}(ke^{\alpha k})^(p(2R_0)-1) & \\ \times\cos(\varepsilon(r-2R_0)+\frac{\pi}{2})(L_1+1), & 2R_0-\frac{\pi}{2\varepsilon}<r\le 2R_0,\\ 0, & 0\le r\le 2R_0-\frac{\pi}{2\varepsilon}, \end{cases}$$

If k is sufficiently large, when $2R_0<r\le R$, then

$$-\Delta_{p(x)}\phi\le -k[\inf p(x)-1-\sup|\nabla p(x)|(\frac{\ln k}{k}+R-r)+\frac{N-1}{kr}]\le -k\alpha.$$

Since α is a constant dependent only on $p(x)$, if k is a big enough, such that $-ka<b$, and since $\phi(x)\ge 0$ and a,f are monotone, this implies

$$-\Delta_{p(x)}\phi\le a(0)L_1+f(0)\le g(x)a(\phi)+f(\phi),\quad 2R_0<|x|\le R\,. \tag{1.5}$$

If k is sufficiently large, then

$$a(e^{\alpha k}-1)\ge 1,\quad f(e^{\alpha k}-1)\ge 1,\quad b(e^{\alpha k}-1)\ge 1,\quad h(e^{\alpha k}-1)\ge 1$$

where k is dependent on a,f,b,h,p, and independent on R. Since

$$\begin{aligned}-\Delta_{p(x)}\phi&=\varepsilon(\frac{2R_0}{r})^{N-1}(ke^{\alpha k})^(p(2R_0)-1)\cos(\varepsilon(r-2R_0)+\frac{\pi}{2})(L_1+1)\\ &\le\varepsilon(L_1+1)2^Nk^{p^+}e^{\alpha kp^+},2R_0-\frac{\pi}{2\varepsilon}<|x|<2R_0\,.\end{aligned}$$

Let $\varepsilon=2^{-N}k^{-p^+}e^{-\alpha kp^+}$. Then

$$-\Delta_{p(x)}\phi\le L_1+1\le g(x)a(\phi)+f(\phi),2R_0-\frac{\pi}{2\varepsilon}<|x|<2R_0. \tag{1.6}$$

Obviously,

$$-\Delta_{p(x)}\phi = 0 \le L_1 + 1 \le g(x)a(\phi) + f(\phi), |x| < 2R_0 - \frac{\pi}{2\varepsilon}. \tag{1.7}$$

Since $\phi(x) \in C^1(\Omega)$, combining (1.5), (1.6), (1.7), we have

$$-\Delta_{p(x)}\phi \le g(x)a(\phi) + f(\phi)$$

for a.e. $x \in \Omega$. Similarly we have

$$-\Delta_{p(x)}\phi \le g(x)b(\phi) + h(\phi),$$

for a.e. $x \in \Omega$. Let $(\phi_1, \phi_2) = (\phi, \phi)$, since $\phi(x) \in C^1(\bar{\Omega})$, it is easy to see that (ϕ_1, ϕ_2) is a subsolution of (1.3).

Step 2. We construct a supersolution of (1.3) Let z_1 be a radial solution of

$$\begin{gathered} -\Delta_{p(x)} z_1(x) = (L_2 + 1)\mu, \quad \text{in } \Omega, \\ z_1 = 0 \quad \text{on } \partial\Omega\,. \end{gathered}$$

We denote $z_1 = z_1(r) = z_1(|x|)$, then z_1 satisfies

$$-(r^{N-1}|z_1'|^{p(r)-2}z_1')' = r^{N-1}(L_2 + 1)\mu, z_1(R) = 0, z_1'(0) = 0\,.$$

Then

$$z_1' = -|\frac{r(L_2 + 1)\mu}{N}|^{\frac{1}{p(r)-1}}, \tag{1.8}$$

and

$$z_1 = \int_r^R |\frac{r(L_2 + 1)\mu}{N}|^{\frac{1}{p(r)-1}} dr.$$

We denote $\beta = \beta((L_2 + 1)\mu) = \max_{0\le r\le R} z_1(r)$, then

$$\beta((L_2 + 1)\mu) = \int_0^R |\frac{r(L_2 + 1)\mu}{N}|^{\frac{1}{p(r)-1}} dr = ((L_2 + 1)\mu)^{\frac{1}{p(q)-1}} \int_0^R |\frac{r}{N}|^{\frac{1}{p(r)-1}} dr,$$

where $q \in [0, 1]$. Since $\int_0^R |\frac{r}{N}|^{\frac{1}{p(r)-1}} dr$ is a constant, then there exists a positive constant $C \ge 1$ such that

$$\frac{1}{C}((L_2 + 1)\mu)^{\frac{1}{p^+-1}} \le \beta((L_2 + 1)\mu) = \max_{0\le r\le R} z_1(r) \le C((L_2 + 1)\mu)^{\frac{1}{p^--1}}\,. \tag{1.9}$$

We consider

$$\begin{gathered} -\Delta_{p(x)} z_1 = (L_2 + 1)\mu \quad \text{in } \Omega \\ -\Delta_{p(x)} z_2 = (L_2 + 1)h(\beta((L_2 + 1)\mu)) \quad \text{in } \Omega \\ z_1 = z_2 = 0 \quad \text{on } \partial\Omega\,. \end{gathered}$$

Then we shall prove that (z_1, z_2) is a supersolution for (1.3). For $\xi \in W^{1,p(x)}(\Omega)$ with $\xi \ge 0$, it is easy to see that

$$\begin{aligned} \int_\Omega |\nabla z_2|^{p(x)-2}\nabla z_2 \cdot \nabla\xi dx &= \int_\Omega (L_2 + 1)h(\beta((L_2 + 1)\mu))\xi dx \\ &\ge \int_\Omega L_2 h(\beta((L_2 + 1)\mu))\xi dx + \int_\Omega h(z_1)\xi dx. \end{aligned}$$

Similar to (1.9), we have

$$\max_{0\le r\le R} z_2(r) \le C[(L_2 + 1)h(\beta((L_2 + 1)\mu))]^{\frac{1}{(p^--1)}}.$$

By (H3), for μ large enough we have

$$h(\beta((L_2+1)\mu)) \geq b(C[(L_2+1)h(\beta((L_2+1)\mu))]^{\frac{1}{p^--1}}) \geq b(z_2).$$

Hence

$$\int_\Omega |\nabla z_2|^{p(x)-2}\nabla z_2 \cdot \nabla \xi dx \geq \int_\Omega g(x)b(z_2)\xi dx + \int_\Omega h(z_1)\xi dx, \tag{1.10}$$

Also

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla \xi dx = \int_\Omega (L_2+1)\mu\xi dx.$$

By (H3), (H4), when μ is sufficiently large, according to (1.9), we have

$$\begin{aligned}
(L_2+1)\mu &\geq [\frac{1}{C}\beta((L_2+1)\mu)]^{p^--1} \\
&\geq L_2 a(\beta((L_2+1)\mu)) + f[C[(L_2+1)^{\frac{1}{(p^--1)}}(h(\beta((L_2+1)\mu)))^{\frac{1}{(p^--1)}}] \\
&\geq g(x)a(z_1) + f(z_2),
\end{aligned}$$

then

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla \xi dx \geq \int_\Omega g(x)a(z_1)\xi dx + \int_\Omega f(z_2)\xi dx. \tag{1.11}$$

According to (1.10) and (1.11), we can conclude that (z_1, z_2) is a supersolution of (1.3).

Let μ be sufficiently large, then from (1.8) and the definition of (ϕ_1, ϕ_2), it is easy to see that $\phi_1 \leq z_1$ and $\phi_2 \leq z_2$. This completes the proof. $\square$

Now we consider the problem

$$\begin{gathered}
-\Delta_{p(x)}u = \lambda F(x,u,v) \quad \text{in } \Omega \\
-\Delta_{p(x)}v = \lambda G(x,u,v) \quad \text{in } \Omega \\
u = v = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{1.12}$$

If $p(x) \equiv p$ (a constant), because of the homogenity of p-Laplacian, (1.3) and (1.4) can be transformed into each other; but, if $p(x)$ is a general function, since $p(x)$-Laplacian is nonhomogeneous, they cannot be transformed into each other. So we can see that $p(x)$-Laplacian problem is more complicated than than that of p-Laplacian, and it is necessary to discuss the problem (1.4) separately.

Theorem 1.2. *If $p(x) \in C^1(\bar{\Omega})$, $\Omega = B(0,R)$, and (H3)–(H5) hold, then there exists a λ^* which is sufficiently large, such that (1.4) possesses a positive solution for any $\lambda \geq \lambda^*$.*

Proof. We construct a subsolution of (1.4). Let $\beta \leq \frac{R}{4}$ satisfy

$$|p(r_1) - p(r_2)| \leq \frac{1}{2}, \forall r_1, r_2 \in [R-2\beta, R]. \tag{1.13}$$

In the following we denote

$$\begin{gathered}
\delta = \min\{\frac{\inf p(x) - 1}{4(\sup|\nabla p(x)| + 1)}\}, \quad p_*^+ = \sup_{R-2\beta\leq|x|\leq R} p(x), \quad p_*^- = \inf_{R-2\beta\leq|x|\leq R} p(x), \\
b = \min\{a(0)L_1 + f(0), b(0)L_1 + h(0), -1\}.
\end{gathered} \tag{1.14}$$

Let $\alpha \in (0, \beta]$, and set

$$\phi(r) = \begin{cases} e^{-k(r-R)} - 1, & R-\alpha < r \le R, \\ e^{\alpha k} - 1 + \int_r^{R-\alpha} (ke^{\alpha k})^{\frac{p(R-\alpha)-1}{p(r)-1}} [\frac{(R-\alpha)^{N-1}}{r^{N-1}} & \\ \times \sin(\varepsilon(r-(R-\alpha)) + \frac{\pi}{2})(L_1+1)]^{\frac{1}{p(r)-1}} dr, & R-2\beta < r \le R-\alpha, \\ e^{\alpha k} - 1 + \int_{R-\alpha-\frac{\pi}{2\varepsilon}}^{R-\alpha} (ke^{\alpha k})^{\frac{p(R-\alpha)-1}{p(r)-1}} [\frac{(R-\alpha)^{N-1}}{r^{N-1}} & \\ \times \sin(\varepsilon(r-(R-\alpha)) + \frac{\pi}{2})(L_1+1)]^{\frac{1}{p(r)-1}} dr, & r \le R-2\beta, \end{cases}$$

where $\varepsilon = \frac{\pi}{2(2\beta-\alpha)}$ which satisfies $\varepsilon(R-2\beta-(R-\alpha)) + \frac{\pi}{2} = 0$.

In the following, we will prove that (ϕ, ϕ) is a subsolution of (1.4). Since

$$\phi'(r) = \begin{cases} e^{-k(r-R)} - 1, & R-\alpha < r \le R, \\ -(ke^{\alpha k})^{\frac{p(R-\alpha)-1}{p(r)-1}} & \\ [\frac{(R-\alpha)^{N-1}}{r^{N-1}} & \\ \times \sin(\varepsilon(r-(R-\alpha)) + \frac{\pi}{2})(L_1+1)]^{\frac{1}{p(r)-1}} dr, & R-2\beta < r \le R-\alpha, \\ 0, & r \le R-2\beta. \end{cases}$$

It is easy to see that $\phi \ge 0$ is decreasing and $\phi \in C^1([0,R]), \phi(x) = \phi(|x|) \in C^1(\Omega)$.

Let $r = |x|$. By computation,

$$-\Delta_{p(x)}\phi(x) = \begin{cases} (ke^{-k(r-R)})^{p(r)-1}[-k(p(r)-1) & \\ +p'(r)\ln k - kp'(r)(r-R) + \frac{N-1}{r}], & R-\alpha < r \le R, \\ \varepsilon(\frac{R-\alpha}{r})^{N-1}(ke^{\alpha k})^{(p(R-\alpha)-1)} & \\ \times \cos(\varepsilon(r-(R-\alpha)) + \frac{\pi}{2})(L_1+1), & R-2\beta < r \le R-\alpha, \\ 0, & r \le R-2\beta. \end{cases}$$

If k is sufficiently large, when $R-\alpha < r \le R$, then we have

$$-\Delta_{p(x)}\phi \le -k^{p(r)}[\inf p(x) - 1 - \sup|\nabla p(x)|(\frac{\ln k}{k} + R - r) + \frac{N-1}{kr}] \le -k^{p(r)}\delta.$$

If k satisfies

$$k^{p_*^-}\delta = -\lambda b, \tag{1.15}$$

and since $\phi(x) \ge 0$ and a, f is monotone, it means that

$$-\Delta_{p(x)}\phi \le \lambda(a(0)L_1 + f(0)) \le \lambda(g(x)a(\phi) + f(\phi)), R-\alpha < |x| \le R. \tag{1.16}$$

From (H3), (H4) there exists a positive constant M such that $a(M-1) \ge 1$, $f(M-1) \ge 1$, $b(M-1) \ge 1$, $h(M-1) \ge 1$. Let

$$\alpha k = \ln M. \tag{1.17}$$

Since

$$\begin{aligned} -\Delta_{p(x)}\phi(x) &= \varepsilon(\frac{R-\alpha}{r})^{N-1}(ke^{\alpha k})^(p(R-\alpha)-1)\cos(\varepsilon(r-(R-\alpha)) + \frac{\pi}{2})(L_1+1) \\ &\le \varepsilon(L_1+1)2^N(ke^{\alpha k})^{p_*^+-1}, R-2\beta < |x| < R-\alpha, \end{aligned}$$

if

$$\varepsilon 2^N(ke^{\alpha k})^{p_*^+-1} \le \lambda, \tag{1.18}$$

then

$$-\Delta_{p(x)}\phi(x) \le \lambda(L_1+1) \le \lambda(g(x)a(\phi) + f(\phi)), \quad R-2\beta < |x| < R-\alpha. \tag{1.19}$$

Obviously

$$-\Delta_{p(x)}\phi(x) = 0 \le \lambda L_1 + 1 \le \lambda(g(x)a(\phi) + f(\phi)), \quad |x| < R - 2\beta\,. \tag{1.20}$$

Combining (1.15), (1.17) and (1.18), we only need

$$\varepsilon 2^N |\frac{-b}{\delta}\lambda|^{\frac{p_*^+-1}{p_*^-}} M^{p_*^+-1} \le \lambda,$$

and according to (1.13), (1.14), we only need

$$(\frac{\pi}{\beta}2^N M^{p_*^+-1}|\frac{-b}{\delta}|^{\frac{p_*^+-1}{p_*^-}})2p_*^- \le \lambda\,.$$

Let

$$\lambda^* = (\frac{\pi}{\beta}2^N M^{p_*^+-1}|\frac{-b}{\delta}|^{\frac{p_*^+-1}{p_*^-}})^{2p_*^-}\,.$$

If $\lambda \ge \lambda^*$ is sufficiently large, then (1.18) is satisfied.

Since $\phi(x) = \phi(|x|) \in C^1(\Omega)$, according to (1.16), (1.19) and (1.20), it is easy to see that if λ is sufficiently large, then (ϕ_1, ϕ_2) is a subsolution of (1.4).

Step 2. We construct a supersolution of (1.4). Similar to the proof of Theorem 1.1, we consider

$$\begin{gathered}
-\Delta_{p(x)}z_1 = \lambda(L_2+1)\mu \quad \text{in } \Omega\\
-\Delta_{p(x)}z_2 = \lambda(L_2+1)h(\beta(\lambda(L_2+1)\mu)) \quad \text{in } \Omega\\
z_1 = z_2 = 0 \quad \text{on } \partial\Omega\,,
\end{gathered}$$

where $\beta = \beta(\lambda(L_2+1)\mu) = \max_{0\le r\le R} z_1(r)$. It is easy to see that

$$\begin{aligned}
\int_\Omega |\nabla z_2|^{p(x)-2}\nabla z_2 \cdot \nabla\xi dx &= \int_\Omega \lambda(L_2+1)h(\beta(\lambda(L_2+1)\mu))\xi dx\\
&\ge \int_\Omega \lambda L_2 h(\beta(\lambda(L_2+1)\mu))\xi dx + \int_\Omega \lambda h(z_1)\xi dx.
\end{aligned}$$

Similar to (1.9), we have

$$\max_{0\le r\le R} z_2(r) \le C[\lambda(L_2+1)h(\beta(\lambda(L_2+1)\mu))]^{\frac{1}{(p^--1)}}.$$

By (H3) for μ large enough we have

$$h(\beta(\lambda(L_2+1)\mu)) \ge b(C[\lambda(L_2+1)h(\beta(\lambda(L_2+1)\mu))]^{\frac{1}{p^--1}}) \ge b(z_2).$$

Hence

$$\int_\Omega |\nabla z_2|^{p(x)-2}\nabla z_2 \cdot \nabla\xi dx \ge \int_\Omega \lambda g(x)b(z_2)\xi dx + \int_\Omega \lambda h(z_1)\xi dx. \tag{1.21}$$

Also

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla\xi dx = \int_\Omega \lambda(L_2+1)\mu\xi dx.$$

By (H3), (H4), when μ is sufficiently large, according to (1.9), we have

$$\begin{aligned}
(L_2+1)\mu &\ge \frac{1}{\lambda}[\frac{1}{C}\beta(\lambda(L_2+1)\mu)]^{p^--1}\\
&\ge L_2 a(\beta(\lambda(L_2+1)\mu)) + f(C[\lambda(L_2+1)h(\beta(\lambda(L_2+1)\mu))]^{\frac{1}{(p^--1)}})\,.
\end{aligned}$$

Then

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla\xi dx \geq \int_\Omega \lambda g(x)a(z_1)\xi dx + \int_\Omega \lambda f(z_2)\xi dx. \tag{1.22}$$

According to (1.21) and (1.22), we can conclude that (z_1, z_2) is a supersolution of (1.4).

Similar to the proof of Theorem 1.1, if μ is sufficiently large, we have $\phi_1 \leq z_1$ and $\phi_2 \leq z_2$. This completes the proof. □

References

[1] J. Ali, R. Shivaji, *Positive solutions for a class of p-Laplacian systems with multiple parametes*, J. Math. Anal. Appl. Article In Press.

[2] C. H. Chen, *On positive weak solutions for a class of quasilinear elliptic systems*, Nonlinear Anal. 62 (2005) 751-756.

[3] X. L. Fan, H. Q. Wu, F. Z. Wang, *Hartman-type results for p(t)-Laplacian systems*, Nonlinear Anal. 52 (2003) 585-594.

[4] X. L. Fan, D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. 263 (2001) 424-446.

[5] X. L. Fan, D. Zhao, *A class of De Giorgi type and Hölder continuity*, Nonlinear Anal. TMA 36 (1999) 295-318.

[6] X. L. Fan, D. Zhao, *The quasi-minimizer of integral functionals with $m(x)$ growth conditions*, Nonlinear Anal. TMA 39 (2000) 807-816.

[7] X. L. Fan, Q. H. Zhang, *Existence of solutions for $p(x)$-Laplacian Dirichlet problem*, Nonlinear Anal. 52 (2003) 1843-1852.

[8] X. L. Fan, Q. H. Zhang, D. Zhao *Eigenvalues of $p(x)$-Laplacian Dirichlet problem*, J. Math. Anal. Appl. 302 (2005) 306-317.

[9] D. D. Hai, R. Shivaji, *An existence result on positive solutions for a class of p-Laplacian systems*, Nonlinear Anal. 56 (2024) 1007-1010.

[10] M. Rûzicka, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Math, vol. 1784, Springer-Verlag, Berlin, 2000.

[11] Q. H. Zhang, *Existence of positive solutions for elliptic systems with nonstandard $p(x)$-growth conditions via sub-supersolution method*, Nonlinear Anal. 67 (2007) 1055-1067.

[12] Q. H. Zhang, *Existence of positive solutions for a class of $p(x)$-Laplacian systems*, J. Math. Anal. Appl. 302 (2005) 306-317.

[13] V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR Izv. 29 (1987) 33-36.

Ghasem A. Afrouzi

Department of Mathematics, Faculty of Basic Sciences, Mazandaran University, Babolsar, Iran

E-mail address: afrouzi@umz.ac.ir

Horieh Ghorbani

Department of Mathematics, Faculty of Basic Sciences, Mazandaran University, Babolsar, Iran

E-mail address: seyed86@yahoo.com