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EXISTENCE OF GLOBAL SOLUTIONS TO NONLINEAR MIXED VOLTERRA-FREDHOLM INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

HARIBHAU L. TIDKE

Abstract. In this paper, we investigate the existence of global solutions to first-order initial-value problems, with nonlocal condition for nonlinear mixed Volterra-Fredholm integrodifferential equations in Banach spaces. The technique used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative and rely on a priori bounds of solution.

1. Introduction

Let $\mathbb{R}^n$ be the Euclidean n-space with norm $\|\cdot\|$. Let $B = C([0,b],\mathbb{R}^n)$ be the Banach space of all continuous functions from $[0,b]$ into $\mathbb{R}^n$ endowed with supremum norm

$$\|x\|_B = \sup\{\|x(t)\| : t \in [0,b]\}.$$

Now we study the mixed Volterra-Fredholm integrodifferential equations of the form

$$x'(t) = f\Big(t, x(t), \int_0^t k(t,s,x(s))ds, \int_0^b h(t,s,x(s))ds\Big), \quad t \in [0,b], \tag{1.1}$$

$$x(0) + g(x) = x_0, \tag{1.2}$$

where $f : [0,b]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, is a function, $k,h : [0,b]\times[0,b]\times\mathbb{R}^n \to \mathbb{R}^n$ are continuous functions and $g : B \to \mathbb{R}^n$ is given function, and x_0 is a given element of $\mathbb{R}^n$.

Several authors have investigated the problems of existence, uniqueness and other properties of solutions of the special forms of (1.1)-(1.2), see [1, 3, 4, 5, 7, 9, 10, 11] and some of the references given therein. The equations of the form (1.1)-(1.2) play an important role for abstract formulation of many initial, boundary value problems of perturbed differential equations, partial differential equations and partial integrodifferential equations which arise in various applications like chemical reaction kinetics. population dynamics, heat-flow in material with memory, viscoelastic and reaction diffusion problems.

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The work in nonlocal initial value problem(IVP for short) was initiated by Byszewski. In [2] Byszewski using the method of semigroups and the Banach fixed point theorem proved the existence and uniqueness of mild, strong and classical solution of first order IVP. For the importance of nonlocal conditions in different fields, the interesting reader is referred to [2] and the references cited therein. Recently, in an interesting paper Dhakne and Kendre [5] studied the existence of global solutions to (1.1) when $g = 0, x(0) = 0$.

The aim of the this paper is to study the existence of global solutions to (1.1)-(1.2). The main tool used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative, rely on a priori bounds of solutions. The interesting and useful aspect of the method employed here is that it yields simultaneously the global existence of solutions and the maximal interval of existence. We are motivated by the work of Dhakne and Kendre [5] and influenced by the work of Byszewski [2].

The paper is organized as follows: In Section 2, we present the preliminaries and hypotheses. Section 3 deals with the main result. Finally, in Section 4, we give an example to illustrate the application of our theorem.

2. Preliminaries and Hypotheses

Before proceeding to the main result, we shall set forth some preliminaries and hypotheses that will be used in our subsequent discussion.

Definition 2.1. Let $f \in L^1(0, b; \mathbb{R}^n)$. The function $x \in B$ given by

$$x(t) = x_0 - g(x) + \int_0^t f(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau, \int_0^b h(s, \tau, x(\tau))d\tau)ds, \quad (2.1)$$

for $t \in [0, b]$ is called the solution of the initial value problem (1.1)-(1.2).

Our results are based on the following lemma, which is a version of the topological transversality theorem given by Granas [6, p. 61].

Lemma 2.2 (Leray-Schauder Alternative). *Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F : S \to S$ be a completely continuous operator, and let*

$$\varepsilon(F) = \{x \in S : x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.$$

Then either $\varepsilon(F)$ is unbounded or F has a fixed point.

We list the following hypotheses for our convenience.

(H1) There exists a constant G such that

$$\|g(x)\| \le G, \quad \text{for } x \in \mathbb{R}^n.$$

(H2) There exists a continuous function $p : [0, b] \to \mathbb{R}_+ = [0, \infty)$ such that

$$\|\int_0^t k(t, s, x(s))ds\| \le p(t)\|x\|,$$

for every $t, s \in [0, b]$ and $x \in \mathbb{R}^n$.

(H3) There exists a continuous function $q : [0, b] \to \mathbb{R}_+$ such that

$$\|\int_0^b h(t, s, x(s))ds\| \le q(t)\|x\|,$$

for every $t, s \in [0, b]$ and $x \in \mathbb{R}^n$.

(H4) There exists a continuous function $l : [0, b] \to \mathbb{R}_+$ such that
$$\|f(t, x, y, z)\| \le l(t)K(\|x\| + \|y\| + \|z\|),$$
for every $t \in [0, b]$ and $x, y, z \in \mathbb{R}^n$, where $K : \mathbb{R}_+ \to (0, \infty)$ is continuous nondecreasing function satisfying
$$K(\alpha(t)x) \le \alpha(t)K(x),$$
and $\alpha(t)$ is defined as the function p.

3. Existence of a Solution

Theorem 3.1. *Suppose that* (H1)-(H4) *hold. Then the initial-value problem* (1.1)-(1.2) *has a solution x on $[0, b]$ provided b satisfies*
$$\int_0^b M(s)ds < \int_c^\infty \frac{ds}{K(s)}, \tag{3.1}$$
where $c = \|x_0\| + G$ and $M(t) = l(t)[1 + p(t) + q(t)]$ for $t \in [0, b]$.

Proof. To prove the existence of a solution of nonlinear mixed Volterra-Fredholm integrodifferential equations (1.1)-(1.2), we apply topological transversality theorem. First we establish the priori bounds on the solutions of the initial value problem
$$x'(t) = \lambda f(t, x(t), \int_0^t k(t, s, x(s))ds, \int_0^b h(t, s, x(s))ds), \quad t \in [0, b], \tag{3.2}$$
under the initial condition (1.2) for $\lambda \in (0, 1)$. Let $x(t)$ be a solution of the problem (3.2)-(1.2), then it satisfies the equivalent integral equation
$$x(t) = x_0 - g(x) + \lambda \int_0^t f(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau, \int_0^b h(s, \tau, x(\tau))d\tau)ds, \tag{3.3}$$
for $t \in [0, b]$. Using (3.3), hypotheses (H1)-(H4) and the fact that $\lambda \in (0, 1)$, we have
$$\begin{aligned}
\|x(t)\| &\le \|x_0 - g(x)\| + \|\int_0^t f(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau, \int_0^b h(s, \tau, x(\tau))d\tau)ds\| \\
&\le [\|x_0\| + G] + \int_0^t \|f(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau, \int_0^b h(s, \tau, x(\tau))d\tau)\|ds \\
&\le [\|x_0\| + G] + \int_0^t l(s)K(\|x(s)\| + p(s)\|x(s)\| + q(s)\|x(s)\|)ds \\
&\le [\|x_0\| + G] + \int_0^t l(s)(1 + p(s) + q(s))K(\|x(s)\|)ds.
\end{aligned} \tag{3.4}$$
Denoting by $u(t)$ the right-hand side of the above inequality, we have
$$u(t) = [\|x_0\| + G] + \int_0^t l(s)(1 + p(s) + q(s))K(\|x(s)\|)ds.$$
Then $\|x(t)\| \le u(t)$ and $u(0) = [\|x_0\| + G] = c$. Therefore,
$$\begin{gathered}
u(t) = c + \int_0^t l(s)(1 + p(s) + q(s))K(\|x(s)\|)ds \\
u(t) \le c + \int_0^b l(s)(1 + p(s) + q(s))K(u(s))ds.
\end{gathered}$$

Differentiating $u(t)$ and using the fact that K is increasing continuous, we get

$$\begin{aligned} u'(t) \le l(t)(1+p(t)+q(t))K(u(t)) \le l(t)(1+p(t)+q(t))K(u(t)), \\ \frac{u'(t)}{K(u(t))} \le M(t). \end{aligned} \tag{3.5}$$

Integrating from 0 to t and using change of variables $t \to s = u(t)$ and the condition (3.1), we obtain

$$\int_c^{u(t)} \frac{ds}{K(s)} \le \int_0^t M(s)ds \le \int_0^b M(s)ds < \int_c^\infty \frac{ds}{K(s)}. \tag{3.6}$$

From this inequality and the mean value theorem we observe that there exists a constant γ, independent of $\lambda \in (0,1)$ such that $u(t) \le \gamma$ for $t \in [0,b]$ and hence $\|x(t)\| \le \gamma$ for $t \in [0,b]$ and consequently, we have

$$\|x\|_B = \sup\{\|x(t)\| : t \in [0,b]\} \le \gamma.$$

Now, we rewrite (1.1)-(1.2) as follows: If $y \in B$ and $x(t) = x_0 - g(x) + y(t)$, $t \in [0,b]$, where $y(t)$ satisfies

$$\begin{aligned} y(t) = \int_0^t f(s, y(s) + x_0 - g(y), \int_0^s k(s,\tau, y(\tau) + x_0 - g(y))d\tau, \\ \int_0^b h(s,\tau, y(\tau) + x_0 - g(y))d\tau)ds, \quad t \in [0,b] \end{aligned}$$

if and only if $x(t)$ satisfies

$$x(t) = x_0 - g(x) + \int_0^t f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^b h(s,\tau,x(\tau))d\tau)ds, \quad t \in [0,b].$$

We define the operator $F : B_0 \to B_0$, $B_0 = \{y \in B : y(0) = 0\}$ by

$$\begin{aligned} (Fy)(t) = \int_0^t f(s, y(s) + x_0 - g(y), \int_0^s k(s,\tau, y(\tau) + x_0 - g(y))d\tau, \\ \int_0^b h(s,\tau, y(\tau) + x_0 - g(y))d\tau)ds, \quad t \in [0,b]. \end{aligned} \tag{3.7}$$

Then F is clearly continuous.

Next, we prove that F is completely continuous. Let $\{w_m\}$ be a bounded sequence in B_0, i.e. $\|w_m\|_B \le d$ for all m, where d is a positive constant. From the definition of operator F and using the hypotheses $(H_1) - (H_4)$ and the fact that $\|w_m\|_B \le d$, we obtain

$$\begin{aligned} \|(Fw_m)(t)\| &\le \int_0^t \|f(s, w_m(s) + x_0 - g(w_m), \int_0^s k(s,\tau, w_m(\tau) + x_0 - g(w_m))d\tau, \\ &\quad \int_0^b h(s,\tau, w_m(\tau) + x_0 - g(w_m))d\tau)\|ds \\ &\le \int_0^t l(s)[1 + p(s) + q(s)]K(\|w_m(s) + x_0 - g(w_m)\|)ds \\ &\le \int_0^t l(s)[1 + p(s) + q(s)]K(d + c)ds \\ &\le M^* K(d + c)b, \end{aligned}$$

where $M^* = \sup\{M(t) : t \in [0,b]\}$. This implies that the set $\{(Fw_m)(t) : \|w_m\|_B \leq d, \quad 0 \leq t \leq b\}$ is uniformly bounded in $\mathbb{R}^n$.

Now we shall show that the sequence Fw_m is equicontinuous. Let $t_1, t_2 \in [0,b]$, Then from the definition of operator F and using the hypotheses $(H_1) - (H_4)$ and the fact that $\|w_m\|_B \leq d$, we have

$$\begin{aligned}
&\|(Fw_m)(t_1) - (Fw_m)(t_2)\| \\
&\leq \int_{t_1}^{t_2} \|f(s, w_m(s) + x_0 - g(w_m), \int_0^s k(s,\tau,w_m(\tau) + x_0 - g(w_m))d\tau, \\
&\quad \int_0^b h(s,\tau,w_m(\tau) + x_0 - g(w_m))d\tau)\|ds \\
&\leq \int_{t_1}^{t_2} l(s)[1 + p(s) + q(s)]K(\|w_m(s) + x_0 - g(w_m)\|)ds \\
&\leq \int_{t_1}^{t_2} l(s)[1 + p(s) + q(s)]K(d + c)ds \\
&\leq M^* K(d + c)(t_2 - t_1),
\end{aligned} \tag{3.8}$$

where $M^* = \sup\{M(t) : t \in [0,b]\}$. From (3.8) we conclude that $\{Fw_m\}$ is equicontinuous and hence by Arzela-Ascoli theorem the operator F is completely continuous.

Finally, the set

$$\varepsilon(F) = \{y \in B_0 : y = \lambda F y,\ \lambda \in (0,1)\}$$

is bounded in B, since for every $y \in \varepsilon(F)$, the function $x(t) = x_0 - g(x) + y(t)$ is a solution of (3.2)-(1.2) for which we have proved that $\|x\|_B \leq \gamma$ and hence $\|y\|_B \leq \gamma + c$. Consequently, by Lemma 2.2, the operator F has a fixed point in B_0. This means that the initial value problem (1.1)-(1.2) has a solution. This completes the proof of the theorem. $\square$

Remark 3.2. We note that in the special case, if we take (i) $M(t) = 1$ in condition (3.1) and the integral on the right side in (3.1) is assumed to diverge, then the solutions of equations (1.1)-(1.2) exist for every $b < \infty$.

4. Application

In this section we apply some of the results established in this paper. First, we consider the partial firs-order differential equation with nonlocal condition

$$w_t(u,t) = P(t, w(u,t), \int_0^t k_1(t,s,w(u,s))ds, \int_0^b h_1(t,s,w(u,s))ds), \tag{4.1}$$

$$w(0,t) = w(\pi,t) = 0, \quad 0 \leq t \leq b, \tag{4.2}$$

$$w(u,0) + g(w(u,t)) = w_0(u), \quad 0 \leq u \leq \pi, \tag{4.3}$$

where $P : [0,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is a function and $k_1, h_1 : [0,b] \times [0,b] \times \mathbb{R} \to \mathbb{R}$ are continuous. We assume that the functions P, k_1 and h_1 in (4.1)-(4.3) satisfy the following conditions.

(1) There exists a constant G such that $|g(x)| \leq G$, for $x \in \mathbb{R}$.

(2) There exists a nonnegative function p_1 defined on $[0,b]$ such that

$$\Big|\int_0^t k_1(t,s,x)ds\Big| \le p_1(t)|x|$$

for $t,s \in [0,b]$ and $x \in \mathbb{R}$.

(3) There exists a nonnegative function q_1 defined on $[0,b]$ such that

$$\Big|\int_0^b h_1(t,s,x)ds\Big| \le q_1(t)|x|$$

for $t,s \in [0,b]$ and $x \in \mathbb{R}$.

(4) There exists nonnegative real valued continuous function l_1 defined on $[0,b]$ and a positive continuous increasing function K_1 defined on $\mathbb{R}_+$ such that

$$|P(t,x,y,z)| \le l_1(t)K_1(|x|+|y|+|z|)$$

for $t \in [0,b]$ and $x,y,z \in \mathbb{R}$.

Let us take $X = L^2[0,\pi]$. Suppose that

$$\int_0^b l_1(s)(1+p_1(s)+q_1(s))ds < \int_c^\infty \frac{ds}{K_1(s)}$$

is satisfied, where $c = \|w_0\| + G$. Define the functions $f : [0,b] \times X \times X \times X \to X$, $k,h : [0,b] \times [0,b] \times X \to X$ as follows

$$\begin{gathered} f(t,x,y,z)(u) = P(t,x(u,t),y(u,t),z(u,t)), \\ k(t,s,x)(u) = k_1(t,s,x(u,t)) \quad \text{and} \\ h(t,s,x)(u) = h_1(t,s,x(u,t)) \end{gathered}$$

for $t \in [0,b], x,y,z \in X$ and $0 \le u \le \pi$. With these choices of the functions, the equations (4.1)-(4.3) can be modelled abstractly as nonlinear mixed Volterra-Fredholm integrodifferential equation with nonlocal condition in Banach space X:

$$x'(t) = f(t,x(t),\int_0^t k(t,s,x(s))ds,\int_0^b h(t,s,x(s))ds), \quad t \in [0,b], \tag{4.4}$$

$$x(0) + g(x) = x_0. \tag{4.5}$$

Since all the hypotheses of the Theorem 3.1 are satisfied, the Theorem 3.1 can be applied to guarantee the solution of the nonlinear mixed Volterra-Fredholm partial integrodifferential equation (4.1)-(4.3) with nonlocal condition.

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Haribhau L. Tidke
Department of Mathematics, School of Mathematical Sciences, North Maharashtra University, Jalgaon-425 001, India
E-mail address: tharibhau@gmail.com