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STABILITY PROPERTIES OF DIFFERENTIAL SYSTEMS UNDER CONSTANTLY ACTING PERTURBATIONS

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In memory of Corrado Risito

Abstract. In this article, we find stability criteria for perturbed differential systems, in terms of two measures. Our main tool is a definition of total stability based on two classes of perturbations.

1. Introduction

Let $\mathbb{R}^+$ denote the interval $0 \leq t < \infty$, and $\mathbb{R}^n$ the n dimensional Euclidean space with the corresponding norm $\|x\|$ for $x \in \mathbb{R}^n$. Let us consider the Cauchy problem

$$\dot{x} = X(t, x), \quad x(t_0) = x_0 \tag{1.1}$$

and assume that $X(t, 0) = 0$ for $t \in \mathbb{R}^+$. Note that this differential system has the null solution $x = 0$.

In the classical total stability theory, it is required that the null solution be stable, not only with respect to (small) perturbations of the initial conditions but also with respect to the perturbations of the right-hand side of the equation. To this end, we associate to the unperturbed system (1.1) a corresponding family of perturbed systems

$$\dot{x} = X(t, x) + X_p(t, x), \quad x(t_0) = x_0. \tag{1.2}$$

This differential system may not possess null solution, because we assume only that the right-hand side of (1.2) be suitably smooth in order to ensure existence, uniqueness and continuous dependance of solutions for the initial value problem.

For the convenience of the reader, we recall that the null solution of (1.1) is said to be *totally uniformly stable*, according to Dubosin-Malkin Definition [4, 18], provided that for arbitrary positive ϵ and $t_0 \geq 0$ there are $\delta_1 = \delta_1(\epsilon) > 0$ and $\delta_2 = \delta_2(\epsilon) > 0$ such that whenever $\|x_0\| < \delta_1$ and $\|X_p\| < \delta_2$, the inequality $\|x(t, t_0, x_0)\| < \epsilon$ is satisfied for all $t \geq t_0$. Notice that in the classical total stability theory (and in the present paper) the symbol $x(t) = x(t, t_0, x_0)$ denotes the solution of (1.2) through a point (t_0, x_0).

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We emphasize that our stability criteria in Section 3 generalize two well-known Malkin theorems. In fact, a Malkin theorem [12, 18] on the *total uniform stability* is included as a special case in Theorem 3.1, while Theorem 3.6 improves another Malkin theorem [13, 18]. There under appropriate hypotheses Malkin proves that

> For arbitrary positive ϵ and $t_0 \geq 0$, there are $\delta_1 = \delta_1(\epsilon) > 0$ and $\delta_2 = \delta_2(\epsilon) > 0$ and for any $\eta \in]0, \epsilon[$ there is $\delta_3(\eta) \in]0, \delta_2]$ such that whenever $\|x_0\| < \delta_1$ and $\|X_p\| < \delta_3$ there exists a constant $T_\eta > 0$, such that $\|x(t, t_0, x_0)\| < \epsilon$ is satisfied for all $t \geq t_0 + T_\eta$.

It is worth noting that this property first comes the concept of the *strong stability under perturbations in generalized dynamical systems* introduced by Seibert [21].

The aim of the present article is to introduce and study a new type of total stability in terms of two measures, by splitting the perturbation terms X_p in two parts. Namely, by putting $X_p = Y + Z$. In Sections 3,4, 5, we require the usual upper restriction on the Euclidean norm of vector Z, while we select vector Y by an appropriate scalar product. In Section 6, a mechanical example illustrates our theoretical results.

2. Preliminaries, notation and basic ideas

Let $K := \{a : \mathbb{R}^+ \to \mathbb{R}^+ : \text{continuous, strictly increasing}, a(0) = 0\}$ be the set of functions of class K in the sense of Hahn. We shall define some concepts in terms of two measures [8, 9, 15]. Namely, we denote by $h(t, x)$ and $h_0(t, x)$ two continuous scalar functions satisfying the conditions:

(i) $\inf_x h_0(t, x) = 0$ for every $t \in \mathbb{R}$;
(ii) there exists a positive constant λ and a function $m = m(u) \in K$ such that $h_0(t, x) < \lambda$ implies $h(t, x) \leq m[h_0(t, x)] < m(\lambda)$.

In mathematical language, condition (ii) means that h_0 is *uniformly finer* than h, and it implies that $\inf_x h(t, x) = 0$ for every $t \in \mathbb{R}$.

Putting $Q(s) = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : 0 < h(t, x) \leq s\}$, we observe that $0 < s' < s$ implies $Q(s') \subseteq Q(s)$, and moreover the intersection $\cap Q(s)$ for all $s > 0$ is the empty set. Hence, the set of the sets $\{Q(s)\}$ represents a Cartan-Silov direction or, simply, a direction.

The above theoretical concepts are essential in the following definition: For every scalar $V = V(t, x)$ we say that $\lim_{h \to 0} V(t, x) = 0$ if and only if for every direction such that $\lim h(t, x) = 0$, we have $\lim V(t, x) = 0$, see [22].

Denote by $U = U(t, x)$ and $G = G(t, x)$ respectively a continuous scalar function and a continuous n-vector function such that $\|G\| > 0$ on $\mathbb{R}^+ \times \mathbb{R}^n$. For the unperturbed differential system

$$\dot{x} = X(t, x), \quad x(t_0) = x_0 \tag{2.1}$$

and a correspondent perturbed differential system

$$\dot{x} = X(t, x) + Y(t, x) + Z(t, x) \quad x(t_0) = x_0 \tag{2.2}$$

without further mention, we will assume that $YG \leq U$, where YG denotes the scalar product of the vectors Y and G. Moreover, we assume that the right-hand sides of (2.1) and (2.2), are L-measurable in $t \in \mathbb{R}^+$, continuous in $x \in \mathbb{R}^n$. Also we assume that for every compact subset $A \subset \mathbb{R}^n$ there exists a map $\sigma_A = \sigma_A(t)$ locally integrable such that $\|X(t, x)\|$, $\|Y(t, x)\|$, $\|Z(t, x)\| < \sigma_A(t)$ when $x \in A$.

The previous conditions (Caratheodory's conditions) ensure the existence and the general continuity of solutions for (2.1) and (2.2); see [2, 3]. Then, for every $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ we denote by $x(t) = x(t, t_0, x_0)$ a solution of (2.2), and we assume that $x(t)$ is defined for $t \geq t_0$.

For every continuous scalar function $V = V(t, x)$ having continuous partial derivatives we put

$$V_t = \frac{\partial V}{\partial t}, \quad V_x = \operatorname{grad} V = \frac{\partial V}{\partial x}, \quad \dot{V}_1 = V_t + V_x \cdot X = \dot{V}. \tag{2.3}$$

The function $\dot{V}$ is said to be the derivative of V computed along the solutions of the unperturbed system (2.1). While the related formula given by Malkin [12, 18],

$$\dot{V}_2(t, x) = \dot{V}(t, x) + V_x(t, x)[Y(t, x) + Z(t, x)] \tag{2.4}$$

gives the derivative of V along the solutions of the perturbed system (2.2).

If ϕ and θ are two scalar functions, it easy to prove the following results which will be used in the next sections.

(i) if $V_x = \phi G$ and $YG \leq U$, when $\phi > 0$ we deduce

$$\dot{V}_2(t, x) = \dot{V}(t, x) + \phi(t, x)[GY(t, x) + GZ(t, x)], \tag{2.5}$$

$$\dot{V}_2(t, x) \leq \dot{V}(t, x) + \phi(t, x)[U(t, x) + \|G(t, x)\| \|Z(t, x)\|]; \tag{2.6}$$

(ii) if $U(t, x) \leq 0$ and $\phi(t, x) > 0, V_x = \phi G$ we deduce

$$\dot{V}_2(t, x) \leq \dot{V}(t, x) + \phi(t, x)\|G(t, x)\| \|Z(t, x)\|; \tag{2.7}$$

(iii) if $U(t, x) \geq 0$, $\phi(t, x) > 0$, $\theta(t, x) > 0$, $V_x(t, x) = \phi G(t, x)$ and $\dot{V}(t, x) \leq -\theta U(t, x)$, we deduce

$$\dot{V}_2(t, x) \leq -[\theta(t, x) - \phi(t, x)]U(t, x) + \phi(t, x)\|G(t, x)\| \|Z(t, x)\|. \tag{2.8}$$

We conclude the present section with a list of definitions concerning the several kinds of the stability in terms of two measures and two perturbations.

Definition 2.1. System (2.1) is said to be (h_0, h)-*stable under two persistent perturbations*, also called (h_0, h)-t.bistable, if for every $t_0 \in \mathbb{R}^+$ and every $\epsilon > 0$, there exist a number $\delta_1 = \delta_1(t_0, \epsilon)$ and a function $\delta_2 = \delta_2(t_0, x, \epsilon) > 0$ such that for all $x_0 \in \mathbb{R}^n$ with $h_0(t_0, x_0) < \delta_1$, all $Z(t, x)$ with $\|Z(t, x)\| < \delta_2$, and all Y with $YG \leq U$; we have $h[t, x(t)] < \epsilon$ when $t \geq t_0$.

If $\delta_1 = \delta_1(\epsilon)$ and $\delta_2 = \delta_2(\epsilon)$ are independent of t_0 and x, we have the uniformity.

Definition 2.2. System (2.1) is said to be *strongly weakly* (h_0, h)-*t.bistable*, if in Definition 2.1, $\delta_2(t_0, x, \epsilon) \geq 0$, and the L-measure of set

$$E_t(\delta_2 = 0) = \{x \in \mathbb{R}^n : \delta_2(t_0, x, \epsilon) = 0\} \tag{2.9}$$

is zero for $t_0 \in \mathbb{R}^+$ and $\epsilon > 0$.

In the following we will briefly write $\delta_2 \in GG$ to indicate this condition.

Definition 2.3. System (2.1) is said to be *weakly* $(h_0, h) - t.bistable$ if, for every $t_0 \in \mathbb{R}^+$ and $\epsilon > 0$, there exists at the most one $x \in \mathbb{R}^n$ such that $\delta_2(t_0, x, \epsilon) = 0$.

In the following this condition will be briefly denoted as $\delta_2 \in ZZ$.

Definition 2.4. System (2.1) is said to be (h_0, h)-*eventually stable under two persistent perturbations*, also called eventually (h_0, h)-t.bistable, if: For every $\epsilon > 0$ there exists $T = T(\epsilon) > 0$, for every $t_0 \geq T$ there exist $\delta_1 = \delta_1(t_0, \epsilon)$ and

$\delta_2 = \delta_2(t_0, x, \epsilon) > 0$ such that for every $x_0 \in \mathbb{R}^n$ with $h_0(t_0, x_0) < \delta_1$, for every Z with $\|Z(t,x)\| < \delta_2$ and every Y with $YG \le U$, we have $h[t, x(t)] < \epsilon$ when $t \ge t_0$.

Definition 2.5. System (2.1) is said to be (h_0, h)-*semiattractive*, if it is (h_0, h)-t.bistable and: For every $\eta \in]0, \epsilon[$ there exists a function $\delta_3 > 0$, with $0 < \delta_3 \le \delta_2$, such that for every $Z : \|Z\| < \delta_3$ and every Y with $YG \le U$, there exists $T_\eta > 0$ for which $h[t, x(t)] < \eta$ when $t \ge t_0 + T_\eta$, where $x(t) = x(t, t_0, x_0)$ is a solution of (2.2).

Definition 2.6. System (2.1) is said to be (h_0, h)-*stable on average under two persistent perturbation*, also called (h_0, h)-*t.bistable on average*, if: For every $t_0 \in \mathbb{R}^+$, every $\epsilon > 0$ and every $T > 0$, there exist δ_1 and $\delta_2 > 0$ such that every solution $x(t) = x(t, t_0, x_0)$ of (2.2) with $h_0(t_0, x_0) < \delta_1$, $YG \le U$, and

$$\int_t^{t+T} \sup\{\|Z(u,x)\| : x \in \mathbb{R}^n\} du < \delta_2 \quad \forall t \ge t_0 \tag{2.10}$$

satisfies $h[t, x(t)] < \epsilon$ for all $t \ge t_0$.

3. Theoretical developments

Suppose that the functions X, G, U are the known start point. We will use the technique that is known as family of Liapunov functions introduced by Salvadori [20]. The basic advantage of this method is that the single function needs to satisfy less rigid requirements than in other methods.

Theorem 3.1. *Let $U : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ be given. Assume that for every $\epsilon > 0$, there exist three scalar functions $\Theta = \Theta(t,x)$, $\phi = \phi(t,x) \in C$, and $V = V(t,x) \in C^1$, and exists a constant l such that on the set $\mathbb{R}^+ \times \mathbb{R}^n$ we have:*

(i) $h(t,x) = \epsilon$ *implies* $V(t,x) \ge l > 0$;
(ii) $\lim_{h \to 0} V(t,x) = 0$;
(iii) $\Theta(t,x) > \phi(t,x) > 0$ *and* $(\Theta - \phi)U > 0$;
(iv) $V_x(t,x) = \phi G(t,x)$;
(v) $\dot{V}(t,x) \le -\Theta U(t,x)$.

Then system (2.1) *is* (h_0, h)-*t.bistable.*

Proof. Given $t_0, \epsilon, l, \Theta, \phi, V$, by (ii) there exists $d > 0$ such that $h(t_0, x) < d$ implies $V(t_0, x) < l$. If we select $x_0 \in \mathbb{R}^n$ such that $h_0(t_0, x_0) < \delta_1 = \min[\lambda, m^{-1}(d)]$ for the previous assumptions we recognize that $h(t_0, x_0) < m[h_0(t_0, x_0)] < d$, hence $V(t_0, x_0) < l$. From the Malkin formula (2.4), according to (iv) and (v), we deduce

$$\dot{V}_2(t,x) \le -(\Theta - \phi)U(t,x) + \phi\|G(t,x)\|\|Z(t,x)\|. \tag{3.1}$$

Then by selecting

$$\|Z(t,x)\| \le \frac{(\Theta - \phi)U(t,x)}{\|\phi G(t,x)\|} = \delta_2(t, x, \epsilon)$$

it follows that $\dot{V}_2(t,x) \le 0$.

Consider a solution $x(t) = x(t, t_0, x_0)$ of the perturbed system and the correspondent functions $h_1(t) = h[t, x(t)], V_1(t) = V[t, x(t)]$. If there exists $t' > t_0$ such that $h_1(t') = \epsilon$ with $h_1(t) < \epsilon$ for $t \in [t_0, t'[$ then we should deduce that $V_1(t') \ge l$, which is a contradiction. ☐

Remark. If $U \geq 0$ the system can be strongly weakly (h_0, h)-t.bistable or weakly (h_0, h)-t.bistable.

Corollary 3.2. *Suppose that there exist three scalar functions* $\Theta = \Theta(t, x), \phi = \phi(t, x) \in C$ *and* $V = V(t, x) \in C^1$ *such that on* $\mathbb{R}^+ \times \mathbb{R}^n$ *we have:*

(i) *for each* $\epsilon > 0$ *there exists* $l > 0$ *such that* $h(t, x) = \epsilon$ *implies* $V(t, x) \geq l > 0$*;*
(ii) $\lim_{h \to 0} V(t, x) = 0$*;*
(iii) $\Theta(t, x) > \phi(t, x) > 0$ *and* $(\Theta - \phi)U > 0$*;*
(iv) $V_x(t, x) = \phi G(t, x)$*;*
(v) $\dot{V}(t, x) \leq -\Theta U(t, x)$*.*

Then (2.1) *is* (h_0, h)*-t.bistable.*

Corollary 3.3. *For a scalar function* $U < 0$*, suppose that there exist three scalar functions* $L = L(t, x)$*,* $\phi = 1$*,* $V = V(t, x)$ *such that the conditions (i)–(iv) in Corollary 3.2 hold, and that* $\dot{V}(t, x) \leq -L(t, x) < 0$*. Then* (2.1) *is* $(h_0, h)-$*t.bistable.*

Proof. From (2.7) we have
$$\dot{V}_2(t, x) \leq -L(t, x) + \|G\|\|Z\| \tag{3.2}$$
hence by choosing $\|Z\| \leq L/\|G\|$ we have the proof. □

Theorem 3.4. *Suppose that for every* $\epsilon > 0$ *there exist two scalar functions* $\phi = \phi(t, x)$*,* $\Theta = \Theta(t, x) \in C$*, a map* $N = N(u)$ *L-measurable, the scalar function* $V = V(t, x) \in C^1$*, and a constant* l *such that on* $\mathbb{R}^+ \times \mathbb{R}^n$ *we have:*

(i) $h(t, x) = \epsilon$ *implies* $V(t, x) \geq l > 0$*;*
(ii) $\lim_{h \to 0} V(t, x) = 0$*;*
(iii) $\dot{V}(t, x) \leq -\Theta U(t, x) + N(t)$ *with* $0 < \int_0^{+\infty} N(u)du < +\infty$ *and* $U > 0$ *(a hypothesis of Hatvani's type);*
(iv) $\Theta(t, x) \geq \phi(t, x) > 0$ *and* $(\Theta - \phi)U > 0$*;*
(v) $V_x(t, x) = \phi G(t, x)$*.*

Then (2.1) *is eventually* (h_0, h)*-t.bistable.*

Proof. Given $\epsilon > 0$ we consider the function
$$W(t, x) = V(t, x) + \int_t^{+\infty} N(u)du \quad (t > 0). \tag{3.3}$$
Let $T > 0$ such that $2\int_{t_0}^{+\infty} N(u)du < l$ for $t_0 \geq T$, and let $d > 0$ such that $h(t_0, x) < d$ implies (by (ii)) $2V(t_0, x) < l$. If $x_0 \in \mathbb{R}^n$ and $h_0(t_0, x_0) < \delta_1 = \min[\lambda, m^{-1}(d)]$, we deduce that $h(t_0, x_0) \leq m[h_0(t_0, x_0)] < d$ and $2V(t_0, x_0) < l$. Then it follows that $W(t_0, x_0) < l$. Consider the derivatives
$$\dot{W}(t, x) = \dot{V}(t, x) - N(t) \leq -\Theta U(t, x) < 0, \tag{3.4}$$
$$\dot{W}_2(t, x) \leq -(\Theta - \phi)U(t, x) + \phi\|G(t, x)\|\|Z(t, x)\|. \tag{3.5}$$
Provided that
$$\|Z(t, x)\| \leq \frac{(\Theta - \phi)U(t, x))}{\phi\|G(t, x)\|} = \delta_2(t, x, \epsilon) \tag{3.6}$$
we obtain $\dot{W}_2(t, x) \leq 0$. Selecting $Z = Z(t, x)$ such that $\|Z(t, x)\| \leq \delta_2$, consider $x(t) = x(t, t_0, x_0)$ a solution of the perturbed system (2.2), and put $H(t) = h[t, x(t)]$, $v(t) = V[t, x(t)]$, $w(t) = W[t, x(t)]$. Suppose that there exists $t' > t_0$ such that

$H(t') = \epsilon$ and $H(t) < \epsilon$ for $t_0 \leq t < t'$. So we have $v(t') > l$ hence $w(t') > l$. This is a contradiction which completes the proof. □

Lemma 3.5. *Suppose that there exist four scalar functions $\phi = \phi(t,x)$, $U = U(t,x)$, $\Theta = \Theta(t,x) \in C$, $V = V(t,x) \in C^1$, a scalar function $\Psi = \Psi(t,h)$ L-integrable with respect to $t \in \mathbb{R}^+$, a map $a = a(u) \in K$ such that on $\mathbb{R}^+ \times \mathbb{R}^n$, we have*

- (i) $V(t,x) \geq a[h(t,x)]$;
- (ii) $\lim_{h\to 0} V(t,x) = 0$;
- (iii) $V_x(t,x) = \phi G(t,x)$;
- (iv) $\dot V(t,x) \leq -\Theta U(t,x) < 0$;
- (v) $\Theta(t,x) > \phi(t,x) > 0$ *and* $(\Theta - \phi)U > 0$;
- (vi) $(\Theta - \phi)U(t,x) = \Psi[t, h(t,x)]$;
- (vii) $\Psi(t,h) \geq \Psi(t,\mu)$ *when* $h \geq \mu > 0$;
- (viii) $\int_{t'}^{+\infty} \Psi(\tau,\rho)d\tau = +\infty$, *for all* $t' \in I$, *all* $\rho > 0$.

Then (2.1) is (h_0,h)-t.bistable. Also for every $\epsilon > 0$, for every $\eta \in]0,\epsilon]$, for each $\gamma > 0$ for every $t_0 \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}^n$ with $h_0(t_0,x_0) < \delta_1$, for every Z with

$$\|Z(t,x)\| \leq \big(\frac{1}{1+\gamma}\big)\frac{(\Theta-\phi)U(t,x)}{\phi\|G(t,x)\|} = \delta_3; \tag{3.7}$$

there exists $t_\eta \geq t_0$ for which $h[t_\eta, x(t_\eta)] < \eta$ where $x = x(t,t_0,x_0)$ is solution of (2.2).

Proof. By contradiction let us assume that that there exist $\epsilon_1 > 0$, $\eta_1 \in]0,\epsilon_1]$, $(t_1,x_1) \in \mathbb{R}^+ \times \mathbb{R}^n$: $h_0(t_1,x_1) < \delta_1, \gamma_1 > 0$, $Z_1 : \|Z_1(t,x)\| < \delta_3$ (depending on γ_1) such that $h[t,x_1(t)] \geq \eta_1$ when $t \geq t_1$ where $x_1(t) = x(t,t_1,x_1)$ is obviously a solution of (2.2).

Consider the derivative $\dot V_2(t,x)$: by hypotheses (iii) and (iv) we have

$$\dot V_2(t,x) = \dot V + \phi G Y(t,x) + \phi G Z(t,x) \leq -(\Theta-\phi)U + \phi\|G\|\|Z\| \tag{3.8}$$

Thus selecting

$$\|Z(t,x)\| \leq \frac{1}{1+\gamma_1}\frac{(\Theta-\phi)U(t,x)}{\phi\|G(t,x)\|} = \frac{1}{1+\gamma_1}\delta_2 = \delta_3 \tag{3.9}$$

we obtain

$$\dot V_2(t,x) \leq -\frac{\gamma_1}{1+\gamma_1}(\Theta-\phi)U < 0. \tag{3.10}$$

hence by (vi)

$$\dot V_2(t,x) \leq -\frac{\gamma_1}{1+\gamma_1}\Psi[t,h(t,x)]. \tag{3.11}$$

On the set $\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : h(t,x) \geq \eta_1 > 0\}$ we have $\Psi[t,h(t,x)] \geq \Psi(t,\eta_1)$ and

$$\dot V_2(t,x) \leq -\frac{\gamma_1}{1+\gamma_1}\Psi(t,\eta_1). \tag{3.12}$$

Along the above solution $x_1(t)$ we obtain, for $t \geq t_1$,

$$\int_{t_1}^{t} \dot V_2[u,x_1(u)]du \leq -\frac{\gamma_1}{1+\gamma_1}\int_{t_1}^{t}\Psi(u,\eta_1)du, \tag{3.13}$$

$$V[t,x_1(t)] \leq V(t_1,x_1) - \frac{\gamma_1}{1+\gamma_1}\int_{t_1}^{t}\Psi(u,\eta_1)du \tag{3.14}$$

which is a contradiction. □

Theorem 3.6. *Under the hypotheses of Lemma 3.5 suppose that*

(ix) *There exist* $b = b(u) \in K$ *such that* $b[h_0(t,x)] \leq h(t,x)$ *on* $\mathbb{R}^+ \times \mathbb{R}^n$.

Then, for every $\epsilon > 0$ *and* $\sigma \in]0,\epsilon]$, *there exists a function* $\delta_3 = \delta_3(t,x,\sigma) \in]0,\delta_2]$ *such that: for every* $t_0 \in \mathbb{R}^+$, *for every* $x_0 \in \mathbb{R}^n$ *with* $h_0(t_0,x_0) < \delta_1$, *and for every* $Z : \|Z(t,x)\| < \delta_3$; *there exists* $T_\sigma > 0$ *for which* $h[t,x(t)] < \sigma$ *when* $t \geq t_0 + T_\sigma$ *where* $x(t) = x(t,t_0,x_0)$ *is a solution of* (2.2).

Proof. Since the system (2.1) is (h_0,h)-t.bistable, given $t_0 \in \mathbb{R}^+$ and $\epsilon > 0$ there exist $\delta_1 = \delta_1(t_0,\epsilon)$ and $\delta_2 = \delta_2(t,x,\epsilon) > 0$ such that fixed $x_0 \in \mathbb{R}^n$ for which $h_0(t_0,x_0) < \delta_1$ and select $Z : \|Z(t,x)\| < \delta_2$ we have $h[t,x(t)] < \epsilon$ for $t \geq t_0$ where $x(t) = x(t,t_0,x_0)$ is a solution of (2.2).

It is obvious that for every $\sigma \in]0,\epsilon[$ there exist $d_1 \in]0,\delta_1[$ and $d_2 \in]0,\delta_2[$ such that fixed $(t_1,x_1) \in \mathbb{R}^+ \times \mathbb{R}^n$ for which $h_0(t_1,x_1) < d_1$, select $Z : \|Z(t,x)\| < d_2$ we have $h[t,x_1(t)] < \sigma$ for $t \geq t_1$ where $x_1(t) = x_1(t,t_1,x_1)$ is a solution of (2.2).

From Lemma 3.5, given $\eta \in]0,\sigma[(\subset]0,\epsilon]$ there exists $d_3 \in]0,d_2[$ such that for every $Z : \|Z(t,x)\| \leq d_3$ there exists $t_\eta \geq t_0$ for which $h[t_\eta,x(t_\eta)] < \eta$ where $x(t) = x(t,t_0,x_0)$ is a solution of (2.2).

If we assume that $\eta = b(d_1)$, we obtain

$$b\{h_0[t_\eta,x(t_\eta)]\} \leq h[t_\eta,x(t_\eta)] < \eta = b(d_1); \tag{3.15}$$

i.e., $h_0[t_\eta,x(t_\eta)] < d_1$. Hence when $\|Z(t,x)\| \leq d_3$ we have $h[t,x(t)] \leq \sigma$ for $t \geq t_\eta$. Putting $T_\eta = t_\eta - t_0$ we then obtain the semiattractivity. □

Theorem 3.7. *Suppose that there exist three functions from* $R \times \mathbb{R}^n$ *to* R: $U = U(t,x)$, $\phi = \phi(t,x) \in C$, $V = V(t,x) \in C^1$; *three functions* $a = a(u)$, $b = b(u)$, $c = c(u)$ *belonging to* K; *and a constant* $N > 0$; *such that on* $\mathbb{R}^+ \times \mathbb{R}^n$, *we have:*

(i) $a[h(t,x)] \leq V(t,x) \leq b[h(t,x)]$;
(ii) $\phi(t,x) > 0$;
(iii) $V_x(t,x) = \phi G(t,x)$, $\|V_x(t,x)\| < N$;
(iv) $\dot V(t,x) \leq -c[h(t,x)]$;
(v) *given* $r,T,\epsilon > 0$, *put* $\nu = \frac{r}{T}\epsilon$: *the condition* $\nu < h(t,x) < \epsilon$ *implies* $\frac{\phi U(t,x)}{V(t,x)} < \frac{c(\nu)}{2b(\epsilon)}$.

Then (2.1) *is* (h_0,h)*-t.bistable on average.*

Proof. Given $t_0 \in \mathbb{R}^+$, ϵ and $T > 0$, from (i) $h(t,x) = \epsilon$ implies $V(t,x) \geq a(\epsilon)$. Select $d \in]0,\epsilon[$ such that:

(i) $h(t_0,x) < d$ implies $V(t_0,x) < a(\epsilon)$;
(ii) $b(d) < \frac{1}{2}a(\epsilon)$.

If $x_0 \in \mathbb{R}^n : h_0(t_0,x_0) < \delta_1 = \min[\lambda, m^{-1}(d)]$ we have $h(t_0,x_0) \leq m[h_0(t_0,x_0)] \leq d$ hence $V(t_0,x_0) < a(\epsilon)$. Let $x(t) = x(t,t_0,x_0)$ be a solution of (2.2) and suppose that there exist $t', t'' \in \mathbb{R}^+$ with the following properties:

(iii) $t_0 \leq t' < t''$;
(iv) $h(t'',x'') = h[t'',x(t'')] = \epsilon$;
(v) $h(t',x') = h[t',x(t')] = \min h[t,x(t)]$ and $h(t',x') \leq h[t,x(t)] \leq h(t'',x'')$ on $t' \leq t \leq t''$.

Put $W(t,x) = V(t,x)e^{\beta(t)}$, where $\beta = \beta(t) : \mathbb{R}^+ \to \mathbb{R}$ is a scalar function that will be defined in (3.23), and consider the derivatives

$$\dot{W}_1(t,x) = \dot{W}(t,x) = \dot{V}(t,x)e^{\beta(t)} + V(t,x)e^{\beta(t)}\dot{\beta}(t), \tag{3.16}$$

$$\dot{W}_2(t,x) = \dot{W}(t,x) + W_x[Y(t,x) + Z(t,x)],$$

$$\dot{W}_2(t,x) = \dot{V}(t,x)e^{\beta(t)} + V(t,x)e^{\beta(t)}\dot{\beta}(t) + e^{\beta(t)}[V_xY(t,x) + V_xZ(t,x)],$$

$$\dot{W}_2(t,x) = W(t,x)\Big[\frac{\dot{V}(t,x)}{V(t,x)} + \dot{\beta}(t) + \frac{V_xY(t,x)}{V(t,x)} + \frac{V_xZ(t,x)}{V(t,x)}\Big] \tag{3.17}$$

if we select $Y(t,x)$ such that $V_xY \le U$ we obtain

$$\dot{W}_2(t,x) \le W(t,x)\Big\{\frac{\dot{V}(t,x)}{V(t,x)} + \dot{\beta}(t) + \frac{\phi U(t,x)}{V(t,x)} + \frac{\|V_x(t,x)\|}{V(t,x)}\|Z(t,x)\|\Big\}. \tag{3.18}$$

On the set

$$\{A\} = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : d = \frac{r}{T}\epsilon < h(t,x) < \epsilon\}$$

for suitable $r \in]0,T[$ we have

$$(s)\frac{\dot{V}(t,x)}{V(t,x)} < -\frac{c(d)}{b(\epsilon)}; \qquad (ss)\frac{\phi U(t,x)}{V(t,x)} < \frac{c(d)}{2b(\epsilon)} \tag{3.19}$$

hence

$$\dot{W}_2(t,x) \le W(t,x)\big\{\dot{\beta}(t) - \frac{c(d)}{2b(\epsilon)} + \frac{N}{a(d)}\|Z(t,x)\|\big\}. \tag{3.20}$$

Fixed $t \in \mathbb{R}^+$, for every $x \in \mathbb{R}^n$ put $\phi(t) = sup\{\|Z(t,x)\|\}$. Given $q \in]0,1[$ we construct the function $\Psi = \Psi(t) : R \to \mathbb{R}$ such that the equalities

$$\begin{aligned} L(T) &= \int_{\mu T}^{(\mu+1)T} \Psi(u)du \\ &= \int_{\mu T}^{(\mu+1)T} \big\{\frac{(1-q)}{2}\frac{c(d)}{b(\epsilon)} - \frac{N}{a(d)}\phi(u)\big\}du \\ &= \frac{(1-q)c(d)}{2b(\epsilon)}T - \frac{N}{a(d)}\int_{\mu T}^{(\mu+1)T} \phi(u)du \end{aligned} \tag{3.21}$$

are fulfilled for every non negative integer μ.

If, for every $t \in \mathbb{R}^+$, we select $\|Z(t,x)\|$ such that

$$\int_{\mu T}^{(\mu+1)T} \phi(u)du \le \frac{(1-q)c(d)a(d)}{2b(\epsilon)N}T = \delta_2 \tag{3.22}$$

we obtain $L(T) \ge 0$. On the strength of the previous conditions we can take it such that $\Psi(t) \ge 0$ for all $t \ge 0$. We set, for $t \in \mathbb{R}^+$,

$$\beta(t) = \int_0^t \big[-\Psi(u) + \frac{(1-q)c(d)}{2b(\epsilon)} - \frac{N}{a(d)}\phi(u)\big]du. \tag{3.23}$$

consequently we recognize that $\beta(\mu T) = 0$ for every natural number μ. Also

$$\dot{\beta}(t) = -\Psi(t) + \frac{(1-q)c(d)}{2b(\epsilon)} - \frac{N}{a(d)}\phi(t), \tag{3.24}$$

$$\dot{W}_2(t,x) \le W(t,x)[-\Psi(t) - \frac{qc(d)}{2b(\epsilon)}] \le 0. \tag{3.25}$$

Assuming $\mu T \le t \le (\mu+1)T$, put

$$\Gamma(u) = -\Psi(u) + \frac{(1-q)c(d)}{2b(\epsilon)} - \frac{N}{a(d)}\phi(u),$$
$$\Delta(u) = \Psi(u) + \frac{(1-q)c(d)}{2b(\epsilon)} + \frac{N}{a(d)}\phi(u).$$

Consequently,

$$\beta(t) = \int_0^t \Gamma(u)du = \int_{\mu T}^t \Gamma(u)du \le \int_{\mu T}^{(\mu+1)T} \Delta(u)du, \tag{3.26}$$

$$\begin{aligned} |\beta(t)| &\le \Big| \int_{\mu T}^{(\mu+1)T} \big[\Psi(u) + \frac{(1-q)c(d)}{b(\epsilon)} + \frac{N}{a(d)}\phi(u)\big] du \Big| \\ &\le 3\frac{(1-q)c(d)}{b(\epsilon)} T = \Theta. \end{aligned} \tag{3.27}$$

Hence we obtain

$$W[t', x'] = V[t', x(t')]e^{\beta(t')} \le b(d)e^{\Theta} < \frac{1}{2}a(\epsilon)e^{\Theta}, \tag{3.28}$$

$$W[t'', x''] = V[t'', x(t'')]e^{\beta(t'')} \ge a(\epsilon)e^{-\Theta}. \tag{3.29}$$

and so according to (3.25) we have

$$\frac{1}{2}a(\epsilon)e^{\Theta} > b(d)e^{\Theta} \ge a(\epsilon)e^{-\Theta}, \quad \frac{1}{2} \ge e^{-2\Theta}. \tag{3.30}$$

Since $0 < q < 1$ is arbitrary we obtain a contradiction. (Oziraner theorem extension) □

4. Theoretical developments for inequalities of the second kind

In this section we assume, as start points, the functions X, V, and select Y from inequalities of the type (second kind)

$$F(V_x, W_x, \dot{V}, \dot{W}, Y) < 0.$$

This way we deduce some propositions very useful for applications.

Theorem 4.1. *Suppose that there exists a family of scalar functions $V = V(t,x) \in C^1$ such that on $\mathbb{R}^+ \times \mathbb{R}^n$ we have:*

(i) *for all $\epsilon > 0$ there exists $l > 0$ such that $h(t,x) = \epsilon$ implies $V(t,x) > l$;*
(ii) $\lim_{h\to 0} V(t,x) = 0$*;*
(iii) $\dot{V}(t,x) < 0$*;*
(iv) $\|V_x(t,x)\| > 0$.

Then (2.1) is (h_0, h)-t.bistable with respect to the "aim perturbations" (friction?) for which $V_x Y(t,x) \le 0$.

Proof. The proof is very similar to that of Theorem 3.1. We limit ourselves to observe that

$$\dot{V}_2(t,x) \le \dot{V}(t,x) + \|V_x(t,x)\|\|Z(t,x)\| \tag{4.1}$$

and thus if

$$\|Z(t,x)\| \le -\frac{\dot{V}(t,x)}{\|V_x(t,x)\|} = \delta_2 \tag{4.2}$$

we have $\dot{V}_2(t,x) \le 0$. □

Theorem 4.2. *Suppose that there exist three functions $\Phi = \Phi(t,x) \in C$, $V = V(t,x)$ and $W = W(t,x) \in C^1$ such that on $\mathbb{R}^+ \times \mathbb{R}^n$ we have:*

(i) *For all $\epsilon > 0$ there exists $l > 0$ such that $h(t,x) = \epsilon$ implies $V(t,x) - W(t,x) \geq l$;*
(ii) $\lim_{h\to 0}[V(t,x) - W(t,x)] = 0$;
(iii) $\dot V(t,x) < 0$;
(iv) $\|V_x(t,x) - W_x(t,x)\| > 0$;
(v) $0 < \Phi(t,x) < 1$.

Then the system (2.1) is (h_0,h)-t.bistable with respect to the "aim perturbations" such that $(V_x - W_x)Y(t,x) \leq (-\Phi\dot V + \dot W)(t,x)$.

Proof. Let $T(t,x) = V(t,x) - W(t,x)$ be an auxiliary function. From the following two conditions

$$\dot T_2(t,x) = [\dot V - \dot W](t,x) + (V_x - W_x)Y(t,x) + (V_x - W_x)Z(t,x), \tag{4.3}$$
$$\dot T_2 \leq \dot V - \dot W - \Phi\dot V + \dot W + \|V_x - V_x\|\|Z\|$$

if

$$\|Z\| \leq -\frac{(1-\Phi)\dot V}{\|V_x - W_x\|} = \delta_2(t,x), \tag{4.4}$$

we obtain $\dot T_2(t,x) \leq 0$. □

Theorem 4.3. *Suppose that there exist a constant $a > 0$ and two functions $V = V(t,x)$, $W = W(t,x) \in C^1$ such that on $\mathbb{R}^+ \times \mathbb{R}^n$ we have*

(i) $V(t,x) \geq 0$;
(ii) *$W(t,x) \geq -a$ and for every $\epsilon > 0$ there exist two constants $r,b > 0$ for which $h(t,x) = \epsilon$ with $V(t,x) < r$ implies $W(t,x) > b$;*
(iii) $\lim_{h\to 0} V(t,x) = \lim_{h\to 0} W(t,x) = 0$;
(iv) *$\|V_x(t,x) + \mu W_x(t,x)\| > 0$ for every $\mu > 0$.*

Then (2.1) is bistable with respect to the "aim perturbations" for which

$$\dot V(t,x) + \mu\dot W(t,x) + [V_x(t,x) + \mu W_x(t,x)]Y(t,x) < 0. \tag{4.5}$$

Proof. Given ϵ and $r,b > 0$, suppose that $0 < \mu(a+b) < r$ where $\mu > 0$ is a constant (correspondent to ϵ). Consider the family of functions

$$v(t,x) = V(t,x) + \mu W(t,x). \tag{4.6}$$

If we assume that $h(t,x) = \epsilon$ and $V(t,x) \geq r$, we obtain

$$v(t,x) - \mu b \geq r - \mu(a+b), v(t,x) \geq \mu b \tag{4.7}$$

When $h(t,x) = \epsilon$ implies $V(t,x) < r$, we deduce $v(t,x) \geq \mu b$ which condition (i) of Theorem 3.1. Finally, consider the derivative

$$\dot v_2(t,x) = \dot V(t,x) + V_xY(t,x) + \mu[\dot W + W_xY](t,x) + [V_x + \mu W_x]Z(T,x) \tag{4.8}$$

if $\dot V + \mu\dot W + [V_x + \mu W_x]Y < 0$, we obtain the proof by choosing

$$\|Z(t,x)\| \leq -\frac{\dot V + V_xY + \mu[\dot W + W_xY]}{\|V_x + \mu W_x\|}. \tag{4.9}$$

□

5. Liapunov functions in Salvadori's sense

Let us consider a continuous non trivial function $\phi = \phi(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ and a constant $\rho \in]0, \sup \phi]$. We shall set, for $t \in \mathbb{R}^+$,

$$E_t(\phi \le \rho) = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \phi(t,x) \le \rho, t = \text{constant}\} \tag{5.1}$$

The meaning of $E_t(\phi = 0)$ and $E_t(\phi \ge \rho)$ is obvious.

Assumption 5.1. Suppose that there exist two positive numbers m, m' such that for every $\rho \in]0,m]$ we have:

- (i) $E_t(\phi = 0) \subset E_t(\phi < \rho)$, or for short $E_t(0) \subset E_t(\rho)$;
- (ii) $\text{dist}\{\partial E_t(\phi = 0), \partial E_t(\phi \le \rho)\} \ge m'$ for every $t \in \mathbb{R}^+$, where dist is the Euclidean distance of sets, and ∂E is the boundary of E.

Definition 5.2. A function $W = W(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ will be called *definitively positive, negative, not equal to zero* on the sets $E_t(\phi = 0)$ with respect to $h(t,x)$ if there exists a constant $m > 0$ such that for every $\eta \in]0,m]$ there exist $\rho, \beta > 0$ with the property: $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$ with $h(t,x) > \eta$ and $\phi(t,x) < \rho$ imply respectively $W(t,x) > \beta$, $W(t,x) < -\beta$, $|W(t,x| > \beta$.

Theorem 5.3. *Suppose that there exist: two functions $V = V(t,x)$ and $W = W(t,x) \in C^1$ from $\mathbb{R}^+ \times \mathbb{R}^n$ to $\mathbb{R}$, and a constant $a > 0$ such that, on $\mathbb{R}^+ \times \mathbb{R}^n$, we have:*

- (i) *$V(t,x) \ge 0, \sup V(t,x) > 0, W(t,x) \ge -a$;*
- (ii) *for every $\epsilon > 0$ two numbers $r, b > 0$ exist such that $h(t,x) = \epsilon$ and $V(t,x) < r$ imply $W(t,x) > b$.*

Then we can construct a family of functions that verifies hypothesis (i) of Theorem 3.1.

Proof. Given $\epsilon > 0$ with r, b; let $0 < \mu \le r/(a+b)$ and consider the family of functions

$$v_\mu = v_\mu(t,x) = V(t,x) + \mu W(t,x). \tag{5.2}$$

Suppose $h = h(t,x) = \epsilon$ and $V(t,x) \ge r$ hence $v_\mu(t,x) \ge r - \mu a$ and

$$v_\mu(t,x) - \mu b \ge r - \mu(a+b) \ge 0 \tag{5.3}$$

hence we have $v_\mu(t,x) \ge \mu b > 0$. If $h = h(t,x) = \epsilon$ and $V(t,x) < r$ we have $W(t,x) > b$ and

$$v_\mu(t,x) = V(t,x) + \mu W(t,x) \ge \mu b. \tag{5.4}$$

□

Theorem 5.4. *Suppose that there exist three functions of class C^1: $V = V(t,x)$, $W = W(t,x)$ from $\mathbb{R}^+ \times \mathbb{R}^n$ to $\mathbb{R}$ and $\phi = \phi(t,x)$ from $\mathbb{R}^+ \times \mathbb{R}^n$ to $\mathbb{R}^+$, and two constants $M, M' > 0$ such that, on $\mathbb{R}^+ \times \mathbb{R}^n$, we have:*

- (i) *$\phi(t,x) \ge 0; \phi(t,x) = 0$ implies $\dot V(t,x) \le 0, \phi(t,x)$ verifies Assumption 5.1;*
- (ii) *for every $\chi > 0$ there exists $\chi' > 0$ such that for every $t \in \mathbb{R}^+$ when $\text{dist}[(t,x), E_t(\phi = 0)] > \chi$ we have $\dot V(t,x) < -\chi'$;*
- (iii) *$|W(t,x)|$ and $\|WX(t,x)\| < M$ on $\mathbb{R}^+ \times \mathbb{R}^n$;*
- (iv) *$\dot V(t,x) \in GG, \dot W(t,x) \in ZZ, \|\dot W(t,x)\| < M'$ on $\mathbb{R}^+ \times \mathbb{R}^n$;*
- (v) *$\dot W(t,x)$ is definitively not equal to zero with respect to h on the sets $E_t(\phi = 0)$ and $h \in C^1$.*

Then we can construct a function whose derivative belongs to ZZ.

Proof. Since $\dot{W}(t,x)$ is definitively not equal to zero on the sets $E_t(\phi = 0)$ with respect to h there exists $m > 0$ such that given $\eta \in]0, m]$ there exist $\beta, \rho > 0$ and three sets:

$$A_1 = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : h(t,x) \geq \eta, \phi(t,x) \leq \rho, \dot{W}(t,x) < -\beta\}, \tag{5.5}$$

$$A_2 = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : h(t,x) \geq \eta, \phi(t,x) \leq \rho, \dot{W}(t,x) > \beta\}, \tag{5.6}$$

$$A_3 = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : h(t,x) \geq \eta, \phi(t,x) \leq \rho, (t,x) \notin A_1 \cup A_2\} = \emptyset \tag{5.7}$$

since $W(t,x) \in C^1$ when $A_1, A_2 \neq \emptyset$ we have $\operatorname{dist}[A_1, A_2] > 0$.

Now, we shall denote, for a fixed $t = t' \in \mathbb{R}^+$, for $i = 1,2$ and for every $r > 0$:

$$B_i(t') = \{(t',x) \in A_i : \phi(t',x) = 0\}; \operatorname{dist}\{\partial E_{t'}(0), \partial E_{t'}(\rho)\} = 3\alpha \ (> 0) \tag{5.8}$$

$$S = S(r) = \{x \in \mathbb{R}^n : \|x\| < r\}. \tag{5.9}$$

Consider also the sets:

$$\begin{gathered} C_i(t') = S(r) \cap B_i(t'), \\ D_i(t') = \{(t',x) \in \mathbb{R} \times \mathbb{R}^n : \operatorname{dist}[(t',x), C_i(t')] < \alpha\}, \\ D_i'(t') = \{(t',x) \in \mathbb{R} \times \mathbb{R}^n : \operatorname{dist}[(t',x), C_i(t')] < 2\alpha\}, \\ D_i''(t') = \{(t',x) \in \mathbb{R} \times \mathbb{R}^n : \operatorname{dist}[(t',x), C_i(t')] < 3\alpha\}\,. \end{gathered}$$

Put $\psi_i(t',x) = 0$ for $(t',x) \notin D_i'$ and $\psi_i(t',x) = 1$ for $(t',x) \in D_i'$ we consider the functions

$$T_i(t',x) = \int_{\mathbb{R}^n} \psi_i(t',x)\Omega_\alpha(x-u)du \tag{5.10}$$

where Ω_α is the averaging kernel of radius α, $i = 1,2$, and $u \in \mathbb{R}^n$.

Since $h, \phi \in C^1$ we can obtain two functions $T_i = T_i(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ that belong to C^1 with respect to the first variable, and belong to C^∞ with respect to x, and

- (i) $0 \leq T_i(t,x) \leq 1$ for $(t,x) \in D_i''$ and $T_i(t,x) = 0$ for $(t,x) \notin D_i''$;
- (ii) $\big|\frac{\partial T_i(t,x)}{\partial t}\big|, \|\frac{\partial T_i(t,x)}{\partial x}\| \leq N$, suitable strictly positive constant;
- (iii) $\dot{T}_i(t,x) = \frac{\partial T_i(t,x)}{\partial t} + \frac{\partial T_i(t,x)}{\partial x}X$, $\|\dot{T}_i\| \leq N(1 + \|X\|)$;
- (iv) $\dot{T}_i(t,x) = 0$ if $(t,x) \notin D_i''$ or if (t,x) is in the interior of D_i.

Now, let us consider:

(1) the function $T = T(t,x)$ defined on $\mathbb{R}^+ \times \mathbb{R}^n$ such that $T = T_1(t,x)$ for $(t,x) \in D_1''(t)$, $T = -T_2(t,x)$ for $(t,x) \in D_2''(t)$; $T = 0$ when $(t,x) \notin D_1''(t) \cup D_2''(t)$;

(2) the function $\omega = TW$ defined on the set

$$\Gamma = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : h(t,x) \geq \eta, \|x\| \leq r\}. \tag{5.11}$$

Since

$$\dot{\omega}(t,x) = T\dot{W}(t,x) + \dot{T}W(t,x), \tag{5.12}$$

we have

$$\begin{gathered} \|\dot{T}W(t,x)\| = \Big|\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}X\Big||W(t,x)| \leq N(|W| + \|WX\|) \leq 2NM, \\ \|T\dot{W}(t,x)\| \leq \|\dot{W}(t,x)\| < M', \quad \|\dot{\omega}(t,x)\| < 2NM + M'. \end{gathered} \tag{5.13}$$

Let us finally consider the following function, defined on Γ,

$$v_\nu = v_\nu(t,x) = V(t,x) + \nu\omega(t,x), \nu > 0 \tag{5.14}$$

and its derivative

$$\dot{v}_\nu = \dot{v}_\nu(t,x) = \dot{V}(t,x) + \nu\dot{\omega}(t,x) \tag{5.15}$$

It is obvious that for fixed $t \in \mathbb{R}^+, (t,x) \in \Gamma$ with $(t,x) \notin [D_1''(t) \cup D_2''(t)]$, we have $v_\nu(t,x) = V(t,x), \dot{v}_\nu(t,x) = \dot{V}(t,x)$.

In this first case since $\operatorname{dist}[(t,x), E_t(\phi = 0)] \geq 3\alpha$ there exists $\alpha' > 0$ such that $\dot{V}(t,x) < -\alpha' < 0$.

If $(t,x) \in D_1(t) \cap \Gamma$ then $T = 1, \omega(t,x) = W(t,x), v_\nu(t,x) = V(t,x) + \nu W(t,x)$ with $\dot{V}(t,x) \leq 0, \dot{W}(t,x) < -\beta$ hence $\dot{v}_\nu(t,x) < -\nu\beta < 0$ for every $\nu > 0$. When $(t,x) \in D_1''(t) \cap \Gamma$ with $(t,x) \notin D_1(t)$, we have $\alpha \leq \operatorname{dist}[(t,x), E_t(\phi = 0)] \leq 3\alpha$ then there exists $\alpha'' > 0$ such that $\dot{V}(t,x) < -\alpha''$; therefore, $\dot{v}_\nu(t,x) < -\alpha'' + \nu(2MN + M')$. If $0 < 2\nu[2MN + M'] < \alpha''$ we obtain

$$2\dot{v}_\nu(t,x) < -\alpha''. \tag{5.16}$$

The cases $(t,x) \in D_2(t) \cap \Gamma$ and $(t,x) \in D_2''(t) \cap \Gamma$ with $(t,x) \notin D_2(t)$ are trivial as $A_1 = \emptyset$ or $A_2 = \emptyset$. $\square$

6. Application to the motion of rigid bodies

In this section we present an illustrative mechanical example. Putting

$$CD = \{(p,q,r,\gamma_1,\gamma_2) \in \mathbb{R}^5 : \gamma_1^2 + \gamma_2^2 \leq 1\}$$

on the set $\mathbb{R}^+ \times CD$ let us consider the system of equations

$$\begin{gathered}
\dot{A}p + 2A\dot{p} + 2(C - A)qr = 2Pz\gamma_2\gamma_3 - 2f_1p - 2f_4r, \\
\dot{A}q + 2A\dot{q} + 2(A - C)pr = -2Pz\gamma_1\gamma_3 - 2f_2q - 2f_5r, \\
\dot{C}r + 2C\dot{r} = 2f_4p + 2f_5q - 2f_3r, \\
\dot{\gamma}_1 = r\gamma_2 - q\gamma_3, \quad \dot{\gamma}_2 = p\gamma_3 - r\gamma_1, \quad \dot{\gamma}_3 = q\gamma_1 - p\gamma_2, \\
\gamma^2 = 1 - \gamma_3, \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1\,.
\end{gathered} \tag{6.1}$$

This system, with the usual designation and when $Pz = 0$, constitutes the basic dynamical system for the motion of a symmetrical rigid body about a fixed point and variable mass [16]; if $P = 0$ the body is non heavy, if $z = 0$ the center of gravity is a fixed point.

Assumption 6.1. Assume that the given functions $A(t), C(t) \in C^1(\mathbb{R}^+ \to \mathbb{R}^+)$, $P(t) \in C(\mathbb{R}^+ \to \mathbb{R}^+)$, $z(t) \in C(\mathbb{R}^+ \to]0,\infty[)$ and $G(t,p,q,r,\gamma_1,\gamma_2), U(t,\dots\gamma_2)$ satisfy the following properties:

(i) $\inf\{A(t), C(t), P(t), -z(t)\} > 0$, $-Pz = \text{const} > 0$ and $A' = \inf A(t) \leq \sup A(t) = A''$;
(ii) $0 < f' = \inf\{f_i(t,p,q,r,\gamma_1,\gamma_2)\} \leq \sup\{f_i(..)\} = f''$ for $i = 1,2,3$;
(iii) for every $t_0, p_0, q_0, r_0, \gamma_1^0, \gamma_2^0$ there exists only one solution, defined for $t \geq t_0$;
(iv) $G = \{Ap, Aq, Cr, -Pz\gamma_1, -Pz\gamma_2\}$, $U = A^2p^2 + A^2q^2 + C^2r^2$;
(v) as measures of stability we select the following functions h and h_0:

$$4h = A(p^2 + q^2) + Cr^2 - Pz(\gamma_1^2 + \gamma_2^2) = h_0; \tag{6.2}$$

(vi) as auxiliary Liapunov's functions we select the following functions of Matrosov's type

$$V = \frac{1}{2}[A(p^2+q^2)+Cr^2] - \frac{1}{2}Pz(\gamma_1^2+\gamma_2^2) = 2h, \tag{6.3}$$

$$W = A(p\gamma_2 - q\gamma_1). \tag{6.4}$$

(vii) $f_1 = 2A^2,\ f_2 = 2A^2,\ f_3 = 2C^2$.

Theorem 6.2. *Under Assumption 6.1, we deduce that:*

(i) *The measure h_0 is uniformly finer than h, $h = 0$ is equivalent to $p = q = r = \gamma_1 = \gamma_2 = 0$.*

(ii) *$V = 2h$ and $\lim_{h\to 0} V = 0$ hence condition (3.2)(ii) hold, and we obtain $\lim_{h\to 0} W = 0$.*

(iii) *$G = \operatorname{grad} V$ hence, for $\phi = 1$, condition (3.2)(iv) is verified.*

(iv) *$\dot V = -f_1p^2 - f_2q^2 - f_3r^2 = -2U < 0$, so conditions (3.2)(iii) and (3.2)(v) hold for $\theta = 2 > \phi = 1$, $-f''(p^2+q^2+r^2) \le \dot V \le -f'(p^2+q^2+r^2)$.*

(v) *$\dot V = 0$ if and only if $p = q = r = 0$; therefore, the L-measure on $\mathbb{R}^5$ of the set*

$$E_1 = E(\dot V = 0) = \{p = q = r = 0, (\gamma_1,\gamma_2) \in \mathbb{R}_2 : \gamma_1^2 + \gamma_2^2 \le 1\}$$

is equal to zero, hence the system (6.1) is strongly weakly (h_0, h)-t.bistable with respect to perturbation

$$\sigma\{Ap, Aq, Cr, 0, 0\} \tag{6.5}$$

where $\sigma = \sigma(t, p, q, r, \gamma_1, \gamma_2) > 0$ belongs to C^1.

(vi) *Since*

$$\begin{aligned} \dot W &= 2(A-C)qr\gamma_2 - A\dot p\gamma_2 + 2Pz\gamma_2^2 - 2f_1p\gamma_2 - 2f_4r\gamma_2 + Ap\dot\gamma_2 \\ &\quad - 2(C-A)pr\gamma_1 + A\dot q\gamma_1 + 2Pz\gamma_1^2 + 2f_2q\gamma_1 + 2f_5r\gamma_1 + Aq\dot\gamma_1, \end{aligned} \tag{6.6}$$

hence on the set E_1, we obtain $\dot W = 2Pz(\gamma_1^2+\gamma_2^2) \le 0$.

(vii) *If $0 < \eta < 1$ and $\gamma_1^2 + \gamma_2^2 = \gamma \ge \eta$ we deduce $2Pz\gamma \le 2Pz\eta < 0$ i.e. on the set $E_2 = E(\dot V = 0, \eta \le \gamma \le 1)$ we have $\dot W \le 2Pz\eta < 0$ and $4h = -Pz\gamma \ge -Pz\eta$. Since $W \in C^1$ there exists $b > 0$ such that on the set*

$$(CD)_1 = \{(p,q,r) \in \mathbb{R}^3 : p^2+q^2+r^2 \le 9b^2\} \times \{(\gamma_1,\gamma_2) \in \mathbb{R}^2 : \gamma \ge \eta\} \tag{6.7}$$

we have $\dot W \le Pz\eta < 0$, i.e. the function $\dot W$ is definitely negative on the set $E(\dot V = 0)$ with respect to the measure h when $\gamma \ge \eta$. According to Theorem 5.4 we have $A_2 = 0$ and

$$\begin{aligned} A_1 &= \{(p,q,r,\gamma_1,\gamma_2) : h \ge -\frac{1}{2}Pz\gamma \ge -\frac{1}{2}Pz\eta; \phi = p^2+q^2+r^2 \le 9b^2\} \\ &= \{(p,q,r) \in \mathbb{R}^3 : \phi = p^2+q^2+r^2 \le 9b^2\} \times \{(\gamma_1,\gamma_2) \in \mathbb{R}^2 : \gamma \ge \eta\}. \end{aligned} \tag{6.8}$$

(viii) *Consider the function $\psi = \psi(p,q,r)$ from $\mathbb{R}^3$ to $\mathbb{R}^+$ such that $\psi = 0$ when $4b^2 \le \phi$, $\psi = 1$ when $0 \le \psi < 4b^2$ and their regularized function, defined on $\mathbb{R}^3$:*

$$T(x) = \int_{\mathbb{R}^3} \psi_i(x)\Omega_b(x-u)du \tag{6.9}$$

where $x = (p,q,r)$ and Ω_b is the averaging kernel of radius b. It is obvious that $0 \le T \le 1, T \in C^\infty$ and: $T = 0$ when $\phi \ge 9b^2$, $0 < T \le 1$ when

$b^2 \le \phi < 9b^2$, $T = 1$ when $\phi < b^2$, $|\dot{T}| < M''(> 0)$ being M'' a suitable constant. On the set $(CD)_1$ we obtain

$$|TW| = |T||W| \le |W| \le A''[|p\gamma_2| + |q\gamma_1|] \le 6A''b. \tag{6.10}$$

(ix) *Successively consider the family of functions $w_\mu = V + \mu TW$ defined on the set*

$$\mathbb{R}^+ \times (CD)_2 = \{t \in \mathbb{R}^+\} \times \{(p, q, r) \in \mathbb{R}^3; (\gamma_1, \gamma_2) \in \mathbb{R}^2 : \eta \le \gamma < 1\}$$

when $\mu > 0$ and suppose that:

(1) *$\phi \ge 9b^2$ in this case $T = 0$, $w_\mu = V$ hence for $h = s$ we obtain $w_\mu = 2s$;*

(2) *$\phi < 9b^2$ now $|TW| \le 6A''b$ therefore $h = s$ implies*

$$[w_\mu]_{h=s} = [V + \mu TW]_{h=s} \ge 2s - 6\mu A''b > s \leftrightarrow \mu < \frac{s}{6A''b}. \tag{6.11}$$

(x) *Consider the derivatives*

$$\dot{w}_\mu = \dot{V} + \mu\dot{T}W + \mu T\dot{W} \tag{6.12}$$

and suppose that:

(1) *$\phi \ge 9b^2$ then $T = 0$ i.e. $\dot{w}_\mu = \dot{V} = -f_1p^2 - f_2q^2 - fr^3r^2 \le -f'\phi \le -9f'b^2$.*

(2) *$\phi \le b^2$ hence $T = 1, \dot{w}_\mu = \dot{V} + \mu\dot{W}, \dot{w}_\mu < \mu Pz\eta$.*

(3) *$b^2 < \phi < 9b^2$ then $\dot{W} \le 0, \dot{w}_\mu \le \dot{V} + \mu\dot{T}W$ but $|\dot{T}W| \le M''|W| \le 6M''A''b$ and $\dot{V} \le -f'b^2$ therefore we obtain $\dot{w}_\mu \le -f'b^2 + 6\mu M''A''b \le -3\mu M''A''2b$ if and only if $\mu \le \frac{f'b}{9M''A''}$.*

When $\mu \le \mu' = \min[\frac{s}{6A''b}, \frac{f'b}{9M''A''}]$ all the conditions of Theorem 3.1 are verified, hence (6.1) is weakly (h_0, h)-t.bistable with respect to the perturbations

$$Y = \sigma\{-(w_\mu)_p, -(w_\mu)_q, -(w_\mu)_r, 0, 0\} \tag{6.13}$$

where $\mu \le \mu'$ and $\sigma = \sigma(p, q, r, \gamma_1, \gamma_2)$ is an arbitrary continuous function.

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