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ALMOST AUTOMORPHIC FUNCTIONS WITH VALUES IN p-FRÉCHET SPACES

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Abstract. In this paper we develop a theory of almost automorphic functions with values in p-Fréchet spaces, $0 < p < 1$, including the l^p, L^p spaces and the Hardy space H^p. Although the p-norm for $0 < p < 1$ does not have all the properties of an usual norm, the majority of main properties of almost automorphic functions with values in Banach spaces are extended to this case. Applications to semigroups of linear operators and to dynamical systems in p-Fréchet spaces are given.

1. Introduction

Harald Bohr's interest in functions that could be represented by a Dirichlet series led him to devise a theory of almost periodic real (and complex) functions, founding this theory between the years 1923 and 1926. Such functions have the form $\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$, where a_n, $z \in \mathbb{C}$ and $(\lambda_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence of real numbers (series which play an important role in complex analysis and analytic number theory).

The theory of almost periodic functions was strongly extended to abstract spaces; see for example the monographs [10, 24, 25] (for Banach space valued functions), and the works [8, 24, 26] (for Fréchet space valued functions). Also, in the recent paper [3] (see also Chapter 3 in the book [25]), the theory of real-valued almost periodic functions has been extended to the case of fuzzy-number-valued functions.

The concept of almost automorphy is a generalization of almost periodicity. It has been introduced in the literature by Bochner in relation to some aspects of differential geometry [4, 5, 6, 7]. Important contributions to the theory of almost automorphic functions have been obtained, for example, in the papers [22, 29, 30, 31, 32, 33, 34], in the books [24, 25, 33] (concerning almost automorphic functions with values in Banach spaces), and in [28] (concerning almost automorphy on groups). Also, the theory of almost automorphic functions with values in fuzzy-number-type spaces was developed in [15, 19] (see also Chapter 4 in [25]). Recently, in [16], we developed the theory of almost automorphic functions with values in a locally convex space (Fréchet space). In [9], the theory of almost

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automorphic and asymptotically almost automorphic semigroups of linear operators on Banach spaces is studied, while in our very recent article [17], we extended the theory from [9] to complete metrizable locally convex (Fréchet) spaces.

The purpose of this paper is to extend the main properties of almost automorphic functions with values in Banach spaces, to the class of almost automorphic functions with values in other important abstract spaces in functional analysis, namely the p-Fréchet spaces, $0 < p < 1$, which are non-locally convex spaces. The paper is organized as follows. In Section 2, we recall some known facts about Frechet spaces, while in Section 3, we develop a theory of almost automorphic functions with values in a p-Fréchet space, $0 < p < 1$. The main results are given in Section 4. Finally, in Section 5, we present some applications to dynamical systems and semigroups of linear operators in p-Fréchet spaces.

2. Preliminaries

It is well known that an F-space $(X, +, \cdot, \|\cdot\|)$ is a linear space (over the field $K = \mathbb{R}$ or $K = \mathbb{C}$) such that $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$, $\|x\| = 0$ if and only if $x = 0$, $\|\lambda x\| \le \|x\|$, for all scalars λ with $|\lambda| \le 1, x \in X$, and with respect to the metric $D(x, y) = \|x - y\|$, X is a complete metric space (see e.g. [12, p. 52], cf. [23] also). Obviously that D is invariant under translations. In addition, if there exists $0 < p < 1$ with $\|\lambda x\| = |\lambda|^p \|x\|$, for all $\lambda \in K, x \in X$, then $\|\cdot\|$ will be called a p-norm and X will be called p-Fréchet space. (This is only a slight abuse of terminology. Note that in [1], these spaces are called p-Banach spaces). In this case, it is immediate that $D(\lambda x, \lambda y) = |\lambda|^p D(x, y)$, for all $x, y \in X$ and $\lambda \in K$.

It is known that the F-spaces are not necessarily locally convex spaces. Three classical examples of p-Fréchet spaces, non-locally convex, are the Hardy space H^p with $0 < p < 1$ that consists in the class of all analytic functions $f : \mathbb{D} \to \mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ with the property

$$\|f\| = \frac{1}{2\pi} \sup \Big\{ \int_0^{2\pi} |f(re^{it})|^p dt, \quad r \in [0, 1) \Big\} < +\infty, \tag{2.1}$$

the l^p space

$$l^p = \big\{ x = (x_n)_n; \|x\| = \sum_{n=1}^{\infty} |x_n|^p < \infty \big\} \tag{2.2}$$

for $0 < p < 1$, and the $L^p[0, 1]$ space, $0 < p < 1$, given by

$$L^p = L^p[0, 1] = \{ f : [0, 1] \to \mathbb{R}; \|f\| = \int_0^1 |f(t)|^p dt < \infty. \} \tag{2.3}$$

More generally, we may consider $L^p(\Omega, \Sigma, \mu), 0 < p < 1$, based on a general measure space (Ω, Σ, μ), with the p-norm given by $\|f\| = \int_\Omega |f|^p d\mu$. Some important characteristics of the F-spaces are given by the following remarks.

Remarks. (1) Three fundamental results in Functional Analysis hold in F-spaces. They are the Principle of Uniform Boundedness (see e.g. [12, p. 52]), the Open Mapping Theorem and the Closed Graph Theorem (see e.g. [23, p. 9-10]). On the other hand, the Hahn-Banach Theorem fails in non-locally convex F-spaces. More precisely, if in an F-space, the Hahn-Banach theorem holds, then that space is necessarily locally convex space (see e.g. [4, Chapter 4]).

(2) If $(X, +, \cdot, \|\cdot\|)$ is a p-Fréchet space over the field K , $0 < p < 1$, then its dual X^* is defined as the class of all linear functionals $h : X \to K$ which satisfy

$|h(x)| \leq |||h||| \cdot \|x\|^{1/p}$, for all $x \in X$, where $|||h||| = \sup\{|h(x)|; \|x\| \leq 1\}$ (see e.g. [1, pp. 4-5]). Note that $||| \cdot |||$ is in fact a norm on X^*.

For $0 < p < 1$, while $(L^p)^* = 0$, we have that $(l^p)^*$ is isometric to l^∞- the Banach space of all bounded sequences (see e.g. [23, p. 20-21]), therefore $(l^p)^*$ becomes a Banach space. Also, if $\phi \in (H^p)^*$, then there exists a unique g which is analytic on $\mathbb{D}$ and continuous on the closure of $\mathbb{D}$, such that

$$\phi(f) = \frac{1}{2\pi} \lim_{r \to 1} \int_0^{2\pi} f(re^{it}) g(e^{-it}) dt,$$

for all $f \in H^p$ (see e.g. [13, Theorem 7.5]). Moreover, $(H^p)^*$ becomes a Banach space with respect to the usual norm $|||\phi||| = \sup\{|\phi(f)|; \|f\| \leq 1\}$ (cf. e.g. [13]). In both cases of l^p and H^p, $0 < p < 1$, their dual spaces separate the points of corresponding spaces.

(3) The spaces l^p and H^p, $0 < p < 1$, have Schauder bases (see e.g. [13, p. 20] for l^p and [23, 27] for H^p). It is also worth to note that according to [14], every linear isometry T of H^p onto itself has the form

$$T(f)(z) = \alpha[\phi'(z)]^{1/p} f(\phi(z)),$$

where α is some complex number of modulus one and ϕ is some conformal mapping of the unit disc onto itself.

3. Basic Definitions and Properties

In this section, we develop a theory of almost automorphic functions with values in a p-Fréchet space, $0 < p < 1$, denoted by $(X, +, \cdot, \| \cdot \|)$. In the previous section, we pointed out that the metric $D(x, y) = \|x - y\|$ is invariant under translations and satisfies $D(cx, cy) = |c|^p D(x, y)$. In addition, D has the following simple properties.

Theorem 3.1. (i) *$D(cx, cy) \leq D(x, y)$ for $|c| \leq 1$;*

(ii) *$D(x + u, y + v) \leq D(x, y) + D(u, v)$;*

(iii) *$D(kx, ky) \leq D(rx, ry)$ if $k, r \in \mathbb{R}, 0 < k \leq r$;*

(iv) *$D(kx, ky) \leq kD(x, y)$, for all $k \geq 1$;*

(v) *$D(cx, cy) \leq (|c| + 1)D(x, y), \forall c \in \mathbb{R}$.*

Proof. Property (i) and (iii) are obvious. (ii) We have

$$\begin{aligned} D(x + u, y + v) &= D(x + (u - v) + v, y + v) \\ &= D(x + u - v, y) \\ &= D(y, x + u - v) \\ &\leq D(y, x) + D(x, x + u - v) \\ &= D(x, y) + D(x + v, x + u) \\ &= D(x, y) + D(v, u). \end{aligned}$$

(iv) Since $0 < p < 1$, we have $D(kx, ky) = |k|^p D(x, y) \leq kD(x, y)$, for all $k \geq 1$.
(v) If $|c| < 1$ then $D(cx, cy) = |c|^p D(x, y) \leq (|c| + 1)D(x, y)$. If $|c| \geq 1$ then we get

$$D(cx, cy) = |c|^p D(x, y) \leq |c|D(x, y) \leq (|c| + 1)D(x, y),$$

which proves the theorem. □

Now, we start with the following Bochner-kind definition.

Definition 3.2. We say that a continuous function $f : \mathbb{R} \to X$, is almost automorphic, if every sequence of real numbers $(r_n)_n$ contains a subsequence $(s_n)_n$, such that for each $t \in \mathbb{R}$, there exists $g(t) \in X$ with the property

$$\lim_{n\to+\infty} \|g(t) - f(t+s_n)\| = \lim_{n\to+\infty} \|g(t-s_n) - f(t)\| = 0.$$

Note that the above convergence on $\mathbb{R}$ is pointwise. Equivalently, in terms of the metric D, we can write

$$\lim_{n\to+\infty} D(g(t), f(t+s_n)) = \lim_{n\to+\infty} D(g(t-s_n), f(t)) = 0.$$

Remark. The almost automorphy in Definition 3.2 is a more general concept than almost periodicity in p-Fréchet spaces, $0 < p < 1$, defined in [20]. Indeed, by the Bochner's criterion (see [20, Theorem 3.7]), a function with values in a p-Fréchet space is almost periodic if and only if for every sequence of real numbers $(r_n)_n$, there exists a subsequence $(s_n)_n$, such that the sequence $(f(t+s_n))_n$ converges uniformly with respect to $t \in \mathbb{R}$, in the metric D. Obviously this is a stronger condition than the pointwise convergence in Definition 3.2.

Example. The function $f_x : \mathbb{R} \to X$ defined by $f_x(t) = xcos\frac{1}{2-sin\pi t-sint}$, for a given $x \in X$, is almost automorphic but not almost periodic, since it is not uniformly continuous on $\mathbb{R}$.

The following elementary properties hold.

Theorem 3.3. *Let $(X, +, \cdot, \|\cdot\|)$ be a p-Fréchet space, $0 < p < 1$ and $D(x, y) = \|x - y\|$. If $f, f_1, f_2 : \mathbb{R} \to X$ are almost automorphic functions then we have:*

- (i) *$f_1 + f_2$ is almost automorphic;*
- (ii) *cf is almost automorphic for every scalar $c \in \mathbb{R}$;*
- (iii) *$f_a(t) = f(t+a), \forall t \in \mathbb{R}$ is almost automorphic for each fixed $a \in \mathbb{R}$;*
- (iv) *We have $\sup\{\|f(t)\|; t \in \mathbb{R}\} < +\infty$ and $\sup\{\|g(t)\|; t \in \mathbb{R}\} < +\infty$, where g is the function attached to f in Definition 3.2;*
- (v) *The range $R_f = \{f(t); t \in \mathbb{R}\}$ is relatively compact in the complete metric space (X, D);*
- (vi) *The function h defined by $h(t) = f(-t), t \in \mathbb{R}$ is almost automorphic;*
- (vii) *If $f(t) = 0_X$ for all $t > a$ for some real number a, then $f(t) = 0_X$ for all $t \in \mathbb{R}$;*
- (viii) *If $A : X \to Y$ is continuous, where Y is another q -Fréchet space, $0 < q < 1$, then $A(f) : \mathbb{R} \to Y$ also is almost automorphic.*
- (ix) *Let $h_n : \mathbb{R} \to X, n \in \mathbb{N}$ be a sequence of almost automorphic functions such that $h_n(t) \to h(t)$ when $n \to +\infty$, uniformly in $t \in \mathbb{R}$ with respect to the p-norm $\|\cdot\|$ (that is with respect to the metric D). Then h is almost automorphic.*

Proof. Property (i) is immediate from

$$D(u+v, w+e) \le D(u, w) + D(v, e), \quad \forall u, v, w, e \in X$$

and from Definition 3.2.

(ii) It follows easily from the property $D(c\cdot u, c\cdot v) \le (|c|+1)D(u, v)$, for all $u, v \in X$, for all $c \in \mathbb{R}$ (see Theorem 3.1, (v)) and from Definition 3.2.

(iii) The proof is immediate by Definition 3.2.

(iv) Let us suppose that $\sup\{\|f(t)\|; t \in \mathbb{R}\} = +\infty$. Then there exists a sequence of real numbers $(r_n)_n$ such that $\|f(r_n)\| \to +\infty$, when $n \to +\infty$. Since f is almost automorphic, by Definition 3.2 for $t = 0$, we can extract a subsequence $(s_n)_n$ of $(r_n)_n$ such that $\lim_{n\to+\infty} \|g(0) - f(s_n)\| = 0$, where $g(0) \in X$. It follows that

$$\|f(s_n)\| \le \|f(s_n) - g(0)\| + \|g(0)\|,$$

where, by passing to the limit as $n \to \infty$, we obtain the contradiction $+\infty \le \|g(0)\|$. The proof for g is similar, by taking into account the relation $\lim_{n\to+\infty} \|g(-s_n) - f(0)\| = 0$, in Definition 3.2, for $t = 0$.

(v) Let $(f(r_n))_n$ be an arbitrary sequence in X. From Definition 3.2, there exists a subsequence $(s_n)_n$ of $(r_n)_n$ such that $\lim_{n\to+\infty} D(g(0), f(s_n)) = 0$, i.e. $(f(s_n))_n$ is a convergent subsequence of $(f(r_n))_n$ in the complete metric space (X, D), which proves that R_f is relatively compact in (X, D).

(vi) The proof of (vi) is similar to the proof of [24, Theorem 2.1.4]. The proof of (vii) is identical to the proof of [24, Theorem 2.1.8]. Property (viii) is again an immediate consequence of Definition 3.2 and of the continuity of A. We leave the details to the reader. Finally, the proof (ix) is identical to the proof of [24, Theorem 2.1.10], by using the fact that (X, D) is complete metric space and the triangle's inequality holds with D as a metric. The theorem is proved. $\square$

Let us recall now that for $f : \mathbb{R} \to X$, the derivative of f at $x \in \mathbb{R}$ denoted by $f'(x) \in X$, is defined by the relation

$$\lim_{h\to 0} d(f'(x), \frac{f(x+h) - f(x)}{h}) = 0.$$

Also, the integral can be defined in usual way, by using Riemann sums or, by using approximation by step functions (cf. [18]). Unfortunately, because the Leibniz-Newton formula does not hold for functions with values in p-Fré chet spaces X with $0 < p < 1$ (cf. [18]), the classical results concerning the almost automorphy of the derivative and of the (indefinite) integral of an almost automorphic function do not seem to be valid, because the Leibniz-Newton formula seems to be the fundamental tool for the proofs. On the other hand, when studying almost automorphic functions, the following concepts and results can still be useful. We will follow here the ideas in [24, Section 2.2]. In general, the results that we proved in [24], in the case of Banach spaces remain the same for the case of p-Fréchet spaces, with $0 < p < 1$, except for those results that are proved through the use of the Leibniz-Newton formula.

Definition 3.4. Let $(X, +, \cdot, \|\cdot\|)$ be a p-Fréchet space, $0 < p < 1$ and $D(x, y) = \|x - y\|$. A continuous function $f : \mathbb{R} \times X \to X$ is said to be almost automorphic in $t \in \mathbb{R}$ for each $x \in X$, if for every sequence of real numbers $(r_n)_n$, there exists a subsequence $(s_n)_n$ such that for all $t \in \mathbb{R}$ and $x \in X$, there exists $g(t, x)$ with the property

$$\lim_{n\to+\infty} \|f(t + s_n, x) - g(t, x)\| = \lim_{n\to+\infty} \|g(t - s_n, x) - f(t, x)\| = 0.$$

The following simple properties hold.

Theorem 3.5. *Let $(X, +, \cdot, \|\cdot\|)$ be a p-Fréchet space, $0 < p < 1$ and $D(x, y) = \|x - y\|$.*

(i) *If $f_1, f_2 : \mathbb{R} \times X \to X$ are almost automorphic in t for each $x \in X$, then $f_1 + f_2$ and $c \cdot f_1$, where $c \in \mathbb{R}$ are also almost automorphic in t for each $x \in X$.*

(ii) *If $f(t,x)$ is almost automorphic in t for each $x \in X$ then for all $x \in X$, we have* $\sup\{\|f(t,x)\|; t \in \mathbb{R}\} < +\infty$. *Also, for the corresponding function g in Definition 3.2 we have* $\sup\{\|g(t,x)\|; t \in \mathbb{R}\} < +\infty$.

(iii) *If $f(t,x)$ is almost automorphic in t for each $x \in X$ and if $\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \forall x, y \in X$ and $t \in \mathbb{R}$, where L is independent of x, y and t, then for the corresponding g in Definition 3.2 we have $\|g(t,x)-g(t,y)\| \leq L\|x-y\|$, for all $x, y \in X$ and $t \in \mathbb{R}$.*

(iv) *Let $f(t,x)$ be almost automorphic in t for each $x \in X$ and $\varphi : \mathbb{R} \to X$ be almost automorphic. If $\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \forall x, y \in X$ and $t \in \mathbb{R}$, where L is independent of x, y and t then the function $F : \mathbb{R} \to X$ defined by $F(t) = f(t, \varphi(t))$ is almost automorphic.*

Proof. The proof of (i) is similar to that of Theorem 3.3, (i), (ii). The proof of (ii) is similar to the proof of Theorem 3.3, (iv). The proofs of (iii) and (iv) are analogous to the proofs of [24, Theorems 2.2.5 and 2.2.6]. We note that in these proofs, only the triangle inequality of the p-norm was used. □

Analogous to the case of Banach spaces (see e.g. [24, p. 37], the concept in Definition 3.2 can be generalized as follows.

Definition 3.6. Let $(X, +, \cdot, \|\cdot\|)$ be a p-Fréchet space, $0 < p < 1$. A continuous function $f : \mathbb{R}_+ \to X$ is said to be asymptotically almost automorphic if it admits the decomposition $f(t) = g(t) + h(t), t \in \mathbb{R}_+$, where $g : \mathbb{R} \to X$ is almost automorphic and $h : \mathbb{R}_+ \to X$ is a continuous function with $\lim_{t\to+\infty} \|h(t)\| = 0$. Here g and h are called the principal and the corrective terms of f, respectively.

Remark. Every almost automorphic function restricted to $\mathbb{R}_+$ is asymptotically almost automorphic, by taking $h(t) = 0_X$, for all $t \in \mathbb{R}_+$.

Regarding this new concept, the following results hold.

Theorem 3.7. *Let $(X, +, \cdot, \|\cdot\|)$ be a p-Fréchet space, $0 < p < 1$, and let $f, f_1, f_2 : \mathbb{R}_+ \to X$ be asymptotically almost automorphic. Then we have:*

(i) *$f_1 + f_2$ and $c \cdot f, c \in \mathbb{R}$ are asymptotically almost automorphic;*

(ii) *For fixed $a \in \mathbb{R}_+$, the function $f_a(t) = f(t+a)$ is asymptotically almost automorphic;*

(iii) *We have* $\sup\{\|f(t)\|; t \in \mathbb{R}_+\} < +\infty$.

(iv) *Let $(X, +, \cdot, \|\cdot\|_1)$ be a p-Fréchet space $(Y, +, \cdot, \|\cdot\|_2)$ be a q-Fréchet space and $f : \mathbb{R}_+ \to X$ be an asymptotically almost automorphic function, $f = g + h$. Let $\phi : X \to Y$ be continuous and assume there is a compact set B in (X, D) with $D(x,y) = \|x-y\|_1$, which contains the closures of $\{f(t); t \in \mathbb{R}_+\}$ and $\{g(t); t \in \mathbb{R}_+\}$. Then $\phi \circ f : \mathbb{R}_+ \to Y$ is asymptotically almost automorphic;*

(v) *The decomposition of an asymptotically almost automorphic function is unique.*

Proof. (i) Let $c \in \mathbb{R}$, $f_1 = g_1 + h_1$, $f_2 = g_2 + h_2$, $f = g + h$, where the decompositions are like those in Definition 3.6. We clearly have $f_1 + f_2 = [g_1 + g_2] + [h_1 + h_2]$ and $c \cdot f = c \cdot g + c \cdot h$. By Theorem 3.3, (i), (ii), it follows that $g_1 + g_2, c \cdot g$ are almost automorphic. Also, from the properties of the p-norm $\|\cdot\|$, we get

$$\lim_{t\to+\infty} \|h_1(t) + h_2(t)\| \leq \lim_{t\to+\infty} \|h_1(t)\| + \lim_{t\to+\infty} \|h_2(t)\| = 0,$$

and

$$\lim_{t\to+\infty} \|c \cdot h(t)\| = |c|^p \lim_{t\to+\infty} \|h(t)\| = 0.$$

(ii) Let $f = g+h$ be the decomposition in Definition 3.6. Then $f_a(t) = g(t+a)+h(t+a)$, where by Theorem 3.3, (iii), $g(t+a)$ is almost automorphic. By Definition 3.6, we immediately get $\lim_{t\to+\infty} \|h(t+a)\| = 0$.

(iii) Let now $f = g + h$ be the decomposition in Definition 3.6. We have

$$\sup\{\|f(t)\|; t \in \mathbb{R}_+\} \le \sup\{\|g(t)\|; t \in \mathbb{R}_+\} + \sup\{\|h(t)\|; t \in \mathbb{R}_+\}.$$

By Theorem 3.3, (iv), we get $\sup\{\|g(t)\|; t \in \mathbb{R}_+\} < +\infty$. Moreover, denoting $Q(t) = \|h(t)\|$, clearly Q is continuous on $[0, +\infty)$ (since the property $|\ \ \|F\| - \|G\|\ \ | \le \|F - G\|$ is a consequence of the triangle's inequality). By hypothesis, $\lim_{t\to+\infty} \|h(t)\| = 0$, which immediately implies that we have

$$\lim_{t\to+\infty} \|h(t)\| = \lim_{t\to+\infty} Q(t) = 0.$$

Let $\varepsilon > 0$ be fixed. There exists $\delta > 0$, such that $\|h(t)\| < \varepsilon$, for all $t > \delta$. From the continuity of Q on $[0, \delta]$, there exists $M > 0$ such that $Q(t) \le M$, for all $t \in [0, \delta]$. In conclusion, $0 \le Q(t) \le M + \varepsilon, \forall t \in \mathbb{R}_+$, which implies the desired conclusion.

(iv) Let $f = g + h$ be the decomposition in Definition 3.6. By Theorem 3.3, (viii), $\phi \circ g : \mathbb{R} \to Y$ is almost automorphic and also by hypothesis, $\phi \circ f$, $\phi \circ g$, are continuous on $\mathbb{R}_+$. Denote now $\Gamma(t) = \phi(f(t)) - \phi(g(t))$. Let $\varepsilon > 0$. By the uniform continuity of ϕ on the compact set B , there exists $\delta > 0$, such that $\|\phi(x)-\phi(y)\|_2 < \varepsilon$, for all $\|x-y\|_1 < \delta$, $x, y \in B$. On the other hand, by hypothesis, we have $\lim_{t\to+\infty} \|h(t)\|_1 = 0$, therefore there exists t_0 (depending on δ), such that $\|h(t)\|_1 = \|f(t) - g(t)\|_1 < \delta$, for all $t > t_0$. Then, for $t > t_0$ we obtain,

$$\|\Gamma(t)\|_2 = \|\phi(f(t)) - \phi(g(t))\|_2 < \varepsilon,$$

for all $t > t_0$, which means $\lim_{t\to+\infty} \|\Gamma(t)\|_2 = 0$.

(v) Let us suppose now that f has two decompositions $f = g_1 + h_1 = g_2 + h_2$. For all $t \ge 0$ we get $g_1(t) - g_2(t) = h_2(t) - h_1(t)$, which implies

$$\lim_{t\to+\infty} \|g_1(t) - g_2(t)\| \le \lim_{t\to+\infty} \|h_2(t)\| + \lim_{t\to+\infty} \|h_1(t)\| = 0.$$

Consider the sequence (n). Since $g_1 - g_2$ is almost automorphic, there exists a subsequence (n_k) such that

$$\lim_{k\to+\infty} \|[g_1(t + n_k) - g_2(t + n_k)] - F(t)\| = 0$$

and

$$\lim_{k\to+\infty} \|F(t - n_k) - [g_1(t) - g_2(t)]\| = 0,$$

with the convergence holding pointwise on $\mathbb{R}$. But

$$\|F(t)\| \le \|F(t) - [g_1(t + n_k) - g_2(t + n_k)]\| + \|g_1(t + n_k) - g_2(t + n_k)\|.$$

Passing to the limit as $k \to +\infty$ and taking the above relations into account, it follows $\|F(t)\| = 0, \forall t \in \mathbb{R}_+$, which implies $g_1(t) - g_2(t) = 0, \forall t$. Therefore, $h_2(t) - h_1(t) = 0$, for all $t \in \mathbb{R}_+$, which proves the theorem. $\square$

Remark. Concerning the derivative and indefinite integral of asymptotically almost automorphic functions, we have the same negative phenomenon as in the case of almost automorphic functions (see the Remark after the proof of Theorem 3.3).

We also have the following result.

Theorem 3.8. *If $(X, +, \cdot, \|\cdot\|)$ is a p-Fréchet space with $0 < p < 1$, then the space of almost automorphic X-valued functions $AA(X)$, is a p-Fréchet space with respect to the p-norm given by $\|f\|_b = \sup\{\|f(t)\|; t \in \mathbb{R}\}$, which generates the metric D_b on $AA(X)$ defined by $D_b(f, g) = \|f - g\|_b$.*

Proof. First note that the convergence of a sequence $(f_n)_n \in AA(X)$ to $f \in AA(X)$ with respect to D_b, is equivalent to the uniform convergence with respect to $t \in \mathbb{R}$, in the p -norm $\|\cdot\|_b$. Now, by Theorem 3.3, (i), (ii), (iv), $AA(X)$ is a linear subspace of the space of all $f : \mathbb{R} \to X$, continuous, bounded (i.e. $\|f\|_b < +\infty$) functions, denoted by $C_b(\mathbb{R}; X)$. Since $C_b(\mathbb{R}; X)$ is complete metric space with respect to the metric $D_b(f, g) = \|f - g\|_b$, by Theorem 3.3, (ix), $AA(X)$ is closed, which implies that $(AA(X), D_b)$ is complete metric space. $\square$

We also now introduce a more general concept.

Definition 3.9. Let $(X, +, \cdot, \|\cdot\|)$ be a p-Fréchet space with $0 < p < 1$, having the dual space $X^* \neq \{0\}$. We say that a weakly continuous function $f : \mathbb{R} \to X$, is weakly almost automorphic, if every sequence of real numbers $(r_n)_n$, contains a subsequence $(s_n)_n$, such that for each $t \in \mathbb{R}$, there exists $g(t) \in X$ with the property

$$\lim_{n\to+\infty} \varphi[f(t + s_n)] = \varphi[g(t)] \quad \text{and} \quad \lim_{n\to+\infty} \varphi[g(t - s_n)] = \varphi[f(t)],$$

for all $\varphi \in X^*$ (the above convergence on $\mathbb{R}$ is pointwise).

Remarks. (1) A function $f : \mathbb{R} \to X$ is called weakly continuous on $\mathbb{R}$, if we consider that X is endowed with the weak topology induced by X^*. For example, in the case of $X = l^p$ or $X = H^p$, $0 < p < 1$, the weak topology is a locally convex Hausdorff topology (see the considerations above, [20, Definition 3.13]).

(2) The convergence in the p-norm $\|\cdot\|$, obviously implies the weak-convergence, from the inequality

$$|\varphi(x)| \le |||\varphi||| \cdot \|x\|^p,$$

where $\varphi \in X^*$ and $|||\varphi||| = sup\{|\varphi(x)|; \|x\| \le 1\}$. This means that a function which is almost automorphic in the sense of Definition 3.2, also is weakly almost automorphic.

(3) If f is weakly almost automorphic, it is immediate that for any $\varphi \in X^*$, the numerical function $F : \mathbb{R} \to \mathbb{R}$ given by $F(x) = \varphi[f(x)]$, for all $x \in \mathbb{R}$, is almost automorphic.

(4) Obviously that Definition 3.9 has no sense for $X = L^p$, $0 < p < 1$, since in this case $X^* = \{0\}$, but seems to be suitable for $X = l^p$ or $X = H^p$, $0 < p < 1$, which have rich dual spaces.

The following properties hold.

Theorem 3.10. *Let $(X, +, \cdot, \|\cdot\|)$ be a p-Fréchet space with $0 < p < 1$, having the dual space $X^* \neq \{0\}$, and suppose that $f_1, f_2, f : \mathbb{R} \to X$ are weakly almost automorphic.*

- (i) *Then $f_1 + f_2$ is weakly almost automorphic;*
- (ii) *Then cf is weakly almost automorphic;*
- (iii) *If $a \in \mathbb{R}$ is fixed, then f_a given by $f_a(x) = f(x + a)$, for all $x \in \mathbb{R}$, is weakly almost automorphic;*
- (iv) *Then f_- given by $f_-(x) = f(-x)$, for all $x \in \mathbb{R}$, is weakly almost automorphic;*

(v) *If $A : X \to X$ is continuous linear operator, that is*

$$\|A(x)\| \le |||A||| \cdot \|x\|, \text{ for all } x \in X,$$

where $|||A||| = \sup\{\|A(x)\|; \|x\| \le 1\} < +\infty$, then $F : \mathbb{R} \to X$ given by $F(x) = A[f(x)]$, is weakly almost automorphic;

(vi) *If the range of f is relatively compact in X then f is almost automorphic in the sense of Definition 3.2 (that is in "strong" sense).*

Proof. The proofs for (i)-(v) follow easily from Definition 3.9. Also, the proof of (vi) is similar to the proof in the case of Banach spaces. We refer the reader to the proof of [24, Theorem 2.3.7]. Indeed, in that proof, the only property that we used was the fact that any sequence belonging to a compact subset contains a convergent subsequence and that the convergence in the p-norm $\|\cdot\|$ implies the weak convergency (see Remark 2 after Definition 3.9). □

Remark. In the case of Banach space valued functions, if f is weakly automorphic then $sup\{\|f(t)\|; t \in \mathbb{R}\} < +\infty$ (see e.g. [24, Theorem 2.3.4]. In Banach spaces, the main tool for the proof is the fact that any weakly convergent sequence is necessarily bounded in norm. Unfortunately, in the case of p-Fréchet spaces, $0 < p < 1$, this proposition does not hold in general. To see this, take for example the Hardy space H^p, $0 < p < 1$. Indeed, there exists weakly convergent sequences which are unbounded with respect to the p-norm (see [13, Corollary 2]). Consequently, it appears that if $f : \mathbb{R} \to X$ is weakly almost automorphic, then f might be unbounded with respect to the p-norm.

However, a weaker result hold, as follows.

Theorem 3.11. *Let $(X, +, \cdot, \|\cdot\|)$ be a p-Fréchet space with $0 < p < 1$, having the dual space $X^* \ne \{0\}$, and suppose that $f : \mathbb{R} \to X$ is weakly almost automorphic.*

(i) *If $X = l^p$, $0 < p < 1$, then $sup\{\|f(t)\|_{l^1} : t \in \mathbb{R}\} < \infty$, where $\|\cdot\|_{l^1}$ denotes the norm in the Banach space l^1, given by $\|x\|_{l^1} = \sum_{k=1}^{\infty} |x_k|$, $x = (x_k)_{k\in\mathbb{N}}$;*

(ii) *If $X = H^p$, $0 < p < 1$, then $sup\{\|f(t)\|_{B^p}; t \in \mathbb{R}\} < \infty$, where $\|F\|_{B^p}$ denotes the norm in the Banach space B^p, defined as the space of all analytic functions F, in the open unit disk, which satisfy*

$$\|F\|_{B^p} = \int_0^1 (1-r)^{(1/p)-2} M_1(r, F) dr < \infty,$$

where

$$M_1(r, F) = \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})| d\theta.$$

Proof. (i) Obviously $l^p \subset l^1$ and $\|x\|_{l^1} \le C\|x\|^{\frac{1}{p}}$, with C independent of x, where $\|\cdot\|$ denotes the p-norm in X. According to e.g. [23, pp. 20-21, 27-28], the space l^1 is the smallest Banach space containing l^p with $0 < p < 1$, called the envelope of l^p and $[l^1]^* = [l^p]^* = l^\infty$, where l^∞ is the Banach space of all bounded sequences.

In what follows we may argue as in proof of [24, Theorem 2.3.4]. Thus, suppose by contradiction that $sup\{\|f(t)\|_{l^1} : t \in \mathbb{R}\} = \infty$, so there exists a sequence of real numbers $(s'_n)_n$ with $lim_{n\to\infty}\|f(s'_n)\|_{l^1} = \infty$. Since f is weakly almost automorphic, we can extract a subsequence $(s_n)_n$ of $(s'_n)_n$, such that for $n \to \infty$, we have $\varphi[f(s_n)] \to \varphi(\alpha)$, for all $\varphi \in (l^p)^* = (l^1)^*$, with $\alpha \in l^p \subset l^1$. In other words, the sequence $(f[s_n])_n$ is weakly convergent in the Banach space l^1, which implies that it is bounded in the norm in l^1, therefore we have obtained the contradiction.

(ii) According to [13, Sections 3 and 4], B^p is a Banach space (with respect to the norm mentioned in our statement) that is the envelope of H^p, it follows $H^p \subset B^p$, $(H^p)^* = (B^p)^*$ and $\|F\|_{B^p} \leq C\|F\|$. The rest of proof is identical with that in (i). □

4. Semigroups of operators on p-Fréchet spaces, $0 < p < 1$

First let us recall a few notions of semigroups of linear operators in p-Fréchet spaces, $0 < p < 1$, developed in [18], as extensions of the results for Banach spaces in [21].

If $(X, \|\cdot\|)$ is a p-Fréchet space, $0 < p < 1$, by repeating the standard techniques in functional analysis (for the case of usual normed spaces) it follows that a linear operator $A : X \to X$ is continuous (as mapping between two metric spaces) if and only if $|||A||| = \sup\{\|A(x)\|; \|x\| \leq 1\} < +\infty$ and

$$\|A(x)\| \leq |||A||| \cdot \|x\|,$$

for all $x \in X$. (for details see e.g. [1, Example 2 after Theorem 1]). More generally, if $(X, \|\cdot\|_1)$ is a p-Fréchet space and $(Y, \|\cdot\|_2)$ is a q-Fréchet space, with $0 < p < 1$, $0 < q < 1$, then according to [2, p. 93, Definition 2.2, relationships (1) and (2)], the boundedness of the linear operator $A : X \to Y$ is equivalent to

$$\|A(x)\|_2^{1/q} \leq |||A||| \cdot \|x\|_1^{1/p}, x \in X,$$

where

$$|||A||| = \sup\{\frac{\|A(x)\|_2^{1/q}}{\|x\|_1^{1/p}}; x \in X, x \neq 0_X\}.$$

Note that in the case when $X = Y$ (and therefore $p = q$), from [2, p. 93, relationships (1) and (2)], it easily follows that the boundedness becomes as it is stated at the beginning (i.e. as for classical linear operators between Banach spaces).

If we denote by $B(X)$ the space of all linear and continuous (i.e. bounded) operators $A : X \to X$, then $|||A||| = \sup\{\|A(x)\|; \|x\| \leq 1\}$ is a p-norm on $B(X)$, since

$$|||\lambda A||| = \sup\{\|\lambda A(x)\|; \|x\| \leq 1\} = |\lambda|^p \sup\{\|A(x)\|; \|x\| \leq 1\} = |\lambda|^p |||A|||.$$

Also, since X is complete with respect to the metric $D(x, y) = \|x - y\|$, it easily follows that $B(X)$ is complete with respect to the metric $D_O(T, S) = |||T - S|||$, for all $T, S \in B(X)$, i.e. $B(X)$ is a p-Fréchet space.

Definition 4.1 ([18]). A family $(T(t))_{t\geq 0}$ of linear continuous (i.e. bounded) operators on the p-Fréchet space $(X, \|\cdot\|)$, $0 < p < 1$, satisfying the properties $T(t + s) = T(t)[T(s)]$, for all $t, s \geq 0$, $T(0) = I$ (I-the identity operator on X) and $T(\cdot)(x) : \mathbb{R}_+ \to X$ is continuous for each $x \in X$, is called a strongly continuous (one parameter) semigroup on X. If $T(t + s) = T(t)[T(s)]$, for all $t, s \in \mathbb{R}$, then $(T(t))_{t\in\mathbb{R}}$ is called group of linear operators on X. Also, $(T(t))_t$ is called uniformly continuous if $T : K \to B(X)$ is continuous, where $K = \mathbb{R}$ or $K = \mathbb{R}_+$.

An operator $A \in B(X)$ is called the (infinitesimal) generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$, if there exists the limit

$$\lim_{t\searrow 0} \|\frac{T(t)(x) - x}{t} - A(x)\| = 0,$$

for some $x \in X$. The domain $D(A)$ of A is the set of all $x \in X$, such that the above limit exists.

We can state the following result.

Theorem 4.2 ([18]). *Let $(X, +, \cdot, \|\cdot\|)$ be a p-Fréchet space where $0 < p < 1$, and $A \in B(X)$. For $x \in X$ and $t \in \mathbb{R}$ let us define $S_m(t)(x) = \sum_{j=0}^{m} \frac{t^j}{j!} A^j(x), m \in \mathbb{N}$. It follows that*

(i) *For each $x \in X$ and $t \in \mathbb{R}$, the sequence $S_m(t, x), m = 1, 2, \dots,$ is convergent in X, that is there exists an element in X dented by $e^{tA}(x)$, such that*
$$\lim_{m\to+\infty} \|e^{tA}(x) - S_m(t, x)\| = 0,$$
and we write $e^{tA}(x) = \sum_{k=0}^{+\infty} \frac{t^k}{k!} A^k(x)$;
(ii) *For any fixed $t \in \mathbb{R}$, we have $e^{tA} \in B(X)$;*
(iii) *$e^{(t+s)A} = e^{tA} e^{sA}, \forall t, s \in \mathbb{R}$;*
(iv) *The limit*
$$\lim_{t\to 0^+} \|A(x) - \frac{e^{tA}(x) - x}{t}\| = 0,$$
exists for all $x \in X$;
(v) *e^{tA} is continuous as function of $t \in \mathbb{R}$ to $B(X)$ and $e^{0A} = I$. Also, $T(t) = e^{tA}$ is differentiable, $\frac{d}{dt}[e^{tA}(x)] = A[e^{tA}(x)]$ and the function $e^{tA}(x_0) : \mathbb{R} \to X$ is the unique solution of the Cauchy problem $x'(t) = A[x(t)], t \in \mathbb{R}$, $x(0) = x_0$.*

Theorem 4.2 shows that $T(t) = e^{tA}, t \geq 0$ is a strongly continuous group of operators. Also, let us prove the following result.

Theorem 4.3. *Let $(X, +, \cdot, \|\cdot\|)$ be a p-Fréchet space, where $0 < p < 1$.*

(i) *Let $(T(t))_{t\in\mathbb{R}}$ be a strongly continuous group of bounded linear operators on X. Assume that the function $x : \mathbb{R} \to X$, defined by $x(t) = T(t)[x_0]$ is almost automorphic for some $x_0 \in X$. Then $\inf_{t\in\mathbb{R}} \|x(t)\| > 0$, or $x(t) = 0, \forall t \in \mathbb{R}$.*
(ii) *Let $x : \mathbb{R}_+ \to X$ and $f : \mathbb{R} \to X$ be two continuous functions and $T = (T(t))_{t\in\mathbb{R}_+}$ be a strongly continuous semigroup of bounded linear operators on X. Suppose that*
$$x(t) = T(t)(x(0)) + \int_0^t T(t - s)(f(s))ds, t \in \mathbb{R}_+.$$
Then for given t in $\mathbb{R}$ and $b > a > 0$, $a + t > 0$, we have
$$x(t + b) = T(t + a)(x(b - a)) + \int_{-a}^{t} T(t - s)(f(s + b))ds.$$

Proof. (i) Let us suppose that we have $\inf_{t\in\mathbb{R}} \|x(t)\| = 0$. Let $(s'_n)_n$ be a sequence of real numbers such that $\lim_{n\to+\infty} \|x[s'_n]\| = 0$. Since, by hypothesis, as function of t, the function $x(t)$ is almost automorphic, by Definition 3.2, we can extract a subsequence $(s_n)_n$ of $(s'_n)_n$ such that for all $t \in \mathbb{R}$, there exists $y(t) \in X$ with the property
$$\lim_{n\to+\infty} \|y(t) - x(t + s_n)\| = \lim_{n\to+\infty} \|y(t - s_n) - x(t)\| = 0,$$
with the above convergence on $\mathbb{R}$ being pointwise. Also, we can easily deduce that
$$x(t + s_n) = T(t + s_n)[x_0] = T(t)(T(s_n)[x_0]) = T(t)[x(s_n)].$$

From the above limits, we obtain

$$\|y(t)\| \leq \|y(t) - x(t+s_n)\| + \|x(t+s_n)\| \leq \\ \|y(t) - x(t+s_n)\| + \||T(t)\|| \cdot \|x(s_n)\|,$$

thus passing to the limit as $n \to +\infty$, it follows that $\|y(t)\| = 0$, that is, $y(t) = 0_X$, for all $t \in \mathbb{R}$. This immediately implies $x(t) = 0$, for all $t \in \mathbb{R}$.

(ii) As in the proof of [24, Theorem 2.4.7], we obtain

$$x(t+b) = T(t+a)\Big[x(b-a) - \int_0^{b-a} T(b-a-s)(f(s))ds\Big] + \int_0^{t+b} T(t+b-s)(f(s))ds.$$

Then from the above relation we get

$$\begin{aligned} &x(t+b) + T(t+a)\Big[\int_0^{b-a} T(b-a-s)(f(s))ds\Big] \\ &= T(t+a)[x(b-a)] + \int_0^{t+b} T(t+b-s)(f(s))ds. \end{aligned}$$

Taking into account that T commutes with the integral (since it is linear and continuous operator), by the property $T(u+v) = T(u)[T(v)], \forall u, v \in \mathbb{R}_+$ and by the substitution $u = s - b$, we obtain

$$x(t+b) + \int_{-b}^{-a} T(t-u)[f(u+b)]du = T(t+a)[x(b-a)] + \int_{-b}^{t} T(t-u)[f(u+b)]du.$$

But because $t > -a$, we can write

$$\int_{-b}^{t} T(t-u)[f(u+b)]du = \int_{-b}^{-a} T(t-u)[f(u+b)]du + \int_{-a}^{t} T(t-u)[f(u+b)]du,$$

we immediately get the required relation from the statement of theorem. The theorem is proved. □

In what follows, we will be concerned with the behavior of asymptotically almost automorphic semigroups of linear operators $T = T(t), t \in \mathbb{R}_+$ on p-Fréchet spaces, $0 < p < 1$. We present some topological and asymptotic properties based on the Nemytskii and Stepanov theory of dynamical systems.

Definition 4.4. Let $(X, +, \cdot, \|\cdot\|)$ be a p-Fréchet space, where $0 < p < 1$. A mapping $u : \mathbb{R}_+ \times X \to X$ is called a dynamical system if:

(i) $u(0_X, x) = x$, for all $x \in X$;
(ii) $u(\cdot, x) : \mathbb{R}_+ \to X$ is continuous for any $t > 0$ and right-continuous at $t = 0$, for each $x \in X$. The mapping $u(\cdot, x)$ is called a motion originating at $x \in X$.
(iii) $u(t, \cdot) : X \to X$ is continuous for each $t \geq 0$;
(iv) $u(t+s, x) = u(t, u(s, x)), \forall x \in X$, for all $t, s \in \mathbb{R}_+$.

Theorem 4.5. *Let $(X, +, \cdot, \|\cdot\|)$ be a p-Fréchet space, where $0 < p < 1$. Every strongly continuous semigroup $(T(t))_{t\geq 0}$ on X determines a dynamical system and conversely, by defining $u(t, x) = T(t)(x), t \in \mathbb{R}_+, x \in X$.*

The proof of the above theorem is similar to the proof of [24, Theorem 2.7.2]. In the rest of this section, $T = (T(t))_{t\in\mathbb{R}_+}$ will be a strongly continuous semigroup of linear bounded operators on the p-Fréchet space $(X, +, \cdot, \|\cdot\|)$, $0 < p < 1$, such that for fixed $x_0 \in X$, the motion $T(t)(x_0) : \mathbb{R}_+ \to X$ is an asymptotically almost automorphic function with principal term f and the corrective term g.

Definition 4.6. A function $\varphi : \mathbb{R} \to X$ is said to be a complete trajectory of T if it satisfies the functional equation $\varphi(t) = T(t-a)(\varphi(a))$, for all $a \in \mathbb{R}, t \geq a$.

Theorem 4.7. *Let $(X, +, \cdot, \| \cdot \|)$ be a p-Fréchet space, where $0 < p < 1$. The principal term f of $T(t)(x_0)$ is a complete trajectory for T.*

Proof. The proof is similar to that of [24, Theorem 2.7.4]. We would only have to consider the limits here with respect to the metric $D(x, y) = \|x - y\|$. □

Definition 4.8. Let $(X, +, \cdot, \| \cdot \|)$ be a p-Fréchet space, where $0 < p < 1$.

The set $\omega^+(x_0) = \{y \in X; \exists \quad 0 \leq t_n \to +\infty, \lim_{n\to+\infty} \|T(t)(x_0) - y\| = 0\}$ is called the ω-limit set of $T(t)(x_0)$.

$\omega_f^+(x_0) = \{y \in X; \exists \quad 0 \leq t_n \to +\infty, \lim_{n\to+\infty} \|f(t_n) - y\| = 0\}$ is called the ω-limit set of f, the principal term of $T(t)(x_0)$.

$\gamma^+(x_0) = \{T(t)(x_0); t \in \mathbb{R}_+\}$ is the trajectory of $T(t)(x_0)$.

A set $B \subseteq X$ is said to be invariant under the semigroup $T = (T(t))_{t\in\mathbb{R}_+}$, if $T(t)(y) \in B, \forall y \in B, \forall t \in \mathbb{R}_+$.

$e \in X$ is called a rest-point for the semigroup T if $T(t)(e) = e, \forall t \geq 0$.

Also, the following properties hold.

Theorem 4.9.

(i) *$\omega^+(x_0)$ is not empty, $\omega^+(x_0) = \omega_f^+(x_0)$, $\omega^+(x_0)$ is invariant under T and is closed in X (with respect to D), $\omega^+(x_0)$ is compact if $\gamma^+(x_0)$ is relatively compact. Also, if x_0 is a rest-point of the semigroup T then $\omega^+(x_0) = \{x_0\}$.*

(ii) *If we denote $\gamma_f(x_0) = \{f(t); t \in \mathbb{R}\}$ then $\gamma_f(x_0)$ is relatively compact (by Theorem 3.3, (v)) and invariant under the semigroup T.*

(iii) *If we denote $\nu(t) = \inf\{\|T(t)(x_0) - y\|; y \in \omega^+(x_0)\}$, then $\lim_{t\to+\infty} \nu(t) = 0$.*

Proof. We can imitate the proofs as in the case of Banach spaces, reasoning with respect to the p-norm instead of the usual norm.

(i) As in the proof of [24, Theorem 2.7.6], from the almost automorphy of f, let $(t_{n_k})_{k\in\mathbb{N}}$ be the sequence satisfying

$$\lim_{k\to+\infty} \|f(t_{n_k}) - g(0)\| = 0.$$

But

$$\|T(t_{n_k})(x_0) - f(t_{n_k})\| = \|(f(t_{n_k}) + g(t_{n_k})) - f(t_{n_k})\| = \|g(t_{n_k})\|,$$

which implies

$$\lim_{k\to+\infty} \|T(t_{n_k})(x_0) - f(t_{n_k})\| = 0.$$

We then immediately get $\lim_{k\to+\infty} \|T(t_{n_k})(x_0) - g(0)\| = 0$, which means that $g(0) \in \omega^+(x_0)$ i.e. $\omega^+(x_0)$ is non empty.

The equality $\omega^+(x_0) = \omega_f^+(x_0)$ follows immediate from $\lim_{t\to+\infty} \|T(t)(x_0) - f(t)\| = 0$, which can be proved as above.

To prove that $\omega^+(x_0)$ is invariant under T, we reason exactly as in the proof of [24, Theorem 2.7.9].

Reasoning as in the proof of [24, Theorem 2.7.10], we immediately obtain that $\omega^+(x_0)$ is closed in X (there only the triangle inequality of $\| \cdot \|$ is used). Arguing exactly as in the proof of [24, Theorem 2.7.11], we get that $\omega^+(x_0)$ is compact if $\gamma^+(x_0)$ is relatively compact. Also, reasoning as in the proof of [24, Theorem 2.7.16], we immediately obtain that $\omega^+(x_0) = \{x_0\}$ for x_0 a rest-point of the semigroup T.

The claims (ii) and (iii) are similar to the proofs of [24, Theorems 2.7.12 and 2.7.13], respectively. □

5. Almost automorphic groups and semigroups on p-Fréchet spaces, $0 < p < 1$

Everywhere in this section, $(X, +, \cdot, \|\cdot\|)$ will be a p-Fréchet space, with $0 < p < 1$. First we recall some concepts and results from [20] concerning B-almost periodic functions with values in p-Fréchet spaces.

Definition 5.1 ([20]). Let $f : \mathbb{R} \to X$ be continuous on $\mathbb{R}$. We say that f is B-almost periodic if: $\forall \epsilon > 0$, $\exists l(\epsilon) > 0$ such that any interval of length $l(\epsilon)$ of the real line contains at least one point ξ with

$$\|f(t+\xi) - f(t)\| < \epsilon, \quad \forall t \in \mathbb{R}.$$

Remarks. (1) A set $E \subset \mathbb{R}$ is called relatively dense (in $\mathbb{R}$), if there exists a number $l > 0$ such that every interval $(a, a+l)$ contains at least one point of E. By using this concept, we can reformulate Definition 5.1 as follows: $f : \mathbb{R} \to X$ is called B-almost periodic if for every $\varepsilon > 0$, there exists a relatively dense set $\{\tau\}_\varepsilon$, such that

$$\sup_{t\in\mathbb{R}} \|f(t+\tau) - f(t)\| \le \varepsilon, \quad \text{for all } \tau \in \{\tau\}_\varepsilon.$$

Also, each $\tau \in \{\tau\}_\varepsilon$ is called ε-almost period of f.

(2) It was proved in [20, Theorem 3.6] that the range of an B-almost periodic function with values in the p-Fréchet space $(X, +, \cdot, \|\cdot\|)$ is relatively compact (r.c. for short) in the complete metric space (X, D), with $D(x, y) = \|x - y\|$.

Similar to the case of Banach spaces, we have developed a theory of Bochner's transform for p-Fréchet spaces (see [20]), as follows.

Let us denote $AP(X) = \{f : \mathbb{R} \to X;\ f \text{ is B-almost periodic}\}$ and for $f \in AP(X)$, let us define $\|f\|_b = \sup\{\|f(t)\|; t \in \mathbb{R}\}$. By [20, Theorem 3.2], we get $\|f\|_b < +\infty$. It follows that $\|\cdot\|_b$ also is a p-norm on the space

$$C_b(\mathbb{R}, X) = \{f : \mathbb{R} \to X;\ \text{is continuous and bounded on } \mathbb{R}\}.$$

In addition, since (X, D) is a complete metric space, by standard reasonings it follows that $C_b(\mathbb{R}, X)$ becomes complete metric space with respect to the metric $D_b(f, g) = \|f - g\|_b$, that is, $(C_b(\mathbb{R}, X), \|\cdot\|_b)$ becomes a p-Fréchet space. Then, the result in [20, Theorems 3.2 and 3.5] shows that $AP(X)$ is a closed subset of $C_b(\mathbb{R}, X)$, that is, $(AP(X), D_b)$ is complete metric space and therefore $(AP(X), \|\cdot\|_b)$ becomes p-Fréchet space.

The Bochner transform on $C_b(\mathbb{R}, X)$ is defined as in the case of Banach spaces, by

$$\tilde{f} : \mathbb{R} \to C_b(\mathbb{R}, X), \tilde{f}(s)(t) = f(t+s),$$

for all $t \in \mathbb{R}$ and we write $\tilde{f} = B(f)$. The properties of Bochner's transform on p-Fréchet spaces, $0 < p < 1$, can be summarized as follows.

Theorem 5.2 ([20, Theorem 3.11]). (i)

$$\|\tilde{f}(s)\|_b = \|f(\cdot + s)\|_b = \|\tilde{f}(0)\|_b, \quad \textit{for all } s \in \mathbb{R};$$

(ii)
$$\|\tilde{f}(s+\tau)-\tilde{f}(s)\|_b = sup\{\|f(t+\tau)-f(t)\|; t \in \mathbb{R}\} = \|\tilde{f}(\tau)-\tilde{f}(0)\|_b, \quad \textit{for all } s,\tau \in \mathbb{R};$$

(iii) *f is B-almost periodic if and only if, $\tilde{f}$ is B-almost periodic, with the same set of ε-almost periods $\{\tau\}_\varepsilon$;*

(iv) *$\tilde{f}$ is B-almost periodic, if and only if there exists a relatively dense sequence in $\mathbb{R}$, denoted by $\{s_n; n \in \mathbb{N}\}$, such that the set of functions $\{\tilde{f}(s_n); n \in \mathbb{N}\}$, is relatively compact in the complete metric space $(C_b(\mathbb{R}, X), D_b)$;*

(v) *$\tilde{f}$ is B-almost periodic, if and only $\tilde{f}(\mathbb{R})$ is relatively compact in the complete metric space $(C_b(\mathbb{R}, X), D_b)$;*

(vi) *(Bochner's criterion) f is B-almost periodic if and only if $\tilde{f}(\mathbb{R})$ is relatively compact in the complete metric space $(C_b(\mathbb{R}, X), D_b)$.*

The above (vi) Bochner's criterion can be restated as follows.

Theorem 5.3 ([20, Theorem 3.7])**.** *A function $f \in C(\mathbb{R}, X)$ is B-almost periodic if and only if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that $(f(t + s_n))$ is uniformly convergent in $t \in \mathbb{R}$.*

Furthermore, we have the following result.

Theorem 5.4 ([20, Theorem 3.12])**.** *Let $f \in C_b(\mathbb{R}, X)$. Let us suppose that there exists a relatively dense set of real numbers (s_n), such that*

(i) *The set $\{f(s_n); n \in \mathbb{N}\}$ is relatively compact in the metric space (X, D) and*

(ii) *for any $n, m \in \mathbb{N}$, the relation*
$$\|f(s_n) - f(s_m)\| \geq c\|f(\cdot + s_n) - f(\cdot + s_m)\|_b,$$
holds with $c > 0$ independent of n, m.

Then, f is B-almost periodic.

It is clear that $AP(X) \subset AA(X)$, and in general, the concepts of B-almost periodicity and almost automorphy are not equivalent. However Theorem 5.4 allows us to prove the equivalence between the B-almost periodicity and almost automorphy of the "orbits" of a group/semigroup. In this sense, we present the following result.

Theorem 5.5. *Let $(T(t))_{t\in\mathbb{R}}$ be a family of uniformly bounded group of bounded linear operators on a p-Fréchet space $(X, +, \cdot, \|\cdot\|)$, $0 < p < 1$ and let $x_0 \in X$ be given. Then the following are equivalent:*

(i) *$t \to T(t)(x_0)$ is almost automorphic;*

(ii) *$t \to T(t)(x_0)$ is B-almost periodic.*

Proof. It suffices to prove that (i) implies (ii). Since $T(t)_{t\in\mathbb{R}}$ is uniformly bounded, there exists $M > 0$ such that $\|T(t)(x_0)\| \leq M\|x_0\|$, for all $t \in \mathbb{R}$. Also, the range $R_{T(t)(x_0)}$ is relatively compact since $T(t)(x_0)$ is almost automorphic as function of t (see Theorem 3.3, (v)). Thus given an r.d. sequence of real numbers (s'_n), we can find a subsequence (s_n) such that $(T(s_n)(x_0))_{n\in\mathbb{N}}$ is Cauchy. Now, in view of the following inequality
$$c\|[T(t + s_n)(x_0) - T(t + s_m)(x_0)]\| \leq \|[T(s_n)(x_0) - T(s_m)(x_0)]\|,$$
for all $t \in \mathbb{R}$, (where $c = \frac{1}{M}$) we conclude that $T(t)(x_0)$ is B-almost periodic by Theorem 5.4. $\square$

We remark that Theorem 5.5 is an extension of a result [9] in Banach spaces to p-Fréchet spaces, $0 < p < 1$.

Definition 5.6. A motion $x \in C(\mathbb{R}, X)$ is said to be strongly stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|x(t_1)-x(t_2)\| < \delta$ implies $\|x(t+t_1)-x(t+t_2)\| < \epsilon$ for all $t \in \mathbb{R}$.

Example 5.7. If $(T(t))_{t\in\mathbb{R}}$ is a family of uniformly bounded group of continuous linear operators on X, then the function $x(t) := T(t)(e)$ for some $e \in X$ is a strongly stable motion in X.

Theorem 5.8. *If $x \in C(\mathbb{R}, X)$ is a strongly stable motion with a relatively compact range in X, then $x \in AP(X)$.*

The proof of the above theorem is a direct consequence of Theorem 5.3. By Definition 3.6 we have introduced the concept of asymptotically almost automorphic function with values in a p-Fréchet space, $0 < p < 1$. In a similar manner, we can introduce the following concept.

Definition 5.9. A function $f \in C([0,\infty), X)$ is said to be asymptotically B-almost periodic if it admits the (unique) decomposition $f = g + h$ where $g \in AP(X)$ and $h \in C([0,\infty), X)$ with $\lim_{t\to\infty} h(t) = 0$. g and h are called principal term and corrective term of f, respectively.

It is clear that if f is asymptotically B-almost periodic, then it is asymptotically almost automorphic. Although the converse is not true in general, we will prove that in the case of uniformly bounded semigroups, the answer is affirmative.

Theorem 5.10. *Let $(T(t))_{t\in\mathbb{R}^+}$ be a family of uniformly bounded semigroup of continuous linear operators on the p-Fréchet space $(X, \|\cdot\|)$. If $t \to T(t)(x_0)$ is asymptotically almost automorphic then it is asymptotically B-almost periodic.*

Proof. Let (s''_n) be a given sequence in $\mathbb{R}^+$. Then we can extract a subsequence (s'_n) such that $(g(s'_n))$ is convergent, where g is the principal term of $T(t)(x_0)$. Since $h \in C([0,\infty), X)$, we can extract a subsequence (s_n) such that $h(s_n)$ is convergent (the situation when $s_n \to +\infty$ is covered by the property of h as the corrective term of $T(t)(x_0)$, i.e $h(s_n) \to 0$). This implies that $(T(s_n)(x_0))$ is convergent. Finally, by recalling Theorem 5.4 and 5.5, the proof is complete. $\square$

In what follows, let us consider the inhomogeneous Cauchy problem

$$\frac{du(t)}{dt} = Au(t) + f(t,u), \ t \geq a \tag{5.1}$$

$$u(a) = u_a \in \mathbb{X}_p, \tag{5.2}$$

where $(\mathbb{X}_p, \|\cdot\|_p)$ is a p-Frechet space, $0 < p < 1$, $A : \mathbb{X}_p \to \mathbb{X}_p$ is linear and continuous and $f : \mathbb{R}\times\mathbb{X}_p \to \mathbb{X}_p$ such that $f(t,x)$ is almost automorphic in t for each $x \in X_p$ and $\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \forall x,y \in X_p$ and $t \geq a$, where L is independent of x, y and t. For instance, $\mathbb{X}_p$ can be any of the examples of spaces in Section 2, see (2.1)–(2.3). Let now $T = \{T(t)\}_{t\geq 0}$ be a family of strongly continuous semigroups on $\mathbb{X}_p$ with generator A. In the case when $\mathbb{X}_p$ $(p \geq 1)$ is a Banach space, it is a well-known fact that the concept of integral of continuous functions defined on compact intervals, with values in $\mathbb{X}_p$, plays a crucial role in defining so-called mild solutions for (5.1)-(5.2). Thus, using standard arguments from fixed point theory, we would be driven to prove that the mild solution of (5.1)-(5.2) is of the form

$$u(t) = T(t)u(a) + \int_a^t T(t-s)f(s,u(s))ds := Su(t). \tag{5.3}$$

Note that the integral in (5.3) is defined as in [2] (cf. also [18]). In other words, we would have to prove that the integral operator Su on the right hand side of (5.3) has a unique fixed point in a suitable space, which will turn out to be the mild solution of (5.1)-(5.2). However, since the fundamental theorem of calculus does not hold in the spaces $\mathbb{X}_p$, $0 < p < 1$ (cf. [18]), first it follows that a differentiable mild solution is not necessarily a solution of (5.1)-(5.2). Also, in general, we do not get the following estimate

$$\Big\| \int_a^t T(t-s)f(s,u(s))ds \Big\|_p \leq \int_a^t \|T(t-s)f(s,u(s))\|_p ds. \tag{5.4}$$

This inequality is essential in proving that the map S above is a contraction on suitable bounded subsets of $C([a,T];\mathbb{X}_p)$. Furthermore, lacking a Leibniz-Newton formula, the indefinite integral of an almost automorphic function is not an almost automorphic function. Consequently, due to the lack of a rich structure of calculus in such non-locally convex spaces, it seems that one cannot hope for an interesting theory with real world applications of semilinear differential equations (with unrestricted or almost automorphic solutions).

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