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CONSEQUENCES OF TALENTI'S INEQUALITY BECOMING EQUALITY

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Abstract. In this article, we consider the case of equality in a well known inequality for the p-Laplacian due to Giorgio Talenti. Our approach seems to be simpler than the one by Kesavan [6]. We use a result from rearrangement optimization to prove the main result in this article. Some physical interpretations are also presented.

1. Introduction

The initial-boundary value problem

$$\begin{aligned} u_{tt} - \Delta u + c(x)u = f(x), \quad \text{in } D \times [0,T] \\ u = 0, \quad \text{on } \partial D \times [0,T] \\ u = g, \quad u_t = h, \quad \text{on } D \times \{t=0\} \end{aligned} \tag{1.1}$$

models the vibration of a non-isotropic elastic membrane subject to an external force. The steady-state version of (1.1) is

$$\begin{aligned} -\Delta u + c(x)u = f(x) \quad \text{in } D \\ u = 0 \quad \text{on } \partial D. \end{aligned} \tag{1.2}$$

In (1.2) we can suppose $f(x)$ represents a vertical force on the membrane such as a load distribution, and a non-zero function $c(x)$ guarantees the membrane is made of several different materials, hence it is non-isotropic. Our purpose in this note is related to a common generalization of (1.2), namely

$$\begin{aligned} -\Delta_p u + c(x)|u|^{p-2}u = f(x), \quad \text{in } D, \\ u = 0, \quad \text{on } \partial D, \end{aligned} \tag{1.3}$$

where $c(x) \geq 0$ is a bounded function, and $f \in L^q(D)$ is a non-negative function; here q is the conjugate exponent of $1 < p < \infty$; i.e., $\frac{1}{p} + \frac{1}{q} = 1$, and Δ_p denotes the usual p-Laplace operator, i.e. $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$. The "symmetrized" problem corresponding to (1.3) is the boundary value problem

$$\begin{aligned} -\Delta_p v = f^{\sharp}(x), \quad \text{in } B, \\ v = 0, \quad \text{on } \partial B, \end{aligned} \tag{1.4}$$

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where B stands for the ball centered at the origin such that $\mathrm{Vol}(B) = |D|$. Henceforth, for a measurable $E \subset \mathbb{R}^n$, $|E|$ indicates the n-dimensional Lebesgue measure of E. The notation $f^\sharp$ denotes the standard Schwarz symmetrization of f on the ball B, so $f^\sharp$ is radial, $f^\sharp(x) = \xi(\|x\|)$, where ξ is a decreasing function; moreover, $f^\sharp$ and f are equi-measurable:

$$|\{x \in B : f^\sharp(x) \geq \alpha\}| = |\{x \in D : f(x) \geq \alpha\}|,$$

for every $\alpha \geq 0$. The reader may see [5] for details. By $u_f \in W_0^{1,p}(D)$ we denote the unique solution of (1.3). It is well known (and easy to prove) that u_f is the unique minimizer of

$$\Psi(u) = \frac{1}{p}\int_D |\nabla u|^p \,dx + \frac{1}{p}\int_D c(x)|u|^p \,dx - \int_D f(x)u \,dx, \tag{1.5}$$

relative to $u \in W_0^{1,p}(D)$. Note that (1.3) satisfies the maximum principle in the sense that $f \geq 0$ implies $u_f \geq 0$. Indeed, setting $u \equiv u_f$, and using $-u^- \in W_0^{1,p}(D)$ as a test function, from (1.3), we infer that

$$\int_D \Delta_p u u^- \,dx + \int_D c(x)|u|^{p-2}u(-u^-) \,dx = \int_D f(x)(-u^-) \,dx.$$

Observe that $uu^- = -(u^-)^2$, and $\nabla u \cdot \nabla u^- = -|\nabla u^-|^2$ on the set $\{x \in D : u(x) \leq 0\}$. Thus

$$\begin{aligned}\int_{\{u\leq 0\}} |\nabla u^-|^p \,dx &\leq \int_{\{u\leq 0\}} |\nabla u^-|^p \,dx + \int_{\{u\leq 0\}} c(x)|u^-|^p \,dx \\ &= \int_D f(x)(-u^-) \,dx \leq 0.\end{aligned}$$

Whence, u^- is constant. This, in conjunction with the fact that u^- vanishes on ∂D, implies $u^- = 0$, hence $u = u^+ \geq 0$. In fact, $u > 0$, a result which is implied by the strong maximum principle, see for example [3].

The following inequality which is attributed to Giorgio Talenti, see for example [8], has proven to be an instrumental tool in partial differential equations,

$$u_f^\sharp(x) \leq v_{f^\sharp}(x), \quad \forall x \in B, \tag{1.6}$$

where $v_{f^\sharp}$ denotes the solution to (1.4). The objective of this article is to discuss the consequences of having equality in (1.6). Indeed we are able to prove the following result.

Theorem 1.1. *Suppose equality holds in* (1.6). *Then*

(i) $c(x) \equiv 0$,
(ii) *D and f are equal to B and $f^\sharp$, respectively, modulo translations.*

Remark 1.2. In case $c \equiv 0$, Theorem 1.1 has already been proven, see for example [6]. However, our proof is simpler than the known proof. It relies on a result from rearrangement optimization theory which itself is intuitively easy to grasp.

This article is organized as follows. In section 2, we collect known results from the theory of rearrangements specialized to our purpose, and in addition recall the well known Polya-Szego inequality. Section 3 is entirely devoted to the proof of Theorem 1.1. In section 4, a generalization of Theorem 2.2 is presented.

2. Preliminaries

Let us begin by recalling the definition of two functions being rearrangement of each other.

Definition 2.1. Given two measurable functions $f, g : D \to \mathbb{R}$, we say f and g are rearrangements of each other provided

$$|\{x \in D : f(x) \geq \alpha\}| = |\{x \in D : g(x) \geq \alpha\}|, \quad \forall \alpha \in \mathbb{R}.$$

For $f_0 \in L^p(D)$, the set comprising all rearrangements of f_0 is denoted by $\mathcal{R}(f_0)$; i.e.,

$$\mathcal{R}(f_0) = \{f : f \text{ and } f_0 \text{ are rearrangements of each other}\}.$$

Using Fubini's Theorem, it is easy to show that $\mathcal{R}(f_0) \subset L^p(D)$. Let us denote by $w_f \in W_0^{1,p}(D)$ the unique solution of

$$\begin{gathered} -\Delta_p w = f, \quad \text{in } D, \\ w = 0, \quad \text{on } \partial D, \end{gathered} \tag{2.1}$$

and define the *energy* functional $\Phi : L^p(D) \to \mathbb{R}$ by

$$\Phi(f) = \int_D f\, w_f \, dx. \tag{2.2}$$

In [2], amongst other results, the authors proved the following result which plays a crucial role in what follows.

Theorem 2.2. *The maximization problem*

$$\sup_{f \in \mathcal{R}(f_0)} \Phi(f) \tag{2.3}$$

is solvable; i.e., there exists $\hat{f} \in \mathcal{R}(f_0)$ such that

$$\Phi(\hat{f}) = \sup_{f \in \mathcal{R}(f_0)} \Phi(f).$$

Moreover, there exists an increasing function η, unknown a priori, such that

$$\hat{f} = \eta \circ \hat{w}, \tag{2.4}$$

almost everywhere in D, where $\hat{w} := w_{\hat{f}}$.

We are going to need the following result, which can be found in [1].

Theorem 2.3. *Let $u \in W_0^{1,p}(D)$ be non-negative. Then $u^\sharp \in W_0^{1,p}(B)$, and*

$$\int_B |\nabla u^\sharp|^p \, dx \leq \int_D |\nabla u|^p \, dx. \tag{2.5}$$

Moreover, if equality holds in (2.5), then $u^{-1}(\beta, \infty)$ is a translate of $u^{*-1}(\beta, \infty)$, for every $\beta \in [0, M]$, where M is the essential supremum of u over D, modulo sets of measure zero.

Lemma 2.4. *Suppose $0 \leq f_0 \in L^p(B)$. Then the maximization problem*

$$\sup_{f \in \mathcal{R}(f_0)} \Phi(f), \tag{2.6}$$

has a unique solution; namely, $f_0^\sharp$, the Schwarz symmetrization of f_0 on B. That is,

(a) $\Phi(f_0^\sharp) = \sup_{f\in\mathcal{R}(f_0)} \Phi(f)$, *and*
(b) $\Phi(f) < \Phi(f_0^\sharp)$, *for all* $f \in \mathcal{R}(f_0) \setminus \{f_0^\sharp\}$.

Proof. Part (a) is straightforward. Indeed, for any $f \in \mathcal{R}(f_0)$, an application of the Hardy-Littlewood inequality, see [4], (1.6) yield

$$\Phi(f) \le \int_B f^\sharp w_f^\sharp \, dx \le \int_B f^\sharp w_{f^\sharp} \, dx = \Phi(f^\sharp) = \Phi(f_0^\sharp).$$

This proves that $f_0^\sharp$ solves (2.6), hence completes the proof of part (a).

Part (b) is more complicated. By contradiction assume the assertion is false. Hence, there exists $f \in \mathcal{R}(f_0) \setminus \{f_0^\sharp\}$ such that

$$\Phi(f) = \Phi(f_0^\sharp). \tag{2.7}$$

The following inequality follows from the variational formulation of $w_{f^\sharp}$:

$$\frac{1}{p}\int_B |\nabla u|^p \, dx - \int_B f^\sharp u \, dx \ge \frac{1}{p}\int_B |\nabla w_{f^\sharp}|^p \, dx - \int_B f^\sharp w_{f^\sharp} \, dx, \tag{2.8}$$

for every $u \in W_0^{1,p}(B)$. Thus, by substituting $u = w_f^\sharp$ in (2.8), we obtain

$$\frac{1}{p}\int_B |\nabla w_f^\sharp|^p \, dx + \frac{1}{q}\int_B f^\sharp w_{f^\sharp} \, dx \ge \int_B f^\sharp w_f^\sharp \, dx.$$

Applying the Hardy-Littlewood inequality to the right hand side yields

$$\frac{1}{p}\int_B |\nabla w_f^\sharp|^p \, dx + \frac{1}{q}\int_B f^\sharp w_{f^\sharp} \, dx \ge \int_B f w_f \, dx = \Phi(f). \tag{2.9}$$

From (2.7) and (2.9), we infer

$$\int_B |\nabla w_f^\sharp|^p \, dx \ge \Phi(f) = \int_B |\nabla w_f|^p \, dx. \tag{2.10}$$

Inequality (2.10) coupled with (2.5) imply that equality holds in (2.10). We now proceed to show that $w_f = w_f^\sharp$, which follows once we prove the set $\{x \in B : \nabla w_f = 0,\ 0 < w_f(x) < M\}$ has measure zero, according to Theorem 2.3. To this end, let us consider $z \in B$ such that $0 < w(z) < M := \max_B w(x)$, where we are using w in place of w_f for simplicity. Using the strong maximum principle [3] it is easy to show that $z \in \partial S$, the boundary of the set S, where $S := \{x \in B : w(x) \ge w(z)\}$. By Theorem 2.2, S is a ball. Since $z \in \partial S$ we can apply the Hopf boundary point lemma [9] to conclude that $\frac{\partial w}{\partial \nu}(z) \ne 0$, where ν stands for the unit outward normal vector to S at z. Thus, $\nabla w(z) \ne 0$, hence the set $\{x \in B : \nabla w_f = 0,\ 0 < w_f(x) < M\}$ has measure zero, as desired. So, $w = w^\sharp$.

At this stage we utilize Theorem 2.3. According to (2.4), there exists an increasing function η such that $f = \eta \circ w$. Since $w = w^\sharp$, we infer $f = \eta \circ w^\sharp$. This, in turn, implies that $f = f^\sharp = f_0^\sharp$, which is a contradiction. □

Remark 2.5. There is a nice interpretation of Lemma 2.4, when $p = 2$. Indeed, in this case, the boundary value problem (2.1) models the displacement, w_f, of an elastic radial membrane, fixed around the boundary, and subject to a vertical force, f, which can be taught of as a load distribution. The assertions of the theorem then are implying that the average displacement, over the region where the load is positioned, is maximized provided the load is located at the center of the membrane. It is intuitively clear that in order to make the membrane stretch as much as possible we should place the load as far as we can from the boundary.

Definition 2.6. Suppose $z_i : D_i \to \mathbb{R}$ are measurable functions, $i = 1, 2$, and $|D_1| = |D_2|$. We write $z_1 \preceq z_2$ provided

$$z_1^\sharp(x) \le z_2^\sharp(x), \quad \forall x \in B,$$

where B is the ball centered at the origin with $|B| = |D_1| = |D_2|$.

Remark 2.7. Note that based on the definition above we observe that Talenti's inequality (1.6) can be written as $u_f \preceq v_{f^\sharp}$, since $v_{f^\sharp} = (v_{f^\sharp})^\sharp$.

3. Proof of Theorem 1.1

Proof. From the assumption, $u_f^\sharp = v_{f^\sharp}$, we deduce

$$\int_B f^\sharp u_f^\sharp \, dx = \int_B f^\sharp \, v_{f^\sharp} \, dx = \int_B |\nabla v_{f^\sharp}|^p \, dx = \int_B |\nabla u_f^\sharp|^p \, dx \le \int_D |\nabla u_f|^p \, dx, \tag{3.1}$$

where in the last inequality we have applied (2.5). On the other hand, by the Hardy-Littlewood inequality, see [4], we have

$$\int_D f u_f \, dx \le \int_B f^\sharp u_f^\sharp \, dx. \tag{3.2}$$

Multiplying the differential equation in (1.3), by u_f, integrating over D, and remembering the boundary condition, $u_f = 0$, we obtain

$$\int_D f u_f \, dx = \int_D |\nabla u_f|^p \, dx + \int_D c(x) u_f^p \, dx. \tag{3.3}$$

the combination of (3.1), (3.2) and (3.3) yields

$$\int_D |\nabla u_f|^p \, dx + \int_D c(x) u_f^p \, dx \le \int_D |\nabla u_f|^p \, dx,$$

hence $\int_D c(x) u_f^p \, dx \le 0$. This, in turn, recalling that $u_f > 0$, implies $c \equiv 0$. This completes the proof of part (i).

We prove part (ii). Let us first observe that from (i) we infer $v_{f^\sharp} = u_{f^\sharp}$. Hence, the hypothesis of the theorem can be written as $u_f^\sharp = u_{f^\sharp}$. Again, from the Hardy-Littlewood inequality and (2.5), we obtain

$$\begin{aligned} \int_D |\nabla u_f|^p \, dx &= \int_D f u_f \, dx \le \int_B f^\sharp u_f^\sharp \, dx = \int_B f^\sharp u_{f^\sharp} \, dx \\ &= \int_B |\nabla u_{f^\sharp}|^p \, dx = \int_B |\nabla u_f^\sharp|^p \, dx \\ &\le \int_D |\nabla u_f|^p \, dx. \end{aligned} \tag{3.4}$$

Hence, all inequalities in (3.4) are in fact equalities. In particular, we obtain

$$\int_D |\nabla u_f|^p \, dx = \int_B |\nabla u_f^\sharp|^p \, dx.$$

Thus, we are now in a position to apply Theorem 2.3, which implies the sets $\{x \in D : u_f(x) \ge \beta\}$ are translations of $\{x \in B : u_f^\sharp(x) \ge \beta\}$. So, in particular, we deduce D is a translation of B. Henceforth, without loss of generality we assume

$D = B$. Whence, we have $\int_B |\nabla u_f|^p\,dx = \int_B |\nabla u_f^\sharp|^p\,dx$. To complete the proof of (ii), we return to (3.4), and recalling that all inequalities are equalities, we obtain

$$\int_B f u_f\,dx = \int_B f^\sharp u_{f^\sharp}\,dx.$$

Now we can apply Lemma 2.4, which yields $f = f^\sharp$. □

Remark 3.1. When $p = 2$, the boundary value problem (1.3) reduces to

$$\begin{gathered} -\Delta u + c(x)u = f(x), \quad \text{in } D, \\ u = 0, \quad \text{on } \partial D. \end{gathered} \tag{3.5}$$

The boundary value problem (3.5) physically models the displacement of a non-isotropic (assuming c is not identically zero) elastic membrane, fixed around the boundary, and subject to a vertical force such as a distribution load. The result of this paper implies that equality in the (1.6) is only possible if the membrane is isotropic; i.e., $c \equiv 0$. In other words, when the membrane is made of several materials with different densities it is impossible to have equality in (1.6), hence the best result is $u_f \preceq v_{f^\sharp}$.

4. A special case

In the maximization problem (2.3), sometimes the generator of the rearrangement class, f_0, is a "complicated" function. We use the term complicated specifically in the following sense: $f_0 = Q(g_0)$, where $Q : \mathbb{R} \to \mathbb{R}^+$ is an increasing and continuous function, and g_0 is a non-negative function that belongs to $L^p(D)$.

In this section we show that if f_0 is a complicated function in the above sense then it is possible to replace (2.3) with another maximization problem which is formulated with respect to g_0. The main result of this section is as follows.

Theorem 4.1. *Let $0 \le f_0 \in L^p(D)$, and suppose f_0 is a complicated function in the sense described above. Then*

$$\sup_{f \in \mathcal{R}(f_0)} \int_D f u_f\,dx = \sup_{g \in \mathcal{R}(g_0)} \int_D Q(g) u_{Q(g)}\,dx. \tag{4.1}$$

The proof of Theorem 4.1 relies on the fact that $\mathcal{R}$ and T commute, as in the following lemma.

Lemma 4.2. *Let $0 \le h_0 \in L^p(D)$. Let $T : \mathbb{R} \to \mathbb{R}^+$ be an increasing and continuous function. Then*

$$\mathcal{R}(T(h_0)) = T(\mathcal{R}(h_0)). \tag{4.2}$$

Proof. Let us first prove the inclusion $T(\mathcal{R}(h_0)) \subseteq \mathcal{R}(T(h_0))$. To this end, let $l \in T(\mathcal{R}(h_0))$. So, there exists $h \in \mathcal{R}(h_0)$ such that $l = T(h)$. Note that for every $\alpha \ge 0$, there exists $\beta \ge 0$ such that $T^{-1}([\alpha,\infty)) = [\beta,\infty)$, where T^{-1} denotes the inverse image, since T is increasing and continuous. Hence

$$\{x \in D : l(x) \ge \alpha\} = \{x \in D : h(x) \ge \beta\}.$$

This, in turn, implies

$$\begin{aligned} |\{x \in D : l(x) \ge \alpha\}| &= |\{x \in D : h(x) \ge \beta\}| \\ &= \{x \in D : h_0(x) \ge \beta\}| \\ &= |\{x \in D : T(h_0)(x) \ge \alpha\}|. \end{aligned}$$

Thus, $l \in \mathcal{R}(T(h_0))$, as desired.

Now we prove that $\mathcal{R}(T(h_0)) \subset T(\mathcal{R}(h_0))$. Consider $l \in \mathcal{R}(T(h_0))$, so $l^* = T(h_0)^*$. Here, "*" stands for the decreasing rearrangement operator. So, for example,

$$l^*(s) = \inf\{\alpha \in \mathbb{R} : |\{x \in D : l(x) \geq \alpha\}| \leq s\},$$

see [7] for details. Since T is decreasing, we infer $l^* = T(h_0)^* = T(h_0^*)$. At this stage, we use another result from [7]; namely, that there exists a measure preserving map $\psi : D \to D$ such that $l = l^* \circ \psi$. Therefore, $l = T(h_0^*) \circ \psi = T(h_0^* \circ \psi)$. Since ψ is measure preserving, we infer $h_0^* \circ \psi \in \mathcal{R}(h_0)$, hence $l \in T(\mathcal{R}(h_0))$, which finishes the proof of the lemma. □

Proof of Theorem 4.1. Let us set

$$L = \sup_{f \in \mathcal{R}(f_0)} \int_D f u_f \, dx, \quad R = \sup_{g \in \mathcal{R}(g_0)} \int_D Q(g) u_{Q(g)} \, dx.$$

We prove only $L \leq R$, since the proof of $L \geq R$ is similar. Consider $f \in \mathcal{R}(f_0)$. Then $f \in \mathcal{R}(Q(g_0)) = Q(\mathcal{R}(g_0))$, by Lemma 4.2. Thus, $f = Q(g)$, for some $g \in \mathcal{R}(g_0)$. Therefore

$$\int_D f u_f \, dx = \int_D Q(g) u_{Q(g)} \, dx \leq R.$$

Since $f \in \mathcal{R}(f_0)$ is arbitrary we deduce $L \leq R$. □

To illustrate the advantages of Theorem 4.1, we consider the following example. Let $P : \mathbb{R} \to \mathbb{R}^+$ be a bounded function. Consider the boundary value problem

$$\begin{gathered} -\Delta_p u = \int_0^{g(x)} P(s) \, ds, \quad \text{in } D, \\ u = 0, \quad \text{on } \partial D. \end{gathered} \tag{4.3}$$

Setting $f(x) = \int_0^{g(x)} P(s) \, ds$, Theorem 4.1 implies that

$$\sup_{f \in \mathcal{R}(f_0)} \int_D f u_f \, dx = \sup_{g \in \mathcal{R}(g_0)} \int_D \Big(\int_0^{g(x)} P(s) \, ds \Big) u_{(\int_0^{g(x)} P(s) \, ds)} \, dx.$$

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