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VERTICAL BLOW UPS OF CAPILLARY SURFACES IN $\mathbb{R}^3$, PART 2: NONCONVEX CORNERS

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Abstract. The goal of this note is to continue the investigation started in Part One of the structure of "blown up" sets of the form $\mathcal{P} \times \mathbb{R}$ and $\mathcal{N} \times \mathbb{R}$ when $\mathcal{P}, \mathcal{N} \subset \mathbb{R}^2$ and $\mathcal{P}$ (or $\mathcal{N}$) minimizes an appropriate functional and the domain has a nonconvex corner. Sets like $\mathcal{P} \times \mathbb{R}$ can be the limits of the blow ups of subgraphs of solutions of capillary surface or other prescribed mean curvature problems, for example. Danzhu Shi recently proved that in a wedge domain Ω whose boundary has a nonconvex corner at a point O and assuming the correctness of the Concus-Finn Conjecture for contact angles 0 and π, a capillary surface in positive gravity in $\Omega \times \mathbb{R}$ must be discontinuous under certain conditions. As an application, we extend the conclusion of Shi's Theorem to the case where the prescribed mean curvature is zero without any assumption about the Concus-Finn Conjecture.

1. Introduction

Consider the nonparametric prescribed mean curvature problem with contact angle boundary data in the cylinder $\Omega \times \mathbb{R}$

$$Nf = H(x, f) \quad \text{for } x \in \Omega \tag{1.1}$$

$$Tf \cdot \nu = \cos\gamma \quad \text{on } \partial\Omega, \tag{1.2}$$

where $n \geq 2$, $\Omega \subset \mathbb{R}^n$ is bounded and open, $Tf = \nabla f / \sqrt{1 + |\nabla f|^2}$, $Nf = \nabla \cdot Tf$, ν is the exterior unit normal on $\partial\Omega$, $\gamma : \partial\Omega \to [0, \pi]$ and $f \in C^2(\Omega)$. As in Part 1 of [12], we consider variational solutions of (1.1)-(1.2) and sequences $\{f_j\}$ which converge locally to generalized solutions $f_\infty : \Omega_\infty \to [-\infty, \infty]$ of (1.1)-(1.2) of the functional

$$\mathcal{F}_\infty(g) = \int_{\Omega_\infty} \sqrt{1 + |Dg|^2}\, dx - \int_{\partial\Omega_\infty} \cos(\gamma_\infty) g \, dH_n \tag{1.3}$$

in the sense that for each compact subset K of $\mathbb{R}^{n+1}$ with finite perimeter, U_∞ minimizes the functional F_K defined on subsets of $\Omega_\infty \times \mathbb{R}$ by

$$F_K(V) = \int_{K \cap (\Omega_\infty \times \mathbb{R})} |D\phi_V| - \int_{K \cap (\partial\Omega_\infty \times \mathbb{R})} \cos(\gamma_\infty) \phi_V \, dH_n;$$

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here $U_\infty = \{(x,t) \in \Omega_\infty \times \mathbb{R} : t < f_\infty(x)\}$ denotes the subgraph of f_∞. The sets

$$\mathcal{P} = \{x \in \Omega_\infty : f_\infty(x) = \infty\}, \tag{1.4}$$
$$\mathcal{N} = \{x \in \Omega_\infty : f_\infty(x) = -\infty\}, \tag{1.5}$$

have a special structure which is of principal interest to us. The set $\mathcal{P}$ minimizes the functional

$$\Phi(A) = \int_{\Omega_\infty} |D\phi_A| - \int_{\partial\Omega_\infty} \cos(\gamma_\infty)\phi_A \, dH_n. \tag{1.6}$$

and the set $\mathcal{N}$ minimizes the functional

$$\Psi(A) = \int_{\Omega_\infty} |D\phi_A| + \int_{\partial\Omega_\infty} \cos(\gamma_\infty)\phi_A \, dH_n \tag{1.7}$$

in the appropriate sense (e.g. [10], [23]). Set $n = 2$. When Ω_∞ has a corner at $O \in \partial\Omega_\infty$ which is convex, the possible geometries of $\mathcal{P}$ and $\mathcal{N}$ were given in [12, Theorems 2.1 and 2.2]. When Ω_∞ has a corner at $O \in \partial\Omega_\infty$ which is nonconvex, we obtain these geometries in Theorems 2.2 and 2.3.

Our goals here and in [12] are to (i) provide a reference which lists the geometric shapes of all minimizers $\mathcal{P}$ of Φ and $\mathcal{N}$ of Ψ; (ii) illustrate techniques used previously (e.g. [22]) when $\alpha < \pi/2$ and $\gamma_1 = \gamma_2$; and (iii) provide applications of these results by proving restricted (i.e. the mean curvature is zero) versions of the Concus-Finn Conjecture (i.e. [12, Theorem 3.4]) and the conclusion of Shi's [21, Theorem 6] (i.e. Theorem 3.7). In [15], these results are used as a fundamental reference for new proofs of the Concus-Finn Conjecture for convex and nonconvex corners. Additional investigations of variational problems in $\mathbb{R}^3$ which use blow-up techniques, including possibly Dirichlet problems, may find these results valuable. Finally, determining the possible geometries of $\mathcal{P}$ and $\mathcal{N}$ when $n > 2$ would be a difficult task which might have important applications to variational problems in $\mathbb{R}^n$; we hope our results serve as a first step in this direction.

2. Statement of Results

Let Ω be an open subset of $\mathbb{R}^2$ with a corner at $O = (0,0) \in \partial\Omega$ such that, for some $\delta_0 > 0$, $\partial\Omega$ is piecewise smooth in $B_{\delta_0}(O)$ and $\partial\Omega \cap B_{\delta_0}(O)$ consists of two $C^{1,\lambda}$ arcs $\partial^+\Omega$ and $\partial^-\Omega$, with $\lambda \in (0,1)$, whose tangent lines approach the lines $L^+ = \{\theta = \alpha\}$ and $L^- = \{\theta = -\alpha\}$, respectively, as the point O is approached. Let ν^+ and ν^- denote the exterior unit normals on $\partial^+\Omega$ and $\partial^-\Omega$ respectively. Here we assume $\alpha \in (0,\pi)$, polar coordinates relative to O are denoted by r and θ and $B_\delta(O)$ is the ball in $\mathbb{R}^2$ of radius δ about O. Let $(x^+(s), y^+(s))$ be an arclength parametrization of $\partial^+\Omega$ and $(x^-(s), y^-(s))$ be an arclength parametrization of $\partial^-\Omega$, where $s = 0$ corresponds to the point O for both parametrizations. We will assume $\gamma_1 = \lim_{s\downarrow 0} \gamma(x^+(s), y^+(s))$ and $\gamma_2 = \lim_{s\downarrow 0} \gamma(x^-(s), y^-(s))$ both exist and $\gamma_1, \gamma_2 \in (0,\pi)$. In this case,

$$\Omega_\infty = \{(r\cos\theta, r\sin\theta) : r > 0, -\alpha < \theta < \alpha\},$$

$(\partial\Omega_\infty) \setminus \{O\} = \Sigma_1 \cup \Sigma_2$ with

$$\Sigma_j = \{(r\cos\theta, r\sin\theta) : r > 0, \theta = (-1)^{j+1}\alpha\}, \quad j = 1,2,$$
$$\lim_{s\downarrow 0} \nu^+(s) = \nu_1 = (-\sin(\alpha), \cos(\alpha)), \quad \lim_{s\downarrow 0} \nu^-(s) = \nu_2 = (-\sin(\alpha), -\cos(\alpha))$$

and the limiting contact angle γ_∞ equals γ_1 on Σ_1 and γ_2 on Σ_2. A set $\mathcal{P} \subset \Omega_\infty$ minimizes Φ if and only if for each $T > 0$,

$$\Phi_T(\mathcal{P}) \le \Phi_T(\mathcal{P} \cup S) \quad \text{and} \quad \Phi_T(\mathcal{P}) \le \Phi_T(\mathcal{P} \setminus S) \quad \text{for every } S \subset \Omega_\infty^T \ ,$$

where $\Omega_\infty^T = \overline{B_T(O)} \cap \Omega_\infty$, $\Sigma_j^T = \overline{B_T(O)} \cap \Sigma_j$, $j = 1, 2$, and

$$\begin{aligned}\Phi_T(A) &= \int_{\Omega_\infty^T} |D\phi_A| - \cos(\gamma_1) \int_{\Sigma_1^T} \phi_A dH^1 - \cos(\gamma_2) \int_{\Sigma_2^T} \phi_A dH^1 \\ &= H^1(\Omega_\infty^T \cap \partial A) - \cos(\gamma_1) H^1(\Sigma_1^T \cap \partial A) - \cos(\gamma_2) H^1(\Sigma_2^T \cap \partial A) \ .\end{aligned}$$

A set $\mathcal{N} \subset \Omega_\infty$ minimizes Ψ if and only if for each $T > 0$,

$$\Psi_T(\mathcal{N}) \le \Psi_T(\mathcal{N} \cup S) \quad \text{and} \quad \Psi_T(\mathcal{N}) \le \Psi_T(\mathcal{N} \setminus S) \quad \text{for every } S \subset \Omega_\infty^T$$

where

$$\begin{aligned}\Psi_T(A) &= \int_{\Omega_\infty^T} |D\phi_A| + \cos(\gamma_1) \int_{\Sigma_1^T} \phi_A dH^1 + \cos(\gamma_2) \int_{\Sigma_2^T} \phi_A dH^1 \\ &= H^1(\Omega_\infty^T \cap \partial A) + \cos(\gamma_1) H^1(\Sigma_1^T \cap \partial A) + \cos(\gamma_2) H^1(\Sigma_2^T \cap \partial A).\end{aligned}$$

If $\mathcal{P}$ minimizes Φ, then after modification on a set of measure zero, we may assume $\partial\mathcal{P}$ coincides with the essential boundary of $\mathcal{P}$ (e.g. [10, Theorem 1.1]) and $\Omega_\infty \cap \partial\mathcal{P}$ consists of a union of rays. If $\mathcal{N}$ minimizes Ψ, then the same holds for $\partial\mathcal{N}$ and $\Omega_\infty \cap \partial\mathcal{N}$. We may also assume $\mathcal{P}$ and $\mathcal{N}$ are open.

In the following theorems, we determine the geometric shapes of $\mathcal{P}$ (Theorem 2.2) and $\mathcal{N}$ (Theorem 2.3); cases (viii) and (xi) are special cases of (x) and (xiii) respectively and are included separately to assist in the descriptions of cases (ix) and (xii). To illustrate these geometries, we provide Figures 1 and 2; cases (viii) and (xi) in Figure 1 are special cases of (x) and (xiii) respectively and are included separately to illustrate cases (ix) and (xii). The shaded regions in these figures illustrate $\mathcal{P}$ and the unshaded regions illustrate $\mathcal{N}$; we note that these figures should be interpreted independently and, while $\mathcal{P}$ and $\mathcal{N}$ must be disjoint, it is not true in general that $\overline{\mathcal{P}} \cup \overline{\mathcal{N}} \in \{\emptyset, \overline{\Omega_\infty}\}$. This is illustrated by Scherk or skewed Scherk surfaces. For example, let $a > 0$ and set

$$f(x, y) = \frac{1}{a}\big(\ln(\sin(ax)) - \ln(\sin(ay))\big) \quad \text{if } 0 < x < \frac{\pi}{a},\ 0 < y < \frac{\pi}{a}.$$

Consider first $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$, where

$$\Omega_0 = (0, \frac{\pi}{a}] \times (0, \frac{\pi}{a}], \quad \Omega_1 = (0, \frac{\pi}{a}) \times (\frac{\pi}{a}, \frac{2\pi}{a}), \quad \Omega_2 = (\frac{\pi}{a}, \frac{2\pi}{a}) \times (0, \frac{\pi}{a}),$$

$\gamma : \partial\Omega \to [0, \pi]$ defined by

$$\gamma(x, y) = \begin{cases} 0 & \text{if } y = 0, 0 < x < \frac{\pi}{a} \text{ or } y > \frac{\pi}{a} \\ \pi & \text{if } x = 0, 0 < y < \frac{\pi}{a} \text{ or } x > \frac{\pi}{a} \end{cases}$$

and $u : \Omega \to [-\infty, \infty]$ defined by

$$u(x, y) = \begin{cases} \infty & \text{if } (x, y) \in \Omega_1 \\ f(x, y) & \text{if } (x, y) \in \Omega_0 \\ -\infty & \text{if } (x, y) \in \Omega_2. \end{cases}$$

Notice that u is a generalized solution of (1.1)-(1.2) with $H \equiv 0$ and the sets $\mathcal{P} = \{(x, y) : u(x, y) = \infty\}$ and $\mathcal{N} = \{(x, y) : u(x, y) = -\infty\}$ are Ω_1 and Ω_2 respectively (recall that we require $\mathcal{P}$ and $\mathcal{N}$ to be open).

We can, of course, modify this example so the domain Ω is convex. Set $\Omega = \{(x,y) : 0 < x < \frac{\pi}{a},\ |y| < x\}$ and define $u : \Omega \to [-\infty, \infty]$ by

$$u(x,y) = \begin{cases} f(x,y) & \text{if } 0 < x < \frac{\pi}{a},\ 0 < y \le x \\ \infty & \text{if } 0 < x \le \frac{\pi}{a},\ -x < y < 0. \end{cases}$$

Then u is a generalized solution of (1.1)-(1.2) with $H \equiv 0$ for a suitable choice of $\gamma : \partial\Omega \to [0, \pi]$.

Since Ω_∞ is an infinite sector here and in [12], the examples above do not apply. In the special case where $\alpha < \pi/2$ and $\gamma_1 = \gamma_2 = \frac{\pi}{2} - \alpha$, Tam ([24]) shows that if $\mathcal{P} \neq \emptyset$ and $\mathcal{N} \neq \emptyset$, then $\overline{\mathcal{P}} \cup \overline{\mathcal{N}} = \overline{\Omega_\infty}$. On the basis of suggestive, but not conclusive, comparison arguments and interesting discussions with Robert Finn, to whom we offer our thanks, we set the conjecture:

Conjecture 2.1. *Suppose $\mathcal{P} \cup \mathcal{N} \neq \emptyset$. Then $\overline{\mathcal{P}} \cup \overline{\mathcal{N}} = \overline{\Omega_\infty}$.*

Theorem 2.2. *Suppose $\alpha > \pi/2$ and $\mathcal{P} \subset \Omega_\infty$ minimizes Φ. Let (r, θ) be polar coordinates about O. Then one of the following holds:*

(i) *$\mathcal{P} = \emptyset$ or $\mathcal{P} = \Omega_\infty$;*

(ii) *$\gamma_1 - \gamma_2 \le 2\alpha - \pi$, there exists $A \in \Sigma_1$ such that $\partial\Omega_\infty \cap \partial\mathcal{P} = \Sigma_1 \setminus OA$, $\Omega_\infty \cap \partial\mathcal{P}$ is the ray L in Ω_∞ starting at A and making an angle of measure γ_1 with $\Sigma_1 \setminus OA$ and $\mathcal{P}$ is the open sector between $\Sigma_1 \setminus OA$ and L;*

(iii) *$\gamma_1 - \gamma_2 \ge \pi - 2\alpha$, there exists $A \in \Sigma_1$ such that $\partial\Omega_\infty \cap \partial\mathcal{P} = \Sigma_2 \cup \overline{OA}$, $\Omega_\infty \cap \partial\mathcal{P}$ is the ray L in Ω_∞ starting at A and making an angle of measure γ_1 with OA and $\mathcal{P}$ is the open region whose boundary is $\Sigma_2 \cup \overline{OA} \cup L$;*

(iv) *$\gamma_1 - \gamma_2 \le 2\alpha - \pi$, there exists $B \in \Sigma_2$ such that $\partial\Omega_\infty \cap \partial\mathcal{P} = \Sigma_1 \cup \overline{OB}$, $\Omega_\infty \cap \partial\mathcal{P}$ is the ray L in Ω_∞ starting at B and making an angle of measure γ_2 with OB and $\mathcal{P}$ is the open region whose boundary is $\Sigma_1 \cup \overline{OA} \cup L$;*

(v) *$\gamma_1 - \gamma_2 \ge \pi - 2\alpha$, there exists $B \in \Sigma_2$ such that $\partial\Omega_\infty \cap \partial\mathcal{P} = \Sigma_2 \setminus OB$, $\Omega_\infty \cap \partial\mathcal{P}$ is the ray L in Ω_∞ starting at B and making an angle of measure γ_2 with $\Sigma_2 \setminus OB$ and $\mathcal{P}$ is the open sector between $\Sigma_2 \setminus OB$ and L;*

(vi) *$\gamma_1 + \pi - \gamma_2 \le 2\alpha$, $\partial\Omega_\infty \cap \partial\mathcal{P} = \Sigma_1 \cup \{O\}$, $\Omega_\infty \cap \partial\mathcal{P}$ is a ray $L = \{\theta = \beta\}$ in Ω_∞ starting at O which makes an angle of measure greater than or equal to γ_1 with Σ_1 and an angle of measure greater than or equal to $\pi - \gamma_2$ with Σ_2 (i.e. $\pi - \alpha - \gamma_2 \le \beta \le \alpha - \gamma_1$) and $\mathcal{P} = \{\beta < \theta < \alpha\}$;*

(vii) *$\gamma_2 + \pi - \gamma_1 \le 2\alpha$, $\partial\Omega_\infty \cap \partial\mathcal{P} = \Sigma_2 \cup \{O\}$, $\Omega_\infty \cap \partial\mathcal{P}$ is a ray $L = \{\theta = \beta\}$ in Ω_∞ starting at O which makes an angle of measure greater than or equal to $\pi - \gamma_1$ with Σ_1 and an angle of measure greater than or equal to γ_2 with Σ_2 (i.e. $\gamma_2 - \alpha \le \beta \le \alpha + \gamma_1 - \pi$) and $\mathcal{P} = \{-\alpha < \theta < \beta\}$;*

(viii) *$\pi - \gamma_1 + \pi - \gamma_2 \le 2\alpha - \pi$, $\partial\mathcal{P}$ is a line $L = \{\theta = \beta\} \cup \{\theta = \beta + \pi\}$ which passes through O and makes angles of measure greater than or equal to $\pi - \gamma_1$ with Σ_1 and $\pi - \gamma_2$ with Σ_2 (i.e. $\pi - \alpha - \gamma_2 \le \beta \le \alpha + \gamma_1 - 2\pi$) and $\mathcal{P} = \{\beta < \theta < \beta + \pi\}$ is the component of $\Omega_\infty \setminus L$ whose closure is disjoint from $\Sigma_1 \cup \Sigma_2$;*

(ix) *$\pi - \gamma_1 + \pi - \gamma_2 \le 2\alpha - \pi$, $\partial\mathcal{P}$ is a line M in Ω_∞ which is a parallel translate of the line L described in (viii) and $\mathcal{P}$ is the component of $\Omega_\infty \setminus M$ whose closure is disjoint from $\Sigma_1 \cup \Sigma_2$;*

(x) *$\pi - \gamma_1 + \pi - \gamma_2 \le 2\alpha - \pi$, $\partial\Omega_\infty \cap \partial\mathcal{P} = \{O\}$, $\Omega_\infty \cap \partial\mathcal{P}$ is a pair of rays $L = \{\theta = \beta_1\}$ and $M = \{\theta = \beta_2\}$ in Ω_∞, each starting at O, such that $\beta_1 - \beta_2 \ge \pi$, $\alpha - \beta_1 \ge \pi - \gamma_1$, $\beta_2 + \pi \ge \pi - \gamma_2$, and $\mathcal{P} = \{\beta_2 < \theta < \beta_1\}$;*

(xi) $\gamma_1+\gamma_2 \le 2\alpha-\pi$, $\partial\mathcal{P}$ *is a line* $L=\{\theta=\beta\}\cup\{\theta=\beta+\pi\}$ *which passes through* O *and makes angles of measure greater than or equal to* γ_1 *with* Σ_1 *and* γ_2 *with* Σ_2 *and* $\mathcal{P}=\{-\alpha<\theta<\beta\}\cup\{\beta+\pi<\theta<\alpha\}$ *is the union of the (two) components of* $\Omega_\infty\setminus L$ *whose closures intersect* $\Sigma_1\cup\Sigma_2$*;*

(xii) $\gamma_1+\gamma_2 \le 2\alpha-\pi$, $\partial\mathcal{P}$ *is a line* M *in* Ω_∞ *which is a parallel translate of the line* L *described in (xi) and* $\mathcal{P}$ *is the component of* $\Omega_\infty\setminus M$ *whose closure contains* $\Sigma_1\cup\Sigma_2$*;*

(xiii) $\gamma_1+\gamma_2 \le 2\alpha-\pi$, $\partial\Omega_\infty\cap\partial\mathcal{P}=\partial\Omega_\infty$, $\Omega_\infty\cap\partial\mathcal{P}$ *is a pair of rays* $L=\{\theta=\beta_1\}$ *and* $M=\{\theta=\beta_2\}$ *in* Ω_∞*, each starting at* O*, such that* $\beta_1-\beta_2\ge\pi$, $\alpha-\beta_1\ge\gamma_1$, $\beta_2+\pi\ge\gamma_2$*, and* $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2$*, where* $\mathcal{P}_1=\{\beta_1<\theta<\alpha\}$ *and* $\mathcal{P}_2=\{-\alpha<\theta<\beta_2\}$*;*

(xiv) $\gamma_1+\gamma_2 \le 2\alpha-\pi$*, there exist* $A\in\Sigma_1$ *and* $B\in\Sigma_2$ *such that* $\partial\Omega_\infty\cap\partial\mathcal{P}=(\Sigma_1\setminus OA)\cup(\Sigma_2\setminus OB)$, $\Omega_\infty\cap\partial\mathcal{P}$ *is the union of rays* L_1 *and* L_2 *in* Ω_∞*, where* L_1 *starts at* A *and makes an angle of measure* γ_1 *with* $\Sigma_1\setminus OA$ *and* L_2 *starts at* B *and makes an angle of measure* γ_2 *with* $\Sigma_2\setminus OB$*, and* $\mathcal{P}$ *is the union of the open sectors between* $\Sigma_1\setminus OA$ *and* L_1 *and between* $\Sigma_2\setminus OB$ *and* L_2 *respectively; or*

(xv) $\pi-\gamma_1+\pi-\gamma_2\le 2\alpha-\pi$*, there exist* $A\in\Sigma_1$ *and* $B\in\Sigma_2$ *such that* $\partial\Omega_\infty\cap\partial\mathcal{P}=\overline{OA}\cup\overline{OB}$, $\Omega_\infty\cap\partial\mathcal{P}$ *is the union of rays* L_1 *and* L_2 *in* Ω_∞*, where* L_1 *starts at* A *and makes an angle of measure* γ_1 *with* OA *and* L_2 *starts at* B *and makes an angle of measure* γ_2 *with* OB*, and* $\mathcal{P}$ *is the open region in* Ω_∞ *between* L_1 *and* L_2*.*

(xvi) $\gamma_1+\gamma_2 \le 2\alpha-\pi$*, there exists* $A\in\Sigma_1$ *such that* $\partial\Omega_\infty\cap\partial\mathcal{P}=\partial\Omega_\infty\setminus OA$, $\Omega_\infty\cap\partial\mathcal{P}$ *is a pair of rays* L *and* M *in* Ω_∞*, and* $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2$*, where* L *starts at* A *and makes an angle of measure* γ_1 *with* $\Sigma_1\setminus OA$, $M=\{\theta=\beta_2\}$ *starts at* O *with* $-\alpha+\gamma_2\le\beta_2\le\alpha-\gamma_1-\pi$, $\mathcal{P}_1$ *is the open, connected region in* Ω_∞ *with boundary* $L\cup\Sigma_1\setminus OA$ *and* $\mathcal{P}_2=\{-\alpha<\theta<\beta_2\}$*;*

(xvii) $\pi-\gamma_1+\pi-\gamma_2\le 2\alpha-\pi$*, there exists* $A\in\Sigma_1$ *such that* $\partial\Omega_\infty\cap\partial\mathcal{P}=\overline{OA}$, $\Omega_\infty\cap\partial\mathcal{P}$ *is a pair of rays* L *and* M *in* Ω_∞ *and* $\mathcal{P}$ *is the connected open subset of* Ω_∞ *with boundary* $L\cup\overline{OA}\cup M$*, where* L *starts at* A *and makes an angle of measure* γ_1 *with* OA *and* $M=\{\theta=\beta_2\}$ *starts at* O *with* $-\alpha+\pi-\gamma_2\le\beta_2\le\alpha+\gamma_1-2\pi$*;*

(xviii) $\gamma_1+\gamma_2 \le 2\alpha-\pi$*, there exists* $B\in\Sigma_2$ *such that* $\partial\Omega_\infty\cap\partial\mathcal{P}=\partial\Omega_\infty\setminus OB$, $\Omega_\infty\cap\partial\mathcal{P}$ *is a pair of rays* L *and* M *in* Ω_∞ *and* $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2$*, where* $L=\{\theta=\beta_1\}$ *starts at* O *with* $-\alpha+\gamma_2+\pi\le\beta_1\le\alpha-\gamma_1$*,* M *starts at* B *and makes an angle of measure* γ_2 *with* $\Sigma_2\setminus OB$, $\mathcal{P}_1=\{\beta_1<\theta<\alpha\}$ *and* $\mathcal{P}_2$ *is the open, connected region in* Ω_∞ *with boundary* $L\cup\Sigma_2\setminus OB$*;*

(xix) $\pi-\gamma_1+\pi-\gamma_2\le 2\alpha-\pi$*, there exists* $B\in\Sigma_2$ *such that* $\partial\Omega_\infty\cap\partial\mathcal{P}=\overline{OB}$, $\Omega_\infty\cap\partial\mathcal{P}$ *is a pair of rays* L *and* M *in* Ω_∞*, and* $\mathcal{P}$ *is the connected open subset of* Ω_∞ *with boundary* $L\cup\overline{OB}\cup M$*, where* $L=\{\theta=\beta_1\}$ *starts at* O *with* $-\alpha+2\pi-\gamma_2\le\beta_1\le\alpha+\gamma_1-\pi$ *and* M *starts at* B *and makes an angle of measure* γ_2 *with* OB*.*

Theorem 2.3. *Suppose* $\alpha>\pi/2$ *and* $\mathcal{N}\subset\Omega_\infty$ *minimizes* Ψ*. Let* (r,θ) *be polar coordinates about* O*. Then one of the following holds:*

(i) $\mathcal{N}=\emptyset$ *or* $\mathcal{N}=\Omega_\infty$*;*

(ii) $\gamma_1-\gamma_2\le 2\alpha-\pi$*, there exists* $A\in\Sigma_1$ *such that* $\partial\Omega_\infty\cap\partial\mathcal{N}=\Sigma_2\cup\overline{OA}$, $\Omega_\infty\cap\partial\mathcal{N}$ *is the ray* L *in* Ω_∞ *starting at* A *and making an angle of measure* $\pi-\gamma_1$ *with* OA *and* $\mathcal{N}$ *is the open region whose boundary is* $\Sigma_2\cup\overline{OA}\cup L$*;*

(iii) $\gamma_1 - \gamma_2 \geq \pi - 2\alpha$, *there exists* $A \in \Sigma_1$ *such that* $\partial\Omega_\infty \cap \partial\mathcal{N} = \Sigma_1 \setminus OA$, $\Omega_\infty \cap \partial\mathcal{N}$ *is the ray* L *in* Ω_∞ *starting at* A *and making an angle of measure* $\pi - \gamma_1$ *with* $\Sigma_1 \setminus OA$ *and* $\mathcal{N}$ *is the open sector between* $\Sigma_1 \setminus OA$ *and* L*;*

(iv) $\gamma_1 - \gamma_2 \leq 2\alpha - \pi$, *there exists* $B \in \Sigma_2$ *such that* $\partial\Omega_\infty \cap \partial\mathcal{N} = \Sigma_2 \setminus OB$, $\Omega_\infty \cap \partial\mathcal{N}$ *is the ray* L *in* Ω_∞ *starting at* B *and making an angle of measure* $\pi - \gamma_2$ *with* $\Sigma_2 \setminus OB$ *and* $\mathcal{N}$ *is the open sector between* $\Sigma_2 \setminus OB$ *and* L*;*

(v) $\gamma_1 - \gamma_2 \geq \pi - 2\alpha$, *there exists* $B \in \Sigma_2$ *such that* $\partial\Omega_\infty \cap \partial\mathcal{N} = \Sigma_1 \cup \overline{OB}$, $\Omega_\infty \cap \partial\mathcal{N}$ *is the ray* L *in* Ω_∞ *starting at* B *and making an angle of measure* $\pi - \gamma_2$ *with* OB *and* $\mathcal{N}$ *is the open region whose boundary is* $\Sigma_1 \cup \overline{OA} \cup L$*;*

(vi) $\gamma_1 + \pi - \gamma_2 \leq 2\alpha$, $\partial\Omega_\infty \cap \partial\mathcal{N} = \Sigma_2 \cup \{O\}$, $\Omega_\infty \cap \partial\mathcal{N}$ *is a ray* $L = \{\theta = \beta\}$ *in* Ω_∞ *starting at* O *which makes an angle of measure greater than or equal to* γ_1 *with* Σ_1 *and an angle of measure greater than or equal to* $\pi - \gamma_2$ *with* Σ_2 *(i.e.* $\pi - \alpha - \gamma_2 \leq \beta \leq \alpha - \gamma_1$*) and* $\mathcal{N} = \{-\alpha < \theta < \beta\}$*;*

(vii) $\gamma_2 + \pi - \gamma_1 \leq 2\alpha$, $\partial\Omega_\infty \cap \partial\mathcal{N} = \Sigma_1 \cup \{O\}$, $\Omega_\infty \cap \partial\mathcal{N}$ *is a ray* $L = \{\theta = \beta\}$ *in* Ω_∞ *starting at* O *which makes an angle of measure greater than or equal to* $\pi - \gamma_1$ *with* Σ_1 *and an angle of measure greater than or equal to* γ_2 *with* Σ_2 *(i.e.* $\gamma_2 - \alpha \leq \beta \leq \alpha + \gamma_1 - \pi$*) and* $\mathcal{N} = \{\beta < \theta < \alpha\}$*;*

(viii) $\pi - \gamma_1 + \pi - \gamma_2 \leq 2\alpha - \pi$, $\partial\mathcal{N}$ *is a line* $L = \{\theta = \beta\} \cup \{\theta = \beta + \pi\}$ *which passes through* O *and makes angles of measure greater than or equal to* $\pi - \gamma_1$ *with* Σ_1 *and* $\pi - \gamma_2$ *with* Σ_2 *(i.e.* $\pi - \alpha - \gamma_2 \leq \beta \leq \alpha + \gamma_1 - 2\pi$*) and* $\mathcal{N} = \{-\alpha < \theta < \beta\} \cup \{\beta + \pi < \theta < \alpha\}$ *is the union of the (two) components of* $\Omega_\infty \setminus L$ *whose closures intersect* $\Sigma_1 \cup \Sigma_2$*;*

(ix) $\pi - \gamma_1 + \pi - \gamma_2 \leq 2\alpha - \pi$, $\partial\mathcal{N}$ *is a line* M *in* Ω_∞ *which is a parallel translate of the line* L *described in (viii) and* $\mathcal{N}$ *is the component of* $\Omega_\infty \setminus M$ *whose closure contains* $\Sigma_1 \cup \Sigma_2$*;*

(x) $\pi - \gamma_1 + \pi - \gamma_2 \leq 2\alpha - \pi$, $\partial\Omega_\infty \cap \partial\mathcal{N} = \partial\Omega_\infty$, $\Omega_\infty \cap \partial\mathcal{N}$ *is a pair of rays* $L = \{\theta = \beta_1\}$ *and* $M = \{\theta = \beta_2\}$ *in* Ω_∞*, each starting at* O*, such that* $\beta_1 - \beta_2 \geq \pi$, $\alpha - \beta_1 \geq \pi - \gamma_1$, $\beta_2 + \pi \geq \pi - \gamma_2$*, and* $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$*, where* $\mathcal{N}_1 = \{\beta_1 < \theta < \alpha\}$ *and* $\mathcal{N}_2 = \{-\alpha < \theta < \beta_2\}$*;*

(xi) $\gamma_1 + \gamma_2 \leq 2\alpha - \pi$, $\partial\mathcal{N}$ *is a line* $L = \{\theta = \beta\} \cup \{\theta = \beta + \pi\}$ *which passes through* O *and makes angles of measure greater than or equal to* γ_1 *with* Σ_1 *and* γ_2 *with* Σ_2 *(i.e.* $\gamma_2 - \alpha \leq \beta \leq \alpha - \gamma_1 - \pi$*) and* $\mathcal{N} = \{\beta < \theta < \beta + \pi\}$ *is the component of* $\Omega_\infty \setminus L$ *whose closure is disjoint from* $\Sigma_1 \cup \Sigma_2$*;*

(xii) $\gamma_1 + \gamma_2 \leq 2\alpha - \pi$, $\partial\mathcal{N}$ *is a line* M *in* Ω_∞ *which is a parallel translate of the line* L *described in (xi) and* $\mathcal{N}$ *is the component of* $\Omega_\infty \setminus M$ *whose closure is disjoint from* $\Sigma_1 \cup \Sigma_2$*;*

(xiii) $\gamma_1 + \gamma_2 \leq 2\alpha - \pi$, $\partial\Omega_\infty \cap \partial\mathcal{N} = \{O\}$, $\Omega_\infty \cap \partial\mathcal{N}$ *is a pair of rays* $L = \{\theta = \beta_1\}$ *and* $M = \{\theta = \beta_2\}$ *in* Ω_∞*, each starting at* O*, such that* $\beta_1 - \beta_2 \geq \pi$, $\alpha - \beta_1 \geq \gamma_1$, $\beta_2 + \alpha \geq \gamma_2$*, and* $\mathcal{N} = \{\beta_2 < \theta < \beta_1\}$*;*

(xiv) $\gamma_1 + \gamma_2 \leq 2\alpha - \pi$, *there exist* $A \in \Sigma_1$ *and* $B \in \Sigma_2$ *such that* $\partial\Omega_\infty \cap \partial\mathcal{N} = \overline{OA} \cup \overline{OB}$, $\Omega_\infty \cap \partial\mathcal{N}$ *is the union of rays* L_1 *and* L_2 *in* Ω_∞*, where* L_1 *starts at* A *and makes an angle of measure* γ_1 *with* $\Sigma_1 \setminus OA$ *and* L_2 *starts at* B *and makes an angle of measure* γ_2 *with* $\Sigma_2 \setminus OB$*, and* $\mathcal{N}$ *is the open region in* Ω_∞ *between* L_1 *and* L_2*; or*

(xv) $\pi - \gamma_1 + \pi - \gamma_2 \leq 2\alpha - \pi$, *there exist* $A \in \Sigma_1$ *and* $B \in \Sigma_2$ *such that* $\partial\Omega_\infty \cap \partial\mathcal{N} = (\Sigma_1 \setminus OA) \cup (\Sigma_2 \setminus OB)$, $\Omega_\infty \cap \partial\mathcal{N}$ *is the union of rays* L_1 *and* L_2 *in* Ω_∞*, where*

L_1 starts at A and makes an angle of measure γ_1 with OA and L_2 starts at B and makes an angle of measure γ_2 with OB, and $\mathcal{N}$ is the union of the open sectors between $\Sigma_1 \setminus OA$ and L_1 and between $\Sigma_2 \setminus OB$ and L_2 respectively.

(xvi) *$\gamma_1 + \gamma_2 \le 2\alpha - \pi$, there exists $A \in \Sigma_1$ such that $\partial\Omega_\infty \cap \partial\mathcal{N} = \partial\Omega_\infty \setminus OA$, $\Omega_\infty \cap \partial\mathcal{N}$ is a pair of rays L and M in Ω_∞, and $\mathcal{N}$ is the connected open subset of Ω_∞ with boundary $L \cup \overline{OA} \cup M$, where L starts at A and makes an angle of measure γ_1 with $\Sigma_1 \setminus OA$ and $M = \{\theta = \beta_2\}$ starts at O with $-\alpha + \gamma_2 \le \beta_2 \le \alpha - \gamma_1 - \pi$;*

(xvii) *$\pi - \gamma_1 + \pi - \gamma_2 \le 2\alpha - \pi$, there exists $A \in \Sigma_1$ such that $\partial\Omega_\infty \cap \partial\mathcal{N} = \overline{OA}$, $\Omega_\infty \cap \partial\mathcal{N}$ is a pair of rays L and M in Ω_∞ and $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$, where L starts at A and makes an angle of measure γ_1 with OA, $M = \{\theta = \beta_2\}$ starts at O with $-\alpha + \pi - \gamma_2 \le \beta_2 \le \alpha + \gamma_1 - 2\pi$, $\mathcal{N}_1$ is the connected open subset of Ω_∞ with boundary $L \cup \Sigma_2 \setminus OA$ and $\mathcal{N}_2 = \{-\alpha < \theta < \beta_2\}$;*

(xviii) *$\gamma_1 + \gamma_2 \le 2\alpha - \pi$, there exists $B \in \Sigma_2$ such that $\partial\Omega_\infty \cap \partial\mathcal{N} = \partial\Omega_\infty \setminus OB$, $\Omega_\infty \cap \partial\mathcal{N}$ is a pair of rays L and M in Ω_∞ and $\mathcal{N}$ is the connected open subset of Ω_∞ with boundary $L \cup \overline{OB} \cup M$, where $L = \{\theta = \beta_1\}$ starts at O with $-\alpha + \gamma_2 + \pi \le \beta_1 \le \alpha - \gamma_1$ and M starts at B and makes an angle of measure γ_2 with $\Sigma_2 \setminus OB$;*

(xix) *$\pi - \gamma_1 + \pi - \gamma_2 \le 2\alpha - \pi$, there exists $B \in \Sigma_2$ such that $\partial\Omega_\infty \cap \partial\mathcal{N} = \overline{OB}$, $\Omega_\infty \cap \partial\mathcal{N}$ is a pair of rays L and M in Ω_∞, and $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$, where $L = \{\theta = \beta_1\}$ starts at O with $-\alpha + 2\pi - \gamma_2 \le \beta_1 \le \alpha + \gamma_1 - \pi$, M starts at B and makes an angle of measure γ_2 with OB, $\mathcal{N}_1 = \{\beta_2 < \theta < \alpha\}$ and $\mathcal{N}_2$ is the connected open subset of Ω_∞ with boundary $M \cup \Sigma_2 \setminus OB$.*

Corollary 2.4. *Suppose $\alpha > \pi/2$ and (γ_1, γ_2) satisfies $\gamma_1 - \gamma_2 < \pi - 2\alpha$ (i.e. (γ_1, γ_2) lies in the open region denoted D_2^+ in Figure 3). Then only cases (i), (ii), (iv) and (vi) of Theorems 2.2 and 2.3 can hold.*

Corollary 2.5. *Suppose $\alpha > \pi/2$ and (γ_1, γ_2) satisfies $\gamma_1 - \gamma_2 > 2\alpha - \pi$ (i.e. (γ_1, γ_2) lies in the open region denoted D_2^- in Figure 3). Then only cases (i), (iii), (v) and (vii) of Theorems 2.2 and 2.3 can hold.*

Corollary 2.6. *Suppose $\alpha > \pi/2$ and (γ_1, γ_2) satisfies $\gamma_1 + \gamma_2 < 2\alpha - \pi$ (i.e. (γ_1, γ_2) lies in the open region denoted D_1^+ in Figure 3). Then cases (viii), (ix), (x), (xv), (xvii) and (xix) of Theorems 2.2 and 2.3 cannot hold.*

Corollary 2.7. *Suppose $\alpha > \pi/2$ and (γ_1, γ_2) satisfies $\gamma_1 + \gamma_2 > 3\pi - 2\alpha$ (i.e. (γ_1, γ_2) lies in the open region denoted D_1^- in Figure 3). Then cases (xi), (xii), (xiii), (xiv), (xvi) and (xviii) of Theorems 2.2 and 2.3 cannot hold.*

The proofs of these corollaries are simple exercises in checking angles.

3. Applications to capillarity

Consider the stationary liquid-gas interface formed by an incompressible fluid in a vertical cylindrical tube with cross-section Ω. For simplicity, we assume that near $(0,0)$, $\partial\Omega$ has straight sides (as in [21]) and so we may assume

$$\Omega = \{(r\cos(\theta), r\sin(\theta)) : 0 < r < 1, -\alpha < \theta < \alpha\}. \tag{3.1}$$

In a microgravity environment or in a downward-oriented gravitational field, this interface will be a nonparametric surface $z = f(x,y)$ which is a solution of the

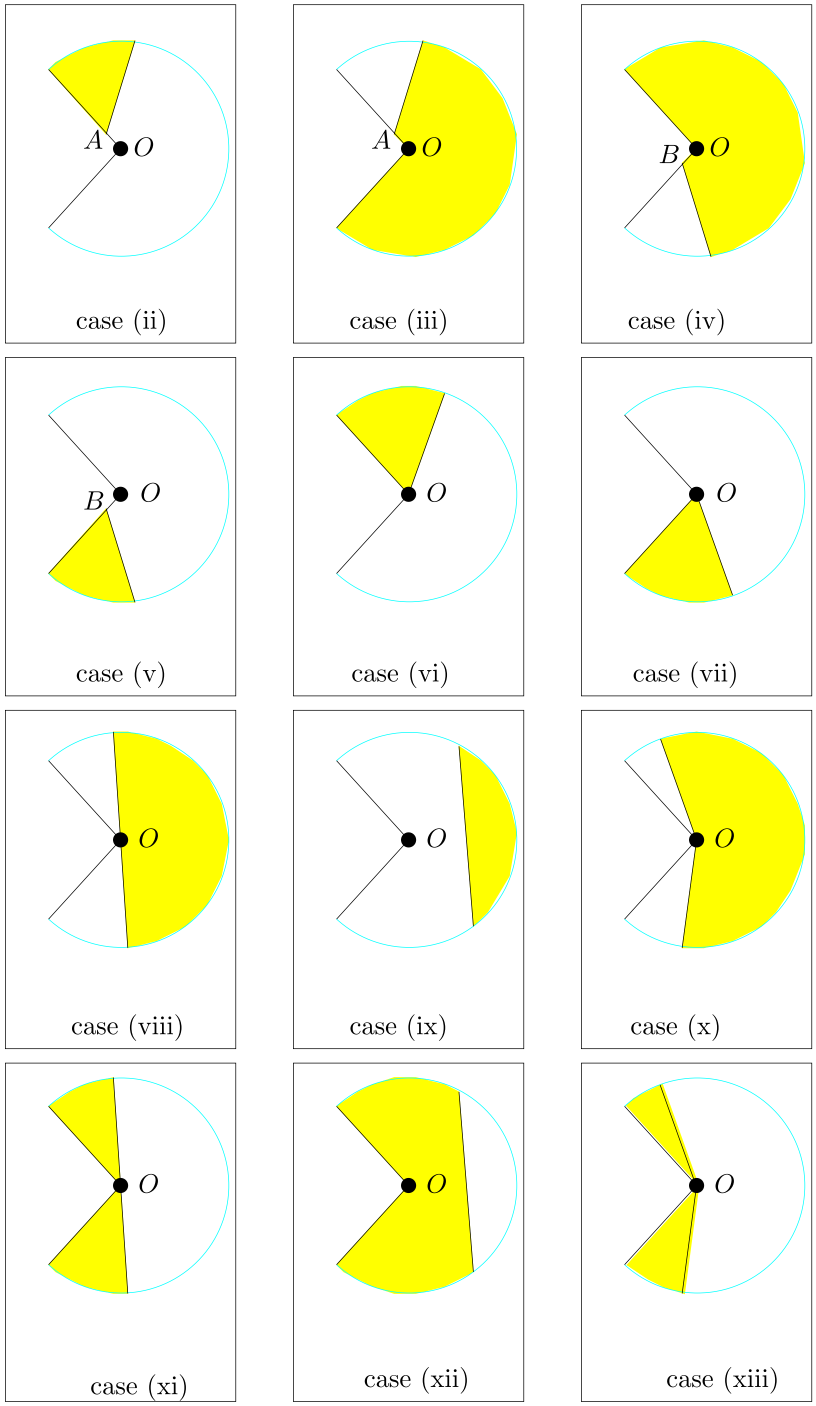

FIGURE 1. Theorems 2.2 and 2.3

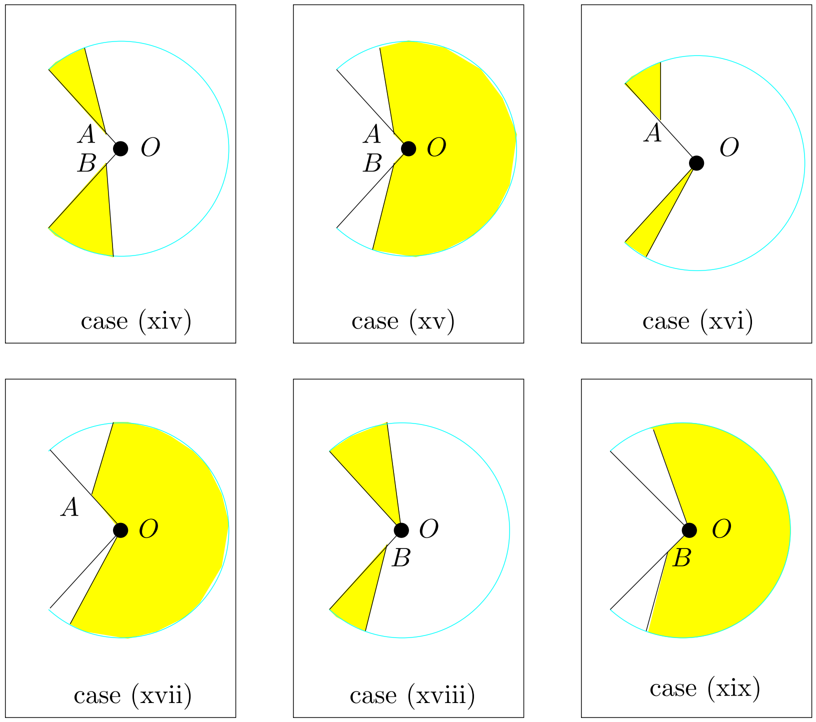

FIGURE 2. Theorems 2.2 and 2.3

boundary value problem (1.1)-(1.2) with $H(z) = \kappa z + \lambda$; that is,

$$Nf = \kappa f + \lambda \quad \text{in } \Omega \tag{3.2}$$

$$Tf \cdot \nu = \cos\gamma \quad \text{a.e. on } \partial\Omega \tag{3.3}$$

where $Tf = \nabla f/\sqrt{1+|\nabla f|^2}$, $Nf = \nabla \cdot Tf$, ν is the exterior unit normal on $\partial\Omega$, κ and λ are constants with $\kappa \geq 0$, $\gamma = \gamma(x,y) \in [0,\pi]$ is the angle at which the liquid-gas interface meets the vertical cylinder ([4]) and $\gamma_1, \gamma_1 \in (0,\pi)$ are as in §2. Many authors have studied the nonparametric capillary problem (3.2)-(3.3), prominently among them are Paul Concus and Robert Finn (e.g. see [2, 4, 5, 6, 7, 8, 9]); the first paper establishing existence was [3] (see also [26]).

We are interested in the behavior of a solution f of (3.2)-(3.3) "at" $(0,0)$. For nonconvex corners, Shi followed the example of an illustration Concus and Finn used for convex corners in [2] and divided the square $(0,\pi)\times(0,\pi)$ into five distinct regions; these regions, illustrated in Figure 3 below, are:

$\mathcal{R} = \{(\gamma_1,\gamma_2) : 2\alpha - \pi \leq \gamma_1 + \gamma_2 \leq 3\pi - 2\alpha, \pi - 2\alpha \leq \gamma_1 - \gamma_2 \leq 2\alpha - \pi\}$
$\mathcal{D}_1^+ = \{(\gamma_1,\gamma_2) : \gamma_1 + \gamma_2 < 2\alpha - \pi\}$
$\mathcal{D}_1^- = \{(\gamma_1,\gamma_2) : \gamma_1 + \gamma_2 > 3\pi - 2\alpha\}$
$\mathcal{D}_2^+ = \{(\gamma_1,\gamma_2) : \gamma_1 - \gamma_2 < \pi - 2\alpha\}$
$\mathcal{D}_2^- = \{(\gamma_1,\gamma_2) : \gamma_1 - \gamma_2 > 2\alpha - \pi\}$.

Shi assumed the Concus-Finn Conjecture was true for $\gamma_1 \in \{0,\pi\}$ and $\gamma_2 \in \{0,\pi\}$) and proved in [20] and [21] that a solution $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ of (3.2) and (3.3) must be discontinuous at O when $(\gamma_1,\gamma_2) \in \mathcal{D}_1^+ \cup \mathcal{D}_1^- \cup \mathcal{D}_2^+ \cup \mathcal{D}_2^-$

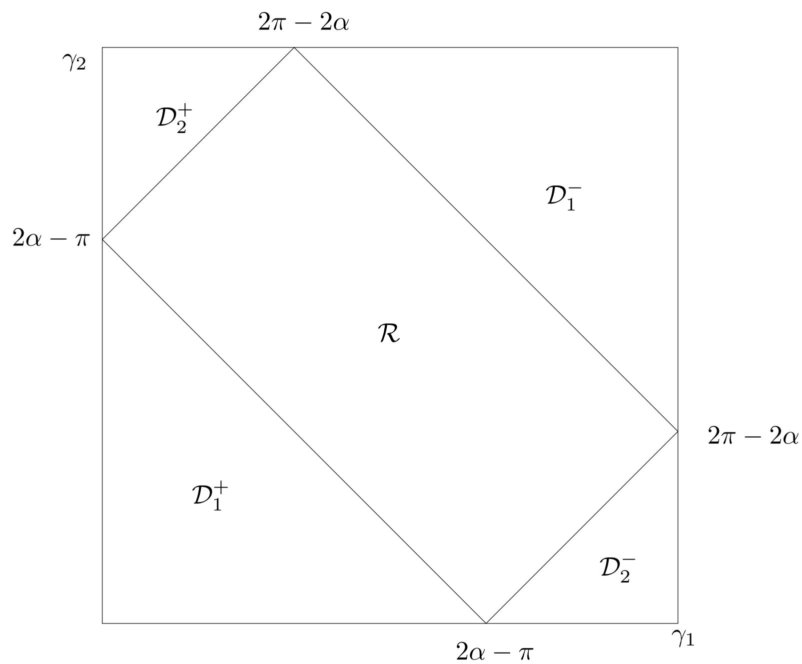

FIGURE 3. Nonconvex Concus-Finn rectangle

and $\kappa > 0$. Our goal here is to reach the same conclusion when $\kappa = \lambda = 0$ and to prepare the necessary background for a direct proof of the "nonconvex Concus-Finn conjecture" in [15].

To determine the behavior of f near $(0,0)$, we need first to determine the behavior of the Gauss map on the edge $\{(0,0,z) : z \in \mathbb{R}\}$. For $\beta \in (-\alpha, \alpha)$, let t_β denote the set of sequences (X_j) in Ω which satisfy

$$\lim_{j\to\infty} X_j = (0,0) \quad \text{and} \quad \lim_{j\to\infty} \frac{X_j}{|X_j|} = (\cos(\beta), \sin(\beta)). \tag{3.4}$$

For a given solution $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ of (3.2) and (3.3), we define

$$\vec{n}(x,y) = \vec{n}_f(x,y) = \big(Tf(x,y), \frac{-1}{\sqrt{1+|\nabla f(x,y)|^2}}\big) \tag{3.5}$$

to be the (downward) unit normal to the graph of f at $(x, y, f(x, y))$. Let $S_0^2 = \{(x,y,0) : x, y \in \mathbb{R},\ x^2 + y^2 = 1\}$.

Lemma 3.1. *Suppose $\alpha > \pi/2$, $(\gamma_1, \gamma_2) \in D_1^+ \cup D_1^- \cup D_2^+ \cup D_2^-$, f is a solution of* (3.2) *and* (3.3)*, $\beta \in (-\alpha, \alpha)$, and $(X_j) \in t_\beta$ such that $\eta = \lim_{j\to\infty} \vec{n}_f(X_j)$ exists. Then $\eta \in S_0^2$.*

It is a fact that no nonvertical plane in $\mathbb{R}^3$ meets $L^+ \times \mathbb{R}$ in an angle of γ_1 and $L^- \times \mathbb{R}$ in an angle of γ_2 when $(\gamma_1, \gamma_2) \in D_1^+ \cup D_1^- \cup D_2^+ \cup D_2^-$ in Figure 3. The proof of the lemma follows as in the proof of [12, Lemma 3.1].

Remark 3.2. As noted in [12, Remark 3.2], we may assume in this section that Ω and γ are as described in §2 and f satisfies (3.2) and (3.3).

Lemma 3.3. *Suppose $\alpha > \pi/2$ and (γ_1, γ_2) lies in D_2^+ (i.e. $\gamma_1 - \gamma_2 < \pi - 2\alpha$). Let $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ satisfy (3.2) and (3.3). Let $\beta \in (-\alpha, \alpha)$ and let $\{(x_j, y_j)\} \in t_\beta$.*

- (i) *If $\beta \in [-\alpha + \pi - \gamma_2, \alpha - \gamma_1]$, then $\lim_{j\to\infty} \vec{n}(x_j, y_j) = (-\sin(\beta), \cos(\beta), 0)$.*
- (ii) *If $\beta \in (-\alpha, -\alpha + \pi - \gamma_2]$, then*
  $\lim_{j\to\infty} \vec{n}(x_j, y_j) = (-\sin(-\alpha + \pi - \gamma_2), \cos(-\alpha + \pi - \gamma_2), 0)$.
- (iii) *If $\beta \in [\alpha - \gamma_1, \alpha)$, $\lim_{j\to\infty} \vec{n}(x_j, y_j) = (-\sin(\alpha - \gamma_1), \cos(\alpha - \gamma_1), 0)$.*

In light of Corollary 2.4, the proof of this lemma is essentially the same as that of [12, Lemma 3.1].

Lemma 3.4. *Suppose $\alpha > \pi/2$ and (γ_1, γ_2) lies in D_2^- (i.e. $\gamma_1 - \gamma_2 > 2\alpha - \pi$). Let $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ satisfy (3.2) and (3.3). Let $\beta \in (-\alpha, \alpha)$ and let $\{(x_j, y_j)\} \in t_\beta$.*

- (i) *If $\beta \in [-\alpha + \gamma_2, \alpha + \gamma_1 - \pi]$, then $\lim_{j\to\infty} \vec{n}(x_j, y_j) = (\sin(\beta), -\cos(\beta), 0)$.*
- (ii) *If $\beta \in (-\alpha, -\alpha + \gamma_2]$, then*
  $\lim_{j\to\infty} \vec{n}(x_j, y_j) = (\sin(-\alpha + \gamma_2), -\cos(-\alpha + \gamma_2), 0)$.
- (iii) *If $\beta \in [\alpha + \gamma_1 - \pi, \alpha)$, then*
  $\lim_{j\to\infty} \vec{n}(x_j, y_j) = (\sin(\alpha + \gamma_1 - \pi), -\cos(\alpha + \gamma_1 - \pi), 0)$.

In light of Corollary 2.5, the proof of this lemma follows using the techniques in the proof of [12, Lemma 3.1].

Lemma 3.5. *Suppose $\alpha > \pi/2$ and (γ_1, γ_2) lies in $D_1^+ \cup D_1^-$. Let $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ satisfy (3.2) and (3.3). Then one of the following conclusions holds:*

- (i) *For each $\beta \in (-\alpha, -\alpha + \min\{\gamma_2, \pi - \gamma_2\})$ and each sequence $(X_j) \in t_\beta$,*
$$\lim_{j\to\infty} \vec{n}_f(x_j, y_j) = (\cos(\theta_1), \sin(\theta_1), 0),$$
  where $\theta_1 = -\alpha - \gamma_2 - \pi/2$.
- (ii) *For each $\beta \in (-\alpha, -\alpha + \min\{\gamma_2, \pi - \gamma_2\})$ and each sequence $(X_j) \in t_\beta$,*
$$\lim_{j\to\infty} \vec{n}_f(x_j, y_j) = (\cos(\theta_2), \sin(\theta_2), 0),$$
  where $\theta_2 = -\alpha + \gamma_2 - \pi/2$.

Proof. We have $\gamma_1 + \gamma_2 \in (0, 2\alpha - \pi) \cup (3\pi - 2\alpha, 2\pi)$. Let us define
$$C_\beta(f) = \{\eta \in S^2 : \eta = \lim_{j\to\infty} \vec{n}_f(X_j) \text{ for some } (X_j) \in t_\beta\}$$
for each $\beta \in (-\alpha, \alpha)$ and set $C(f) = \cup_{\beta\in(-\alpha,\alpha)} C_\beta(f)$. From items (a) of Lemmas 6.1-6.8 (see §6) and a simple computation, we have
$$C_\beta(f) \subset \{(\cos(\theta_1), \sin(\theta_1), 0),\ (\cos(\theta_2), \sin(\theta_2), 0)\},$$
when $\beta \in (-\alpha, -\alpha + \min\{\gamma_2, \pi - \gamma_2\})$. We argue by contradiction and therefore assume there exist $\beta_1, \beta_2 \in (-\alpha, -\alpha + \min\{\gamma_2, \pi - \gamma_2\})$, $(X_j) \in t_{\beta_1}$ and $(Y_j) \in t_{\beta_2}$ such that
$$\lim_{j\to\infty} \vec{n}_f(X_j) = (\cos(\theta_1), \sin(\theta_1), 0),$$
$$\lim_{j\to\infty} \vec{n}_f(Y_j) = (\cos(\theta_2), \sin(\theta_2), 0).$$

For each $j \in \mathbb{N}$, let σ_j be the line segment joining X_j and Y_j. By the Intermediate Value Theorem, there exists $Z_j \in \sigma_j$ such that

$$\vec{n}_f(Z_j) \in \{(r\cos(-\alpha - \frac{\pi}{2}), r\sin(-\alpha - \frac{\pi}{2}), -\sqrt{1-r^2}) : -1 \le r \le 1\} \tag{3.6}$$

for each $j \in \mathbb{N}$. Since $\lim_{j\to\infty} X_j = (0,0)$ and $\lim_{j\to\infty} Y_j = (0,0)$, we see that $\lim_{j\to\infty} Z_j = (0,0)$. Using compactness and the argument in the proof of Lemma 6.1 (see §6), we may replace the sequence (Z_j) by a subsequence such that

$$\lim_{j\to\infty} \frac{Z_j}{|Z_j|} = (\cos(\beta_3), \sin(\beta_3)) \quad \text{for some } \beta_3 \in (-\alpha, -\alpha + \min\{\gamma_2, \pi - \gamma_2\})$$

and $\lim_{j\to\infty} \vec{n}_j(Z_j) = \eta$ exists and $\eta \in C_{\beta_3}(f)$, where

$$f_j(X) = \frac{f(|Z_j|X) - f(Z_j)}{|Z_j|}$$

and $\vec{n}_j(x,y)$ is given by (6.1). Now (3.6) and Lemma 3.1 imply

$$\eta = \pm(\cos(-\alpha - \frac{\pi}{2}), \sin(-\alpha - \frac{\pi}{2}), 0).$$

However, neither of these unit vectors lies in $C_{\beta_3}(f)$ and so a contradiction exists. Thus our claim is established. □

Lemma 3.6. *Suppose $\alpha > \pi/2$ and (γ_1, γ_2) lies in $D_1^+ \cup D_1^-$. Let $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ satisfy* (3.2) *and* (3.3)*. Then one of the following conclusions holds:*

(i) *For each $\beta \in (\alpha - \min\{\gamma_1, \pi - \gamma_1\}, \alpha)$ and each sequence $(X_j) \in t_\beta$,*
$$\lim_{j\to\infty} \vec{n}_f(x_j, y_j) = (\cos(\theta_3), \sin(\theta_3), 0),$$
where $\theta_3 = \alpha - \gamma_1 + \pi/2$.

(ii) *For each $\beta \in (\alpha - \min\{\gamma_1, \pi - \gamma_1\}, \alpha)$, and each sequence $(X_j) \in t_\beta$,*
$$\lim_{j\to\infty} \vec{n}_f(x_j, y_j) = (\cos(\theta_4), \sin(\theta_4), 0),$$
where $\theta_4 = \alpha + \gamma_1 + \pi/2$.

The proof of the lemma above is similar to that of Lemma 4 and uses items (e) of Lemmas 6.1-6.8 in §6 in place of items (a) of Lemmas 6.1-6.8.

Theorem 3.7 ("Nonconvex Concus-Finn Conjecture" with $\kappa = \lambda = 0$)**.** *Suppose $\alpha > \pi/2$ and $\kappa = \lambda = 0$ in* (3.2)*. Suppose further that $(\gamma_1, \gamma_2) \in D_1^+ \cup D_1^- \cup D_2^+ \cup D_2^-$. Then every solution of* (3.2)-(3.3) *must be discontinuous at $O = (0,0)$.*

Using Lemmas 3.1-3.6, we see that the proof of this theorem is the same as that of [12, theorem 3.4].

4. Proofs for nonconvex corners: Theorems 2.2 and 2.3

In this section, we assume $\alpha > \pi/2$ and let $\mathcal{P}$ and $\mathcal{N}$ denote minimizers of Φ and Ψ respectively.

Claim 4.1. Every component of $\Omega_\infty \cap \partial\mathcal{P}$ is unbounded and every component of $\Omega_\infty \cap \partial\mathcal{N}$ is unbounded.

This claim is clear since if L is a bounded component of $\Omega_\infty \cap \partial\mathcal{P}$ (or of $\Omega_\infty \cap \partial\mathcal{N}$), then ∂L must be two distinct points of $\partial\Omega_\infty$ and $L \subset \Omega_\infty$; clearly this is impossible.

Claim 4.2. $\Omega_\infty \cap \partial\mathcal{P}$ and $\Omega_\infty \cap \partial\mathcal{N}$ have at most two components. If $\Omega_\infty \cap \partial\mathcal{P}$ has a component M which satisfies $\overline{M} \cap \partial\Omega_\infty = \emptyset$, then $\Omega_\infty \cap \partial\mathcal{P}$ has only this one component M. If $\Omega_\infty \cap \partial\mathcal{N}$ has a component M which satisfies $\overline{M} \cap \partial\Omega_\infty = \emptyset$, then $\Omega_\infty \cap \partial\mathcal{N} = M$.

Proof. Using the arguments in [12], §5, Claim 4.2, cases (c), (d), (g) and (h), we see that at most one component of $\Omega_\infty \cap \partial\mathcal{P}$ can have a point on Σ_1 in its closure. Using the arguments in [12], §5, Claim 4.2, cases (e), (f), (k) and (l), we see that at most one component of $\Omega_\infty \cap \partial\mathcal{P}$ can have a point on Σ_2 in its closure. From [12, Lemma 4.9], we see that at most two components of $\Omega_\infty \cap \partial\mathcal{P}$ can have O in their closure (and the measure of the angle between them is at least π). If there are two distinct components L and M of $\Omega_\infty \cap \partial\mathcal{P}$ with $\overline{L} \cap \partial\Omega_\infty = \emptyset$ and $\overline{M} \cap \partial\Omega_\infty = \emptyset$, then L and M must be parallel (e.g. [12], §4, (i)-(ii)) and this violates [12, Lemma 4.19]. Hence $\Omega_\infty \cap \partial\mathcal{P}$ has, at most, five components; let us say $\Omega_\infty \cap \partial\mathcal{P} = L_1 \cup L_2 \cup M_1 \cup M_1 \cup Q$, where $\overline{L_1} \cap \Sigma_1 \neq \emptyset$, $\overline{L_2} \cap \Sigma_2 \neq \emptyset$, $M_1 = \{\theta = \beta_1\}$, $M_2 = \{\theta = \beta_2\}$ (with $\beta_1 - \beta_2 \geq \pi$) and $\overline{Q} \cap \partial\Omega_\infty = \emptyset$. We shall show that the actual maximum number of components is two.

Suppose two components of $\Omega_\infty \cap \partial\mathcal{P}$ lie in $\{0 \leq \theta < \alpha\}$ and intersect $\overline{\Sigma_1}$. (In the notation of the previous paragraph, $\Omega_\infty \cap \partial\mathcal{P}$ contains L_1 and M_1 with $\beta_1 \geq 0$.) The arguments in [12], §5, Claim 4.2, cases (c), (d), (g) and (h) then yield a contradiction. Similarly, if two components of $\Omega_\infty \cap \partial\mathcal{P}$ lie in $\{-\alpha < \theta \leq 0\}$ and intersect $\overline{\Sigma_2}$ (i.e. $\Omega_\infty \cap \partial\mathcal{P}$ contains L_2 and M_2 with $\beta_1 \leq 0$.) then [12], §5, Claim 4.2, cases (e), (f), (k) and (l) yield a contradiction. Hence $\Omega_\infty \cap \partial\mathcal{P}$ can have at most two distinct components whose closures intersect $\partial\Omega_\infty$. The same conclusion holds for $\Omega_\infty \cap \partial\mathcal{N}$.

Suppose L and M are components of $\Omega_\infty \cap \partial\mathcal{P}$ such that $L \cap \partial\Omega_\infty \neq \emptyset$ and $M \cap \partial\Omega_\infty = \emptyset$. This will result in a contradiction. From [12, Lemma 4.19], we see that L and M cannot be parallel. Let L^* denote the line which contains L and let E denote the point of intersection of L^* and M. We may suppose $\overline{L} \cap \overline{\Sigma_1} = \{A\}$; notice then that $E \in \{-\alpha < \theta \leq 0\}$. The contradiction is obtained by modifying the proofs of [12], §5, Claim 4.2, cases (g) and (h); we include the details here for the benefit of the reader. If $\Omega_\infty \cap \partial\mathcal{P}$ has only the two components L and M, then $\partial\mathcal{P} = \Sigma_2 \cup \overline{OA} \cup L \cup M$. If $\Omega_\infty \cap \partial\mathcal{P}$ has three components L, M and L_2 (with $\overline{L_2} \cap \Sigma_2 = \{Y\}$), then $\partial\mathcal{P} = L_2 \cup \overline{OY} \cup \overline{OA} \cup L \cup M$; in this case, we note that if L and L_2 are parallel, then M is parallel to both L and L_2 and this violates [12, Lemma 4.19]. We exclude a potential third component L_2 of $\Omega_\infty \cap \partial\mathcal{P}$ in our arguments below since its inclusion would, at most, add a finite number of fixed terms to the right-hand sides of (4.1) and (4.2).

(a) Suppose $\mathcal{P}$ has only one component. Let B be the point of intersection of L^* and Σ_2 and let $D \in L$. Let C be the orthogonal projection of D on line M and pick T so that $T > \max\{OD, OC, OE\}$. Let Δ be the open, nonconvex polygon with boundary $OADCEB$. Since $\mathcal{P}$ minimizes Φ_T, we have $\Phi_T(\mathcal{P}) \leq \Phi_T(\mathcal{P} \setminus \Delta)$. Hence

$$EC + AD - \cos(\gamma_1)OA - \cos(\gamma_2)OB \leq EB + CD.$$

Now OA, OB, EB and EA are fixed and $AD = ED - EA$; rewriting the inequality above yields the following inequality in which the right-hand side is fixed while the left-hand side goes to infinity as the length ED goes to infinity:

$$EC + ED - CD \leq \cos(\gamma_1)OA + \cos(\gamma_2)OB + EB + EA, \tag{4.1}$$

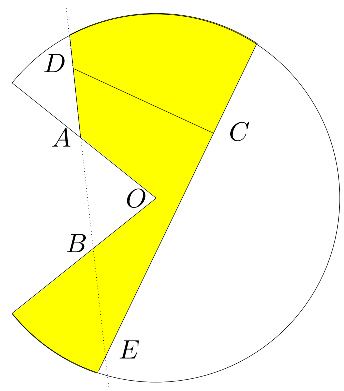

FIGURE 4. Case (a)

which is a contradiction.

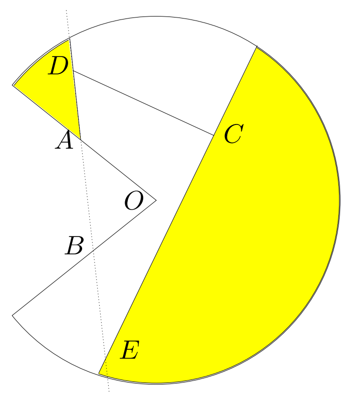

FIGURE 5. Case (b)

(b) Suppose $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the disjoint, convex, open sets with boundaries $\overline{\Sigma_1} \cup L \setminus OA$ and M respectively. Let B, C, T and Δ be as in (a) above and let $D \in L$. Since $\mathcal{P}$ minimizes Φ_T, we have $\Phi_T(\mathcal{P}) \leq \Phi_T(\mathcal{P} \cup \Delta)$. Hence

$$EC + AD \leq EB + CD - \cos(\gamma_1)OA - \cos(\gamma_2)OB.$$

Now OA, OB, EB and EA are fixed and $AD = ED - EA$; rewriting the inequality above yields the following inequality in which the right-hand side is fixed while the left-hand side goes to infinity as the length ED goes to infinity:

$$EC + ED - CD \leq EB + EA - \cos(\gamma_1)OA - \cos(\gamma_2)OB, \tag{4.2}$$

which is a contradiction.

Since cases (a) and (b) and their counterparts when $\overline{L} \cap \overline{\Sigma_2} \neq \emptyset$ represent the only cases in which $\Omega_\infty \cap \partial\mathcal{P}$ could have components L and M with $\overline{L} \cap \partial\Omega_\infty \neq \emptyset$ and $\overline{M} \cap \partial\Omega_\infty = \emptyset$, we see that if a component M of $\Omega_\infty \cap \partial\mathcal{P}$ with $\overline{M} \cap \partial\Omega_\infty = \emptyset$

exists, then $\Omega_\infty \cap \partial\mathcal{P}$ has no other components. If no such component M exists, then $\Omega_\infty \cap \partial\mathcal{P}$ could have two components L_1 and L_2 whose closures intersect $\overline{\Sigma_1}$ and $\overline{\Sigma_2}$ respectively, one component L_1 whose closure intersects $\partial\Omega_\infty$ or $\Omega_\infty \cap \partial\mathcal{P}$ could be empty. A similar argument for $\Omega_\infty \cap \partial\mathcal{N}$ completes the proof of the claim. □

Claim 4.3. Suppose M is a component of $\Omega_\infty \cap \partial\mathcal{P}$ and let ω denote the unit normal to M in the direction of $\mathcal{P}$. Let σ be the measure of the angle between ω and ν_1.

(a) If $\partial\Omega_\infty \subset \partial\mathcal{P}$, then $\sigma \geq \gamma_1$.
(b) If $\partial\Omega_\infty \cap \partial\mathcal{P} = \emptyset$, then $\sigma \leq \gamma_1$.

Proof. Let Σ_1^* denote the line which contains Σ_1 and let C denote the point of intersection of Σ_1^* and M. We will consider the proofs of (a) and (b) separately.

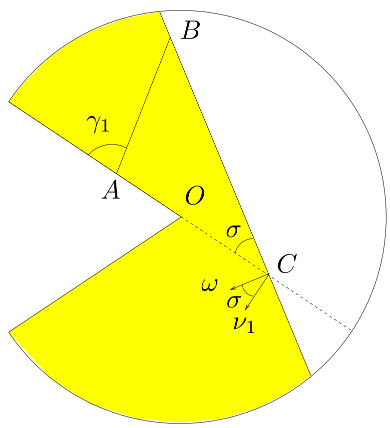

FIGURE 6. $\partial\Omega_\infty \subset \partial\mathcal{P}$

Suppose (a) holds and $\sigma < \gamma_1$. Let $A \in \Sigma_1$ and pick $B \in M$ so that angle OAB has measure $\pi - \gamma_1$. Notice that σ is the measure of angle ACB. Since $\mathcal{P}$ minimizes Φ,

$$\phi_T(\mathcal{P}) \leq \phi_T(\mathcal{P} \setminus \triangle ABC)$$

for T large. Hence

$$BC - \cos(\gamma_1)OA \leq AB + OC$$

or $BC \leq AB + \cos(\gamma_1)AC + OC - \cos(\gamma_1)OC$. If δ is the measure of angle ABC (so $\delta = \gamma_1 - \sigma$), then the law of sines implies $AC = (\sin(\delta)/\sin(\gamma_1))BC$ and $AB = (\sin(\sigma)/\sin(\gamma_1)BC$. Hence

$$1 \leq \cos(\gamma_1 - \sigma) + (1 - \cos(\gamma_1))\frac{OC}{BC},$$

as a short calculation shows. For BC sufficiently large, this yields a contradiction since OC is fixed and $\gamma_1 - \sigma > 0$.

Suppose (b) holds and $\sigma > \gamma_1$. Let $A \in \Sigma_1$ and pick $B \in M$ so that angle OAB has measure γ_1. Since $\mathcal{P}$ minimizes Φ,

$$\phi_T(\mathcal{P}) \leq \phi_T(\mathcal{P} \cup \triangle ABC)$$

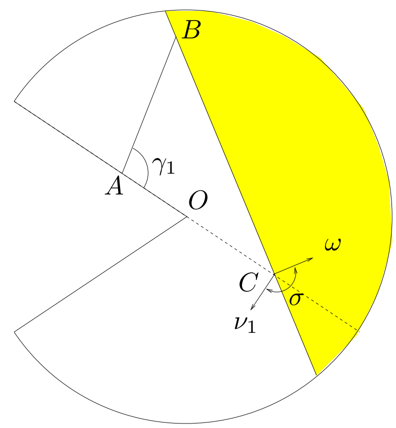

FIGURE 7. $\partial\Omega_\infty \cap \partial\mathcal{P} = \emptyset$

for T large. Hence

$$BC \le AB - \cos(\gamma_1)OA + OC$$

or $BC \le AB - \cos(\gamma_1)AC + OC + \cos(\gamma_1)OC$. If δ is the measure of angle ABC (so $\delta = \sigma - \gamma_1$), then using the law of sines we obtain

$$1 \le -\cos(\pi + \gamma_1 - \sigma) + (1 + \cos(\gamma_1))\frac{OC}{BC},$$

as a short calculation shows. For BC sufficiently large, this yields a contradiction since OC is fixed and $\pi + \gamma_1 - \sigma < \pi$. □

The proofs of the following three claims are similar to the proof above. We leave the details to the reader.

Claim 4.4. Suppose M is a component of $\Omega_\infty \cap \partial\mathcal{P}$ and let ω denote the unit normal to M in the direction of $\mathcal{P}$. Let σ be the measure of the angle between ω and ν_2.

(a) If $\partial\Omega_\infty \subset \partial\mathcal{P}$, then $\sigma \ge \gamma_2$.
(b) If $\partial\Omega_\infty \cap \partial\mathcal{P} = \emptyset$, then $\sigma \le \gamma_2$.

Claim 4.5. Suppose M is a component of $\Omega_\infty \cap \partial\mathcal{N}$ and let ω denote the unit normal to M in the direction of $\Omega_\infty \setminus \mathcal{N}$. Let σ be the measure of the angle between ω and ν_1.

(a) If $\partial\Omega_\infty \cap \partial\mathcal{N} = \emptyset$, then $\sigma \ge \gamma_1$.
(b) If $\partial\Omega_\infty \subset \partial\mathcal{N}$, then $\sigma \le \gamma_1$.

Claim 4.6. Suppose M is a component of $\Omega_\infty \cap \partial\mathcal{N}$ and let ω denote the unit normal to M in the direction of $\Omega_\infty \setminus \mathcal{N}$. Let σ be the measure of the angle between ω and ν_2.

(a) If $\partial\Omega_\infty \cap \partial\mathcal{N} = \emptyset$, $\sigma \ge \gamma_2$.
(b) If $\partial\Omega_\infty \subset \partial\mathcal{N}$, then $\sigma \le \gamma_2$.

Proof of Theorems 2.2 and 2.3. Consider first the case that $\Omega_\infty \cap \partial\mathcal{P}$ has exactly one component, denoted by L. Then one of the following holds:

$$\overline{L} \cap \Sigma_1 \neq \emptyset, \quad \overline{L} \cap \Sigma_2 \neq \emptyset, \quad \overline{L} \cap \partial\Omega_\infty = \{O\} \quad \text{or} \quad \overline{L} \cap \partial\Omega_\infty = \emptyset.$$

Suppose $\overline{L} \cap \Sigma_1 \neq \emptyset$ and let A be the point of intersection of $\overline{L}$ and Σ_1. If $OA \in \overline{\Omega_\infty \setminus \mathcal{P}}$, then [12, Lemma 4.8], implies $\Sigma_1 \setminus OA$ and L meet at A in an angle of measure γ_1 and a slight modification of the argument of the proof of [12, Lemma 4.6], implies $\gamma_1 + \pi - \gamma_2 \leq 2\alpha$. If $OA \in \overline{\mathcal{P}}$, then [12, Lemma 4.8], implies OA and L meet at A in an angle of measure γ_1 and a slight modification of the argument of the proof of [12, Lemma 4.7], implies $\pi - \gamma_1 + \gamma_2 \leq 2\alpha$. Hence either (ii) or (iii) of Theorem 2.2 holds. If $\overline{L} \cap \Sigma_2 \neq \emptyset$, then [12, Lemmas 4.6, 4.7, 4.10] imply that either (iv) or (v) of Theorem 2.2 holds. If $\overline{L} \cap \partial\Omega_\infty = \{O\}$, then either $\Sigma_1 \subset \overline{\mathcal{P}}$ and so (vi) of Theorem 2.2 holds (by [12, Lemma 4.6, 4.11]) or $\Sigma_2 \subset \overline{\mathcal{P}}$ and so (vii) of Theorem 2.2 holds (by [12, Lemma 4.7, 4.9])

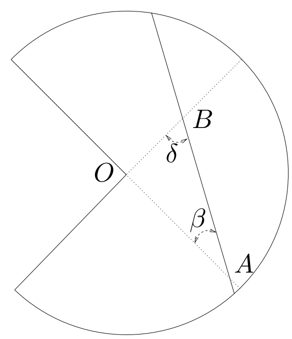

FIGURE 8. Cases (ix) and (xii)

Suppose $\overline{L} \cap \partial\Omega_\infty = \emptyset$. Then either $\partial\Omega_\infty \subset \overline{\mathcal{P}}$ or $\partial\Omega_\infty \cap \overline{\mathcal{P}} = \emptyset$. Let Σ_1^* and Σ_2^* be the lines on which Σ_1 and Σ_2 respectively lie. Now [12, Lemma 4.20] implies L is not parallel to either Σ_1^* or Σ_2^*. Let $A, B \in \Omega_\infty$ satisfy $\Sigma_1^* \cap L = \{A\}$ and $\Sigma_2^* \cap L = \{B\}$. Let β and δ be the measures of the angles OAB and OBA respectively. Let ω denote the unit normal to L in the direction of $\mathcal{P}$. If $\partial\Omega_\infty \subset \overline{\mathcal{P}}$, then β is the measure of the angle between ω and ν_1, δ is the measure of the angle between ω and ν_2, (a) of Claims 4.3 and 4.4 imply $\beta \geq \gamma_1$ and $\delta \geq \gamma_2$ and so (xii) of Theorem 2.2 holds. If $\partial\Omega_\infty \cap \overline{\mathcal{P}} = \emptyset$, then β is the measure of the angle between $-\omega$ and ν_1, δ is the measure of the angle between $-\omega$ and ν_2, (b) of Claims 4.3 and 4.4 imply $\beta \geq \pi - \gamma_1$ and $\delta \geq \pi - \gamma_2$ and so (ix) of Theorem 2.2 holds.

Consider next the case that $\Omega_\infty \cap \partial\mathcal{P}$ has exactly two components, denoted by L and M with $\overline{L} \cap \overline{\Sigma_1} = \{A\}$ and $\overline{M} \cap \overline{\Sigma_2} = \{B\}$. Then the following combinations are possible:

(a) $A \in \Sigma_1$ and $B \in \Sigma_2$;
(b) $A = O$ and $B \in \Sigma_2$;
(c) $A \in \Sigma_1$ and $B = O$;
(d) $A = O$ and $B = O$.

If (a) holds, then (xiv) and (xv) of Theorem 2.2 follow from [12, Lemma 4.8, 4.10]. If (b) holds, then (xviii) and (xix) of Theorem 2.2 follow from [12, Lemmas 4.6, 4.9, 4.10]. If (c) holds, then (xvi) and (xvii) of Theorem 2.2 follow from [12, Lemmas

4.7, 4.8, 4.11]. If (d) holds, then (x) and (xiii) of Theorem 2.2 follow from [12, Lemma 4.6, 4.7, 4.9, 4.11]; we note that (viii) and (xi) are special cases of (x) and (xiii) respectively. This completes the proof of Theorem 2.2. The proof of [12, Theorem 2.2] follows by similar arguments.

5. Some Additional Corollaries

The proofs of the following corollaries are simple exercises in checking angle conditions in Theorems 2.2 and 2.3.

Corollary 5.1. *Suppose $\alpha > \pi/2$, $\gamma_1 + \gamma_2 < 2\alpha - \pi$, $\alpha - (\pi - \gamma_1) \geq -\alpha + (\pi - \gamma_2)$, $\gamma_1 \leq \pi/2$ and $\gamma_2 \leq \pi/2$. Let $r_0 > 0$, $\beta \in (-\alpha, \alpha)$ and $Y = (r_0 \cos(\beta), r_0 \sin(\beta))$ and suppose $Y \in \partial\mathcal{P} \cap \partial\mathcal{N}$.*

- (a) *If $-\alpha < \beta < -\alpha + \gamma_2$, then one of cases (iv), (v), (xiv) or (xviii) of Theorems 2.2 and 2.3 holds.*
- (b) *If $-\alpha + \gamma_2 \leq \beta < -\alpha + \pi - \gamma_2$, then one of cases (iv), (vii), (xi), (xii), (xiii) or (xvi) of Theorems 2.2 and 2.3 holds.*
- (c) *If $-\alpha + \pi - \gamma_2 \leq \beta \leq \alpha - (\pi - \gamma_1)$, then one of cases (vi), (vii) or (xii) of Theorems 2.2 and 2.3 holds.*
- (d) *If $\alpha - (\pi - \gamma_1) < \beta \leq \alpha - \gamma_1$, then one of cases (iii), (vi), (xi), (xii) ,(xiii) or (xviii) of Theorems 2.2 and 2.3 holds.*
- (e) *If $\alpha - \gamma_1 < \beta < \alpha$, then one of cases (ii), (iii), (xiv) or (xvi) of Theorems 2.2 and 2.3 holds.*

Corollary 5.2. *Suppose $\alpha > \pi/2$, $\gamma_1 + \gamma_2 < 2\alpha - \pi$, $\alpha - (\pi - \gamma_1) < -\alpha + (\pi - \gamma_2)$, $\gamma_1 \leq \pi/2$ and $\gamma_2 \leq \pi/2$. Let $r_0 > 0$, $\beta \in (-\alpha, \alpha)$ and $Y = (r_0 \cos(\beta), r_0 \sin(\beta))$ and suppose $Y \in \partial\mathcal{P} \cap \partial\mathcal{N}$.*

- (a) *If $-\alpha < \beta < -\alpha + \gamma_2$, then one of cases (iv), (v), (xiv) or (xviii) of Theorems 2.2 and 2.3 holds.*
- (b) *If $-\alpha + \gamma_2 \leq \beta < \alpha - (\pi - \gamma_1)$, then one of cases (iv), (vii), (xi), (xii), (xiii) or (xvi) of Theorems 2.2 and 2.3 holds.*
- (c) *If $\alpha - (\pi - \gamma_1) \leq \beta \leq -\alpha + (\pi - \gamma_2)$, then one of cases (iii), (iv) or (xii) of Theorems 2.2 and 2.3 holds.*
- (d) *If $-\alpha + (\pi - \gamma_2) < \beta \leq \alpha - \gamma_1$, then one of cases (iii), (vi), (xi), (xii), (xiii) or (xviii) of Theorems 2.2 and 2.3 holds.*
- (e) *If $\alpha - \gamma_1 < \beta < \alpha$, then one of cases (ii), (iii), (xiv) or (xvi) of Theorems 2.2 and 2.3 holds.*

Corollary 5.3. *Suppose $\alpha > \pi/2$, $\gamma_1 + \gamma_2 < 2\alpha - \pi$, $\gamma_1 > \pi/2$ and $\gamma_2 \leq \pi/2$. Let $r_0 > 0$, $\beta \in (-\alpha, \alpha)$ and $Y = (r_0 \cos(\beta), r_0 \sin(\beta))$ and suppose $Y \in \partial\mathcal{P} \cap \partial\mathcal{N}$.*

- (a) *If $-\alpha < \beta < -\alpha + \gamma_2$, then one of cases (iv), (v), (xiv) or (xviii) of Theorems 2.2 and 2.3 holds.*
- (b) *If $-\alpha + \gamma_2 \leq \beta < -\alpha + (\pi - \gamma_2)$, then one of cases (iv), (vii), (xi), (xii), (xiii) or (xvi) of Theorems 2.2 and 2.3 holds.*
- (c) *If $-\alpha + (\pi - \gamma_2) \leq \beta \leq \alpha - \gamma_1$, then one of cases (vi), (vii) or (xii) of Theorems 2.2 and 2.3 holds.*
- (d) *If $\alpha - \gamma_1 < \beta \leq \alpha - (\pi - \gamma_1)$, then one of cases (ii), (vii), (xiv) or (xvi) of Theorems 2.2 and 2.3 holds.*
- (e) *If $\alpha - (\pi - \gamma_1) < \beta < \alpha$, then one of cases (ii), (iii), (xiv) or (xvi) of Theorems 2.2 and 2.3 holds.*

Corollary 5.4. *Suppose $\alpha > \pi/2$, $\gamma_1 + \gamma_2 < 2\alpha - \pi$, $\gamma_1 \le \pi/2$ and $\gamma_2 > \pi/2$. Let $r_0 > 0$, $\beta \in (-\alpha, \alpha)$ and $Y = (r_0 \cos(\beta), r_0 \sin(\beta))$ and suppose $Y \in \partial\mathcal{P} \cap \partial\mathcal{N}$.*

(a) *If $-\alpha < \beta < -\alpha + (\pi - \gamma_2)$, then one of cases (iv), (v), (xiv) or (xviii) of Theorems 2.2 and 2.3 holds.*
(b) *If $-\alpha + (\pi - \gamma_2) \le \beta < -\alpha + \gamma_2$, then one of cases (v), (vi), (xiv) or (xviii) of Theorems 2.2 and 2.3 holds.*
(c) *If $-\alpha + \gamma_2 \le \beta \le \alpha - (\pi - \gamma_1)$, then one of cases (vi), (vii) or (xii) of Theorems 2.2 and 2.3 holds.*
(d) *If $\alpha - (\pi - \gamma_1) < \beta \le \alpha - \gamma_1$, then one of cases (iii), (vi), (xi), (xii), (xiii) or (xviii) of Theorems 2.2 and 2.3 holds.*
(e) *If $\alpha - \gamma_1 < \beta < \alpha$, then one of cases (ii), (iii), (xiv) or (xvi) of Theorems 2.2 and 2.3 holds.*

Corollary 5.5. *Suppose $\alpha > \pi/2$, $\gamma_1 + \gamma_2 > 3\pi - 2\alpha$, $\alpha - \gamma_1 \ge -\alpha + \gamma_2$, $\gamma_1 \ge \pi/2$ and $\gamma_2 \ge \pi/2$. Let $r_0 > 0$, $\beta \in (-\alpha, \alpha)$ and $Y = (r_0 \cos(\beta), r_0 \sin(\beta))$ and suppose $Y \in \partial\mathcal{P} \cap \partial\mathcal{N}$.*

(a) *If $-\alpha < \beta < -\alpha + (\pi - \gamma_2)$, then one of cases (iv), (v), (xv) or (xix) of Theorems 2.2 and 2.3 holds.*
(b) *If $-\alpha + (\pi - \gamma_2) \le \beta < -\alpha + \gamma_2$, then one of cases (v), (vi), (viii), (ix), (x) or (xvii) of Theorems 2.2 and 2.3 holds.*
(c) *If $-\alpha + \gamma_2 \le \beta \le \alpha - \gamma_1$, then one of cases (vi), (vii) or (ix) of Theorems 2.2 and 2.3 holds.*
(d) *If $\alpha - \gamma_1 < \beta \le \alpha - (\pi - \gamma_1)$, then one of cases (ii), (vii), (viii), (ix), (x) or (xix) of Theorems 2.2 and 2.3 holds.*
(e) *If $\alpha - (\pi - \gamma_1) < \beta < \alpha$, then one of cases (ii), (iii), (xv) or (xvii) of Theorems 2.2 and 2.3 holds.*

Corollary 5.6. *Suppose $\alpha > \pi/2$, $\gamma_1 + \gamma_2 > 3\pi - 2\alpha$, $\alpha - \gamma_1 < -\alpha + \gamma_2$, $\gamma_1 \ge \pi/2$ and $\gamma_2 \ge \pi/2$. Let $r_0 > 0$, $\beta \in (-\alpha, \alpha)$ and $Y = (r_0 \cos(\beta), r_0 \sin(\beta))$ and suppose $Y \in \partial\mathcal{P} \cap \partial\mathcal{N}$.*

(a) *If $-\alpha < \beta < -\alpha + (\pi - \gamma_2)$, then one of cases (iv), (v), (xv) or (xix) of Theorems 2.2 and 2.3 holds.*
(b) *If $-\alpha + (\pi - \gamma_2) \le \beta < \alpha - \gamma_1$, then one of cases (v), (vi), (viii), (ix), (x) or (xvii) of Theorems 2.2 and 2.3 holds.*
(c) *If $\alpha - \gamma_1 \le \beta \le -\alpha + \gamma_2$, then one of cases (ii), (v) or (ix) of Theorems 2.2 and 2.3 holds.*
(d) *If $-\alpha + \gamma_2 < \beta \le \alpha - (\pi - \gamma_1)$, then one of cases (ii), (vii), (viii), (ix), (x) or (xix) of Theorems 2.2 and 2.3 holds.*
(e) *If $\alpha - (\pi - \gamma_1) < \beta < \alpha$, then one of cases (ii), (iii), (xv) or (xvii) of Theorems 2.2 and 2.3 holds.*

Corollary 5.7. *Suppose $\alpha > \pi/2$, $\gamma_1 + \gamma_2 > 3\pi - 2\alpha$, $\gamma_1 < \pi/2$ and $\gamma_2 \ge \pi/2$. Let $r_0 > 0$, $\beta \in (-\alpha, \alpha)$ and $Y = (r_0 \cos(\beta), r_0 \sin(\beta))$ and suppose $Y \in \partial\mathcal{P} \cap \partial\mathcal{N}$.*

(a) *If $-\alpha < \beta < -\alpha + (\pi - \gamma_2)$, then one of cases (iv), (v), (xv) or (xix) of Theorems 2.2 and 2.3 holds.*
(b) *If $-\alpha + (\pi - \gamma_2) \le \beta < -\alpha + \gamma_2$, then one of cases (v), (vi), (viii), (ix), (x) or (xvii) of Theorems 2.2 and 2.3 holds.*
(c) *If $-\alpha + \gamma_2 \le \beta \le \alpha - (\pi - \gamma_1)$, then one of cases (vi), (vii) or (ix) of Theorems 2.2 and 2.3 holds.*

(d) *If* $\alpha - (\pi - \gamma_1) < \beta \leq \alpha - \gamma_1$, *then one of cases (iii), (vi), (xv) or (xvii) of Theorems 2.2 and 2.3 holds.*
(e) *If* $\alpha - \gamma_1 < \beta < \alpha$, *then one of cases (ii), (iii), (xv) or (xvii) of Theorems 2.2 and 2.3 holds.*

Corollary 5.8. *Suppose* $\alpha > \pi/2$, $\gamma_1 + \gamma_2 > 3\pi - 2\alpha$, $\gamma_1 \geq \pi/2$ *and* $\gamma_2 < \pi/2$. *Let* $r_0 > 0$, $\beta \in (-\alpha, \alpha)$ *and* $Y = (r_0 \cos(\beta), r_0 \sin(\beta))$ *and suppose* $Y \in \partial\mathcal{P} \cap \partial\mathcal{N}$.
(a) *If* $-\alpha < \beta < -\alpha + \gamma_2$, *then one of cases (iv), (v), (xv) or (xix) of Theorems 2.2 and 2.3 holds.*
(b) *If* $-\alpha + \gamma_2 \leq \beta < -\alpha + (\pi - \gamma_2)$, *then one of cases (iv), (vii), (xv) and (xix) of Theorems 2.2 and 2.3 holds.*
(c) *If* $-\alpha + (\pi - \gamma_2) \leq \beta \leq \alpha - \gamma_1$, *then one of cases (vi), (vii) or (ix) of Theorems 2.2 and 2.3 holds.*
(d) *If* $\alpha - \gamma_1 < \beta \leq \alpha - (\pi - \gamma_1)$, *then one of cases (ii), (vii), (viii), (ix), (x) or (xix) of Theorems 2.2 and 2.3 holds.*
(e) *If* $\alpha - (\pi - \gamma_1) < \beta < \alpha$, *then one of cases (ii), (iii), (xv) or (xvii) of Theorems 2.2 and 2.3 holds.*

Remark 5.9. If $2\alpha < \gamma_1 + \gamma_2 + \pi$, then cases (xi), (xii) and (xiii) cannot occur. If $2\alpha + \gamma_1 + \gamma_2 < 3\pi$, then cases (viii), (ix) and (x) cannot occur.

6. Lemmas 6.1-6.8

The following lemmas are used in the proofs of Lemmas 3.5 and 3.6. Let us recall that we defined

$$C_\beta(f) = \{\eta \in S^2 : \eta = \lim_{j\to\infty} \vec{n}_f(X_j) \text{ for some } (X_j) \in t_\beta\}$$

for each $\beta \in (-\alpha, \alpha)$ and set $C(f) = \cup_{\beta\in(-\alpha,\alpha)} C_\beta(f)$.

Lemma 6.1. *Suppose* $\alpha > \pi/2$ *and* (γ_1, γ_2) *lies in* D_1^+ *(i.e.* $\gamma_1 + \gamma_2 < 2\alpha - \pi$*). Suppose further that* $\alpha - (\pi - \gamma_1) \geq -\alpha + (\pi - \gamma_2)$, $\gamma_1 \leq \pi/2$ *and* $\gamma_2 \leq \pi/2$. *Let* $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ *satisfy* (3.2) *and* (3.3) *and define* $\vec{n}(x, y)$ *by (3.5). Then:*
(a) *If* $\beta \in (-\alpha, -\alpha + \gamma_2)$ *and* $\eta \in C_\beta(f)$, *then* η *is one of the following:* $(-\sin(\alpha - \gamma_2), -\cos(\alpha - \gamma_2), 0)$ *or* $(-\sin(\alpha + \gamma_2), -\cos(\alpha + \gamma_2), 0)$.
(b) *If* $\beta \in [-\alpha+\gamma_2, -\alpha+(\pi-\gamma_2))$ *and* $\eta \in C_\beta(f)$, *then* η *is one of the following:* $(\cos(\beta - \frac{\pi}{2}), \sin(\beta - \frac{\pi}{2}), 0)$, $(\cos(\omega), \sin(\omega), 0)$ *or* $(\cos(\theta), \sin(\theta), 0)$ *for some* $\theta \in [2\pi - \alpha + \gamma_2 - \frac{\pi}{2}, \alpha - \gamma_1 + \frac{\pi}{2}]$, *where* $\omega = \frac{3\pi}{2} - \alpha - \gamma_2$.
(c) *If* $\beta \in [-\alpha + (\pi - \gamma_2), \alpha - (\pi - \gamma_1)]$ *and* $\eta \in C_\beta(f)$, *then* η *is one of the following:* $(-\sin(\beta), \cos(\beta), 0)$, $(\sin(\beta), -\cos(\beta), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some* θ *with* $\alpha + \pi - \gamma_1 \leq \theta \leq 2\pi - \alpha + \gamma_2$.
(d) *If* $\beta \in (\alpha - (\pi - \gamma_1), \alpha - \gamma_1]$ *and* $\eta \in C_\beta(f)$, *then* η *is one of the following:* $(-\sin(\beta), \cos(\beta), 0)$, $(-\sin(\alpha + \gamma_1), \cos(\alpha + \gamma_1), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some* θ *with* $\alpha + \pi - \gamma_1 \leq \theta \leq 2\pi - \alpha + \gamma_2$.
(e) *If* $\beta \in (\alpha - \gamma_1, \alpha)$ *and* $\eta \in C_\beta(f)$, *then* η *is one of the following:* $(-\sin(\alpha - \gamma_1), \cos(\alpha - \gamma_1), 0)$ *or* $(-\sin(\alpha + \gamma_1), \cos(\alpha + \gamma_1), 0)$.

Proof. Let $\beta \in (-\alpha, \alpha)$ and $\eta \in C_\beta(f)$. Let $\{(x_j, y_j) : j \in \mathbb{N}\}$ be a sequence in Ω satisfying (3.4) such that $\vec{n}_f(x_j, y_j) \to \eta$ as $j \to \infty$. For each $j \in \mathbb{N}$, define $f_j \in C^2(\Omega_j) \cap C^1(\overline{\Omega_j} \setminus \{O\})$ by

$$f_j(x, y) = \frac{f(\epsilon_j x, \epsilon_j y) - f(x_j, y_j)}{\epsilon_j},$$

where $\epsilon_j = \sqrt{x_j^2 + y_j^2}$. Let $U_j = \{(x,t) \in \Omega_j \times \mathbb{R} : t < f_j(x)\}$ denote the subgraph of f_j and $\vec{n}_j$ be the downward unit normal to the graph of f_j; that is,

$$\vec{n}_j(x,y) = \big(Tf_j(x,y), \frac{-1}{\sqrt{1+|\nabla f_j(x,y)|^2}}\big), \quad (x,y) \in \Omega_j. \tag{6.1}$$

Notice that $\vec{n}_j(\frac{x}{\epsilon_j}, \frac{y}{\epsilon_j}) = \vec{n}_f(x,y)$. As in §1 of [12], there exists a subsequence of $\{(x_j,y_j)\}$, still denoted $\{(x_j,y_j)\}$, and a generalized solution $f_\infty : \Omega_\infty \to [-\infty,\infty]$ of (1.3) such that f_j converges to f_∞ in the sense that $\phi_{U_j} \to \phi_{U_\infty}$ in $L^1_{loc}(\Omega_\infty \times \mathbb{R})$ as $j \to \infty$. Let $\mathcal{P}$ and $\mathcal{N}$ be given by (1.4) and (1.5) respectively. Notice that $f_j(x_j/\epsilon_j, y_j/\epsilon_j) = 0$ for all $j \in \mathbb{N}$ and so $f_\infty(\cos(\beta),\sin(\beta)) = 0$. As in the proof of [12, Lemma 3.1], we see that $(\Omega_\infty \times \mathbb{R}) \cap \partial U_\infty$ is the portion of a plane Π in $\Omega_\infty \times \mathbb{R}$ with $(\cos(\beta),\sin(\beta),0) \in \Pi$, $U_\infty = \mathcal{P} \times \mathbb{R}$ and $(\cos(\beta),\sin(\beta)) \in \partial\mathcal{P} \cap \partial\mathcal{N}$. Now

$$\vec{n}_\infty(\cos(\beta),\sin(\beta)) = \lim_{j\to\infty} \vec{n}_j(\frac{x_j}{\epsilon_j}, \frac{y_j}{\epsilon_j}) = \lim_{j\to\infty} \vec{n}_f(x_j,y_j) = \eta.$$

and so η is the unit normal to $\partial\mathcal{P} \times \mathbb{R}$ at $(\cos(\beta),\sin(\beta),0)$ pointing into $\mathcal{P} \times \mathbb{R}$ and to $\partial\mathcal{N} \times \mathbb{R}$ at $(\cos(\beta),\sin(\beta),0)$ pointing away from $\mathcal{N} \times \mathbb{R}$. The conclusions of the lemma follow from Corollary 5.1. □

The proofs of the following lemmas follow by a similar argument using Corollaries 6-12 (e.g. Lemma 6.2 uses Corollary 5.2).

Lemma 6.2. *Suppose $\alpha > \pi/2$ and (γ_1,γ_2) lies in D_1^+ (i.e. $\gamma_1 + \gamma_2 < 2\alpha - \pi$). Suppose further that $\alpha - (\pi - \gamma_1) < -\alpha + (\pi - \gamma_2)$, $\gamma_1 \le \pi/2$ and $\gamma_2 \le \pi/2$. Let $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ satisfy* (3.2) *and* (3.3) *and define $\vec{n}(x,y)$ by (3.5). For each $\beta \in (-\alpha,\alpha)$, there exists a unit vector $\vec{n}_\beta = \vec{n}_\beta(f)$ such that if $\{(x_j,y_j)\}$ is any sequence in Ω satisfying $\lim_{j\to\infty}(x_j,y_j) = (0,0)$ and (3.4), then $\lim_{j\to\infty} \vec{n}(x_j,y_j) = \vec{n}_\beta$. In addition:*

- (a) *If $\beta \in (-\alpha, -\alpha+\gamma_2)$, then $\vec{n}_\beta$ is one of the following:* $(-\sin(\alpha-\gamma_2), -\cos(\alpha-\gamma_2), 0)$ *or* $(-\sin(\alpha+\gamma_2), -\cos(\alpha+\gamma_2), 0)$.
- (b) *If $\beta \in [-\alpha+\gamma_2, \alpha-(\pi-\gamma_1))$, then $\vec{n}_\beta$ is one of the following:* $(\sin(\beta), -\cos(\beta), 0)$, $(-\sin(\alpha+\gamma_2), -\cos(\alpha+\gamma_2), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some θ with $\alpha+\pi-\gamma_1 \le \theta \le 2\pi-\alpha+\gamma_2$.*
- (c) *If $\beta \in [\alpha-(\pi-\gamma_1), -\alpha+(\pi-\gamma_2)]$, then $\vec{n}_\beta$ is one of the following:* $(-\sin(\alpha+\gamma_1), \cos(\alpha+\gamma_1), 0)$, $(-\sin(\alpha+\gamma_2), -\cos(\alpha+\gamma_2), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some θ with $\alpha+\pi-\gamma_1 \le \theta \le 2\pi-\alpha+\gamma_2$.*
- (d) *If $\beta \in (-\alpha+(\pi-\gamma_2), \alpha-\gamma_1]$, then $\vec{n}_\beta$ is one of the following:* $(-\sin(\beta), \cos(\beta), 0)$, $(-\sin(\alpha+\gamma_1), \cos(\alpha+\gamma_1), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some θ with $\alpha+\pi-\gamma_1 \le \theta \le 2\pi-\alpha+\gamma_2$.*
- (e) *If $\beta \in (\alpha-\gamma_1, \alpha)$, then $\vec{n}_\beta$ is one of the following:* $(-\sin(\alpha-\gamma_1), \cos(\alpha-\gamma_1), 0)$ *or* $(-\sin(\alpha+\gamma_1), \cos(\alpha+\gamma_1), 0)$.

Lemma 6.3. *Suppose $\alpha > \pi/2$, (γ_1,γ_2) lies in D_1^+ (i.e. $\gamma_1+\gamma_2 < 2\alpha-\pi$), $\gamma_1 > \pi/2$ and $\gamma_2 \le \pi/2$. Let $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ satisfy* (3.2) *and* (3.3) *and define $\vec{n}(x,y)$ by (3.5). For each $\beta \in (-\alpha,\alpha)$, there exists a unit vector $\vec{n}_\beta = \vec{n}_\beta(f)$ such that if $\{(x_j,y_j)\}$ is any sequence in Ω satisfying $\lim_{j\to\infty}(x_j,y_j) = (0,0)$ and (3.4), then $\lim_{j\to\infty} \vec{n}(x_j,y_j) = \vec{n}_\beta$. In addition:*

- (a) *If $\beta \in (-\alpha, -\alpha+\gamma_2)$, then $\vec{n}_\beta$ is one of the following:* $(-\sin(\alpha-\gamma_2), -\cos(\alpha-\gamma_2), 0)$ *or* $(-\sin(\alpha+\gamma_2), -\cos(\alpha+\gamma_2), 0)$.

(b) *If* $\beta \in [-\alpha+\gamma_2, -\alpha+(\pi-\gamma_2))$*, then* $\vec{n}_\beta$ *is one of the following:* $(\sin(\beta), -\cos(\beta), 0)$*,* $(-\sin(\alpha+\gamma_2), -\cos(\alpha+\gamma_2), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some* θ *with* $\alpha+\pi-\gamma_1 \le \theta \le 2\pi-\alpha+\gamma_2$.

(c) *If* $\beta \in [-\alpha+(\pi-\gamma_2), \alpha-\gamma_1]$*, then* $\vec{n}_\beta$ *is one of the following:* $(-\sin(\beta), \cos(\beta), 0)$*,* $(\sin(\beta), -\cos(\beta), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some* θ *with* $\alpha+\pi-\gamma_1 \le \theta \le 2\pi-\alpha+\gamma_2$.

(d) *If* $\beta \in (\alpha-\gamma_1, \alpha-(\pi-\gamma_1)]$*, then* $\vec{n}_\beta$ *is one of the following:* $(\sin(\beta), -\cos(\beta), 0)$ *or* $(-\sin(\alpha-\gamma_1), \cos(\alpha-\gamma_1), 0)$.

(e) *If* $\beta \in (\alpha-(\pi-\gamma_1), \alpha)$*, then* $\vec{n}_\beta$ *is one of the following:* $(-\sin(\alpha-\gamma_1), \cos(\alpha-\gamma_1), 0)$ *or* $(-\sin(\alpha+\gamma_1), \cos(\alpha+\gamma_1), 0)$.

Lemma 6.4. *Suppose* $\alpha > \pi/2$ *and* (γ_1, γ_2) *lies in* D_1^+ *(i.e.* $\gamma_1+\gamma_2 < 2\alpha-\pi$*). Suppose further that* $\alpha-(\pi-\gamma_1) \ge -\alpha+(\pi-\gamma_2)$*,* $\gamma_1 \le \pi/2$ *and* $\gamma_2 \le \pi/2$*. Let* $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ *satisfy* (3.2) *and* (3.3) *and define* $\vec{n}(x,y)$ *by (3.5). For each* $\beta \in (-\alpha, \alpha)$*, there exists a unit vector* $\vec{n}_\beta = \vec{n}_\beta(f)$ *such that if* $\{(x_j, y_j)\}$ *is any sequence in* Ω *satisfying* $\lim_{j\to\infty}(x_j, y_j) = (0,0)$ *and (3.4), then* $\lim_{j\to\infty} \vec{n}(x_j, y_j) = \vec{n}_\beta$*. In addition:*

(a) *If* $\beta \in (-\alpha, -\alpha+(\pi-\gamma_2))$*, then* $\vec{n}_\beta$ *is one of the following:* $(-\sin(\alpha-\gamma_2), -\cos(\alpha-\gamma_2), 0)$ *or* $(-\sin(\alpha+\gamma_2), -\cos(\alpha+\gamma_2), 0)$.

(b) *If* $\beta \in [-\alpha+(\pi-\gamma_2), -\alpha+\gamma_2)$*, then* $\vec{n}_\beta$ *is one of the following:* $(-\sin(\beta), \cos(\beta), 0)$ *or* $(-\sin(\alpha-\gamma_2), -\cos(\alpha-\gamma_2), 0)$.

(c) *If* $\beta \in [-\alpha+\gamma_2, \alpha-(\pi-\gamma_1)]$*, then* $\vec{n}_\beta$ *is one of the following:* $(-\sin(\beta), \cos(\beta), 0)$*,* $(\sin(\beta), -\cos(\beta), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some* θ *with* $\alpha+\pi-\gamma_1 \le \theta \le 2\pi-\alpha+\gamma_2$.

(d) *If* $\beta \in (\alpha-(\pi-\gamma_1), \alpha-\gamma_1]$*, then* $\vec{n}_\beta$ *is one of the following:* $(-\sin(\beta), \cos(\beta), 0)$*,* $(-\sin(\alpha+\gamma_1), \cos(\alpha+\gamma_1), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some* θ *with* $\alpha+\pi-\gamma_1 \le \theta \le 2\pi-\alpha+\gamma_2$.

(e) *If* $\beta \in (\alpha-\gamma_1, \alpha)$*, then* $\vec{n}_\beta$ *is one of the following:* $(-\sin(\alpha-\gamma_1), \cos(\alpha-\gamma_1), 0)$ *or* $(-\sin(\alpha+\gamma_1), \cos(\alpha+\gamma_1), 0)$.

Lemma 6.5. *Suppose* $\alpha > \pi/2$ *and* (γ_1, γ_2) *lies in* D_1^- *(i.e.* $\gamma_1+\gamma_2 > 3\pi-2\alpha$*). Suppose further that* $\alpha-\gamma_1 \ge -\alpha-\gamma_2$*,* $\gamma_1 \ge \pi/2$ *and* $\gamma_2 \ge \pi/2$*. Let* $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ *satisfy* (3.2) *and* (3.3) *and define* $\vec{n}(x,y)$ *by (3.5). For each* $\beta \in (-\alpha, \alpha)$*, there exists a unit vector* $\vec{n}_\beta = \vec{n}_\beta(f)$ *such that if* $\{(x_j, y_j)\}$ *is any sequence in* Ω *satisfying* $\lim_{j\to\infty}(x_j, y_j) = (0,0)$ *and (3.4), then* $\lim_{j\to\infty} \vec{n}(x_j, y_j) = \vec{n}_\beta$*. In addition:*

(a) *If* $\beta \in (-\alpha, -\alpha+(\pi-\gamma_2))$*, then* $\vec{n}_\beta$ *is one of the following:* $(-\sin(\alpha-\gamma_2), -\cos(\alpha-\gamma_2), 0)$ *or* $(-\sin(\alpha+\gamma_2), -\cos(\alpha+\gamma_2), 0)$.

(b) *If* $\beta \in [-\alpha+(\pi-\gamma_2), -\alpha+\gamma_2)$*, then* $\vec{n}_\beta$ *is one of the following:* $(-\sin(\alpha-\gamma_2), -\cos(\alpha-\gamma_2), 0)$*,* $(-\sin(\beta), \cos(\beta), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some* θ *with* $\alpha+\pi-\gamma_1 \le \theta \le 2\pi-\alpha+\gamma_2$.

(c) *If* $\beta \in [-\alpha+\gamma_2, \alpha-\gamma_1]$*, then* $\vec{n}_\beta$ *is one of the following:* $(-\sin(\beta), \cos(\beta), 0)$*,* $(\sin(\beta), -\cos(\beta), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some* θ *with* $\alpha+\pi-\gamma_1 \le \theta \le 2\pi-\alpha+\gamma_2$.

(d) *If* $\beta \in (\alpha-\gamma_1, \alpha-(\pi-\gamma_1)]$*, then* $\vec{n}_\beta$ *is one of the following:* $(\sin(\beta), -\cos(\beta), 0)$*,* $(-\sin(\alpha+\gamma_1), \cos(\alpha+\gamma_1), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some* θ *with* $\alpha+\pi-\gamma_1 \le \theta \le 2\pi-\alpha+\gamma_2$.

(e) *If* $\beta \in (\alpha-(\pi-\gamma_1), \alpha)$*, then* $\vec{n}_\beta$ *is one of the following:* $(-\sin(\alpha-\gamma_1), \cos(\alpha-\gamma_1), 0)$ *or* $(-\sin(\alpha+\gamma_1), \cos(\alpha+\gamma_1), 0)$.

Lemma 6.6. *Suppose $\alpha > \pi/2$ and (γ_1, γ_2) lies in D_1^- (i.e. $\gamma_1 + \gamma_2 > 3\pi - 2\alpha$). Suppose further that $\alpha - \gamma_1 < -\alpha - \gamma_2$, $\gamma_1 \geq \pi/2$ and $\gamma_2 \geq \pi/2$. Let $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ satisfy* (3.2) *and* (3.3) *and define $\vec{n}(x, y)$ by (3.5). For each $\beta \in (-\alpha, \alpha)$, there exists a unit vector $\vec{n}_\beta = \vec{n}_\beta(f)$ such that if $\{(x_j, y_j)\}$ is any sequence in Ω satisfying $\lim_{j\to\infty}(x_j, y_j) = (0, 0)$ and (3.4), then $\lim_{j\to\infty} \vec{n}(x_j, y_j) = \vec{n}_\beta$. In addition:*

(a) *If $\beta \in (-\alpha, -\alpha + (\pi - \gamma_2))$, then $\vec{n}_\beta$ is one of the following: $(-\sin(\alpha - \gamma_2), -\cos(\alpha - \gamma_2), 0)$ or $(-\sin(\alpha + \gamma_2), -\cos(\alpha + \gamma_2), 0)$.*
(b) *If $\beta \in [-\alpha + (\pi - \gamma_2), \alpha - \gamma_1)$, then $\vec{n}_\beta$ is one of the following: $(-\sin(\alpha - \gamma_2), -\cos(\alpha - \gamma_2), 0)$, $(-\sin(\beta), \cos(\beta), 0)$ or $(\sin(\theta), -\cos(\theta), 0)$ for some θ with $\alpha + \pi - \gamma_1 \leq \theta \leq 2\pi - \alpha + \gamma_2$.*
(c) *If $\beta \in [\alpha - \gamma_1, -\alpha + \gamma_2]$, then $\vec{n}_\beta$ is one of the following: $(-\sin(\alpha - \gamma_1), \cos(\alpha - \gamma_1), 0)$, $(-\sin(\alpha - \gamma_2), -\cos(\alpha - \gamma_2), 0)$ or $(\sin(\theta), -\cos(\theta), 0)$ for some θ with $\alpha + \pi - \gamma_1 \leq \theta \leq 2\pi - \alpha + \gamma_2$.*
(d) *If $\beta \in (-\alpha + \gamma_2, \alpha - (\pi - \gamma_1)]$, then $\vec{n}_\beta$ is one of the following: $(\sin(\beta), -\cos(\beta), 0)$, $(-\sin(\alpha + \gamma_1), \cos(\alpha + \gamma_1), 0)$ or $(\sin(\theta), -\cos(\theta), 0)$ for some θ with $\alpha + \pi - \gamma_1 \leq \theta \leq 2\pi - \alpha + \gamma_2$.*
(e) *If $\beta \in (\alpha - (\pi - \gamma_1), \alpha)$, then $\vec{n}_\beta$ is one of the following: $(-\sin(\alpha - \gamma_1), \cos(\alpha - \gamma_1), 0)$ or $(-\sin(\alpha + \gamma_1), \cos(\alpha + \gamma_1), 0)$.*

Lemma 6.7. *Suppose $\alpha > \pi/2$, (γ_1, γ_2) lies in D_1^- (i.e. $\gamma_1 + \gamma_2 > 3\pi - 2\alpha$). $\gamma_1 < \pi/2$ and $\gamma_2 \geq \pi/2$. Let $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ satisfy* (3.2) *and* (3.3) *and define $\vec{n}(x, y)$ by (3.5). For each $\beta \in (-\alpha, \alpha)$, there exists a unit vector $\vec{n}_\beta = \vec{n}_\beta(f)$ such that if $\{(x_j, y_j)\}$ is any sequence in Ω satisfying $\lim_{j\to\infty}(x_j, y_j) = (0, 0)$ and (3.4), then $\lim_{j\to\infty} \vec{n}(x_j, y_j) = \vec{n}_\beta$. In addition:*

(a) *If $\beta \in (-\alpha, -\alpha + (\pi - \gamma_2))$, then $\vec{n}_\beta$ is one of the following: $(-\sin(\alpha - \gamma_2), -\cos(\alpha - \gamma_2), 0)$ or $(-\sin(\alpha + \gamma_2), -\cos(\alpha + \gamma_2), 0)$.*
(b) *If $\beta \in [-\alpha + (\pi - \gamma_2), -\alpha + \gamma_2)$, then $\vec{n}_\beta$ is one of the following: $(-\sin(\alpha - \gamma_2), -\cos(\alpha - \gamma_2), 0)$, $(-\sin(\beta), \cos(\beta), 0)$ or $(\sin(\theta), -\cos(\theta), 0)$ for some θ with $\alpha + \pi - \gamma_1 \leq \theta \leq 2\pi - \alpha + \gamma_2$.*
(c) *If $\beta \in [-\alpha + \gamma_2, \alpha - (\pi - \gamma_1)]$, then $\vec{n}_\beta$ is one of the following: $(-\sin(\beta), \cos(\beta), 0)$, $(\sin(\beta), -\cos(\beta), 0)$ or $(\sin(\theta), -\cos(\theta), 0)$ for some θ with $\alpha + \pi - \gamma_1 \leq \theta \leq 2\pi - \alpha + \gamma_2$.*
(d) *If $\beta \in (\alpha - (\pi - \gamma_1), \alpha - \gamma_1]$, then $\vec{n}_\beta$ is one of the following: $(-\sin(\alpha + \gamma_1), \cos(\alpha + \gamma_1), 0)$ or $(-\sin(\beta), \cos(\beta), 0)$.*
(e) *If $\beta \in (\alpha - \gamma_1, \alpha)$, then $\vec{n}_\beta$ is one of the following: $(-\sin(\alpha - \gamma_1), \cos(\alpha - \gamma_1), 0)$ or $(-\sin(\alpha + \gamma_1), \cos(\alpha + \gamma_1), 0)$.*

Lemma 6.8. *Suppose $\alpha > \pi/2$, (γ_1, γ_2) lies in D_1^- (i.e. $\gamma_1 + \gamma_2 > 3\pi - 2\alpha$), $\gamma_1 \geq \pi/2$ and $\gamma_2 < \pi/2$. Let $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\})$ satisfy* (3.2) *and* (3.3) *and define $\vec{n}(x, y)$ by (3.5). For each $\beta \in (-\alpha, \alpha)$, there exists a unit vector $\vec{n}_\beta = \vec{n}_\beta(f)$ such that if $\{(x_j, y_j)\}$ is any sequence in Ω satisfying $\lim_{j\to\infty}(x_j, y_j) = (0, 0)$ and (3.4), then $\lim_{j\to\infty} \vec{n}(x_j, y_j) = \vec{n}_\beta$. In addition:*

(a) *If $\beta \in (-\alpha, -\alpha + \gamma_2)$, then $\vec{n}_\beta$ is one of the following: $(-\sin(\alpha - \gamma_2), -\cos(\alpha - \gamma_2), 0)$ or $(-\sin(\alpha + \gamma_2), -\cos(\alpha + \gamma_2), 0)$.*
(b) *If $\beta \in [-\alpha + \gamma_2, -\alpha + (\pi - \gamma_2))$, then $\vec{n}_\beta$ is one of the following: $(-\sin(\alpha + \gamma_2), -\cos(\alpha + \gamma_2), 0)$ or $(\sin(\beta), -\cos(\beta), 0)$.*

(c) *If* $\beta \in [-\alpha + (\pi - \gamma_2), \alpha - \gamma_1]$, *then* $\vec{n}_\beta$ *is one of the following:* $(-\sin(\beta), \cos(\beta), 0)$, $(\sin(\beta), -\cos(\beta), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some* θ *with* $\alpha + \pi - \gamma_1 \le \theta \le 2\pi - \alpha + \gamma_2$.

(d) *If* $\beta \in (\alpha - \gamma_1, \alpha - (\pi - \gamma_1)]$, *then* $\vec{n}_\beta$ *is one of the following:* $(\sin(\beta), -\cos(\beta), 0)$, $(-\sin(\alpha + \gamma_1), \cos(\alpha + \gamma_1), 0)$ *or* $(\sin(\theta), -\cos(\theta), 0)$ *for some* θ *with* $\alpha + \pi - \gamma_1 \le \theta \le 2\pi - \alpha + \gamma_2$.

(e) *If* $\beta \in (\alpha - (\pi - \gamma_1), \alpha)$, *then* $\vec{n}_\beta$ *is one of the following:* $(-\sin(\alpha - \gamma_1), \cos(\alpha - \gamma_1), 0)$ *or* $(-\sin(\alpha + \gamma_1), \cos(\alpha + \gamma_1), 0)$.

References

[1] H. Bear and G. Hile; *Behavior of Solutions of Elliptic Differential Inequalities near a point of Discontinuous Boundary Data*, Comm. PDE **8** (1983), 1175–1197.

[2] P. Concus and R. Finn; *Capillary wedges revisited*, SIAM J. Math. Anal. **27** (1996), no. 1, 56–69.

[3] M. Emmer; *Esistenza, unicit e regolarit nelle superfici de equilibrio nei capillari*, Ann. Univ. Ferrara Sez. VII (N.S.) **18** (1973), 79–94.

[4] R. Finn; *Equilibrium Capillary Surfaces*, Springer-Verlag, New York, 1987.

[5] R. Finn; *Local and global existence criteria for capillary surfaces in wedges*, Calc. Var. Partial Differential Equations 4 (1996), no. 4, 305–322.

[6] R. Finn; *Capillary Surfaces Interfaces*, Notices A.M.S. **46** (1999), 770-781.

[7] R. Finn; *On the equations of capillarity*, J. Math. Fluid Mech. **3** (2001), no. 2, 139–151.

[8] R. Finn; *Some Properties of Capillary Surfaces*, Milan J. Math. **70** (2002), 1–23.

[9] R. Finn; *Eight remarkable properties of capillary surfaces*, Math. Intel. **24**, No. 3, (2002), 21–33.

[10] E. Giusti; *Generalized solutions of prescribed mean curvature equations*, Pacific J. Math. **88** (1980), 297–321.

[11] E. Guisti; *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston, 1984.

[12] T. Jeffres and K. Lancaster; *Vertical Blow Ups of Capillary Surfaces in* $\mathbb{R}^3$ *Part 1: Convex Corners*, Electron. J. Differential Equations, Vol. 2007 (2007), No. 152, pp. 1-24.

[13] K. E. Lancaster; *Boundary behavior of a nonparametric minimal surface in* R^3 *at a nonconvex point*, Analysis **5** (1985), 61–69.

[14] K. E. Lancaster; *A Proof of the Concus-Finn Conjecture*, preprint.

[15] K. E. Lancaster; *New Proofs of the Concus-Finn Conjectures*, in preparation.

[16] K. E. Lancaster and D. Siegel; *Existence and Behavior of the Radial Limits of a Bounded Capillary Surface at a Corner*, Pacific J. Math. Vol. **176**, No. 1 (1996), 165-194. Correction to figures, Pacific J. Math. Vol. **179**, No. 2 (1997), 397-402.

[17] U. Massari and L. Pepe; *Successioni convergenti di ipersuperfici di curvatura media assegnata*, Rend. Sem. Univ. Padova **53**, (1975), 53-68.

[18] M. Miranda; *Un principio di massimo forte per le frontiere minimali e una sua applicazione alla risoluzione del problema al contorno per l'equazione delle superfici di area minima*, Rend. Sem. Mat. Univ. Padova **45** (1971), 355–366.

[19] M. Miranda; *Superfici minime illimitate*, Ann. Scuola Norm. Sup. Pisa **4** (1977), 313–322.

[20] Danzhu Shi; *Behavior of Capillary Surfaces at a Reentrant Corner*, Ph.D. Dissertation, Stanford University, February 2005.

[21] Danzhu Shi; *Capillary Surfaces at a Reentrant Corner*, Pacific J. Math. Vol. **224**, No. 2 (2006), 321-353.

[22] L. F. Tam; *The Behavior of Capillary Surfaces as Gravity Tends to Zero*, PhD Dissertation, Stanford University 1984.

[23] L. F. Tam; *The behavior of a capillary surface as gravity tends to zero*, Comm. Partial Differential Equations **11** (1986), 851–901.

[24] L. F. Tam; *Regularity of capillary surfaces over domains with corners: borderline case*, Pacific J. Math. **124** (1986), 469–482.

[25] L. F. Tam; *On Existence Criteria for Capillary Free Surfaces without gravity*, Pacific J. Math. **125**, No. 2 (1986), 469-485.

[26] J. Taylor; *Boundary regularity for solutions to various capillarity and free boundary problems*, Comm. Partial Differential Equations **2** (1977), no. 4, 323–357.

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