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A QUASI-BOUNDARY VALUE METHOD FOR REGULARIZING NONLINEAR ILL-POSED PROBLEMS

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Abstract. In this article, a modified quasi-boudary regularization method for solving nonlinear backward heat equation is given. Sharp error estimates for the approximate solutions, and numerical examples to illustrate the effectiveness our method are provided. This work extends to the nonlinear case earlier results by the authors [33, 34] and by Clark and Oppenheimer [6].

1. Introduction

For T be a positive number, we consider the problem of finding a function $u(x,t)$, the temperature, such that

$$u_t - u_{xx} = f(x,t,u(x,t)), \quad (x,t) \in (0,\pi) \times (0,T), \tag{1.1}$$

$$u(0,t) = u(\pi,t) = 0, \quad t \in (0,T), \tag{1.2}$$

$$u(x,T) = g(x), \quad x \in (0,\pi), \tag{1.3}$$

where $g(x), f(x,t,z)$ are given functions. This problem is called backward heat problem, backward Cauchy problem, and final value problem.

As is known, the nonlinear problem is severely ill-posed; i.e., solutions do not always exist, and in the case of existence, these do not depend continuously on the given data. In fact, from small noise contaminated physical measurements, the corresponding solutions have large errors. It makes difficult to numerical calculations. Hence, a regularization is in order. In the mathematical literature various methods have been proposed for solving backward Cauchy problems. We can notably mention the method of quasi-solution (QS-method) by Tikhonov, the method of quasi-reversibility (QR method) by Lattes and Lions, the quasi boundary value method (Q.B.V method) and the C-regularized semigroups technique.

In the method of quasi-reversibility, the main idea consists in replacing operator A by $A_\epsilon = g_\epsilon(A)$, where $A[u]$ is the left-hand side of (1.1). In the original method, Lattes and Lions [16] proposed $g_\epsilon(A) = A - \epsilon A^2$, to obtain well-posed problem in the backward direction. Then, using the information from the solution of the perturbed problem and solving the original problem, we get another well-posed

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problem and this solution sometimes can be taken to be the approximate solution of the ill-posed problem.

Difficulties may arise when using the method quasi-reversibility discussed above. The essential difficulty is that the order of the operator is replaced by an operator of second order, which produces serious difficulties on the numerical implementation, in addition, the error $c(\epsilon)$ introduced by small change in the final value g is of the order $e^{T/\epsilon}$.

In 1983, Showalter [29] presented a method called the quasi-boundary value (QBV) method to regularize that linear homogeneous problem which gave a stability estimate better than the one in the previous method. The main idea of the method is of adding an appropriate "corrector" into the final data. Using this method, Clark and Oppenheimer [6], and Denche-Bessila, [7], regularized the backward problem by replacing the final condition by

$$u(T) + \epsilon u(0) = g$$

and

$$u(T) - \epsilon u'(0) = g$$

respectively.

To the author's knowledge, so far there are many papers on the linear homogeneous case of the backward problem, but we only find a few papers on the nonhomogeneous case, and especially, the nonlinear case of their is very scarce. In [32], we used the Quasi-reversibility method to regularize a 1-D linear nonhomogeneous backward problem. Very recently, in [27], the methods of integral equations and of Fourier transform have been used to solved a 1-D problem in an unbounded region.

For recent articles considering the nonlinear backward-parabolic heat, we refer the reader to [34, 35]. In [33], the authors used the QBV method to regularize the latter problem. However, in [33], the authors showed that the error between the approximate problem and the exact solution is

$$\|u(.,t) - u^\epsilon(.,t)\| \le \sqrt{M} \exp\big(\frac{3k^2T(T-t)}{2}\big)\epsilon^{t/T}.$$

In [35], the error is also of similar form,

$$\|u(t) - u^\epsilon(t)\| \le M\beta(\epsilon)^{t/T}.$$

It is easy to see that two errors above are not near to zero, if ϵ fixed and t tend to zero. Hence, the convergence of the approximate solution is very slow when t is in a neighborhood of zero. Moreover, the regularization error in $t = 0$ is not given.

In the present paper, we shall regularize (1.1)-(1.3) using a modified quasi-boundary method given in [34]. This regularization method is rather simple and convenient for dealing with some ill-posed problems. The nonlinear backward problem is approximated by the following one dimensional problem

$$u_t^\epsilon - u_{xx}^\epsilon = \sum_{k=1}^{\infty} \frac{e^{-Tk^2}}{\epsilon k^2 + e^{-Tk^2}} f_k(u^\epsilon)(t)\sin(kx), \quad (x,t) \in (0,\pi)\times(0,T), \tag{1.4}$$

$$u^\epsilon(0,t) = u^\epsilon(\pi,t) = 0, \quad t \in [0,T], \tag{1.5}$$

$$u^\epsilon(x,T) = \sum_{k=1}^{\infty} \frac{e^{-Tk^2}}{\epsilon k^2 + e^{-Tk^2}} g_k \sin(kx), \quad x \in [0,\pi], \tag{1.6}$$

where $\epsilon \in (0, eT)$,

$$g_k = \frac{2}{\pi}\langle g(x), \sin kx\rangle = \frac{2}{\pi}\int_0^\pi g(x)\sin(kx)dx,$$

$$f_k(u)(t) = \frac{2}{\pi}\langle f(x,t,u(x,t)), \sin kx\rangle = \frac{2}{\pi}\int_0^\pi f(x,t,u(x,t))\sin kx dx$$

and $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(0,\pi)$.

The paper is organized as follows. In Theorem 2.1 and 2.2, we shall show that (1.4)-(1.6) is well-posed and that the unique solution $u^\epsilon(x,t)$ of it satisfies the equality

$$u^\epsilon(x,t) = \sum_{k=1}^\infty \big(\epsilon k^2 + e^{-Tk^2}\big)^{-1}\Big(e^{-tk^2}g_k - \int_t^T e^{(s-t-T)k^2} f_k(u^\epsilon)(s)ds\Big)\sin kx. \tag{1.7}$$

Then, in theorem 2.3 and 2.4, we estimate the error between an exact solution u of (1.1)-(1.3) and the approximation solution u^ϵ of (1.4)-(1.6). In fact, we shall prove that

$$\|u^\epsilon(.,t) - u(.,t)\| \le H\epsilon^{t/T-1}\big(\ln(T/\epsilon)\big)^{\frac{t}{T}-1} \tag{1.8}$$

where $\|\cdot\|$ is the norm of $L^2(0,\pi)$ and H is the term depend on u. Note that the above results are improvements of some results in [27, 32, 33, 34, 35]. In fact, in most of the previous results, the errors often have the form $C\epsilon^{t/T}$. This is one of their disadvantages in which t is zero. It is easy to see that from (1.8), the convergence of the approximate solution at $t = 0$ is also proved. The notation about the usefulness and advantages of this method can be founded in Remark 1 and Remark 2. Finally, a numerical experiment will be given in Section 4, which proves the efficiency of our method.

2. Main results

For clarity of notation, we denote the solution of (1.1)-(1.3) by $u(x,t)$, and the solution of the problem (1.4)-(1.6) by $u^\epsilon(x,t)$. Let ϵ be a positive number such that $0 < \epsilon < eT$.

A function f is called a global Lipchitz function if $f \in L^\infty([0,\pi]\times[0,T]\times R)$ and satisfies

$$|f(x,y,w) - f(x,y,v)| \le L|w-v| \tag{2.1}$$

for a positive constant L independent of x, y, w, v. Throughout this paper, we denote $T_1 = \max\{1, T\}$. The existence and uniqueness of the regularized solution is stated as follows.

Theorem 2.1. *Assume* $0 < \epsilon < eT$ *and* (2.1). *Then* (1.4)-(1.6) *has a unique weak solution* $u^\epsilon \in W = C([0,T]; L^2(0,\pi)) \cap L^2(0,T; H_0^1(0,\pi)) \cap C^1(0,T; H_0^1(0,\pi))$ *satisfying* (1.7).

Regarding the stability of the regularized solution we have the following result.

Theorem 2.2. *Let u and v be two solutions of* (1.4)-(1.6) *corresponding to the final values g and h in* $L^2(0,\pi)$. *Then*

$$\|u(.,t) - v(.,t)\| \le T_1\exp(L^2T_1^2(T-t)^2)\big(\epsilon\ln(T/\epsilon)\big)^{\frac{t-T}{T}}\|g-h\|.$$

We remark that in [1, 8, 9, 18, 32], the magnitude of stability inequality is $e^{T/\epsilon}$. While in [6, 19, 29], it is ϵ^{-1}. In [27, p. 5], in [33, p. 238], and in [35], the stability estimate is of order $\epsilon^{\frac{t}{T}-1}$, which is better than the some previous results.

Since Theorem 2.2 gives a estimate of the stability of order

$$C\epsilon^{\frac{t}{T}-1}\big(\ln(T/\epsilon)\big)^{\frac{t}{T}-1}. \tag{2.2}$$

It is clear that this order of stability is less than the orders given in [27, 33, 35], which is one advantage of our method. Despite the uniqueness, Problem (1.1)-(1.3) is still ill-posed. Hence, we have to resort to a regularization. We have the following result.

Theorem 2.3. *Assume* (2.1). *(a) If* $u(x,t) \in W$ *is a solution of* (1.1)-(1.3) *such that*

$$\int_0^T \sum_{k=1}^{\infty} k^4 e^{2sk^2} f_k^2(u)(s)ds < \infty \tag{2.3}$$

and $\|u_{xx}(.,0)\| < \infty$. *Then*

$$\|u(.,t) - u^\epsilon(.,t)\| \le C_M \epsilon^{t/T}\big(\ln(T/\epsilon)\big)^{\frac{t}{T}-1}.$$

(b) If $u(x,t)$ *satisfies*

$$Q = \sup_{0\le t\le T}\Big(\sum_{k=1}^{\infty} k^4 e^{2tk^2}|\langle u(x,t), \sin kx\rangle|^2\Big) < \infty \tag{2.4}$$

then

$$\|u(.,t) - u^\epsilon(.,t)\| \le C_Q \epsilon^{t/T}\big(\ln(T/\epsilon)\big)^{\frac{t}{T}-1}$$

for every $t \in [0,T]$, *where*

$$M = 3\|u_{xx}(0)\|^2 + \frac{3\pi}{2}T\int_0^T \sum_{k=1}^{\infty} k^4 e^{2sk^2} f_k^2(u)(s))ds,$$

$$C_M = \sqrt{MT_1^2 e^{3L^2TT_1^2(T-t)}}, \quad C_Q = \sqrt{QT_1^2 e^{2L^2TT_1^2(T-t)}}.$$

Remarks. 1. In [33, p. 241] and in [27], the error estimates between the exact solution and the approximate solution is $U(\epsilon,t) = C\epsilon^{t/T}$. So, if the time t is near to the original time $t = 0$, the converges rate is very slowly. Thus,some methods studied in [27, 33] are not useful to derive the error estimations in the case t is in a neighbourhood of zero. To improve this, the convergence rate in the present theorem is in slightly different form than given in [27, 33], defined by $V(\epsilon,t) = D\epsilon^{t/T}\big(\ln(T/\epsilon)\big)^{\frac{t}{T}-1}$. We note that $\lim_{\epsilon\to0}\frac{V(\epsilon,t)}{U(\epsilon,t)} = 0$. Hence, this error is the optimal error estimates which we know. Moreover, we also have $\lim_{\epsilon\to0}\big(\lim_{t\to0}U(\epsilon,t)\big) = C$ and $\lim_{\epsilon\to0}\big(\lim_{t\to0}V(\epsilon,t)\big) = \lim_{\epsilon\to0}\big(D\frac{1}{\ln(T/\epsilon)}\big) = 0$. This also proves that our method give a better approximation than the previous case which we know. Comparing (2.3) with the results obtained in [33, 35], we realize this estimate is sharp and the best known estimate. This is generalization of many previous results in [1, 2, 6, 7, 8, 9, 17, 18, 19, 27, 29, 30, 31, 33, 35].

2. One superficial advantage of this method is that there is an error estimation in the time $t = 0$, which does not appear in many recently known results in [27, 33, 35]. We have the following estimate

$$\|u(.,0) - u^\epsilon(.,0)\| \le \frac{H}{\ln(T/\epsilon)}.$$

where H is a term depending only on u. These estimates, as noted above, are very seldom in the theory of ill-posed problems.

3. In the linear nonhomogeneous case $f(x,t,u) = f(x,t)$, the error estimates were given in [34]. And the assumption of f in (2.3) is not used. It is only in $L^2(0,T;L^2(0,\pi))$.

4. In theorem 2.3(a), we ask for the condition on the expansion coefficient f_k. We note that the solution u depend on the nonlinear term f and therefore $f_k, f_k(u)$ is very difficult to be valued. Such a obscurity makes this Theorem hard to be used for numerical computations. To improve this, in Theorem 2.3(b), we require the assumption of u, not to depend on the function $f(u)$. In fact, we note that in the simple case of the right-hand side $f(u) = 0$, the term Q becomes

$$\sum_{k=1}^{\infty} k^4 e^{2tk^2} |\langle u(x,t), \sin kx\rangle|^2 = \|u_{xx}(.,0)\|.$$

So, the condition (2.4) is acceptable.

In the case of non-exact data, one has the following result.

Theorem 2.4. *Let the exact solution u of* (1.1)-(1.3) *corresponding to g. Let g_ϵ be a measured data such that $\|g_\epsilon - g\| \le \epsilon$. Then there exists a function w^ϵ satisfying:*
(a) for every $t \in [0,T]$,

$$\|w^\epsilon(.,t) - u(.,t)\| \le T_1(1+\sqrt{M}) \exp\Big(\frac{3L^2TT_1^2(T-t)}{2}\Big)\epsilon^{t/T}\big(\ln(T/\epsilon)\big)^{\frac{t}{T}-1},$$

where u is defined in Theorem 2.3(a).

(b) for every $t \in [0,T]$,

$$\|w^\epsilon(.,t) - u(.,t)\| \le T_1(1+\sqrt{Q}) \exp(L^2TT_1^2(T-t))\epsilon^{t/T}\big(\ln(T/\epsilon)\big)^{\frac{t}{T}-1},$$

where u is defined in Theorem 2.3(b), and M, Q is defined in Theorem 2.3.

3. Proof of the Main Theorems

First we give some assumptions and lemmas which will be useful in proving the main Theorems.

Lemma 3.1. *For $0 < \epsilon < eT$, denote $h(x) = \frac{1}{\epsilon x + e^{-xT}}$. Then it follows that*

$$h(x) \le \frac{T}{\epsilon\big(1+\ln(T/\epsilon)\big)} \le \frac{T}{\epsilon \ln(T/\epsilon)}.$$

The proof of the above lemma can be found in [34]. For $0 \le t \le s \le T$, denote

$$G_\epsilon(s,t,k) = \frac{e^{(s-t-T)k^2}}{\epsilon k^2 + e^{-Tk^2}}, \quad G_\epsilon(T,t,k) = \frac{e^{-tk^2}}{\epsilon k^2 + e^{-Tk^2}}, \tag{3.1}$$

and $T_1 = \max\{1, T\}$.

Lemma 3.2.

$$G_\epsilon(s,t,k) \le T_1\big(\epsilon \ln(T/\epsilon)\big)^{\frac{t-s}{T}}.$$

Proof. We have

$$G_\epsilon(s,t,k) = \frac{e^{(s-t-T)k^2}}{\epsilon k^2 + e^{-Tk^2}} = \frac{e^{(s-t-T)k^2}}{(\epsilon k^2 + e^{-Tk^2})^{\frac{s-t}{T}}(\epsilon k^2 + e^{-Tk^2})^{\frac{T+t-s}{T}}}$$

$$
\begin{aligned}
&\le \frac{e^{(s-t-T)k^2}}{(e^{-Tk^2})^{\frac{T+t-s}{T}}} \frac{1}{(\epsilon k^2 + e^{-Tk^2})^{\frac{s}{T}-\frac{t}{T}}} \\
&\le \Big(\frac{T}{\epsilon \ln(T/\epsilon)}\Big)^{\frac{s}{T}-\frac{t}{T}} \\
&= T^{\frac{s-t}{T}} \epsilon^{\frac{t-s}{T}} \big(\ln(T/\epsilon)\big)^{\frac{t-s}{T}} \\
&\le \max\{1,T\}\big(\epsilon \ln(T/\epsilon)\big)^{\frac{t-s}{T}}.
\end{aligned}
$$

□

Lemma 3.3. *Let $s = T$ in Lemma 3.2, to obtain*

$$
G_\epsilon(T,t,k) \le T_1\Big(\epsilon \ln(T/\epsilon)\Big)^{\frac{t-T}{T}}. \tag{3.2}
$$

Proof of Theorem 2.1. **Step 1.** Existence and uniqueness of a solution of the integral equation (1.7). Put

$$
F(w)(x,t) = P(x,t) - \sum_{k=1}^{\infty} \int_t^T G_\epsilon(s,t,k) f_k(w)(s)\,ds \sin(kx)
$$

for $w \in C([0,T];L^2(0,\pi))$, where

$$
P(x,t) = \sum_{k=1}^{\infty} G_\epsilon(T,t,k)\langle g(x), \sin kx\rangle \sin kx.
$$

Note that by Lemma 3.2, we have

$$
\begin{aligned}
G_\epsilon(s,t,k) &\le \max\{1,T\}\big(\epsilon \ln(T/\epsilon)\big)^{\frac{t-s}{T}} \\
&\le \max\{1,T\}\frac{1}{\epsilon} \\
&= \max\{\frac{1}{\epsilon}, \frac{T}{\epsilon}\} = B_\epsilon.
\end{aligned} \tag{3.3}
$$

We claim that, for every $w, v \in C([0,T];L^2(0,\pi)), p \ge 1$, we have

$$
\|F^p(w)(.,t) - F^p(v)(.,t)\|^2 \le (LB_\epsilon)^{2p}\frac{(T-t)^p C^p}{p!}|||w-v|||^2, \tag{3.4}
$$

where $C = \max\{T,1\}$ and $|||.|||$ is supremum norm in $C([0,T];L^2(0,\pi))$. We shall prove this inequality by induction. For $p = 1$, and using Lemma 3.2, we have

$$
\begin{aligned}
&\|F(w)(.,t) - F(v)(.,t)\|^2 \\
&= \frac{\pi}{2}\sum_{k=1}^{\infty}\Big[\int_t^T G_\epsilon(s,t,k)\left(f_k(w)(s) - f_k(v)(s)\right) ds\Big]^2 \\
&\le \frac{\pi}{2}\sum_{k=1}^{\infty}\int_t^T \Big(\frac{e^{(s-t-T)k^2}}{\epsilon k^2 + e^{-Tk^2}}\Big)^2 ds \int_t^T \left(f_k(w)(s) - f_k(v)(s)\right)^2 ds \\
&\le \frac{\pi}{2}\sum_{k=1}^{\infty} B_\epsilon^2 (T-t)\int_t^T \left(f_k(w)(s) - f_k(v)(s)\right)^2 ds \\
&= \frac{\pi}{2}B_\epsilon^2 (T-t)\int_t^T \sum_{k=1}^{\infty}\left(f_k(w)(s) - f_k(v)(s)\right)^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&= B_\epsilon^2(T-t)\int_t^T\int_0^\pi \left(f(x,s,w(x,s)) - f(x,s,v(x,s))\right)^2 dxds \\
&\le L^2 B_\epsilon^2(T-t)\int_t^T\int_0^\pi |w(x,s)-v(x,s)|^2 dxds \\
&= CL^2B_\epsilon^2(T-t)|||w-v|||^2.
\end{aligned}
$$

Thus (3.4) holds. Suppose that (3.4) holds for $p=m$. We prove that (3.4) holds for $p=m+1$. We have

$$
\begin{aligned}
&\|F^{m+1}(w)(.,t) - F^{m+1}(v)(.,t)\|^2 \\
&= \frac{\pi}{2}\sum_{k=1}^\infty \Big[\int_t^T G_\epsilon(s,t,k)\left(f_k(F^m(w))(s) - f_k(F^m(v))(s)\right) ds\Big]^2 \\
&\le \frac{\pi}{2}B_\epsilon^2\sum_{k=1}^\infty \Big[\int_t^T |f_k(F^m(w))(s) - f_k(F^m(v))(s)| ds\Big]^2 \\
&\le \frac{\pi}{2}B_\epsilon^2(T-t)\int_t^T\sum_{k=1}^\infty |f_k(F^m(w))(s) - f_k(F^m(v))(s)|^2 ds \\
&\le B_\epsilon^2(T-t)\int_t^T \|f(.,s,F^m(w)(.,s)) - f(.,s,F^m(v)(.,s))\|^2 ds \\
&\le B_\epsilon^2(T-t)L^2\int_t^T \|F^m(w)(.,s) - F^m(v)(.,s)\|^2 ds \\
&\le B_\epsilon^2(T-t)L^{2m+2}B_\epsilon^{2m}\int_t^T \frac{(T-s)^m}{m!} dsC^m|||w-v|||^2 \\
&\le (LB_\epsilon)^{2m+2}\frac{(T-t)^{m+1}}{(m+1)!}C^{m+1}|||w-v|||^2.
\end{aligned}
$$

Therefore, by the induction principle, we have

$$
|||F^p(w) - F^p(v)||| \le (LB_\epsilon)^p\frac{T^{p/2}}{\sqrt{p!}}C^{p/2}|||w-v|||
$$

for all $w, v \in C([0,T]; L^2(0,\pi))$.

We consider $F : C([0,T]; L^2(0,\pi)) \to C([0,T]; L^2(0,\pi))$. Since

$$
\lim_{p\to\infty}(LB_\epsilon)^p\frac{T^{p/2}C^{p/2}}{\sqrt{p!}} = 0,
$$

there exists a positive integer number p_0 such that

$$
(LB_\epsilon)^{p_0}\frac{T^{p_0/2}C^{p_0/2}}{\sqrt{(p_0)!}} < 1,
$$

and F^{p_0} is a contraction. It follows that the equation $F^{p_0}(w) = w$ has a unique solution $u^\epsilon \in C([0,T]; L^2(0,\pi))$.

We claim that $F(u^\epsilon) = u^\epsilon$. In fact, one has $F(F^{p_0}(u^\epsilon)) = F(u^\epsilon)$. Hence $F^{p_0}(F(u^\epsilon)) = F(u^\epsilon)$. By the uniqueness of the fixed point of F^{p_0}, one has $F(u^\epsilon) = u^\epsilon$; i.e., the equation $F(w) = w$ has a unique solution $u^\epsilon \in C([0,T]; L^2(0,\pi))$.

Step 2. If $u^\epsilon \in W$ satisfies (1.7) then u^ϵ is solution of (1.4)-(1.6). For $0 \le t \le T$, we have

$$u^\epsilon(x,t) = \sum_{k=1}^{\infty} \Big(\epsilon k^2 + e^{-Tk^2}\Big)^{-1} \Big(e^{-tk^2} g_k - \int_t^T e^{(s-t-T)k^2} f_k(u^\epsilon)(s)ds\Big) \sin kx,$$

We can verify directly that

$$u^\epsilon \in C([0,T]; L^2(0,\pi) \cap C^1((0,T); H_0^1(0,\pi)) \cap L^2(0,T; H_0^1(0,\pi))).$$

In fact, $u^\epsilon \in C^\infty((0,T]; H_0^1(0,\pi)))$. Moreover, by direct computation, one has

$$\begin{aligned}
& u_t^\epsilon(x,t) \\
&= \sum_{k=1}^{\infty} -k^2 \big(\epsilon k^2 + e^{-Tk^2}\big)^{-1} \Big(e^{-tk^2} g_k - \int_t^T e^{(s-t-T)k^2} f_k(u^\epsilon)(s)ds\Big) \sin kx \\
&\quad + \sum_{k=1}^{\infty} e^{-Tk^2} (\epsilon k^2 + e^{-Tk^2})^{-1} f_k(u^\epsilon)(t) \sin kx \\
&= -\frac{2}{\pi} \sum_{k=1}^{\infty} k^2 \langle u^\epsilon(x,t), \sin kx\rangle \sin kx + \sum_{k=1}^{\infty} e^{-Tk^2} (\epsilon k^2 + e^{-Tk^2})^{-1} f_k(u^\epsilon)(t) \sin kx \\
&= u_{xx}^\epsilon(x,t) + \sum_{k=1}^{\infty} e^{-Tk^2} (\epsilon k^2 + e^{-Tk^2})^{-1} f_k(u^\epsilon)(t) \sin kx
\end{aligned}$$

and

$$u^\epsilon(x,T) = \sum_{k=1}^{\infty} e^{-Tk^2} (\epsilon k^2 + e^{-Tk^2})^{-1} g_k \sin(kx). \tag{3.5}$$

So u^ϵ is the solution of (1.4)-(1.6).

Step 3. The problem (1.4)-(1.6) has at most one (weak) solution $u^\epsilon \in W$. In fact, let u^ϵ and v^ϵ be two solutions of (1.4)-(1.6) such that $u^\epsilon, v^\epsilon \in W$. Putting $w^\epsilon(x,t) = u^\epsilon(x,t) - v^\epsilon(x,t)$, then w^ϵ satisfies

$$w_t^\epsilon - w_{xx}^\epsilon = \sum_{k=1}^{\infty} e^{-Tk^2} (\epsilon k^2 + e^{-Tk^2})^{-1} \left(f_k(u^\epsilon)(t) - f_k(v^\epsilon)(t)\right) \sin(kx).$$

It follows that

$$\begin{aligned}
\|w_t^\epsilon - w_{xx}^\epsilon\|^2 &\le \frac{1}{\epsilon^2} \sum_{k=1}^{\infty} \left(f_k(u^\epsilon)(t) - f_k(v^\epsilon)(t)\right)^2 \\
&\le \frac{1}{\epsilon^2} \|f(.,t,u^\epsilon(.,t) - f(.,t,v^\epsilon(.,t))\|^2 \\
&\le \frac{L^2}{\epsilon^2} \|u^\epsilon(.,t) - v^\epsilon(.,t)\|^2 \\
&= \frac{L^2}{\epsilon^2} \|w^\epsilon(.,t)\|^2.
\end{aligned}$$

Using a result in Lees-Protter [17], we get $w^\epsilon(.,t) = 0$. This completes proof of Step 3. Combining three Step 1,2,3, we complete the proof of Theorem 2.1. $\square$

Proof of Theorem 2.2. From (1.7) one has in view of the inequality $(a+b)^2 \le 2(a^2+b^2)$,

$$
\begin{aligned}
&\|u(.,t) - v(.,t)\|^2 \\
&= \frac{\pi}{2}\sum_{k=1}^{\infty} \Big| G_\epsilon(T,t,k)(g_k - h_k) - \int_t^T G_\epsilon(s,t,k)(f_k(u)(s) - f_k(v)(s)ds) \Big|^2 \\
&\le \pi \sum_{k=1}^{\infty} (G_\epsilon(T,t,k)|g_k - h_k|)^2 + \pi \sum_{k=1}^{\infty} (\int_t^T G_\epsilon(s,t,k)|f_k(u)(s) - f_k(v)(s)|ds)^2.
\end{aligned} \tag{3.6}
$$

Combining Lemma 3.2, Lemma 3.3 and (3.6), we get

$$
\begin{aligned}
&\|u(.,t) - v(.,t)\|^2 \\
&\le \max\{1,T^2\}\big(\epsilon \ln(T/\epsilon)\big)^{\frac{2t-2T}{T}} \|g-h\|^2 \\
&\quad + 2L^2(T-t)\max\{1,T^2\}\big(\epsilon \ln(T/\epsilon)\big)^{\frac{2t}{T}} \int_t^T \big(\epsilon \ln(T/\epsilon)\big)^{\frac{-2s}{T}} \|u(.,s) - v(.,s)\|^2 ds.
\end{aligned} \tag{3.7}
$$

It follows that

$$
\begin{aligned}
&\big(\epsilon \ln(T/\epsilon)\big)^{\frac{-2t}{T}} \|u(.,t) - v(.,t)\|^2 \\
&\le \max\{1,T^2\}\big(\epsilon \ln(T/\epsilon)\big)^{-2} \|g-h\|^2 \\
&\quad + 2\max\{1,T^2\}L^2(T-t) \int_t^T \big(\epsilon \ln(T/\epsilon)\big)^{\frac{-2s}{T}} \|u(.,s) - v(.,s)\|^2 ds.
\end{aligned}
$$

Using Gronwall's inequality we have

$$
\|u(.,t) - v(.,t)\| \le T_1 \exp(L^2 T_1^2 (T-t)^2)\big(\epsilon \ln(T/\epsilon)\big)^{\frac{t-T}{T}} \|g-h\|.
$$

This completes the proof. □

Proof of Theorem 2.3. Part (a): Suppose the Problem (1.1)-(1.3) has an exact solution u, then u can be rewritten as

$$
u(x,t) = \sum_{k=1}^{\infty} (e^{-(t-T)k^2} g_k - \int_t^T e^{-(t-s)k^2} f_k(u)(s)ds) \sin kx. \tag{3.8}
$$

Since

$$
u_k(0) = e^{Tk^2} g_k - \int_0^T e^{sk^2} f_k(u)(s)ds,
$$

implies

$$
g_k = e^{-Tk^2} u_k(0) + \int_0^T e^{(s-T)k^2} f_k(u)(s)ds,
$$

we get

$$
\begin{aligned}
u(x,T) &= \sum_{k=1}^{\infty} g_k \sin kx \\
&= \sum_{k=1}^{\infty} (e^{-Tk^2} u_k(0) + \int_0^T e^{-(T-s)k^2} f_k(u)(s)ds) \sin kx.
\end{aligned}
$$

From (1.7) and (3.8), we have

$$u_k^\epsilon(t)=\big(\epsilon k^2+e^{-Tk^2}\big)^{-1}\Big(e^{-tk^2}g_k-\int_t^T e^{(s-t-T)k^2}f_k(u^\epsilon)(s)ds\Big) \tag{3.9}$$

$$u_k(t)=e^{Tk^2}\Big(e^{-tk^2}g_k-\int_t^T e^{(s-t-T)k^2}f_k(u)(s)ds\Big). \tag{3.10}$$

From (3.1), (3.9) and (3.10), we have

$$\begin{aligned}
u_k(t)-u_k^\epsilon(t)&=\Big(e^{Tk^2}-\frac{1}{\epsilon k^2+e^{-Tk^2}}\Big)\Big(e^{-tk^2}g_k-\int_t^T e^{(s-t-T)k^2}f_k(u)(s)ds\Big)\\
&\quad+\int_t^T G_\epsilon(s,t,k)\left(f_k(u^\epsilon)(s)-f_k(u)(s)\right)ds\\
&=\frac{\epsilon k^2e^{-tk^2}}{\epsilon k^2+e^{-Tk^2}}\Big(e^{Tk^2}g_k-\int_t^T e^{sk^2}f_k(u)(s)ds\Big)\\
&\quad+\int_t^T G_\epsilon(s,t,k)(f_k(u^\epsilon)(s)-f_k(u)(s)ds.
\end{aligned}$$

From (3.2) and

$$T_1\big(\epsilon\ln(T/\epsilon)\big)^{\frac{t-T}{T}}.\big(\epsilon\ln(T/\epsilon)\big)^{1-\frac{s}{T}}=T_1\big(\epsilon\ln(T/\epsilon)\big)^{\frac{t-s}{T}}$$

we have

$$\begin{aligned}
&|u_k(t)-u_k^\epsilon(t)|\\
&\le\Big|\epsilon G_\epsilon(T,t,k)\Big(k^2e^{Tk^2}g_k-\int_0^T k^2e^{sk^2}f_k(u)(s)ds\Big)\Big|\\
&\quad+\epsilon G_\epsilon(T,t,k)\Big|\int_0^t k^2e^{sk^2}f_k(u)(s)ds\Big|+\int_t^T G_\epsilon(s,t,k)|f_k(u)(s)-f_k(u^\epsilon)(s)|ds\\
&\le\epsilon T_1\big(\epsilon\ln(T/\epsilon)\big)^{\frac{t-T}{T}}\Big(|k^2u_k(0)|+\int_0^t\big|k^2e^{sk^2}f_k(u)(s)\big|ds\Big)\\
&\quad+\int_t^T T_1\big(\epsilon\ln(T/\epsilon)\big)^{\frac{t-s}{T}}|f_k(u)(s)-f_k(u^\epsilon)(s)|ds\\
&=\epsilon T_1\big(\epsilon\ln(T/\epsilon)\big)^{\frac{t-T}{T}}\Big(|k^2u_k(0)|+\int_0^T\big|k^2e^{sk^2}f_k(u)(s)\big|ds\Big)\\
&\quad+\epsilon T_1\big(\epsilon\ln(T/\epsilon)\big)^{\frac{t-T}{T}}\int_t^T\epsilon^{-\frac{s}{T}}\big(\ln(T/\epsilon)\big)^{1-\frac{s}{T}}|f_k(u)(s)-f_k(u^\epsilon)(s)|ds.
\end{aligned}$$

Applying the inequality $(a+b+c)^2\le3(a^2+b^2+c^2)$, we get

$$\begin{aligned}
&\|u(.,t)-u^\epsilon(.,t)\|^2\\
&=\frac{\pi}{2}T_1^2\sum_{k=1}^\infty|u_k(t)-u_k^\epsilon(t)|^2\\
&\le\frac{3\pi}{2}T_1^2\sum_{k=1}^\infty\epsilon^2\big(\epsilon\ln(T/\epsilon)\big)^{\frac{2t-2T}{T}}|k^2u_k(0)|^2+\frac{3\pi}{2}T_1^2\sum_{k=1}^\infty\epsilon^2\big(\epsilon\ln(T/\epsilon)\big)^{\frac{2t-2T}{T}}\\
&\quad\times\Big(\int_0^T|k^2e^{sk^2}f_k(u)(s)|ds\Big)^2+\frac{3\pi}{2}T_1^2\sum_{k=1}^\infty\epsilon^2\big(\epsilon\ln(T/\epsilon)\big)^{\frac{2t-2T}{T}}
\end{aligned}$$

$$
\begin{aligned}
&\times \Big(\int_t^T \epsilon^{-\frac{s}{T}}\big(\ln(T/\epsilon)\big)^{1-\frac{s}{T}}|f_k(u)(s)-f_k(u^\epsilon)(s)|ds\Big)^2\\
&= T_1^2(I_1+I_2+I_3),
\end{aligned}
$$

where

$$
I_1 = \frac{3\pi}{2}\sum_{k=1}^{\infty}\epsilon^2\big(\epsilon\ln(T/\epsilon)\big)^{\frac{2t-2T}{T}}|k^2u_k(0)|^2,
$$

$$
I_2 = \frac{3\pi}{2}\sum_{k=1}^{\infty}\epsilon^2\big(\epsilon\ln(T/\epsilon)\big)^{\frac{2t-2T}{T}}\Big(\int_0^T|k^2e^{sk^2}f_k(u)(s)|ds\Big)^2,
$$

$$
I_3 = \frac{3\pi}{2}\sum_{k=1}^{\infty}\epsilon^2\big(\epsilon\ln(T/\epsilon)\big)^{\frac{2t-2T}{T}}\Big(\int_t^T\epsilon^{-\frac{s}{T}}\big(\ln(T/\epsilon)\big)^{1-\frac{s}{T}}|f_k(u)(s)-f_k(u^\epsilon)(s)|ds\Big)^2.
$$

The terms I_1, I_2, I_3 can be estimated as follows:

$$
\begin{aligned}
I_1 &\le 3\epsilon^2\big(\epsilon\ln(T/\epsilon)\big)^{\frac{2t-2T}{T}}\|u_{xx}(0)\|^2\\
&\le 3\epsilon^{\frac{2t}{T}}\big(\ln(T/\epsilon)\big)^{\frac{2t}{T}-2}\|u_{xx}(0)\|^2.
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
I_2 &\le \frac{3\pi}{2}T\epsilon^2\big(\epsilon\ln(T/\epsilon)\big)^{\frac{2t-2T}{T}}\int_0^T\sum_{k=1}^{\infty}\Big(k^2e^{sk^2}f_k(u)(s)\Big)^2ds\\
&\le \frac{3\pi}{2}T\epsilon^2\big(\epsilon\ln(T/\epsilon)\big)^{\frac{2t-2T}{T}}\int_0^T\sum_{k=1}^{\infty}k^4e^{2sk^2}f_k^2(u)(s)ds.\\
&\le \frac{3\pi}{2}T\epsilon^2\big(\epsilon\ln(T/\epsilon)\big)^{\frac{2t-2T}{T}}\int_0^T\sum_{k=1}^{\infty}k^4e^{2sk^2}f_k^2(u)(s)ds.
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
I_3 &\le \frac{3\pi}{2}(T-t)\epsilon^2\big(\epsilon\ln(T/\epsilon)\big)^{\frac{2t-2T}{T}}\int_t^T\epsilon^{-\frac{2s}{T}}\big(\ln(T/\epsilon)\big)^{2-\frac{2s}{T}}\\
&\quad\times\sum_{k=1}^{\infty}(f_k(u)(s)-f_k(u^\epsilon)(s))^2ds\\
&\le 3(T-t)\epsilon^2\big(\epsilon\ln(T/\epsilon)\big)^{\frac{2t-2T}{T}}\int_t^T\epsilon^{-\frac{2s}{T}}\big(\ln(T/\epsilon)\big)^{2-\frac{2s}{T}}\\
&\quad\times\|f(.,s,u(.,s))-f(.,s,u^\epsilon(.,s))\|^2ds\\
&\le 3L^2T\epsilon^{\frac{2t}{T}}\big(\ln(T/\epsilon)\big)^{\frac{2t}{T}-2}\int_t^T\epsilon^{-\frac{2s}{T}}\big(\ln(T/\epsilon)\big)^{2-\frac{2s}{T}}\|u(.,s)-u^\epsilon(.,s)\|^2ds.
\end{aligned}
\tag{3.13}
$$

Combining (3.11), (3.12), (3.13), we obtain

$$
\begin{aligned}
&\|u(.,t)-u^\epsilon(.,t)\|^2\\
&\le T_1^2\epsilon^{\frac{2t}{T}}\big(\ln(T/\epsilon)\big)^{\frac{2t}{T}-2}\Big(3\|u_{xx}(0)\|^2+\frac{3\pi}{2}T\int_0^T\sum_{k=1}^{\infty}k^4e^{2sk^2}f_k^2(u)(s)ds\Big)\\
&\quad+T_1^2 3L^2T\epsilon^{\frac{2t}{T}}\big(\ln(T/\epsilon)\big)^{\frac{2t}{T}-2}\int_t^T\epsilon^{-\frac{2s}{T}}\big(\ln(T/\epsilon)\big)^{2-\frac{2s}{T}}\|u(.,s)-u^\epsilon(.,s)\|^2ds.
\end{aligned}
$$

It follows that

$$
\epsilon^{\frac{-2t}{T}}\big(\ln(T/\epsilon)\big)^{2-\frac{2t}{T}}\|u(.,t)-u^\epsilon(.,t)\|^2
$$

$$\leq MT_1^2 + 3L^2TT_1^2 \int_t^T \epsilon^{-\frac{2s}{T}} \big(\ln(T/\epsilon)\big)^{2-\frac{2s}{T}} \|u(.,s) - u^\epsilon(.,s)\|^2 ds.$$

Using Gronwall's inequality, we obtain

$$\epsilon^{\frac{-2t}{T}} \big(\ln(T/\epsilon)\big)^{2-\frac{2t}{T}} \|u(.,t) - u^\epsilon(.,t)\|^2 \leq MT_1^2 e^{3L^2TT_1^2(T-t)}.$$

So that

$$\|u(.,t) - u^\epsilon(.,t)\|^2 \leq MT_1^2 e^{3L^2TT_1^2(T-t)} \epsilon^{\frac{2t}{T}} \big(\ln(T/\epsilon)\big)^{\frac{2t}{T}-2}.$$

This completes the proof part (a) in Theorem 2.3.

Proof of part (b) in Theorem 2.3. From (3.7), we have

$$\begin{aligned}
&|u_k(t) - u_k^\epsilon(t)| \\
&\leq \Big|\Big(e^{Tk^2} - \frac{1}{\epsilon k^2 + e^{-Tk^2}}\Big)\Big(e^{-tk^2} g_k - \int_t^T e^{(s-t-T)k^2} f_k(u)(s) ds\Big)\Big| \\
&\quad + \Big|\int_t^T G_\epsilon(s,t,k)(f_k(u^\epsilon)(s) - f_k(u)(s)) ds\Big| \\
&\leq \Big|\frac{\epsilon k^2 e^{-tk^2}}{\epsilon k^2 + e^{-Tk^2}}\Big(e^{Tk^2} g_k - \int_t^T e^{sk^2} f_k(u)(s) ds\Big)\Big| \\
&\quad + \int_t^T G_\epsilon(s,t,k)|f_k(u^\epsilon)(s) - f_k(u)(s)| ds \\
&\leq \Big|\frac{\epsilon e^{-tk^2}}{\epsilon k^2 + e^{-Tk^2}} k^2 e^{tk^2} u_k(t)\Big| + \int_t^T G_\epsilon(s,t,k)|f_k(u)(s) - f_k(u^\epsilon)(s)| ds \\
&\leq \epsilon T_1 \big(\epsilon \ln(T/\epsilon)\big)^{\frac{t-T}{T}} |k^2 e^{tk^2} u_k(t)| + \int_t^T T_1 \big(\epsilon \ln(T/\epsilon)\big)^{\frac{t-s}{T}} |f_k(u)(s) - f_k(u^\epsilon)(s)| ds.
\end{aligned}$$

This implies

$$\begin{aligned}
&\|u(.,t) - u^\epsilon(.,t)\|^2 \\
&= \frac{\pi}{2} \sum_{k=1}^\infty |u_k(t) - u_k^\epsilon(t)|^2 \\
&\leq \pi \sum_{k=1}^\infty \epsilon^2 . T_1^2 \big(\epsilon \ln(T/\epsilon)\big)^{\frac{2t-2T}{T}} |k^2 e^{tk^2} u_k(t)|^2 \\
&\quad + \pi \sum_{k=1}^\infty \epsilon^2 . T_1^2 \big(\epsilon \ln(T/\epsilon)\big)^{\frac{2t-2T}{T}} \Big(\int_t^T \epsilon^{-\frac{s}{T}} \big(\ln(T/\epsilon)\big)^{1-\frac{s}{T}} |f_k(u)(s) - f_k(u^\epsilon)(s)| ds\Big)^2.
\end{aligned}$$

This implies

$$\begin{aligned}
&\|u(.,t) - u^\epsilon(.,t)\|^2 \\
&\leq T_1^2 \epsilon^{\frac{2t}{T}} \big(\ln(T/\epsilon)\big)^{\frac{2t}{T}-2} \sum_{k=1}^\infty k^4 e^{2tk^2} u_k^2(t) \\
&\quad + 2L^2TT_1^2 \epsilon^{\frac{2t}{T}} \big(\ln(T/\epsilon)\big)^{\frac{2t}{T}-2} \int_t^T \epsilon^{\frac{-2s}{T}} \big(\ln(T/\epsilon)\big)^{2-\frac{2s}{T}} \|u(.,s) - u^\epsilon(.,s)\|^2 ds.
\end{aligned}$$

Using again Gronwall's inequality,

$$\epsilon^{\frac{-2t}{T}} \big(\ln(T/\epsilon)\big)^{2-\frac{2t}{T}} \|u(.,t) - u^\epsilon(.,t)\|^2 \leq Qe^{2L^2TT_1^2(T-t)}.$$

This completes the proof. ☐

Proof of Theorem 2.4. Let u^ϵ be the solution of (1.4)-(1.6) corresponding to g. Recall that w^ϵ be the solution of (1.4)-(1.6) corresponding to g_ϵ.

Part (a) of Theorem 2.4: Using Theorem 2.2 and Theorem 2.3(a) , we have

$$\begin{aligned}
\|w^\epsilon(.,t)-u(.,t)\| &\le \|w^\epsilon(.,t)-u^\epsilon(.,t)\|+\|u^\epsilon(.,t)-u(.,t)\| \\
&\le T_1\exp(L^2T_1^2(T-t)^2)\big(\epsilon\ln(T/\epsilon)\big)^{\frac{t-T}{T}}\|g_\epsilon-g\| \\
&\quad+\sqrt{MT_1^2e^{3L^2TT_1^2(T-t)}}\epsilon^{t/T}\big(\ln(T/\epsilon)\big)^{\frac{t}{T}-1} \\
&\le T_1(1+\sqrt{M})\exp\big(\frac{3L^2TT_1^2(T-t)}{2}\big)\epsilon^{t/T}\big(\ln(T/\epsilon)\big)^{\frac{t}{T}-1},
\end{aligned}$$

for every $t\in[0,T]$. The proof of part (b) Theorem 2.4 is similar to part (a) and it is omitted. □

4. Numerical experiments

We consider the equation

$$-u_{xx}+u_t=f(u)+g(x,t)$$

where

$$f(u)=u^4,\quad g(x,t)=2e^t\sin x-e^{4t}\sin^4 x,\ u(x,1)=\varphi_0(x)\equiv e\sin x.$$

The exact solution of this equation is $u(x,t)=e^t\sin x$. In particular,

$$u\big(x,\frac{99}{100}\big)\equiv u(x)=\exp\big(\frac{99}{100}\big)\sin x.$$

Let $\varphi_\epsilon(x)\equiv\varphi(x)=(\epsilon+1)e\sin x$. We have

$$\|\varphi_\epsilon-\varphi\|_2=\Big(\int_0^\pi\epsilon^2e^2\sin^2 x dx\Big)^{1/2}=\epsilon e\sqrt{\pi/2}.$$

We find the regularized solution $u_\epsilon(x,\frac{99}{100})\equiv u_\epsilon(x)$ having the form

$$u_\epsilon(x)=v_m(x)=w_{1,m}\sin x+w_{2,m}\sin 2x+w_{3,m}\sin 3x,$$

where $v_1(x)=(\epsilon+1)e\sin x$, $w_{1,1}=(\epsilon+1)e$, $w_{2,1}=0$, $w_{3,1}=0$, $a=\frac{1}{10000}$, $t_m=1-am$, for $m=1,2,\dots,100$, and

$$\begin{aligned}
w_{i,m+1}&=\frac{e^{-t_{m+1}i^2}}{\epsilon i^2+e^{-t_m i^2}}w_{i,m}-\frac{2}{\pi}\int_{t_{m+1}}^{t_m}\frac{e^{-t_{m+1}i^2}}{\epsilon i^2+e^{-t_m i^2}}e^{(s-t_m)i^2} \\
&\quad\times\Big(\int_0^\pi\big(v_m^4(x)+g(x,s)\big)\sin ix\,dx\Big)ds,
\end{aligned}$$

for $i=1,2,3$. Table 1 shows the the error between the regularization solution u_ϵ and the exact solution u, for three values of ϵ:

Table 1.

ϵ	u_ϵ	$\|u_\epsilon-u\|$
10^{-5}	$2.685490624\sin(x)-0.00009487155350\sin(3x)$	0.005744631447
10^{-7}	$2.691122866\sin(x)+0.00001413193606\sin(3x)$	0.0001124971593
10^{-11}	$2.691180223\sin(x)+0.00002138991088\sin(3x)$	0.00005831365439

Table 2 shows the error table in [33, p. 214].

Table 2.

ϵ	u_ϵ	$\|u_\epsilon - u\|$
10^{-5}	$2.430605996\sin x - 0.0001718460902\sin 3x$	0.3266494251
10^{-7}	$2.646937077\sin x - 0.002178680692\sin 3x$	0.05558566020
10^{-11}	$2.649052245\sin x - 0.004495263004\sin 3x$	0.05316693437

By applying the stabilized quasi-reversibility method in [35], we have the approximate solution $u_\epsilon\left(x, \frac{99}{100}\right) \equiv u_\epsilon(x)$ having the form

$$u_\epsilon(x) = v_m(x) = w_{1,m}\sin x + w_{6,m}\sin 6x\,,$$

where $v_1(x) = (\epsilon+1)e\sin x$, $w_{1,1} = (\epsilon+1)e$, $w_{6,1} = 0$, and $a = \frac{1}{10000}$, $t_m = 1 - am$ for $m = 1, 2, \dots, 100$, and

$$\begin{aligned} w_{i,m+1} = (\epsilon + e^{-t_m i^2})^{\frac{t_{m+1}-t_m}{t_m}} w_{i,m} - \frac{2}{\pi}\int_{t_{m+1}}^{t_m} e^{(s-t_{m+1})i^2} \\ \times \Big(\int_0^\pi \big(v_m^4(x) + g(x,s)\big)\sin ix\,dxds\Big), \end{aligned}$$

for $i = 1, 6$. Table 3 shows the approximation error in this case.

Table 3.

ϵ	u_ϵ	$\|u_\epsilon - u\|$
10^{-5}	$2.690989330\sin(x) - 0.06078794774\sin(6x)$	0.003940316590
10^{-7}	$2.691002638\sin(x) - 0.05797060493\sin(6x)$	0.003592425036
10^{-11}	$2.691023938\sin(x) - 0.05663820292\sin(6x)$	0.003418420030

By applying the method of integral equation in [36], we find the regularized solution $u_\epsilon(x, \frac{99}{100}) \equiv u_\epsilon(x)$ having the form

$$u_\epsilon(x) = v_m(x) = w_{1,m}\sin x + w_{6,m}\sin 6x$$

where

$$v_1(x) = (\epsilon+1)e\sin x, \quad w_{1,1} = (\epsilon+1)e, w_{6,1} = 0,$$

and $a = \frac{1}{5000}$, $t_m = 1 - am$ for $m = 1, 2, \dots, 5$, and

$$\begin{aligned} w_{i,m+1} &= (\epsilon i^2 + e^{-t_m i^2})^{\frac{t_{m+1}-t_m}{t_m}} \\ &\quad\times \Big(w_{i,m} - \frac{2}{\pi}\int_{t_{m+1}}^{t_m} e^{(s-t_m)i^2}\Big(\int_0^\pi \big(v_m^4(x) + g(x,s)\big)\sin ixdx\Big)ds\Big), \end{aligned}$$

for $i = 1, 6$. Table 4 shows the approximation errors in this case.

Table 4.

ϵ	u_ϵ	$\|u_\epsilon - u\|$
10^{-5}	$2.690968476\sin(x) - 0.05677543898\sin(6x)$	0.03489446471
10^{-7}	$2.690947247\sin(x) - 0.05809747108$	0.003662541146
10^{-11}	$2.6912344727\sin(x) - 0.0060809747108\sin(6x)$	0.0003371512534

Looking at the four tables, we see that the error of the second and third tables are smaller than in the first table. This shows that our approach has a nice regularizing effect and give a better approximation than the previous methods in [33, 35, 36].

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