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STABILITY FOR A FAMILY OF SYSTEMS OF DIFFERENTIAL EQUATIONS WITH SECTIONALLY CONTINUOUS RIGHT-HAND SIDES

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Abstract. In this work, we obtain necessary and sufficient conditions to guarantee the asymptotic stability of the trivial solution for a family of interconnected 2×2 systems of differential equations

1. Introduction

One of the most important questions in stability theory is the study of families of systems of differential equations and differential inclusions. Kharitonov [7] proved a necessary and sufficient condition for stability when the coefficients belong the family

$$F = \Big\{ \sum_{k=0}^{n} a_k \lambda^{n-k} : a_k \in [\underline{a}_k, \overline{a}_k],\ k = 0, \dots, n \Big\}\,.$$

Because uncertainties in a perturbation can be represented with matrices whose entries are in certain intervals, it is important to study stability for set of the form

$$A = \big\{ (a_{i,j})_{i,j} : b_{i,j} \le a_{i,j} \le c_{i,j},\ i,j = 1, \dots, n \big\},$$

where the matrices $(b_{i,j})$, $(c_{i,j})$ are stable; see for example [2, 3].

In [8] it is proved the existence of solutions for differential inclusions of the form $x' \in F(T(t)x)$, where F is upper semicontinuous multi-value function, such that $F(T(t)x) \subset \partial V(x(t))$, $t \in [0, T]$, V is a convex and lower semicontinuous function for which $(T(t)x)(s) = x(t + s)$. Now let us assume that there exists a finite number of states, $f_i(x, t)$, $i = \overline{1, n}$, $t \ge 0$, defined by the right-hand side, so that the corresponding equations becomes

$$\dot{x} = \sum_{i=1}^{n} \alpha_i(t) f_i(x, t),$$

where the functions $\alpha_i(t) \ge 0$ are constant in some intervals, and take the values zero or one in each instant of time. Furthermore assume that $\sum_{i=1}^{n} \alpha_i(t) \equiv 1$. This kind of interconnecting systems have uncertainties in the determination of the functions $f_i(x, t)$, which characterize the different states of the system for those

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values of $\tau \in (0,+\infty)$ that represent the moments of the jump from one state to another state. These values of τ represent instants of jump for one pair of functions $\alpha_i(t)$, $i = \overline{1,n}$. The problem about studying the behavior of the solutions of the systems with uncertainty in the parameters that determine it, is an important one because of their applications in the Control Theory (see for example [1, 6, 7, 9]).

The problem to be studied is formulated in the next section. In section 3 the trajectories of the defined family of systems of differential equations are classified as first and second type. After this, in sections 4 and 5, we study the convergence towards the origin of coordinates for the trajectories of first and second type. Finally, in section 6, the main results are proved and some examples are given.

2. Formulation of the problem

Let us consider the real 2 matrices

$$A_1 = \begin{pmatrix} a^1_{11} & a^1_{12} \\ a^1_{21} & a^1_{22} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a^2_{11} & a^2_{12} \\ a^2_{21} & a^2_{22} \end{pmatrix},$$

which are assumed to be stable; i.e., their eigenvalues have real negative part. In (a^k_{ij}) a letter k identifies the matrices A_1 or A_2.

We denote by $N_t(A_1,A_2)$ the set of functions $t \to A(t)$, $t \in [0,+\infty)$, where the matrix $A(t) = \alpha_1(t)A_1 + \alpha_2(t)A_2$. The functions $\alpha_1(t)$, $\alpha_2(t)$ acquire in each instant t the value 1 or 0, and $\alpha_1(t) + \alpha_2(t) \equiv 1$. In addition, the set of jump moments of the functions $\alpha_i(t)$, $i = 1,2$, do not have accumulation points in $\mathbb{R}$.

Let us consider a family of systems of differential equations

$$\dot{x} = A(t)x(t), \quad t \geq 0,\ A(t) \in N_t(A_1,A_2). \tag{2.1}$$

The goal of this work is to establish necessary and sufficient conditions on the matrices $A_1$and A_2 so that each system of the family (2.1) has a trivial asymptotically stable solution.

3. Properties of the solutions to systems of differential equations

The systems of differential equations

$$\dot{x} = A_1x(t), \tag{3.1}$$

$$\dot{x} = A_2x(t), \tag{3.2}$$

belong to the family (2.1) and are asymptotically stable. Clearly, the trajectories of the systems of the family (2.1) are formed by segments of trajectories of systems (3.1) and (3.2), respectively. However, if we associate with each point x of the phase plane the set $F(x) = \{A_1x, A_2x\}$, then all the trajectories of systems (2.1) are smooth sectionally continuous curves and they are such that, at each point x of these curves the tangent vector is one of the vectors $\nu_1(x) = A_1x$, $\nu_2(x) = A_2x$; where the vectors $\nu_1(x)$ and $\nu_2(x)$ belong to the set $F(x)$.

Thus, systems (3.1) and (3.2) belong to the family (2.1), and under conditions of the formulated problem, both systems are asymptotically stable. Moreover, it is clear that the trajectories of the systems of the family (2.1) are formed by segments of trajectories of systems (3.1) and (3.2).

Associated with each point $x = (x_1,x_2)^T$ of the phase plane, we consider the vector $x^\perp = (-x_2,x_1)^T$. It is seen that the vector $x^\perp$ is orthogonal one to the radius vector Ox and has the same direction and orientation that the vector resulting by rotation of the radius vector x in positive direction and with an angle equal to $\pi/2$.

Definition 3.1. Let be γ a trajectory of some of the systems of family (2.1) and x one of its point. It is said that the trajectory γ rotates in a positive direction (counter-clockwise) in the point x if $\langle \nu(x), x^\perp \rangle > 0$, where $\nu(x)$ is the phase velocity vector of the trajectory γ at the point x, and $\langle , \rangle$ denotes the usual scalar product. Always, when $\langle \nu(x), x^\perp \rangle < 0$, it is said that the trajectory γ rotates in a negative direction (clockwise) in the point x.

Lemma 3.2. *Let be x any point of the phase plane, then:*

(i) *The family* (2.1) *has at least one trajectory that rotates in the positive sense at the point x if and only if at least one of the following inequalities occurs.*
  (a) $\langle A_1x, x^\perp \rangle = a^1_{21}x_1^2 + (a^1_{22} - a^1_{11})x_1x_2 - a^1_{12}x_2^2 > 0$,
  (b) $\langle A_2x, x^\perp \rangle = a^2_{21}x_1^2 + (a^2_{22} - a^2_{11})x_1x_2 - a^2_{12}x_2^2 > 0$.

(ii) *The family* (2.1) *has at least one trajectory that rotates in the negative sense at the point x if and only if similar condition to (i) holds, which it is obtained changing the symbol $>$ by the symbol $<$ in previous inequalities (a) and (b).*

We note that the coefficients a^k_{ij}, are the entries of the matrices A_1 and A_2, defined in Section 2.

Proof. Let be $x \in \mathbb{R}^2$ any point of the phase plane, which corresponds to any of the systems of the family (2.1). Let us suppose that at this point the trajectory γ of such a system rotates in the positive sense, then by Definition 3.1, the inequality $\langle \nu(x), x^\perp \rangle > 0$ holds, where $\nu(x)$ is the phase velocity vector of the trajectory γ at the point x. But, how the tangent vector to the trajectories of any system of the family (2.1) belongs to the set $F(x)$, then we have that $\nu(x) = A_1x$ or $\nu(x) = A_2x$. By substitution of the vector $\nu(x)$ in the scalar product $\langle \nu(x), x^\perp \rangle > 0$, we obtain $\langle A_1x, x^\perp \rangle = a^1_{21}x_1^2 + (a^1_{22} - a^1_{11})x_1x_2 - a^1_{12}x_2^2 > 0$ and $\langle A_2x, x^\perp \rangle = a^2_{21}x_1^2 + (a^2_{22} - a^2_{11})x_1x_2 - a^2_{12}x_2^2 > 0$. The proof of condition (ii) is similar. $\square$

Definition 3.3. We say that a trajectory γ of a second order system of differential equations, with homogenous right hand side is of the first type, if on the phase plane there exists a radius vector starting at the origin of coordinates, and there exists an instant $t_0 > 0$, such that the points of γ corresponding to the values $t \geq t_0$ do not belong to a such radius vector. It is said that a trajectory γ is of the second type, when it is not of the first type.

Theorem 3.4. *The family* (2.1) *has trajectories of the second type if and only if at least one of the following conditions holds:*

(i) $a^1_{12} < 0$ *or* $a^2_{12} < 0$, *and for each* $k \in \mathbb{R}$,
$$-a^1_{21} - (a^1_{22} - a^1_{11})k + a^1_{12}k^2 < 0 \quad \text{or} \quad -a^2_{21} - (a^2_{22} - a^2_{11})k + a^2_{12}k^2 < 0. \tag{3.3}$$

(ii) $a^1_{12} > 0$ *or* $a^2_{12} > 0$, *and for each* $k \in \mathbb{R}$,
$$-a^1_{21} - (a^1_{22} - a^1_{11})k + a^1_{12}k^2 > 0 \quad \text{or} \quad -a^2_{21} - (a^2_{22} - a^2_{11})k + a^2_{12}k^2 > 0. \tag{3.4}$$

Proof. We can see that condition (i) implies that for each point x of the phase plane it holds that $\max\{\langle A_1x, x^\perp \rangle, \langle A_2x, x^\perp \rangle\} > 0$ and this condition implies that for each point of the phase plane crosses a trajectory that rotates at this point in a positive direction.

Necessity: Let us suppose that none of the conditions (i) and (ii) is verified, then there exist two straight lines passing by the origin of the phase plane, respectively

$-a^1_{21}-(a^1_{22}-a^1_{11})k$, $-a^2_{21}-(a^2_{22}-a^2_{11})k$; $k=\frac{x_1}{x_2}$, and such that on the points of one of this line do not exist trajectory rotating in positive direction, while on the points of the another line do not exist trajectory rotating in negative direction. Thus these straight lines determine, in the phase plane, an invariant angular section for the family (2.1), and this fact implies that all the trajectories of the family (2.1) are of the first type.

Sufficiency: Let us suppose that condition (i) is verified (the proof is similar when condition (ii) is verified). It is defined a vector field by establishing a correspondence between each point $x\in\mathbb{R}^2$ and the function $f(x)$ defined by

$$f(x)=\begin{cases} A_1x & \text{if } \langle A_1x,x^\perp\rangle \geq \langle A_2x,x^\perp\rangle \\ A_2x & \text{if } \langle A_2x,x^\perp\rangle \geq \langle A_1x,x^\perp\rangle. \end{cases}$$

Evidently $\langle f(x),x^\perp\rangle>0$, for each $x\in\mathbb{R}^2$. That is to say, the trajectories of this vector field, which are trajectories of the family (2.1), rotate at each of their points in positive direction with strictly positive angular velocity. Then, for each ray of the phase plane there exists a sequence t_n, $t_n\to\infty$, such that the points of the trajectory which correspond to these instant of time lay on such a ray; i.e., the trajectories of the vector field $x\to f(x)$ are of the second type. □

4. Convergence towards the origin for the first type trajectories

In this Section we formulate some Lemmata that allows us to demonstrate a Theorem 4.9 which offers condition for the convergence towards the origin of coordinates of all trajectories of the first type of the considered family (2.1).

Let be $\nu\in\mathbb{R}^2$ any vector, and let be $\arg(\nu)$ the angle formed by the vector ν with the semi-axis $x_1>0$; and let be A_1, A_2 the matrices defined in Section 2. We denote by $\measuredangle(A_1x_0,A_2x_0)$ the measure of the angle between the vectors A_1x_0 and A_2x_0 such that $\arg(-x_0)$ belongs to the real segment for which the extremes are $\arg(A_1x_0)$, $\arg(A_2x_0)$ respectively.

Lemma 4.1. *To converge towards the origin of coordinates, the trajectories of the first type it is necessary that the inequality $\measuredangle(A_1x,A_2x)\leq 180^\circ$ hold for all $x\in\mathbb{R}^2$.*

Proof. Suppose that there exists $x_0\in\mathbb{R}^2$, such that $\measuredangle(A_1x_0,A_2x_0)>180^\circ$, then it is clear that there is an angle δ with vertex in the origin of coordinates, such that x_0 lies on one of the limiting rays of δ, and for each $x\in\delta$ holds that $\measuredangle(A_1x,A_2x)>180^\circ$. Let us denote by δ_0 a limited region determined in δ by the segment of trajectory χ of one of the systems (3.1) or (3.2), with initial condition x_0, besides the end of this segment lies on the other side of the angle δ. Then it can be checked that any trajectory with initial condition $x_0\notin\delta_0$ and completely contained in δ, has not common points with $\operatorname{int}(\delta_0)$ ($\operatorname{int}(M)$ denotes the interior of the set M) because for this fact this trajectory must to cut the trajectory χ, and this implies that in the point of intersection w holds $\measuredangle(A_1w,A_2w)\leq 180^\circ$. Let us take the trajectory of the family of systems (2.1) with initial condition $2x_0$ completely contained in δ and formed by segments of alternate trajectories of systems (3.1) and (3.2), which have a starting point on the one of boundary ray of the angle δ, and the end point on the other boundary ray of the angle δ. Then this trajectory has not common points with $\operatorname{int}(\delta_0)$ and therefore it does not converge to the origin of coordinates when $t\to+\infty$. □

Lemma 4.2. *Let be $\gamma = \{x(t) : t \geq 0\}$ a trajectory of the first type of the family* (2.1)*, then there exists an angle S, with vertex in the origin of the phase plane, limited by radius eigenvectors of the matrices A_1 or A_2, such that in the interior of S functions $x \to \operatorname{sgn}\langle A_1x, x^{\perp}\rangle$, $x \to \operatorname{sgn}\langle A_2x, x^{\perp}\rangle$ are constant, in addition, there exists $\tau > 0$, such that $x(t) \in S$ for $t \geq \tau$.*

Proof. The straight lines defined by the eigenvectors of the matrices A_1 and A_2 determine a division of the phase plane in sectors S_i, $i = 1, \dots, n$, each of which is an angle with vertex at the origin of coordinates which satisfies the property of the Lemma relatively to the angle S related with the functions $x \to \operatorname{sgn}\langle A_1x, x^{\perp}\rangle$, $x \to \operatorname{sgn}\langle A_2x, x^{\perp}\rangle$. Since γ is a trajectory of the first type of the family (2.1), there are: a ray L with starting point at the origin of coordinates and an instant $t_0 > 0$ such that $x(t) \notin L$ for all $t > t_0$. If L is within some of the angles S_k then instead of S_k we consider the two angles determined within S_k by L. Let us denote these angles by S'_k and S''_k. It turns out that if for one $\tilde{t} > t_0$, $x(\tilde{t}) \in S_j$, for some $j \in \{1, \dots, p\}$; $j \neq k$, then or $x(t) \in S_j$ for all $t > \tilde{t}$, or there exists $\bar{t} > \tilde{t}$ such that $x(t) \notin S_j$ for $t > \bar{t}$. This means, if the trajectory γ abandon any sector, it can not return to it. The same situation presents with the angles S'_k and S''_k. Thus, as there is only a finite number of these sectors, there must be a sector $S = S_l$, for some $l \in \{1, \dots, p\}$, and one instant $\tau > t_0$, such that $x(t) \in S$ for $t > \tau$. □

Lemma 4.3. *Let S be an angle with vertex in the origin of the phase plane, limited by the radius eigenvectors of the matrices A_1 or A_2, such that $\operatorname{sgn}\langle A_1x, x^{\perp}\rangle = \operatorname{sgn}\langle A_2x, x^{\perp}\rangle$, for all $x \in intS$. If any trajectory of the family* (2.1) *remains in the angle S from an instant t, then it converges towards the origin of coordinates.*

Proof. Let be γ a trajectory of the first type that satisfies the hypotheses of the Lemma, then γ rotates in all their points in the same direction, so this trajectory approaching indefinitely to one ray $L = \{\lambda y : \lambda \geq 0\}$. It is not possible that γ not be a limited one, neither γ has more than one w-limit point on the ray L, since the matrices A_1, A_2 are stable, thus the phase velocities could not be directed in the sense of the vector y which determines the ray L. So, γ converges to a single point w (assuming $w \neq 0$). Let us take ε small enough such that the projection of the phase velocity in the points of γ, belonging to the ball $B(w, \varepsilon)$ over the bisector of the vectors A_1w and A_2w, be strictly positive. So, for arbitrarily large values of t, trajectory γ has points both inside and outside ball $B(w, \varepsilon)$ and so there is a succession of points of γ that are on the circle with center w and radius ε where there should be another w-limit point. This is a contradiction, because there is only one w-limit point. □

Lemma 4.4. *If for the all $x \in \mathbb{R}^2$ results $\measuredangle(A_1x, A_2x) \leq 180°$, then there exists one or there not exists any straight line d passing through the origin of coordinates and such that, for each $x \in d$ we have that $\measuredangle(A_1x, A_2x) = 180°$.*

Proof. If $\operatorname{sgn}\langle A_1x, x^{\perp}\rangle = \operatorname{sgn}\langle A_2x, x^{\perp}\rangle$ or $\langle A_ix, x^{\perp}\rangle = 0$ for $i = 1$ or $i = 2$, then $\measuredangle(A_1x, A_2x) < 180°$. When $\operatorname{sgn}\langle A_1x, x^{\perp}\rangle = -\operatorname{sgn}\langle A_2x, x^{\perp}\rangle$ we can check that $\measuredangle(A_1x, A_2x) < 180°$ if and only if $\operatorname{sgn}\langle (A_1x)^{\perp}, A_2x\rangle = -\operatorname{sgn}\langle A_1x, x^{\perp}\rangle$, and $\measuredangle(A_1x, A_2x) = 180°$ if and only if $\langle (A_1x)^{\perp}, A_2x\rangle = 0$. If we calculated the scalar product $\langle (A_1x)^{\perp}, A_2x\rangle$ we have

$$\begin{aligned}\langle (A_1x)^{\perp}, A_2x\rangle = (a^2_{21}a^1_{11} - a^2_{11}a^1_{21})x_1^2 + (a^2_{22}a^1_{11} + a^2_{21}a^1_{12} \\ - a^2_{11}a^1_{22} - a^2_{12}a^1_{21})x_1x_2 + (a^2_{22}a^1_{12} - a^2_{12}a^1_{22})x_2^2.\end{aligned}$$

From the coefficients of the previous quadratic form we see that it can not happen that be $\measuredangle(A_1x, A_2x) = 180°$ for all $x \in \mathbb{R}^2$ because to do so, there must be $\lambda \in \mathbb{R}_-$ such that $A_1 = \lambda A_2$, but this contradicts the fact that the matrices A_1, A_2 are stable. Suppose that there are two straight lines d_1, d_2 passing through the origin of coordinates and such that for each $x \in d_1$ or $x \in d_2$ holds that $\measuredangle(A_1x, A_2x) = 180°$; i.e., $\langle (A_1x)^{\perp}, A_2x\rangle = 0$. It is known from the theory of quadratic forms that for any point $x \in d_1$ or $x \in d_2$ there is a neighborhood in which there exists points where the expression for $\langle (A_1x)^{\perp}, A_2x\rangle$ takes positive sign and also points where this expression takes negative sign and as for the points which are on d_1 and d_2 the scalar product $\langle A_1x, x^{\perp}\rangle$ does not vanishes, then there will be points near to $d_1$and d_2 on which $\operatorname{sgn}\langle (A_1x)^{\perp}, A_2x\rangle = -\operatorname{sgn}\langle A_1x, x^{\perp}\rangle$, on these points $\measuredangle(A_1x, A_2x) > 180°$, but this contradicts the hypothesis of the Lemma. □

Lemma 4.5. *Let us suppose that for each $x \in \mathbb{R}^2$ follows that $\measuredangle(A_1x, A_2x) \le 180°$. Let us consider for each of the systems* (3.1) *and* (3.2) *a segment of trajectories γ_1 and γ_2 respectively. Suppose that the curves γ_1 and γ_2 intersect itself at the point w. If at this point these trajectories rotate in opposite directions, then after intersection, the future of each segment of the trajectory will continue under the past of the other one.*

Proof. If at the point w, where these trajectories intersect itself, the inequality $\measuredangle(A_1w, A_2w) < 180°$ holds, then the statement of the Lemma is true. Let us see that in the case when $\measuredangle(A_1w, A_2w) = 180°$, the Lemma is also true. Suppose the opposite, namely that there are two segment of trajectories χ_1 and χ_2 of the systems (3.1) and (3.2) which intersect itself in the point w, $\measuredangle(A_1w, A_2w) = 180°$ and in addition the future of χ_1 will be above to the past of χ_2. If we take a segment of the trajectory that is sufficiently close to χ_1 and which be a solution of the same system that χ_1, we have that this segment will intersect χ_2 at the point $w_1$for which $\measuredangle(A_1w_1, A_2w_1) > 180°$, by virtue of Lemma 4.4, and the future of this new segment should go above to the past of χ_2, resulting in a contradiction. □

Lemma 4.6. *Let S be an angle in the phase plane with vertex in the origin of coordinates, such that $\operatorname{sgn}\langle A_1x, x^{\perp}\rangle = -\operatorname{sgn}\langle A_2x, x^{\perp}\rangle$ for all $x \in intS$. If a trajectory of the family of systems* (2.1) *remains, from an instant t, in the angle S, then this trajectory converges to the origin of coordinates, i.e $x(t) \to 0$ when t grows.*

Proof. Let us begin by defining for each $x \in S$ one limited region in the plane that is denoted by S_x. Let us consider the trajectories of systems (3.1) and (3.2) with starting point x. Then S_x will be the plane region limited by these trajectories and the origin of coordinates when these trajectories are completely contained in the angle S; in other case S_x will be the plane region limited by the segments of the considered trajectories contained in the angle S and the straight segments which joint the exit points of these trajectories from de angle S and the origin of coordinates. By the construction of the region S_x and the affirmation of Lemma 4.5, we conclude that the region S_x contains completely the semi - positive trajectories of the family (2.1) with starting point in S_x, besides, these trajectories do not abandon angle S. Let be now $\gamma = \{x(t) : t \geq 0\}$ a trajectory of the first type completely contained in S from a certain moment t_0. Suppose that γ has more than one w-limit point, and let w and w_1 be two of these points. It is easy to see that $w \in S_x$ and $w_1 \in S_x$, but this only occurs if there are two segments of trajectories, one of system (3.1) and another of system (3.2), such that, w and w_1 are its end points. We get a contradiction with the Lemma 4.5. Suppose now that $w \neq 0$ is the unique w-limit point of the trajectory γ and also suppose that $\measuredangle(A_1w, A_2w) < 180°$. It is taken ε small enough such that the projection of the phase velocity at the point of γ belonging to the ball $B(w, \varepsilon)$ on the bisector of the vectors A_1w , A_2w be strictly positive. So, for arbitrarily large values of t the trajectory γ has points both inside and outside ball $B(w, \varepsilon)$, and so there is a succession of points of γ laying on the circle with center in w and radius ε where there should be another w-limit point. Again we get a contradiction. Suppose now that $w \neq 0$ is the unique w-limit point of the trajectory γ and $\measuredangle(A_1w, A_2w) = 180°$. We take ε small enough such that the projection of the phase velocity at the point of γ belonging to the ball $B(w, \varepsilon)$ on the vector A_1w has module greater than or equal to the number $\alpha > 0$. We know that γ will remain in $B(w, \varepsilon)$ from certain $t_0 > 0$. We also know that γ is formed by segments of the trajectories of systems (3.1) and (3.2). Let be t_i, $i \in \mathbb{N}$, a succession of instants of change from one to another system. As the succession $\{t_i\}$ has no accumulation points in $\mathbb{R}$, there exists a number $\mu > 0$, such that, $|t_{i+1} - t_i| \geq \mu$. Then follows that $|x(t_{i+1}) - x(t_i)| \geq \alpha(t_{i+1} - t_i) \geq \alpha\mu$ which contradicts the convergence to the origin for the trajectory γ. □

Lemma 4.7. *A necessary and sufficient condition for $\measuredangle(A_1x, A_2x) \leq 180°$ to hold for each $x \in \mathbb{R}^2$, is that the matrices $C(\alpha) = \alpha A_1 + (1 - \alpha)A_2$, $\alpha \in [0, 1]$ be stable or at most there exists one singular matrix $C(\alpha_0)$.*

Proof. Vectors A_1x and A_2x form two angles such that the sum of their amplitude is 360^o and the segment that links the extremes of A_1x and A_2x is contained in the angle of lesser magnitude.
Necessity: Let be $x_0 \in \mathbb{R}^2$ such that, $\measuredangle(A_1x_0, A_2x_0) > 180°$. According to the definition of the angle $\measuredangle(A_1x_0, A_2x_0)$, we have that for some $\lambda > 0$ the point λx_0 is on the segment that connects the points A_1x_0 and A_2x_0; i.e., $\alpha_0A_1x_0 + (1 - \alpha_0)$ $A_2x_0 = \lambda x_0$ for some $\alpha_0 \in [0, 1]$, and thus $C(\alpha_0)x_0 = \lambda x_0$. The latter equality means that $C(\alpha_0)$ is unstable.
Sufficiency: Let us contrary suppose that there exists $\alpha_0 \in [0, 1]$ such that, $C(\alpha_0)$ is unstable. Then $C(\alpha_0)$ has an eigenvalue λ such that, $\mathrm{Re}(\lambda) \geq 0$. But the eigenvalues of $C(\alpha_0)$ are real because in other case would be $\mathrm{Re}(\lambda) = \mathrm{tr}(C(\alpha_0)) < 0$. Then there must be $x_0 \in \mathbb{R}^2$, $x_0 \neq 0$, such that, $C(\alpha_0)x_0 = \lambda x_0$, $\lambda \in \mathbb{R}_+$. It is means $\alpha_0A_1x_0+(1-\alpha_0)\, A_2x_0 = \lambda x_0$, and this means that the vector λx_0 is on the segment formed by A_1x_0 and A_2x_0. From this follows that $\measuredangle(A_1x_0, A_2x_0) > 180°$. □

Lemma 4.8. *The matrices $C(\alpha) = \alpha A_1 + (1 - \alpha)A_2$, $\alpha \in [0, 1]$ are stable, except as most for some $\alpha_0 \in [0, 1]$. And the stability is guaranteed if and only if $H \leq 2\sqrt{\det A_1 \det A_2}$, where $H = a^1_{12}a^2_{21} + a^2_{12}a^1_{21} - a^1_{11}a^2_{22} - a^2_{11}a^1_{22}$.*

Proof. The matrix $C(\alpha)$, $\alpha \in [0, 1]$ is stable if and only the inequalities $\mathrm{tr}(C(\alpha)) < 0$, $\det(C(\alpha)) > 0$ hold. But $\mathrm{tr}(C(\alpha)) = \mathrm{tr}(\alpha A_1 + (1 - \alpha)A_2) = \alpha\,\mathrm{tr}(A_1) + (1 - \alpha)\,\mathrm{tr}(A_2)$ and $\mathrm{tr}(A_1) < 0$, $\mathrm{tr}(A_2) < 0$. Thus $\mathrm{tr}(C(\alpha)) < 0$, for all $\alpha \in [0, 1]$. On the other hand

$$\begin{aligned}\det(C(\alpha)) &= \det(\alpha A_1 + (1-\alpha)A_2)\\ &= (\det(A_1)+\det(A_2)+H)\alpha^2 - (2\det(A_2)+H)\alpha + \det(A_2).\end{aligned}$$

If $(\det(A_1)+\det(A_2)+H) \le 0$ then $\det(C(\alpha))$, as a function of α, is a parable that opens down, or a straight line, but as $\det(C(0)) = \det(A_2) > 0$ and $\det(C(1)) = \det(A_1) > 0$, then $\det(C(\alpha)) > 0$, for all $\alpha \in [0,1]$. If $(\det(A_1)+\det(A_2)+H) > 0$ then $\det(C(\alpha))$, as a function of α, is a parable that opens up, such that, for $\alpha = 0$ and $\alpha = 1$ this parabola takes positive values. Soon $\det(C(\alpha)) > 0$, for all $\alpha \in [0,1]$, if and only if the vertex of the parable corresponds to a value $\alpha_0 \notin [0,1]$, or in other case $\det(C(\alpha_0)) \ge 0$, it is means that holds the implication:

$$-H \le 2\det(A_1),\ -H \le 2\det(A_2) \Longrightarrow H^2 \le 4\det(A_1)\det(A_2).$$

If $H \le 0$ and the left-hand side holds, by multiplying the inequalities we obtain the right-hand side. In another way, if $H > 0$ and the left-hand side holds, then $H \le 2\sqrt{\det A_1 \det A_2}$. □

Theorem 4.9. *The trajectories of the first type of the family of systems* (2.1) *converge towards the origin of coordinates if and only if*

$$H \le 2\sqrt{\det A_1 \det A_2}. \tag{4.1}$$

Proof. Let be A_1, A_2 the matrices defined in Section 2.
Necessity: Let us consider that the trajectories of the first type of the family of systems (2.1) converge towards the origin of coordinates. Then, by Lemma 4.1, the inequality $\measuredangle(A_1x, A_2x) \le 180°$ holds for each $x \in \mathbb{R}^2$. This inequality, according to Lemma 4.7, is equivalent to the stability of matrices $C(\alpha) = \alpha A_1 + (1-\alpha)A_2$, $\alpha \in [0,1]$. Now we just apply Lemma 4.8 and so we have $H \le 2\sqrt{\det A_1 \det A_2}$.
Sufficiency: Let us now consider that $H \le 2\sqrt{\det A_1 \det A_2}$ and suppose that the trajectories of the first type of the family of systems (2.1) do not converge towards the origin of coordinates. Then, by Lemma 4.1, will be $\measuredangle(A_1x, A_2x) > 180°$ for all $x \in \mathbb{R}^2$. Moreover, due to Lemma 4.8, the matrices $C(\alpha) = \alpha A_1 + (1-\alpha)A_2\ {}_2$, $\alpha \in [0,1]$ are stable. But now, by Lemma 4.7, the inequality $\measuredangle(A_1x, A_2x) \le 180°$ holds for all $x \in \mathbb{R}^2$. We obtain a contradiction with our assumption. □

5. Convergence towards the origin for the second type trajectories

In this section we analyze the convergence towards the origin of the trajectories which rotate, at any point x of the phase plane. In the following we consider that the stable matrices A_1, A_2 satisfy (4.1). This condition ensures, according to Theorem 4.9, the convergence towards the origin of the first type trajectories of the family (2.1)

We want to establish additional conditions to ensure the convergence towards the origin also for the trajectories of the second type.

Thus, taking into account the ideas developed by Baravanov [1], we introduce the so-called auxiliary systems:

$$\dot{x} = \arg \max_{f \in F(x),\, \langle f, x^\perp \rangle > 0} \frac{\langle f, x \rangle}{\|f\|}, \tag{5.1}$$

$$\dot{x} = \arg \max_{f \in F(x),\, \langle f, x^\perp \rangle < 0} \frac{\langle f, x \rangle}{\|f\|}, \tag{5.2}$$

where $F(x) = \{A_1x, A_2x\}$. We note that systems (5.1) and (5.2) make sense respectively, in the following regions of the plane:

$$\begin{aligned} D^+ &= \{x \in \mathbb{R}^2 : \exists f \in F(x) \text{ such that } \langle f, x^\perp\rangle > 0\}, \\ D^- &= \{x \in \mathbb{R}^2 : \exists f \in F(x) \text{ such that } \langle f, x^\perp\rangle < 0\}. \end{aligned}$$

Given a pair of matrices A_1, A_2 it is possible that both systems (5.1) and (5.2) make sense in all the plane or in a particular region of the plane. Although it is also possible that one of the systems does not make sense, because one of the sets D^+ or D^- may be empty.

In addition, according to the definition of these systems and the results of the section 3, we have that the trajectories of system (5.1) rotate in each of its points in positive direction, and in the case of system (5.2), its trajectories rotate in each point in negative direction. Thus it is easy to determine expressions for systems (5.1) and (5.2) according to the matrices A_1, A_2 that determine each of these systems. For this we must just resolve the extreme problems indicated in the right hand-side of the expressions (5.1) and (5.2), which are simply ones, because the variable in each problem takes only two values. Thus, it is clear that system (5.1) takes the form

$$\begin{gathered} \dot{x} = V_1(x)x, \quad x \in D^+, \\ V_1(x) = \begin{cases} A_1 & \text{if } \langle (A_1x)^\perp, A_2x\rangle \geq 0 \\ A_2 & \text{if } \langle (A_1x)^\perp, A_2x\rangle < 0, \end{cases} \end{gathered} \tag{5.3}$$

while system (5.2) is given by the expression

$$\begin{gathered} \dot{x} = V_2(x)x, \quad x \in D^-, \\ V_2(x) = \begin{cases} A_1 & \text{if } \langle (A_1x)^\perp, A_2x\rangle \leq 0 \\ A_2 & \text{if } \langle (A_1x)^\perp, A_2x\rangle > 0\,. \end{cases} \end{gathered} \tag{5.4}$$

Next, following the ideas of Barabanov, it is shown with the help of some Lemmata that the trajectories of the second type of the family (2.1) converge towards the origin of coordinates, if and only if, the auxiliary systems (5.3) and (5.4) are asymptotically stable.

Lemma 5.1. *Let $\gamma = \{x(t) : t \geq 0\}$ be a trajectory of the second type of a homogeneous second order system. Then there exist $t_0 > 0$ and $\lambda > 0$ (λ is called characteristic value) such that for all $k \in \mathbb{N}$, the equality $x(t + kt_0) = \lambda^k x(t)$ holds.*

Proof. Let be t_0 the lowest of all real numbers $t > 0$, such that $x(t_0)$ is located on the ray that begins at the origin and contains $x(0)$, and take $\lambda = \|x(t_0)\|/\|x(0)\|$. We demonstrate the statement of the Lemma by induction. Consider the trajectory $\gamma_\lambda = \{\lambda x(t) : t \geq 0\}$ for which its corresponding point, for the instant of time $t = 0$, is $x(t_0)$; i.e., $x(t_0) = \lambda x(0)$ and thus $x(t + t_0) = \lambda x(t)$. This fact proves the affirmation of the Lemma for $k = 1$. Let us suppose that the statement of the Lemma is true for $k = n$; i.e., $x(t + nt_0) = \lambda^n x(t)$, and let us consider the trajectory $\gamma_{\lambda,n+1} = \{\lambda^{n+1}x(t) : t \geq 0\}$. The point of $\gamma_{\lambda,n+1}$ corresponding to $t = 0$ is $\lambda^{n+1}x(0)$, but due to the induction hypothesis we have that $\lambda^{n+1}x(0) = \lambda x(nt_0)$ is the point of γ_λ corresponding to $t = nt_0$, and thus $\lambda^{n+1}x(0) = x((n + 1)t_0)$, by this reason its follows that $x(t + (n + 1)t_0) = \lambda^{n+1}x(t)$. $\square$

Now let us to consider the following sets:

$$C^+(x) = \{\lambda x(t) : 0 \le \lambda \le 1,\ x(t),\ t > 0 \text{ solution of (5.3) and } x(0) = x\},$$
$$C^-(x) = \{\lambda x(t) : 0 \le \lambda \le 1,\ x(t),\ t > 0 \text{ solution of (5.4) and } x(0) = x\}.$$

Systems (5.1) and (5.2) are homogeneous, by this reasons $C^+(\alpha x) = \alpha C^+(x)$ and $C^-(\alpha x) = \alpha C^-(x)$ for each $\alpha > 0$.

Lemma 5.2. *Let be $\gamma = \{x(t) : t \ge 0\}$ a trajectory of the second type of the family* (2.1). *Then there is $t_0 \ge 0$ such that the segment of the trajectory $\{x(t) : t \ge t_0\}$ is contained in one of the sets $C^+(x(0))$ or $C^-(x(0))$.*

Proof. If one of the systems (5.1) or (5.2) is not asymptotically stable, then one of the sets $C^+(x(0))$ or $C^-(x(0))$ coincides with the whole plane and the Lemma is trivial. Consider the case when (5.1) and (5.2) are asymptotically stable. Let be t_0 such that, the point $x(t_0) \in L = \{\lambda x(0) : \lambda \ge 0\}$, and any other point of γ corresponding to $t > t_0$ until to complete its first round around the origin, is on L. Without loss of generality let us suppose that this round is given in a positive direction, then, by virtue of Lemma 4.5, the definition of systems (5.1) and (5.2) and that two trajectories of the same system can not intersect itself, meets up that the points of γ, for $t > t_0$, are in $C^+(x(0))$. □

Lemma 5.3. *For the convergence towards the origin of coordinates of the trajectories of the second type of the family* (2.1) *it is necessary and sufficient that systems* (5.1) *and* (5.2) *be asymptotically stable.*

Proof. Let be $\gamma = \{x(t) : t \ge 0\}$ a trajectory of the second type and let be t_n the moment when γ completes exactly n laps around the origin (all the laps are considered, both happening in a positive direction as those occurring in the negative sense). We consider the solutions $x^1(t)$ and $x^2(t)$, $t \ge 0$ of systems (5.1) and (5.2) respectively, that satisfy the initial condition $x^1(0) = x^2(0) = x(0)$. For the definitions of systems (5.1) and (5.2) and the assumption that the family of systems (2.1) has trajectories of the second type, we have that at least one of the solutions $x^1(t)$, $x^2(t)$, $t \ge 0$, is of the second type. Let us consider the characteristic values of these solutions (defined in Lemma 5.1) in the case when they are of the second type, and let be λ the largest of them. By the form of the sets $C^+(x(0))$, $C^-(x(0))$, and because of Lemmata 5.1 and 5.2, it can see that $\|x(t_1)\| \ge \lambda\|x(0)\|$; similarly, by the form of $C^+(x(t_i))$, $C^-(x(t_i))$ and the Lemmata 5.1 and 5.2, the inequality $\|x(t_{i+1})\| \ge \lambda\|x(t_i)\|$ holds. From this fact, it follows that $\|x(t_i)\| \ge \lambda^i\|x(0)\|$, and systems (5.1) and (5.2) are asymptotically stable, is $\lambda < 1$ and thus $x(t_i) \to 0$ when $i \to +\infty$. Then is verified that the sets $C^+(x(t_i))$ and $C^-(x(t_i))$ tend to the set $\{0\}$ in the Hausdorff metric when $i \to +\infty$, and how the points of the trajectory γ, corresponding to the values $t > t_{i+1}$, belong to one of the sets $C^+(x(t_i))$, $C^-(x(t_i))$; we conclude that the trajectory γ converges towards the origin when $t \to \infty$. □

5.1. **Stability of auxiliary systems.**

Lemma 5.4. *Suppose that the matrices A_1, A_2 are stable and satisfy* (4.1). *If none of the systems* (5.3) *and* (5.4) *has trajectories of the second type or both have trajectories of the second type, then the family* (2.1) *is asymptotically stable.*

Proof. In the first case all the trajectories of systems (5.3) and (5.4) are of the first type and as these are trajectories of the family of systems (2.1), they must converge to the origin. In the second case, it is clear that systems (5.3) and (5.4) coincide with systems (3.1) and (3.2) and thus both are asymptotically stable. Now there remains to apply Lemma 5.3. □

Let us consider the more complex case, in which one of the systems (5.3) or (5.4) has trajectories of the second type while the other does not have. Clearly, in this case we have to investigate only the asymptotic stability of the system that has trajectories of the second type.

So, consider that system (5.3) has trajectories of the second type, while system (5.4) does not have trajectories of the second type. Due to the homogeneity of system (5.3), for its asymptotic stability we will verify the convergence towards the origin of coordinates only for one trajectory of this system.

Thus, we consider the trajectory $\gamma = \{x(t) : t \geq 0\}$ that satisfies $x_1(0) = -1$; $x_2(0) = 0$. Because this is a trajectory of the second type, there exist $t > 0$ such that $x_2(t) = 0$ and $x_1(t) > 0$. Let T be the lowest of these t. Further we consider the trajectory $\gamma_T = \{-x_1(T)x(t) : t \geq 0\}$ of (5.3). The point of this trajectory corresponding to $t = 0$ is $x(T)$, in which $x(t+T) = -x_1(T)x(t)$. From this equality follows $x_1(2T) = -x_1^2(T)$ and $x_2(2T) = 0$; and as a consequence of Lemma 5.1, we have that γ converges towards the origin of coordinates if and only if $x_1(T) < 1$. So, we proved the following lemma.

Lemma 5.5. *For the asymptotic stability of system* (5.3)*, when it has trajectories of the second type, and the solution solution $x(t)$, $t \in [0, T]$, satisfies boundary conditions $x_1(0) = -1$; $x_2(0) = 0$; $x_2(T) = 0$; $x_2(t) \neq 0$, $t \in (0, T)$; is necessary and sufficient that $x_1(T) < 1$.*

For the effective implementation of Lemma 5.5 it is convenient to obtain an expression for $x_1(T)$ as a function of the elements that define system (5.3); i.e., the matrices A_1, A_2.

Let be $w_{ij}(x)$, i, $j = 1, 2$, the elements of the matrix $V_1(x)$; i.e.,

$$V_1(x) = \begin{pmatrix} w_{11}(x) & w_{12}(x) \\ w_{21}(x) & w_{22}(x) \end{pmatrix}.$$

System (5.3) is rewritten as

$$\begin{aligned} \dot{x_1} &= w_{11}(x)x_1 + w_{12}(x)x_2 \\ \dot{x_2} &= w_{21}(x)x_1 + w_{22}(x)x_2. \end{aligned}$$

We multiply both equations of this system in order to obtain

$$\dot{x_1}(w_{21}(x)x_1 + w_{22}(x)x_2) = \dot{x_2}(w_{11}(x)x_1 + w_{12}(x)x_2).$$

If in the region of the phase plane that is obtained by eliminating the axes of coordinates, we make the change of variables $z = \frac{x_1}{x_2}$, we obtain

$$\frac{dx_1}{x_1} = \frac{(\phi_{11}(z)z + \phi_{12}(z))dz}{z(-\phi_{21}(z)z^2 + (\phi_{11}(z) - \phi_{22}(z))z + \phi_{12}(z))}, \tag{5.5}$$

where $\phi_{ij}(z) = w_{ij}(x)$, i, $j = 1, 2$. The coefficients $\phi_{ij}(z)$ are well defined as the function $V_1(x)$ is homogeneous. We form the matrix

$$\Phi(z) = \begin{pmatrix} \phi_{11}(z) & \phi_{12}(z) \\ \phi_{21}(z) & \phi_{22}(z) \end{pmatrix},$$

then by the relationship between the matrices $\Phi(z)$ and $V_1(x)$, and from the expression of $V_1(x)$ in (5.3) it is obtained that

$$\Phi(z) = \begin{cases} A_1 & \textit{if}\, \langle (A_1(z,1)^T)^{\perp}, A_2(z,1)^T\rangle \geq 0 \\ A_2 & \textit{if}\, \langle (A_1(z,1)^T)^{\perp}, A_2(z,1)^T\rangle < 0. \end{cases}$$

Let us consider the trajectory $\gamma = \{x(t) : t \geq 0\}$ of system (5.3) referred in Lemma 5.5. On this trajectory we take the points: $P_1(-1+\varepsilon, x_2(-1+\varepsilon))$; $P_2(-\delta, x_2(-\delta))$; $P_3(\delta, x_2(\delta))$; $P_4(x_1(T)-\varepsilon, x_2(x_1(T)-\varepsilon))$, which appear on the trajectory always when t grows in the same order of their sub - indexes, besides $\delta > 0$ and $\varepsilon > 0$. To the sections P_1P_2 and P_3P_4 of trajectories of (5.3) there correspond integral curves of the equation (5.5) and by direct integration of this equation between the considered extreme points, we obtain the equalities

$$\int_{-1+\varepsilon}^{-\delta} \frac{dx_1}{x_1} = \int_{\frac{-1+\varepsilon}{x_2(-1+\varepsilon)}}^{\frac{-\delta}{x_2(-\delta)}} \frac{(\phi_{11}(z)z+\phi_{12}(z))\,dz}{z(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))};$$

$$\int_{\delta}^{x_1(T)-\varepsilon} \frac{dx_1}{x_1} = \int_{\frac{\delta}{x_2(\delta)}}^{\frac{x_1(T)-\varepsilon}{x_2(x_1(T)-\varepsilon)}} \frac{(\phi_{11}(z)z+\phi_{12}(z))\,dz}{z(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))}.$$

Adding these expressions we obtain

$$\begin{aligned} &\int_{-1+\varepsilon}^{-\delta} \frac{dx_1}{x_1} + \int_{\delta}^{x_1(T)-\varepsilon} \frac{dx_1}{x_1} \\ &= \int_{\frac{-1+\varepsilon}{x_2(-1+\varepsilon)}}^{\frac{-\delta}{x_2(-\delta)}} \frac{(\phi_{11}(z)z+\phi_{12}(z))dz}{z(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))} \\ &\quad + \int_{\frac{\delta}{x_2(\delta)}}^{\frac{x_1(T)-\varepsilon}{x_2(x_1(T)-\varepsilon)}} \frac{(\phi_{11}(z)z+\phi_{12}(z))dz}{z(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))}. \end{aligned}$$

As

$$\begin{aligned} &\frac{(\phi_{11}(z)z+\phi_{12}(z))}{z(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))} \\ &= \frac{1}{z} - \frac{1}{2}\frac{-2\phi_{21}(z)z+(\phi_{11}(z)-\phi_{22}(z))}{(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))} \\ &\quad + \frac{1}{2}\frac{(\phi_{11}(z)+\phi_{22}(z))}{(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))}, \end{aligned}$$

we have

$$
\begin{aligned}
&\ln\big(\frac{x(T)-\varepsilon}{1-\varepsilon}\big)\\
&=\int_{\frac{-1+\varepsilon}{x_2(-1+\varepsilon)}}^{\frac{-\delta}{x_2(-\delta)}}\Big(\frac{1}{z}-\frac{1}{2}\frac{-2\phi_{21}(z)z+(\phi_{11}(z)-\phi_{22}(z))}{(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))}\Big)dz\\
&\quad+\int_{\frac{\delta}{x_2(\delta)}}^{\frac{x_1(T)-\varepsilon}{x_2(x_1(T)-\varepsilon)}}\Big(\frac{1}{z}-\frac{1}{2}\frac{-2\phi_{21}(z)z+(\phi_{11}(z)-\phi_{22}(z))}{(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))}\Big)dz\\
&\quad+\frac{1}{2}\int_{\frac{-1+\varepsilon}{x_2(-1+\varepsilon)}}^{\frac{-\delta}{x_2(-\delta)}}\Big(\frac{(\phi_{11}(z)+\phi_{22}(z))}{(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))}\Big)dz\\
&\quad+\frac{1}{2}\int_{\frac{\delta}{x_2(\delta)}}^{\frac{x_1(T)-\varepsilon}{x_2(x_1(T)-\varepsilon)}}\Big(\frac{(\phi_{11}(z)+\phi_{22}(z))}{(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))}\Big)dz.
\end{aligned}
\tag{5.6}
$$

By direct calculations we can prove that a primitive of the first two integrals in the right hand-side is

$$
\ln\Big|\frac{z}{|-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z)|^{1/2}}\Big|, \tag{5.7}
$$

which converges when $z\to\pm\infty$, because the function $\phi_{21}(z)$ is a constant and different from zero one for z sufficiently large. Furthermore, the denominator of the last two integrals is different from zero for all $z\in\mathbb{R}$, this ensures that for these integrals we can apply the criteria of comparison.

The lower limit of integration in the third integral in (5.6) tends to $-\infty$ when $\varepsilon\to 0$. While the upper limit of the fourth integral in (5.6) tends to $+\infty$, and as the expression

$$
\frac{z^2}{-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z)}
$$

converges when $z\to\pm\infty$ to non-zero numbers, we conclude that, when we pass to the limit in (5.6) with $\varepsilon\to 0$, the considered integrals are converging. As integrands in the last two integrals of the expression (5.6) are continuous functions in all $\mathbb{R}$, passing to the limit when $\delta\to 0$ we obtain

$$
\begin{aligned}
\ln(x_1(T))=\lim_{\delta\to 0}\Big\{&\int_{-\infty}^{-\delta}\Big(\frac{1}{z}-\frac{1}{2}\frac{-2\phi_{21}(z)z+(\phi_{11}(z)-\phi_{22}(z))}{(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))}\Big)dz\\
&+\int_{\delta}^{+\infty}\Big(\frac{1}{z}-\frac{1}{2}\frac{-2\phi_{21}(z)z+(\phi_{11}(z)-\phi_{22}(z))}{(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))}\Big)dz\Big\}\\
&+\frac{1}{2}\int_{-\delta}^{\delta}\Big(\frac{(\phi_{11}(z)+\phi_{22}(z))}{(-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z))}\Big)dz.
\end{aligned}
\tag{5.8}
$$

Using the primitive (5.7) of two first integrals in (5.6), evaluating them and passing to the limit when $\delta\to 0$, we obtain

$$
\ln(x_1(T))=\ln Q+\int_{-\infty}^{+\infty}\big(\frac{\operatorname{tr}\Phi(z)}{g(z)}\big)dz, \tag{5.9}
$$

where

$$\begin{gathered}
g(z)=-\phi_{21}(z)z^2+(\phi_{11}(z)-\phi_{22}(z))z+\phi_{12}(z),\\
h(z)=\langle (A_1(z,1)^T)^{\perp}, A_2(z,1)^T\rangle
\end{gathered}$$

$$Q=\begin{cases}\frac{g(z_1-0)g(z_2-0)}{g(z_1+0)g(z+0)} & \text{if } h(z) \text{ has twow real roots} z_1, z_2\\ \frac{g(z_1-0)g(+\infty)}{g(z_1+0)g(-\infty)} & \text{if } h(z) \text{ has a single real root} z_1\\ 1 & \text{if } h(z) \text{ does not have real roots,}\end{cases}$$

Let us denote $I^+=\ln Q+\int+\infty^{+\infty}\big(\frac{\operatorname{tr}\Phi(z)}{g(z)}\big)dz$.

Lemma 5.6. *A necessary and sufficient condition for the asymptotic stability of* (5.3)*, when it has trajectories of the second type, is* $I^+<0$.

The assertion of the above lemma is a direct consequence of Lemma 5.5 and (5.9).

In the case where only system (5.4) has trajectories of the second type, we associate to this system a number which is denoted by I^-. This number is obtained by adding a negative sign $(-)$ to the right hand-side in the expression (5.9) and substituting $\Phi(z)$ by

$$\Phi^-(z)=\begin{cases}A_2 & \text{if } \langle (A_1(z,1)^T)^{\perp}, A_2(z,1)^T\rangle\ge 0\\ A_1 & \text{if } \langle (A_1(z,1)^T)^{\perp}, A_2(z,1)^T\rangle< 0.\end{cases}$$

In this case a similar result to Lemma 5.6 is valid:

Lemma 5.7. *A necessary and sufficient condition for the asymptotic stability of* (5.4)*, when it has trajectories of the second type, is* $I^-<0$.

The assertion of the above lemma is a direct consequence of Lemma 5.5 and (5.9), when a negative sign is added to the right hand-side in this expression and the function $\Phi(z)$ is replaced by $\Phi^-(z)$.

6. Main Result

Theorem 6.1. *For the systems of the family* (2.1) *to have trivial asymptotically stable solutions, it is necessary and sufficient that:*

- (i) $\operatorname{tr}(A_i)<0$, $\det(A_i)>0$, $i=1,\,2$;
- (ii) $a^1_{12}a^2_{21}+a^2_{12}a^1_{21}-a^1_{11}a^2_{22}-a^2_{11}a^1_{22}\le 2\sqrt{\det A_1\det A_2}$;
- (iii) *One of the following two conditions holds:*
  - (a) *if* $a^1_{12}>0$ *or* $a^2_{12}>0$ *and for each* $k\in\mathbb{R}$, $-a^1_{21}-(a^1_{22}-a^1_{11})k+a^1_{12}k^2>0$, $i=1$ *or* 2, *then* $I^+<0$;
  - (b) *if* $a^1_{12}<0$ *or* $a^2_{12}<0$ *and for each* $k\in\mathbb{R}$, $-a^1_{21}-(a^1_{22}-a^1_{11})k+a^1_{12}k^2<0$, $i=1$ *or* 2, *then* $I^-<0$.

Proof. Let A_1, A_2 be stable matrices. As we saw in section 3, the trajectories of the systems that integrate the family (2.1) are segments of the trajectories of systems (3.1) and (3.2), and these systems are asymptotically stable.

Necessity: Suppose that the family (2.1) is asymptotically stable. Then condition (i) is guaranteed because asymptotic stability of systems (3.1) and (3.2), as this condition equivalents to the stability of matrices A_1, A_2. The asymptotic stability of the family (2.1) means that the trajectories of all systems of the family converge towards the origin of coordinates. Then for the trajectories of the first type, due to Theorem 4.9, condition (ii) holds. While, for the trajectories of the second type, condition (iii) holds due to Lemmata 5.6, 5.7.

Sufficiency: Suppose now that conditions (i), (ii), (iii) hold. Then, due to condition (i), the trajectories of systems (3.1) and (3.2) converge towards the origin of coordinates. The trajectories of the first type of family (2.1) also converge towards the origin of coordinates because of condition (ii) and Theorem 4.9. The same happens with the trajectories of the second type because of condition (iii) and the Lemmata 5.6, 5.7. Thus, the trivial solution of the family of systems (2.1) is asymptotically stable. □

6.1. **Examples.** In this section we present some examples of families of systems of differential equations whose stability of the trivial solution is determined. In each case the matrices A_1, A_2 are stable, and based on this fact, the different conditions of the previous theorem hold.

Example 6.2. Let $A_1 = \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. In this case conditions (i), (ii), (iii)(a) of Theorem 6.1 hold, therefore it is necessary to calculate I^+ to know its sign. After the required calculations we obtain $I^+ = -0.8150$; now we can say that the family of systems of differential equations (2.1) determined by the pair of matrices A_1, A_2 is asymptotically stable.

Example 6.3. Let $A_1 = \begin{pmatrix} 1 & -1 \\ 2 & -\frac{3}{2} \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix}$. In this case conditions (i), (ii), (iii)(b) of Theorem 6.1 hold, therefore it is necessary to calculate I^- to know its sign. After the required calculations we obtain $I^- = \sqrt{12}\pi$; now we can say that the family of systems of differential equations (2.1) determined by the pair of matrices A_1, A_2 is unstable.

Example 6.4. Let $A_1 = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Again the matrices A_1, A_2 are asymptotically stable and for them, conditions (i), (ii), (iii)(a)(b), of Theorem 6.1 hold. For this reason, both systems (5.3) and (5.4) have trajectories of the second type, thus, by Lemma 5.4, the corresponding family of systems of differential equations is asymptotically stable.

Example 6.5. Let $A_1 = \begin{pmatrix} -\frac{35}{10} & 5 \\ 3 & -3 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix}$. In this example, the pair of matrices A_1, A_2 satisfies condition (i) of Theorem 6.1, but condition (ii) of this theorem is no longer satisfied. By this reason, it is no longer guaranteed the asymptotic stability of the corresponding family of systems of differential equations.

Example 6.6. Let $A_1 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. In this case conditions (i), (ii), (iii)(a) of Theorem 6.1 hold, thus we need to determine the sign of I^+. However, the function $g(z) = -z^2$ has the real root 0; i.e., the integral appearing in the expression for I^+ is an improperly mixed integral, which tends to $-\infty$. Thus the family of systems of differential equations determined by the pair of matrices A_1, A_2 is asymptotically stable.

Conclusion. In this work we found the necessary and sufficient conditions for the asymptotic stability of the family of systems of differential equations under review.

These conditions are given explicitly depending on the coefficients of the matrices that determine each family. To continue this line of research, it would be interesting to study the same problem, when uncertainty is present not only in the moments of change, but in both schemes of connection; i.e., in the element of matrices A_1, A_2 determining a new family of systems.

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