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GLOBAL CLASSICAL SOLUTIONS FOR REACTION-DIFFUSION SYSTEMS WITH A TRIANGULAR MATRIX OF DIFFUSION COEFFICIENTS

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Abstract. The goal of this article is to study the existence of classical solutions global in time for reaction-diffusion systems with strong coupling in the diffusion and with exponential growth (or without any growth) conditions on the nonlinear reactive terms. This extends some similar results in the case of a diagonal diffusion-operator associated with nonlinearities preserving the positivity and the total mass of the solutions or for which the total mass is a priori bounded.

1. Introduction

In this study, we are interested in the existence of classical global solutions to the reaction-diffusion system

$$\frac{\partial u}{\partial t} - a\Delta u = \lambda - f(u,v) - \mu u \quad \text{in } (0,+\infty)\times\Omega, \tag{1.1}$$

$$\frac{\partial v}{\partial t} - c\Delta u - d\Delta v = f(u,v) - \mu v \quad \text{in } (0,+\infty)\times\Omega, \tag{1.2}$$

where Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$ with boundary $\partial\Omega$, the constants a, c, d, λ, μ are such that

- (A1) $a > 0$, $c > 0$, $d > a$, $2\sqrt{ad} > c$, $\lambda \geq 0$ and $\mu > 0$,

and the function f is a nonnegative and continuously differentiable on $[0,+\infty)$ such that

- (A2) $f(0,\eta) = 0$ and $f(\xi,\eta) \geq 0$, with $f(\xi, \frac{c}{d-a}(\frac{\lambda}{\mu} - \xi)) = 0$ when $a + c \geq d$,
- (A3) $f(\xi,\eta) \leq C\varphi(\xi)\eta^r e^{\alpha\eta}$ for some constants $C > 0$ and $\alpha > 0$ when $a + c < d$, where r is a positive constant such that $r \geq 1$ and φ is any nonnegative continuously differentiable function on $[0,+\infty)$ such that $\varphi(0) = 0$.

We assume that the solutions of (1.1)–(1.2) also satisfy: the boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } (0,+\infty)\times\partial\Omega, \tag{1.3}$$

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where $\frac{\partial}{\partial\nu}$ is the outward normal derivative to $\partial\Omega$; and the initial conditions

$$u(0,x)=u_0(x),\quad v(0,x)=v_0(x)\quad \text{in } \Omega, \tag{1.4}$$

where u_0, v_0 are nonnegative and bounded functions satisfying the following restrictions:

$$\begin{gathered} \|u_0\|_\infty \le \frac{\lambda}{\mu} < \frac{8ad}{\alpha n(a-d)^2},\quad \text{when } a+c<d,\\ \|u_0\|_\infty \le \frac{\lambda}{\mu},\quad \text{when } a+c\ge d, \end{gathered} \tag{1.5}$$

$$v_0 \ge \frac{c}{d-a}(\frac{\lambda}{\mu}-u_0). \tag{1.6}$$

Problem (1.1)–(1.4) may be viewed as a diffusive epidemic model where u and v represent the nondimensional population densities of susceptibles and infectives, respectively. This problem can be represent a model describing the spread of an infection disease (such as AIDS for instance) within a population assumed to be divided into the susceptible and infective classes as precised (for further motivation see for instance [6, 7, 10] and the references therein).

When $\lambda=\mu=0$, Kouachi and Youkana [18] generalized the method of Haraux and Youkana [12] with the reaction term $f(\xi,\eta)$ requiring the condition

$$\lim_{\eta\to+\infty}\frac{\ln(1+f(\xi,\eta))}{\eta} < \alpha^*,\quad \text{for any } \xi\ge 0,$$

with

$$\alpha^* = \frac{2ad}{n(a-d)^2\|u_0\|_\infty}\min\{1,\frac{a-d}{c}\},$$

where a, c and d satisfy $a>0$, $c>0$, $d>0$ and $a>d$. This condition reflects the weak exponential growth of the function f.

Kanel and Kirane [15] proved the existence of a classical global solutions for a coupled reaction-diffusion system without any conditions on the growth of the function f under the following conditions:

- $a>d$ and $c\ge d-a>0$,
- $f(\xi,\eta)=F(\xi)G(\eta)$.

Later they improved their results in [16] where they extended the result of Herrero et al [14] to the case of a bounded domain under the following assumptions:

- $a<d$ and $0\le c<d-a$,
- $f(\xi,\eta)\le C\varphi(\xi)e^{\alpha\eta}$, for some $C>0$, $\alpha>0$ and any nonnegative continuous and locally Lipschitzian function φ on $\mathbb{R}$ such that $\varphi(0)=0$.

In the case, where $\lambda\ge 0$ and $\mu>0$, Abdelmalek and Youkana [1] proved the global existence of nonnegative classical solutions for the nonlinearities of weakly exponential growth under the following assumptions:

- $a>0$, $c>0$, $d-a\ge c$, and $2\sqrt{ad}>c$,
- $\|u_0\|_\infty\le \lambda/\mu$.

We note that solving problem (1.1)–(1.4) is quite difficult. As a consequence of the blow-up examples found in [23], we can prove that there is blow-up of the solutions in finite time for such triangular systems even though the initial data are regular, the solutions are positive and the nonlinear terms are negative, a structure that ensured the global existence in the diagonal case.

The aim of this article is to prove the existence of global classical solutions to (1.1)–(1.4) without any restrictions on the growth of the function f when $a+c \geq d$ and with possibility of exponential growth for this function when $a+c < d$. For this purpose, we demonstrate that for any initial conditions satisfying (1.5) and (1.6), the problem (1.1)–(1.4) is equivalent to a problem for which the global existence follows from a similar Lyapunov functionals appeared in [12, 4, 18, 24] under the assumptions (A1)–(A3).

2. Existence of local and positive solutions

The study of existence and uniqueness of local solutions (u, v) of (1.1)–(1.4) follows from the basic existence theory for parabolic semilinear equations (see, e.g., [3, 9, 13, 22]). As a consequence, for any initial data in $C(\overline{\Omega})$ or $L^\infty(\Omega)$ there exists a $T^* \in (0, +\infty]$ such that (1.1)–(1.4) has a unique classical solution on $[0, T^*) \times \Omega$. Furthermore, if $T^* < +\infty$, then

$$\lim_{t \uparrow T^*} \left(\|u(t)\|_\infty + \|v(t)\|_\infty \right) = +\infty.$$

Therefore, if there exists a positive constant C such that

$$\|u(t)\|_\infty + \|v(t)\|_\infty \leq C \quad \forall t \in [0, T^*),$$

then $T^* = +\infty$.

Since the initial conditions (1.5) and (1.6) are satisfied under assumptions (A1) and (A3), the next lemma says that the classical solution of (1.1)–(1.4) on $[0, T^*) \times \Omega$ remains nonnegative on $[0, T^*) \times \Omega$.

Lemma 2.1. *Assume* (A1), (A3). *Then for any initial conditions u_0 and v_0 satisfying* (1.5) *and* (1.6), *the classical solution (u, v) of problem* (1.1)–(1.4) *on* $[0, T^*) \times \Omega$ *satisfies*

$$0 \leq u \leq \frac{\lambda}{\mu}, \quad v \geq \frac{c}{d-a}\left(\frac{\lambda}{\mu} - u\right).$$

Proof. In system (1.1)-(1.2) the change of variables

$$w = v - \frac{c}{d-a}\left(\frac{\lambda}{\mu} - u\right),$$
$$F(u, w) = f\left(u, w + \frac{c}{d-a}\left(\frac{\lambda}{\mu} - u\right)\right)$$

leads to the system

$$\frac{\partial u}{\partial t} - a\Delta u = \lambda - F(u, w) - \mu u \quad \text{in } (0, +\infty) \times \Omega, \tag{2.1}$$

$$\frac{\partial w}{\partial t} - d\Delta w = \frac{d-a-c}{d-a} F(u, w) - \mu w \quad \text{in } (0, +\infty) \times \Omega, \tag{2.2}$$

with boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad \text{on } (0, +\infty) \times \partial\Omega, \tag{2.3}$$

and initial conditions

$$u(0, x) = u_0(x), \quad w(0, x) = w_0(x) \quad \text{in } \Omega, \tag{2.4}$$

If we assume (A1) and (A3), then a simple application of comparison theorem [26, Theorem 10.1] to system (2.1)-(2.2) implies that for any initial conditions u_0 and v_0 satisfying (1.5) and (1.6), we have

$$0 \le u(t,x) \le \frac{\lambda}{\mu}, \quad v(t,x) \ge \frac{c}{d-a}(\frac{\lambda}{\mu} - u(t,x)) \quad \forall (t,x) \in [0,T^*) \times \Omega.$$

□

3. Existence of global solutions

It is clear that to prove the existence of global solutions for problem (1.1)–(1.4) we need to prove it for problem (2.1)-(2.4).

At first, when $a + c \ge d$, the local classical solutions of (1.1)–(1.4) may be extended as a classical and uniformly bounded solutions on $[0, +\infty) \times \Omega$ for any nonnegative initial data u_0 and v_0 satisfying (1.5) and (1.6) without any restrictions on the growth of the function f under the assumptions (A1) and (A2). Indeed, since u, w and f are nonnegative, then from (A1) and (A2), we have by the comparison theorem that

$$0 \le u(t,x) \le \frac{\lambda}{\mu}, \quad 0 \le w(t,x) \le \|w_0\|_\infty \quad \forall (t,x) \in [0,T^*) \times \Omega.$$

Now, the main result for the case $a + c < d$ is stated in the following theorem.

Theorem 3.1. *Under assumptions (A1)-(A3) and restrictions* (1.5) *and* (1.6)*, the solutions of problem* (1.1)–(1.4) *are global and uniformly bounded on* $[0, +\infty) \times \Omega$.

Since $0 \le u \le \frac{\lambda}{\mu}$, then the problem of global existence reduces to establish the uniform boundedness of w on $[0, T^*)$. By L^p-regularity theory for parabolic operator (see, e.g., [19, 25]) it follows that it is sufficient to derive a uniform estimate of $\|\rho F(u,w) - \mu w\|_p$ on $[0,T^*)$ for some $p > \frac{n}{2}$ where $\rho = \frac{d-a-c}{d-a}$. The proof of Theorem 3.1 is based on the following key proposition.

Proposition 3.1. *Suppose* (A1)-(A3)*,* (1.5) *and* (1.6)*. For every classical solution* (u, w) *of* (2.1)-(2.4) *on* $[0, T^*) \times \Omega$*, consider the function*

$$L(t) = \int_\Omega [\delta u + (M-u)^{-\gamma}(w+1)^{\beta p} e^{\alpha p w}](t,x)dx,$$

where α, β, γ, δ, p *and* M *are positive constants such that*

$$\frac{\lambda}{\mu} < M < \frac{2\gamma}{\alpha n}, \quad \gamma < \overline{\gamma} = \frac{4ad}{(a-d)^2}, \quad \beta = \max\{r, \frac{\overline{\gamma}(1+\gamma)}{p(\overline{\gamma}-\gamma)}\}. \tag{3.1}$$

Then, there exists $\delta > 0$, $\sigma > 0$ *and* $p > \frac{n}{2}$ *such that*

$$\frac{d}{dt}L(t) \le -\mu L(t) + \sigma \quad \forall t \in [0, T^*). \tag{3.2}$$

Before proving this proposition we need the following lemmas.

Lemma 3.1. *Let* (u, w) *be a solution of* (2.1)-(2.4) *on* $[0, T^*) \times \Omega$*, then under assumption (A3), we have*

$$\int_\Omega F(u(t,x), w(t,x))dx \le \lambda|\Omega| - \frac{d}{dt}\int_\Omega u(t,x)dx. \tag{3.3}$$

Proof. Since u is a nonnegative function, it suffices to integrate both sides of (2.1) on Ω, to obtain (3.3), which competes the proof. □

Lemma 3.2. *Let ϕ and ψ be two nonnegative continuous functions on $[0,+\infty)$ with $\phi(\eta)$ goes to $+\infty$ as $\eta \to +\infty$. Then there exists a positive constant A such that*

$$\big(1-\phi(\eta)\big)\psi(\eta) \le A \quad \forall \eta \ge 0. \tag{3.4}$$

Proof. Since $\phi(\eta)$ goes to $+\infty$ as $\eta \to +\infty$, there exists $\eta_0 > 0$ such that for all $\eta > \eta_0$, we obtain

$$(1-\phi(\eta))\psi(\eta) \le 0.$$

On the other hand, if η is in the compact interval $[0,\eta_0]$, then the continuous function

$$\eta \longmapsto \big(1-\phi(\eta)\big)\psi(\eta)$$

is bounded. So that (3.4) immediately follows. □

Proof of Proposition 3.1. Differentiating L with respect to t, one obtains

$$\frac{d}{dt}L(t) = \delta\frac{d}{dt}\int_\Omega u(t,x)dx + I + J, \tag{3.5}$$

where

$$\begin{aligned} I = \int_\Omega \Big(& a\gamma(M-u)^{-\gamma-1}(w+1)^{\beta p}e^{\alpha p w}\Delta u \\ &+ dp(M-u)^{-\gamma}[\alpha(w+1)^{\beta p} + \beta(w+1)^{\beta p-1}]e^{\alpha p w}\Delta w\Big)\,dx, \end{aligned}$$

and

$$\begin{aligned} J = \int_\Omega & \mu\Big(\gamma(M-u)^{-1}(\frac{\lambda}{\mu} - u) - p[\alpha + \beta(w+1)^{-1}]w\Big) \\ &\times (M-u)^{-\gamma}(w+1)^{\beta p}e^{\alpha p w}dx \\ &+ \int_\Omega \Big(\rho p(M-u)[\alpha+\beta(w+1)^{-1}] - \gamma\Big) \\ &\times F(u,w)(M-u)^{-\gamma-1}(w+1)^{\beta p}e^{\alpha p w}dx. \end{aligned}$$

Using Green's formula in the first integral and taking into account (2.3), we obtain

$$I \le -\int_\Omega Q(\nabla u, \nabla w)(M-u)^{-\gamma-2}(w+1)^{\beta p}e^{\alpha p w}dx,$$

where

$$\begin{aligned} Q(\nabla u,\nabla w) &= a\gamma(1+\gamma)|\nabla u|^2 + \gamma p(a+b)(M-u)[\alpha+\beta(w+1)^{-1}]\nabla u\nabla w \\ &\quad + dp(M-u)^2[\alpha^2 p + 2\alpha\beta p(w+1)^{-1} + \beta(\beta p-1)(w+1)^{-2}]|\nabla w|^2, \end{aligned}$$

is a quadratic form with respect to ∇u and ∇w.

The discriminant of Q is given by

$$\begin{aligned} D = \gamma p(d-a)^2(M-u)^2\Big(&\beta[\beta p(\gamma-\overline{\gamma}) + \overline{\gamma}(1+\gamma)](w+1)^{-2} \\ &+ \alpha^2 p(\gamma-\overline{\gamma})[\alpha + 2\beta(w+1)^{-1}]\Big). \end{aligned}$$

From conditions (3.1), we have $D \le 0$, then we obtain $Q(\nabla u, \nabla v) \ge 0$ and consequently

$$I \le 0. \tag{3.6}$$

Concerning the term J, since $0 \le u \le \frac{\lambda}{\mu} < M$, we observe that

$$\begin{aligned} J \le & \int_\Omega \mu\big(\gamma - p[\alpha + \beta(w+1)^{-1}]w\big)(M-u)^{-\gamma}(w+1)^{\beta p} e^{\alpha p w} dx \\ & + (M - \frac{\lambda}{\mu})^{-\gamma-1} \int_\Omega \big(pM[\alpha + \beta(w+1)^{-1}] - \gamma\big) F(u,w)(w+1)^{\beta p} e^{\alpha p w} dx, \end{aligned}$$

or

$$\begin{aligned} J \le & -\mu L(t) + \lambda\delta|\Omega| \\ & + \mu(M - \frac{\lambda}{\mu})^{-\gamma} \int_\Omega \big(\gamma + 1 - p[\alpha + \beta(w+1)^{-1}]w\big)(w+1)^{\beta p} e^{\alpha p w} dx \\ & + (M - \frac{\lambda}{\mu})^{-\gamma-1} \int_\Omega [\beta p M - (\gamma - \alpha p M)(w+1)] F(u,w)(w+1)^{\beta p - 1} e^{\alpha p w} dx. \end{aligned}$$

From (3.1), we obtain $\frac{n}{2} < \frac{\gamma}{\alpha M}$. Then we can choose p such that $\frac{n}{2} < p < \frac{\gamma}{\alpha M}$. Using Lemma 3.2, we get $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$J \le -\mu L(t) + (\lambda\delta + \mu\delta_1(M - \frac{\lambda}{\mu})^{-\gamma})|\Omega| + \delta_2(M - \frac{\lambda}{\mu})^{-\gamma-1} \int_\Omega F(u,w) dx.$$

Let $\delta = \delta_2(M - \frac{\lambda}{\mu})^{-\gamma-1}$ and using Lemma 3.1, we obtain

$$J \le -\mu L(t) + (2\lambda\delta + \mu\delta_1(M - \frac{\lambda}{\mu})^{-\gamma})|\Omega| - \delta \frac{d}{dt} \int_\Omega u(t,x) dx. \tag{3.7}$$

From (3.6) and (3.7), we conclude that

$$\frac{d}{dt} L(t) \le -\mu L(t) + \sigma,$$

where $\sigma = (2\lambda\delta + \mu\delta_1(M - \frac{\lambda}{\mu})^{-\gamma})|\Omega|$. This concludes the proof. □

Proof of Theorem 3.1. Let p be the same as in Proposition 3.1. Since $M^{-\gamma} \le (M-u)^{-\gamma}$, from (3.1) and (A3) it follows that

$$\begin{gathered} \|w\|_p^p = \int_\Omega |w|^p dx \le M^\gamma L(t), \\ \|F(u,w)\|_p^p = \int_\Omega |F(u,w)|^p dx \le M^\gamma K^p L(t), \end{gathered}$$

where

$$K = \max_{0 \le \xi \le \frac{\lambda}{\mu}} \varphi(\xi).$$

By (3.2), it is seen that there exists a positive constant B such that

$$L(t) \le B \quad \forall t \in [0, T^*),$$

and consequently

$$\begin{gathered} \|w\|_p^p \le B M^\gamma, \\ \|F(u,w)\|_p^p \le B M^\gamma K^p. \end{gathered}$$

Hence $\rho F(u(t,.), w(t,.)) - \mu w(t,.)$ is uniformly bounded in $L^p(\Omega)$ for all $t \in [0, T^*)$ with $p > \frac{n}{2}$. Using the regularity results for solutions of parabolic equations in [19, 25], we conclude that the solutions of the problem (1.1)–(1.4) are uniformly bounded on $[0, +\infty) \times \Omega$. □

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References

[1] S. Abdelmalek and A. Youkana; *Global existence of solutions for some coupled systems of reaction-diffusion*, Int. Journal of Math. Analysis 5 (2011), 425–432.

[2] N. D. Alikakos; *L^p-bounds of solutions of reaction-diffusion equations*, Comm. Partial Differential Equations 4 (1979), 827–868.

[3] H. Amann; *Dynamic theory of quasilinear parabolic equations - I. Abstract evolution equations*, Nonlinear Anal. 12 (1988), 895–919.

[4] A. Barabanova; *On the global existence of solutions of a reaction-diffusion equation with exponential nonlinearity*, Proc. Amer. Math. Soc. 122 (1994), 827–831.

[5] N. Boudiba and M. Pierre; *Global existence for coupled reaction-diffusion systems*, J. Math. Ana. and Appl. 250 (2000), pp. 1–12.

[6] C. Castillo-Chavez, K. Cooke, W. Huang and S. A. Levin; *On the role of long incubation periods in the dynamics of acquired immunodeficiency syndrome (AIDS)*, J. Math. Biol. 27 (1989), 373–398.

[7] E. L. Cussler, *Diffusion, mass transfer in fluid systems*, Second edition, Cambridge University Press, 1997.

[8] E. H. Daddiouaissa; *Existence of global solutions for a system of reaction-diffusion equations with exponential nonlinearity*, Electron. J. Qual. Theory Differ. Equ. 73 (2009), pp. 1–7.

[9] A. Friedman; *Partial differential equation of parabolic type*, Prentice-Hall, Englewood Chiffs, Inc., N.J., 1964.

[10] Y. Hamaya; *On the asymptotic behavior of a diffusive epidemic model (AIDS)*, Nonlinear Analysis 36 (1999)

[11] A. Haraux and M. Kirane; *Estimations C^1 pour des problèmes paraboliques non-linéaires*, Ann. Fac. Sci. Toulouse Math. 5 (1983), 265–280.

[12] A. Haraux and A. Youkana; *On a result of K. Masuda concerning reaction-diffusion equations*, Tohoku. Math. J. 40 (1988), 159–163.

[13] D. Henry; *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics 840, Springer-Verlag, Berlin-New York, 1981.

[14] M. A. Herrero, A. A. Lacey and J. J. L. Velazquez; *Global existence for reaction-diffusion systems modelling ignition*, Arch. Rational Mech. Anal. 142 (1998), 219–251.

[15] J. I. Kanel and M. Kirane; *Global existence and large time behavior of positive solutions to a reaction-diffusion system*, J. Differential and Integral Equations 13 (2000), 255–264.

[16] J. I. Kanel and M. Kirane; *Global solutions of reaction-diffusion systems with a balance law and nonlinearities of exponential growth*, J. Differential Equations 165 (2000), 24–41.

[17] M. Kirane and S. Kouachi; *Global solutions to a system of strongly coupled reaction-diffusion equations*, Nonlinear Anal. 26 (1996), 1387–1396.

[18] S. Kouachi and A. Youkana; *Global existence for a class of reaction-diffusion system*, Bull. Polish Acad. Sci. Math. 49 (2001), 303–308.

[19] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva; *Linear and quasi-linear equations of parabolic type*, Amer. Math. Soc. 1968.

[20] K. Masuda; *On the global existence and asymptotic behaviour of reaction-diffusion equations*, Hokkaido Math. J. 12 (1983), 360–370.

[21] L. Melkemi, A. Z. Mokrane and A. Youkana; *Boundedness and large-time behavior results for a diffusive epedimic model*, J. Applied Math. 2007 (2007), pp. 1–12.

[22] A. Pazy; *Semigroups of linear operators and applications to partial differential equations*, Appl. Math. Sci. 44, Springer-Verlag, New York 1983.

[23] M. Pierre and D. Schmitt; *Blowup in reaction-diffusion systems with dissipation of mass*, SIAM J. Math. Anal. 28 (1997), 259–269.

[24] B. Rebiai and S. Benachour; *Global classical solutions for reaction-diffusion systems with nonlinearities of exponential growth*, J. Evol. Equ. 10 (2010), 511–527.

[25] F. Rothe; *Global solutions of reaction-diffusion systems*, Lecture Notes in Mathematics 1072, Springer-Verlag, Berlin-New York, 1984.

[26] J. A. Smoller; *Shock waves and reaction-diffusion equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science] 258, Springer-Verlag, Berlin-New York, 1983.

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