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DISSIPATIVE INITIAL BOUNDARY VALUE PROBLEM FOR THE BBM-EQUATION

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Abstract. This paper concerns a dissipative initial boundary value problem for the Benjamin-Bona-Mahony (BBM) equation. We prove the existence and uniqueness of global solutions and the decay of the energy as time tends to infinity.

1. Introduction

This paper concerns the dissipative initial boundary value problems for the Benjamin-Bona-Mahoney (BBM) equation

$$u_t - u_{txx} + uu_x = 0 \tag{1.1}$$

which was derived by Benjamin-Bona-Mahony, [2, 3], and usually is called the alternative Korteweg-de Vries (KdV) equation. In spite of the fact that both (1.1) and the KdV equation,

$$u_t + au_{xxx} + uu_x = 0 \tag{1.2}$$

are dispersive equations and have almost the same names, formulations of initial boundary value problems for them are completely different. Considering (1.1) and (1.2) in a rectangle $Q = (0,1) \times (0,T)$, $T > 0$, one must put for (1.1) one condition at $x = 0$ and one condition at $x = 1$. On the other hand, for (1.2) one must put three conditions at the ends of the interval $(0,1)$. A number of conditions at $x = 0$ and $x = 1$ depends on a sign of the coefficient a: if $a > 0$, then we pose one condition at $x = 0$ and two conditions at $x = 1$. If $a < 0$, then we pose two conditions at $x = 0$ and one condition at $x = 1$.

Historically, interest in dispersive-type evolution equations dates from the 19th century when Russel [20], Airy [1], Boussinesq [9] and later Korteweg and de Vries [15] studied propagation of waves in dispersive media. Due to physical reasons, these and posterior studies mostly dealt with one-dimensional problems posed on the entire real line, see [2, 4, 7, 8, 14, 21] and references therein. Moreover, the emphasis in these works was mainly focused on the existence and qualitative structure of the solitary, cnoidal and other specific types of waves, whereas correctness of the corresponding mathematical problems attracted minor interest.

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Initial boundary value problems for (1.1) with Dirichlet boundary conditions were considered in [8, 11, 13, 19, 18, 6]. Bubnov in [10] studied general boundary conditions and proved existence of local solutions to a corresponding mixed problem. Mixed problems for multi-dimensional versions of (1.1) were considered in [13, 19, 18]. It is easy to see that mixed problems for (1.1) with Dirichlet boundary conditions imply conservation of the energy:

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_0^1 \{u(x,t)^2 + u_x(x,t)^2\}dx = 0.$$

It means that the energy can not decay with time. Differently, the KdV equation itself has dissipative properties and solutions of initial boundary value problems for it decay with time see [12, 16, 17].

The goal of our paper is to find such boundary conditions which guarantee existence of global regular solutions and decay of the energy for the BBM equation. For this purpose we pose dissipative nonlinear boundary conditions (2.2). From the physical point of view, if to consider dynamics of a fluid in a cylinder, the Dirichlet boundary conditions mean that the walls of a cylinder are impermeable: a fluid cannot enter or exit the cylinder. On the other hand, nonlinear boundary conditions (2.2) allow a fluid to exit, for a example, when a cylinder has porous walls. This effect stabilizes the system and dissipates the energy.

This paper has the following structure: in Chapter 2 we formulate a nonlinear problem and consider decay properties of linearized problems. In Chapter 3, first we prove local existence of regular solutions to the nonlinear problem, using the theory of elliptic equations with a parameter t, then global existence and uniqueness of regular solutions. In Chapter 4, decay properties of the energy, as $t \to \infty$, are proved.

2. Formulation of the problem

In $Q = (0,1) \times (0,T)$ we consider the following initial boundary value problem:

$$u_t - u_{txx} + uu_x = 0, \quad x \in (0,1),\ t \in (0,T), \tag{2.1}$$

$$u(0,t) = 0, \quad u_{tx}(1,t) = \frac{1}{3}u^2(1,t) - u(1,t), \quad t > 0, \tag{2.2}$$

$$u(x,0) = u_0(x),\ x \in (0,1). \tag{2.3}$$

2.1. **Linear problem.**

First we study the linearized version of (2.1)-(2.3):

$$u_t - u_{txx} = 0, \quad (x,t) \in Q, \tag{2.4}$$

$$u(0,t) = 0, \quad u_{tx}(1,t) = -u(1,t),\ t > 0, \tag{2.5}$$

$$u(x,0) = u_0(x), \quad x \in (0,1). \tag{2.6}$$

We also assume that the initial data admits the compatibility condition $u_0(0) = 0$. It is easy to see that problem (2.4)-(2.6) has a unique solution.

Considering solutions of the form $u(x,t) = v(x)w(t)$, we obtain

$$w_t(t)(v(x) - v(x)_{xx}) = 0, \quad x \in (0,1),\ t \in R^+, \tag{2.7}$$

$$v(0)w(t) = 0, \quad w_t(t)v_x(1) = -w(t)v(1). \tag{2.8}$$

This problem has two type of solutions:

$$\lambda_1 = 0, \quad w_1(t) = C_1\exp(\lambda_1 t), \quad v_1(x) \in C^2[0,1], \quad v_1(0) = v_1(1) = 0$$

and

$$\lambda_2 = -\frac{e^2-1}{e^2+1}, \quad w_2(t) = C_2 \exp(\lambda_2 t), \quad v_2(x) = \frac{e^x - e^{-x}}{2}.$$

Since $u_0(0) = 0$,

$$\phi(x) = u_0(x) - C\frac{e^x - e^{-x}}{2},$$

where

$$C = \frac{2eu_0(1)}{e^2-1},$$

is a stationary solution of (2.7),(2.8) corresponding to $\lambda_1 = 0$. This implies that

$$u(x,t) = \phi(x) + \frac{2eu_0(1)}{e^2-1}[\frac{e^x - e^{-x}}{2}] \exp(-\frac{e^2-1}{e^2+1}t)$$

is a unique solution of (2.4)-(2.6) and

$$|u(x,t) - \phi(x)| \le |u_0(1)| \exp(-\frac{e^2-1}{e^2+1}t).$$

These results can be summarized as follows.

Theorem 2.1. *Problem* (2.4)-(2.6) *has a continuum of stationary solutions and any nonstationary solution converges to a stationary one exponentially as $t \to \infty$.*

Remark 2.2. Consider the linearized problem with the Dirichlet boundary conditions,

$$u_t - u_{txx} = 0, \quad (x,t) \in Q, \tag{2.9}$$

$$u(0,t) = u(1,t) = 0,, \quad t > 0, \tag{2.10}$$

$$u(x,0) = u_0(x), \tag{2.11}$$

it is easy to show that this problem has only stationary solutions.

3. Nonlinear Problem

3.1. **Local Solutions.** We start with the linear problem

$$u_t - u_{txx} = f(x,t), \quad (x,t) \in Q, \tag{3.1}$$

$$u(0,t) = 0, \quad u_{tx}(1,t) = g(t), \quad t > 0, \tag{3.2}$$

$$u(x,0) = u_0(x), \quad x \in (0,1). \tag{3.3}$$

Denote $w(x,t) = u_t(x,t)$, then the problem becomes

$$w - w_{xx} = f(x,t), \quad x \in (0,1),\ t > 0, \tag{3.4}$$

$$w(0,t) = 0, \quad w_x(1,t) = g(t), \quad t > 0 \tag{3.5}$$

which is an elliptic problem with a parameter t.

Lemma 3.1. *Regular solutions of* (3.4)–(3.5) *satisfy the inequality*

$$\|w(t)\|_{H^2(0,1)} \le C(\|f(t)\|_{L^2(0,1)} + |g(t)|). \tag{3.6}$$

Here and in the sequel the constants C do not depend on $g(t)$, $f(x,t)$.

Proof. Considering
$$w(x,t) = z(x,t) - g(t)(1-x)x, \tag{3.7}$$
we rewrite (3.4)-(3.5) as the following elliptic problem with a parameter t:
$$z - z_{xx} = f_1(x,t) \equiv f(x,t) + g(t)x(1-x) - g(t), \quad x \in (0,1), \tag{3.8}$$
$$z(0,t) = z_x(1,t) = 0, \quad t > 0. \tag{3.9}$$
Standard elliptic estimates [5] give
$$\|z(.,t)\|_{H^2(0,1)} \le C\|f_1(t)\|_{L^2(0,1)} \le C(\|f(.,t)\|_{L^2(0,1)} + |g(t)|).$$
This and (3.7) imply (3.6). □

Remark 3.2. Let $u(x,t)$ be a solution to the problem
$$u_t - u_{txx} = f(x,t), \quad (x,t) \in Q,$$
$$u(0,t) = u_t(0,t) = 0, \quad u_{tx}(1,t) = \frac{1}{3}u^2(1,t) - u(1,t), \quad t > 0,$$
$$u(x,0) = u_0(x).$$
Then (3.4)-(3.6) imply
$$\|u_t(.,t)\|_{H^2(0,1)} \le C(\|f(.,t)\|_{L^2(0,1)} + |u(1,t)| + |u(1,t)|^2). \tag{3.10}$$

Lemma 3.3. *Regular solutions of* (3.1)-(3.3) *in the cylinder* $Q_T = (0,1) \times (0,T)$, $T > 0$*, satisfy the inequality*
$$\begin{aligned} &\|u\|_{C([0,T];H^1(0,1))} \\ &\le \|u_0\|_{H^1(0,1)} + CT\Big(\|f\|_{C([0,T];L^2(0,1))} + |u(1,t)|_{C[0,T]} + |u(1,t)|^2_{C[0,T]}\Big). \end{aligned} \tag{3.11}$$

Proof. Because
$$u(x,t) = u_0(x) + \int_0^t u_s(x,s)ds,$$
we have
$$\|u(t)\|_{H^1(0,1)} \le \|u_0\|_{H^1(0,1)} + \sqrt{t}(\int_0^t \|u_s(.,s)\|^2_{H^1(0,1)}ds)^{1/2},$$
$$\|u(x,t)\|_{C([0,T];H^1(0,1))} \le \|u_0(x)\|_{H^1(0,1)} + T\|u_t(x,t)\|_{C([0,1];H^1(0,1))}.$$
Using (3.10), we obtain
$$\begin{aligned} &\max_{(x,t)\in\overline{Q_T}}(|u_t(x,t)|) \le \|u_t(x,t)\|_{C([0,T];H^1(0,1))} \\ &\qquad \le C(\|f(x,t)\|_{C([0,T];L^2(0,1))} + |u(1,t)|_{C[0,T]} + |u(1,t)|^2_{C[0,T]}) \end{aligned}$$
and
$$\|u\|_{C(0,t;H^1(0,1))} \le \|u_0\|_{H^1(0,1)} + TC\{\|f\|_{C(0,t;L^2(0,1))} + |u(1,t)|_{C(0,t)} + |u(1,t)|^2_{C(0,t)}\}. \tag{3.12}$$
This completes the proof. □

Using the estimates of Lemmas 3.1 and 3.3, we can solve locally in t the nonlinear problem (2.1)-(2.3).

Theorem 3.4. *Let* $u_0 \in H^1(0,1)$*. Then there is* $T_0 > 0$ *such that for all* $t \in (0,T_0)$ *there exists* $u(x,t)$ *such that* $u \in C(0,T_0;H^1(0,1))$, $u_t \in C(0,T_0;H^2(0,1))$, $u_{tt} \in C(0,T_0;H^2(0,1))$*, which is a unique regular solution of* (2.1)-(2.3).

Remark 3.5. If $u_0 \in H^2(0,1)$, then $u \in C(0,T_0;H^2(0,1))$.

Proof of Theorem 3.4. We use the contraction mapping theorem. Let $\|u_0\|_{H^1(0,1)} < R$, $R > 1$ and B_R be a ball of functions $w(x,t)$ such that

$$w \in C(0,T_0;H^1(0,1)),\ T_0 > 0, \quad \|w\|_{C(0,T_0;H^1(0,1))} < 2R,$$
$$w(x,0) = u_0(x), \quad w(0,t) = 0, \quad t \in (0,T_0),$$

where the constant T_0 will be defined later. For $w \in B_R$ consider the linear problem

$$v_t - v_{txx} = -ww_x, \quad (x,t) \in (0,1) \times (0,T_0), \tag{3.13}$$
$$v(0,t) = 0, \quad v_{tx} = \frac{1}{3}w^2(1,t) - w(1,t), \quad t \in (0,T_0), \tag{3.14}$$
$$v(x,0) = u_0(x), \quad x \in (0,1). \tag{3.15}$$

Since (3.13)-(3.15) is a linear, elliptic problem for v_t, solvability of this problem follows from Lemmas 3.1 and 3.3. Therefore, we can define the operator $P : v(x,t) = P(w(x,t))$ in B_R. The proof will be completed after proving the following two propositions. $\square$

Proposition 3.6. *The operator P maps B_R into B_R for $T_0 > 0$ sufficiently small.*

Proof. Fixing $1 < R < \infty$ and taking into account (3.6),(3.10),(3.12) and the obvious inequality

$$\|w(1,t)\|_{C[0,T_0)} \le \|w\|_{C[0,T_0;H^1(0,1))},$$

we find

$$\begin{aligned}\|ww_x\|_{C(0,T_0;L^2(0,1))} &\le \max_{[0,T_0]}\Big(\int_0^1 w^2(x,t)w_x^2(x,t)dx\Big)^{1/2}\\ &\le \|w\|_{C[0,T_0;H^1(0,1))}\|w_x\|_{C[0,T_0;L^2(0,1))}\\ &\le \|w\|^2_{C[0,T_0;H^1(0,1))}.\end{aligned}$$

Using Lemma 3.3, we obtain

$$\begin{aligned}\|v\|_{C[0,T_0;H^1(0,1))} &\le \|u_0\|_{H^1(0,1)} + C_0T_0\{1 + \|w\|_{C[0,T_0;H^1(0,1))}\}\\ &\le \|u_0\|_{H^1(0,1)} + C_0T_0\|w\|^2_{C[0,T_0;H^1(0,1))}\\ &\le \|u_0\|_{H^1(0,1)} + C_0T_0R^2.\end{aligned}$$

Taking $0 < T_0 < 1/(4C_0R^2)$, we get

$$\|v\|_{C[0,T_0;H^1(0,1))} \le R + \frac{R}{4} < 2R$$

which completes the proof. $\square$

Proposition 3.7. *For $T_0 > 0$ sufficiently small the operator P is a contraction mapping in B_R.*

Proof. For any $w_1, w_2 \in B_R$ denote $v_i = P(w_i)$, $i = 1,2$; $s = w_1 - w_2$, $z = v_1 - v_2$. From (3.13)-(3.15), we obtain

$$z_t - z_{txx} = -(w_2s_x + w_{1x}s), \quad (x,t) \in (0,1) \times (0,T_0),$$
$$z(0,t) = 0,\ z_{tx}(1,t) = -\frac{1}{3}(w_1(1,t) + w_2(1,t)) - 1)s(1,t), \quad t \in (0,T_0),$$
$$z(x,0) = 0, \quad x \in (0,1).$$

By Lemma 3.3,

$$\|z\|_{C(0,T_0;H^1(0,1))} \leq C_0 T_0 R \|s\|_{C(0,T_0;H^1(0,1))},$$

where the constant C_0 does not depend on s. Taking $0 < T_0 < 1/(C_0 R)$, we obtain

$$\|z\|_{C(0,T_0;H^1(0,1))} \leq \gamma \|s\|_{C(0,T_0;H^1(0,1))}$$

with $0 < \gamma < 1$. This completes the proof. □

Propositions 3.6 and 3.7 imply that the operator $P : B_R \to B_R$ is a contraction mapping provided $T_0 > 0$ sufficiently small. Hence, there exists a unique function $u(x,t) : u \in C(0, T_0; H^1(0,1))$ such that $u = Pu$. More regularity follows directly from (2.1)-(2.3) and estimates of elliptic problems for u_t, u_{tt}, see Lemmas 3.1 and 3.3. This proves Theorem 3.4.

3.2. **Global Solutions.**

Theorem 3.8. *Let $u_0 \in H^1(0,1)$. Then there exists a function $u(x,t)$ such that*

$$u \in L^\infty(0,\infty; H^1(0,1)), \quad u_t \in L^\infty(0,\infty; H^2(0,1)), \quad u_{tt} \in L^\infty(0,\infty; H^2(0,1))$$

which is a unique solution of (2.1)-(2.3).

Proof. Due to Theorem 2.1, it is sufficient to extend local solutions to any finite interval $(0,T)$. For this purpose we need a priori estimate independent of t.

Multiplying (2.1) by u and integrating over $(0,1) \times (0,t)$, $t \in (0,T_0)$, we get

$$E(t) = \frac{1}{2}\int_0^1 (u^2(x,t) + u_x^2(x,t))dx = E(0) - \int_0^t u^2(1,s)ds \leq E(0). \tag{3.16}$$

This estimate guarantees prolongation of local solutions, provided by Theorem 2.1, for any finite interval $(0,T_0)$. Moreover, since it does not depend on T_0, the interval of the existence is $(0,\infty) : u \in L^\infty(0,\infty; H^1(0,1))$. Returning to (2.1)-(2.4), we rewrite it as an elliptic problem for u_t:

$$(I - \partial_{xx}^2)u_t = -uu_x \in L^\infty(0,\infty; L^2(0,1)), \tag{3.17}$$

$$u(0,t) = 0, \quad u_{tx} = \frac{1}{3}u^2(1,t) - u(1,t) \in L^\infty(0,\infty; L^2(0,1)). \tag{3.18}$$

By Lemmas 3.1 and 3.3,

$$u_t \in L^\infty(0,\infty; H^2(0,1)). \tag{3.19}$$

Differentiating (3.17), (3.18) with respect to t, we get

$$(I - \partial_{xx}^2)u_{tt} = -uu_{tx} - u_t u_x \in L^\infty(0,\infty; L^2(0,1)),$$

$$u_{tt}(0,t) = 0,\ u_{ttx}(1,t) = \frac{2}{3}u(1,t)u_t(1,t) - u_t(1,t) \in L^\infty(0,\infty; L^2(0,1)).$$

Hence,

$$u_{tt} \in L^\infty(0,\infty; H^2(0,1)). \tag{3.20}$$

This proves the existence part of Theorem 3.8.

To prove uniqueness of solutions, assume that there exist two different solutions u_1, u_2 of (2.1)-(2.3). For $z = u_1 - u_2$ we have the following problem:

$$Lz = z_t - z_{txx} = -\frac{1}{2}(u_{1x} + u_{2x})z - \frac{1}{2}(u_1 + u_2)z_x, \quad (x,t) \in (0,1) \times (0,\infty),$$
$$z(0,t) = 0, \quad z_{tx}(1,t) = \frac{1}{3}(u_1(1,t) + u_2(1,t))z(1,t) - z(1,t), \quad t > 0,$$
$$z(x,0) = 0, \quad x \in (0,1).$$

Multiplying Lz by z and integrating over $(0,1) \times (0,t)$, we obtain

$$\begin{aligned}\int_0^1 (z^2(x,t) + z_x^2(x,t))dx = &- \int_0^t \int_0^1 \{(u_{1x}(x,s) + u_{2x}(x,s))z^2(x,s) \\ &- \frac{1}{2}(u_1(x,s) + u_2(x,s))z(x,s)z_x(x,s)\}dxds \\ &- 2\int_0^t \{\frac{1}{3}[u_1(1,s) + u_2(1,s)]z^2(1,s) + z^2(1,s)\}ds.\end{aligned}$$

Since $|z(1,s)|^2 \le \|z_x(s)\|^2_{L^2(0,1)}$, we arrive to the inequality

$$\begin{aligned}&\int_0^1 (z^2(x,t) + z_x^2(x,t))dx \\ &\le C\int_0^t (1 + \|u_{1x}(s)\|_{L^2(0,1)} + \|u_{2x}(s)\|_{L^2(0,1)})\|z_x(s)\|^2_{L^2(0,1)}ds.\end{aligned}$$

Because $u_i \in L^\infty(0,\infty; H^1(0,1))$, $i = 1,2$,

$$\int_0^1 (z^2(x,t) + z_x^2(x,t))dx \le C\int_0^t \int_0^1 (z^2(x,s) + z_x^2(x,s))dxds.$$

By the Gronwall lemma,

$$\int_0^1 (z^2(x,t) + z_x^2(x,t))dx = 0, \quad t > 0.$$

Then

$$z(x,t) = 0, \quad (x,t) \in (0,1) \times (0,\infty)$$

that completes the proof. □

4. Uniform Decay of Solutions as $t \to \infty$

Lemma 4.1. *For regular solutions of* (2.1)-(2.3), $\lim_{t\to+\infty} u(1,t) = 0$.

Proof. From (3.16),

$$\int_0^t u^2(1,s)ds \le E(0), \quad \text{and} \quad E(t) \le E(0) \quad \text{for all } t > 0. \tag{4.1}$$

Due to (3.19),

$$\sup_{t>0} \|u_t(t)\|_{H^2(0,1)} \le C_1, \quad \sup_{t>0} |\frac{d}{dt}u^2(x,t)| \le C_2. \tag{4.2}$$

Assume that

$$\lim_{t\to+\infty} u^2(x,t) \ne 0.$$

This implies that there exist a positive number $\varepsilon_1 > 0$ and a sequence of $t_n \to +\infty$ such that $u^2(1,t_n) \geq \varepsilon_1$ for all $n \in N$. Since

$$\sup_{t>0} |\frac{d}{dt} u^2(x,t)| \leq C_2,$$

it follows that

$$u^2(1,t) > \frac{1}{2}\varepsilon_1 \quad \text{for } t \in [t_n - \frac{\varepsilon_1}{2C_2}, t_n + \frac{\varepsilon_1}{2C_1}] \text{ and all } n \in N.$$

We may assume that $t_n - \frac{\varepsilon_1}{2C_2} > 0$. Therefore,

$$\int_0^{t_n + \frac{\varepsilon_1}{2C_1}} u^2(1,s)ds \geq \sum_{i=1}^n \int_{t_i - \frac{\varepsilon_1}{2C_2}}^{t_i + \frac{\varepsilon_1}{2C_1}} u^2(1,s)ds > \varepsilon_1 \frac{n\varepsilon_1}{C_2} \to +\infty.$$

Thus we have a contradiction with (4.1) which completest the proof. □

Theorem 4.2. *For regular solutions of* (2.1)-(2.3), $\lim_{t\to+\infty} E(t) = 0$.

Proof. Let $\tau > 0$ and $t_n \to +\infty$. Consider a sequence

$$u^n(x,t) = u(x, t_n + t), \quad (x,t) \in \bar{Q}_\tau = [0,1] \times [0,\tau].$$

It follows from (3.16), (3.19), (3.20) that from the sequence $u^n(x,t)$ we can extract a subsequence, which we again denote by $u^n(x,t)$, such that

$$u^n(x,t) \to w(x,t) \quad \text{in } C^{\alpha_1}(\bar{Q}_\tau),\ \alpha_1 \in (0, \frac{1}{2}); \tag{4.3}$$

$$u_t^n(x,t) \to w_t(x,t) \quad \text{in } C^{\alpha_1}(\bar{Q}_\tau); \tag{4.4}$$

$$u_x^n(x,t) \rightharpoonup w_x(x,t) \quad \text{weakly in } L^2(0,\tau; L^2(0,1)), \tag{4.5}$$

$$u_{txx}^n(x,t) \rightharpoonup w_{txx}(x,t) \quad \text{weakly in } L^2(0,\tau; L^2(0,1)). \tag{4.6}$$

We will return to this proof after the following proposition.

Proposition 4.3. *It holds*

$$u^n(x,t)u_x^n(x,t) \rightharpoonup w(x,t)w_x(x,t) \quad \textit{weakly in } L^2(0,\tau; L^2(0,1)).$$

Proof. Writing

$$u^n u_x^n - w w_x = u_x^n(u^n - w) + w(u_x^n - w_x),$$

from (4.3), we have

$$\lim_{n\to\infty} \|u_x^n(u^n - w)\|_{L^2(Q_\tau)} = 0.$$

A function $w(x,t)$ is bounded in $C^{\alpha_1}(\bar{Q}_\tau)$, whence by (4.5),

$$w(u_x^n - w_x) \rightharpoonup 0 \quad \text{weakly in } L^2(0,\tau; L^2(0,1)).$$

This completes the proof of Proposition 4.3 □

Due to (4.4), (4.6), Proposition 4.3 implies

$$w_t - w_{txx} + w w_x = 0,\ (x,t) \in Q_\tau \text{ and } w(0,t) = 0. \tag{4.7}$$

By Lemma 4.1 and (4.3), $w(1,t) = 0$, but since

$$w_{tx}(1,t) = \frac{1}{3} w^2(1,t) - w(1,t),$$

we have

$$w(1,t) = w_{tx}(1,t) = 0.$$

Denoting $v(x,t) = w_t(x,t)$, from (4.7), we get

$$v - v_{xx} + ww_x = 0,\ x \in (0,1),$$
$$v(0) = v(1) = v_x(1) = 0.$$

Let $g(x,y)$ be a Green function of the problem

$$z_{xx} - z = 0, \quad x \in (0,1), \quad z(0) = z(1) = 0.$$

It is known that

$$g(x,y) = \frac{1}{D(0)} \begin{cases} v_1(x)v_2(y), & 0 \le x \le y; \\ v_1(y)v_2(x), & y \le x \le 1, \end{cases}$$

where

$$v_{1xx} - v_1 = 0, \quad v_1(0) = 0, v_{1x}(0) = 1;$$
$$v_{2xx} - v_2 = 0, \quad v_2(1) = 0, v_{2x}(1) = -1;$$
$$\begin{vmatrix} v_1(x) & v_2(x) \\ v_{1x}(x) & v_{2x}(x) \end{vmatrix} = D(x).$$

Simple calculations give

$$v_1(x) = \frac{e^x - e^{-x}}{2}, \quad v_2(x) = \frac{e^{2-x} - e^x}{2e}$$

and

$$v(x,t) = -\int_0^1 g(x,y)w(y,t)w_y(y,t)dy = \frac{1}{2}\int_0^1 g_y(x,y)w^2(y,t)dy.$$

From here,

$$v_x(x,t) = \frac{1}{2}\int_0^1 g_{xy}(x,y)w^2(y,t)dy.$$

The function $g_{xy}(x,y)$ is negative for $0 < x, y < 1$. On the other hand,

$$v_x(1,t) = \frac{1}{2}\int_0^1 g_{xy}(1,y)w^2(y,t)dy = \frac{1}{2}\int_0^1 (\frac{-e^{2-y} - e^y}{2})w^2(y,t)dy = 0.$$

Hence,

$$\int_0^1 (\frac{e^{2-y} + e^y}{2})w^2(y,t)dy = 0$$

and, consequently, $w^2(y,t) = 0$. It implies that $E(t)$ tends to zero when $t \in [t_n, t_n + \tau]$ and $n \to \infty$. Due to monotonicity of $E(t)$, we have $\lim_{t\to+\infty} E(t) = 0$. This completes the proof of Theorem 4.2. $\square$

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