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HARMONIC SOLUTIONS TO A CLASS OF DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH SEPARATED VARIABLES

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Abstract. We study the properties of periodic solutions of a class of periodically perturbed Differential-Algebraic Equations with separated variables. Under suitable hypotheses, these equations are equivalent to separated variables ODEs on a manifold. By combining known results on Differential-Algebraic Equations, with an argument about ODEs on manifolds, we obtain a global continuation result for the T-periodic solutions to the considered equations. As an application of our method, a multiplicity result is provided.

1. Introduction

This article concerns the properties of harmonic solutions to periodic perturbations of a class of T-periodic Differential-Algebraic Equations (DAEs). More precisely, we will consider the following separated variables DAE in semi-explicit form

$$\begin{aligned} \dot{x} &= a(t)f(x,y), \\ g(x,y) &= 0, \end{aligned} \tag{1.1}$$

where $f : U \to \mathbb{R}^k$ and $g : U \to \mathbb{R}^s$ are given continuous maps defined on an open connected set $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ and $a : \mathbb{R} \to \mathbb{R}$ is continuous. Assume that $g \in C^\infty(U, \mathbb{R}^s)$, with the property that the Jacobian matrix $\partial_2 g(p,q)$ of g, with respect to the last s variables, is invertible for any $(p,q) \in U$. Given $T > 0$, we also assume that the function a is T-periodic and such that

$$\frac{1}{T}\int_0^T a(t)\,dt \neq 0, \tag{1.2}$$

and we consider the perturbed problem

$$\begin{aligned} \dot{x} &= a(t)f(x,y) + \lambda h(t,x,y), \quad \lambda \geq 0, \\ g(x,y) &= 0, \end{aligned} \tag{1.3}$$

where $h : \mathbb{R} \times U \to \mathbb{R}^k$ is continuous and T-periodic in the first variable. Here, the function a has the meaning of a perturbation factor and, up to dividing a by its average, we can assume that $a = 1 + \alpha$, where $\alpha : \mathbb{R} \to \mathbb{R}$ is continuous, T-periodic

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and with zero average. Sometimes, for convenience, we will refer to the pair of maps (α, h) as the *T-perturbation pair*.

Since $\partial_2 g(p,q)$ is invertible for any $(p,q) \in U$, then 0 is a regular value for g, and so $g^{-1}(0)$ is a closed C^∞ submanifold of U of dimension k. Throughout this article the letter M will be used to denote the submanifold $g^{-1}(0)$ and, if not otherwise stated, the points of M will be written as pairs $(p,q) \in M$.

As a direct consequence of the Implicit Function Theorem, M can be locally represented as a graph of some map from an open subset of $\mathbb{R}^k$ to $\mathbb{R}^s$ and, hence, Equation (1.3) can be locally decoupled. However, globally, this might be false or not convenient for our purpose (see, e.g. [2, 13]). For instance, even in the case when M is a graph of some map γ, the analytical expression of γ might be so complicated that the decoupled version of (1.3) turns out impractical.

Following the ideas in [3, 13] (see also [11]), we will show that the Equation (1.3) is equivalent to an ODE on M. Therefore, to study the T-periodic solutions to (1.3) we will be able to use topological methods based on the degree of tangent vector fields, as well as results on periodic solutions to ODEs on manifolds.

Our aim is to get information on the structure of the set of solutions to (1.3). Thus, denoted by $C_T(U)$ the metric subspace of $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ of all the continuous T-periodic functions taking values in U, we will give conditions ensuring the existence of a connected set of pairs $(\lambda, (x,y)) \in [0,\infty) \times C_T(U)$, with $\lambda > 0$ and (x,y) a T-periodic solution to (1.3), whose closure is not compact and meets the set of pairs formed by constant solutions corresponding to $\lambda = 0$ (the so called *trivial* pairs).

The plan for this article is as follows: In Section 2 we introduce some preliminaries and we give an overview of our approach to treat the above class of DAEs. In section 3, proceeding as in [12, 13], we obtain information about the set of T-pairs to (1.3); examples of applications of our method are given. Thereafter, in Section 4, a multiplicity result for (1.3) is provided.

2. Preliminaries and basic notion

We first recall some facts and definitions about the function spaces used in the sequel. Let $I \subseteq \mathbb{R}$ be an interval and $V \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open. Given $r \in \mathbb{N}$, the set of all V-valued C^r-functions defined on I is denoted by $C^r(I,V)$. Frequently, we also use the notation $C(I,V)$ in place of $C^0(I,V)$ and, when $I = \mathbb{R}$, we write $C^r(V) = C^r(\mathbb{R}, V)$. We will denote by the symbol $C_T(V)$ the metric subspace of the Banach space $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ of all the continuous T-periodic functions assuming values in V.

Let us now give the precise concept of solution to equations of the form (1.3). Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected, let $g : U \to \mathbb{R}^s$, $f : U \to \mathbb{R}^k$, $a : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \times U \to \mathbb{R}^k$ be given continuous maps, where $g \in C^\infty(U, \mathbb{R}^s)$ has the property that $\partial_2 g(p,q)$ is invertible for any $(p,q) \in U$. For a given $\lambda \geq 0$, a solution of (1.3) consists of a pair of functions $(x,y) \in C^1(I,U)$, with $I \subseteq \mathbb{R}$ an open interval, such that

$$\begin{gathered}\dot{x}(t) = a(t) f\big(x(t), y(t)\big) + \lambda h(t, x(t), y(t)),\\ g(x(t), y(t)) = 0,\end{gathered}$$

for all $t \in I$. Existence and uniqueness results for the initial value problems related to DAEs of type (1.3) will be discussed in the next section: they will be deduced as consequences of known theorems about ODEs on manifolds.

Let (x, y) be a solution of (1.3) defined on an interval I, and corresponding to a given $\lambda \geq 0$. We say that (z, w) is a continuation (or extension) of (x, y) if there exists an interval $\tilde{I} \supset I$, such that (z, w) coincides with (x, y) on I, and (z, w) satisfies (1.3). A solution (x, y) for which there are no extensions is called *maximal*. As in the case of ODEs, Zorn's lemma implies that any solution is the restriction of a maximal solution.

Remark 2.1. Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected. Consider the initial value problems

$$\begin{aligned} \dot{x} &= a(t) f(x, y), \\ g(x, y) &= 0, \\ (x(0), y(0)) &= (x_0, y_0), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \dot{x} &= f(x, y), \\ g(x, y) &= 0, \\ (x(0), y(0)) &= (x_0, y_0), \end{aligned} \tag{2.2}$$

where $(x_0, y_0) \in U$, $f : U \to \mathbb{R}^k$, $g : U \to \mathbb{R}^s$ are defined above, and the map $a : \mathbb{R} \to \mathbb{R}$ is continuous, T-periodic with $1/T \int_0^T a(t)\, dt = 1$. Suppose also that f is C^1, so that the uniqueness of solutions for the problems (2.1) and (2.2) is guaranteed (see below for details). Let $(\xi, \sigma) : J \to U$ and $(u, v) : I \to U$ be the maximal solutions of (2.1) and (2.2) respectively, with I and J the related maximal intervals of existence.

Let $t > 0$ be such that $\int_0^l a(s) ds \in I$ for all $l \in [0, t]$, then it follows that

$$(\xi(t), \sigma(t)) = \Big(u\Big(\int_0^t a(s) ds \Big), v\Big(\int_0^t a(s) ds \Big) \Big),$$

and hence $t \in J$. Conversely, by using a standard maximality argument, one can prove that $t \in J$ implies $\int_0^t a(s) ds \in I$. Define the map $\phi_a : J \to I$, $t \mapsto \phi_a(t) = \int_0^t a(s) ds$. Notice that, if $T \in J$, then $\phi_a(T) = T \in I$, and so $(\xi(T), \sigma(T)) = (x(T), y(T))$.

The argument in Remark 2.1 plays a fundamental role in our approach to treat Equation (1.3). As we will formally prove, the study of the T-periodic solutions to (1.3) can be reduced to that of the T-periodic solutions for a particular autonomous semi-explicit DAE. As a consequence, the main results in [13] can be applied to the present case.

2.1. **Associated vector fields and ODEs on M.** Proceeding as in [2, 13], we associate to (1.3) an ODE on $g^{-1}(0) = M$.

Let us first recall that, given a differentiable manifold $N \subseteq \mathbb{R}^n$, a continuous map $w : N \to \mathbb{R}^n$ with the property that for any $p \in N$, $w(p)$ belongs to the tangent space T_pN of N at p is called a *tangent vector field on N*.

With our hypotheses it is always possible to associate a pair Ψ, Υ of tangent vector fields on M to the functions f and h in (1.3). In fact, consider the maps $\Psi : M \to \mathbb{R}^k \times \mathbb{R}^s$ and $\Upsilon : \mathbb{R} \times M \to \mathbb{R}^k \times \mathbb{R}^s$ defined as follows:

$$(p, q) = \big(f(p, q), -[\partial_2 g(p, q)]^{-1} \partial_1 g(p, q) f(p, q) \big), \text{ and} \tag{2.3a}$$

$$\Upsilon(t, p, q) = \big(h(t, p, q), -[\partial_2 g(p, q)]^{-1} \partial_1 g(p, q) h(t, p, q) \big). \tag{2.3b}$$

Given a point $(p,q) \in M \subseteq \mathbb{R}^k \times \mathbb{R}^s$, since $T_{(p,q)}M$ coincides with the kernel $\ker d_{(p,q)}g$ of the differential $d_{(p,q)}g$ of g at (p,q), it can be easily seen that Ψ is tangent to M in the sense that $\Psi(p,q)$ belongs to $T_{(p,q)}M$ for all $(p,q) \in M$ (see, e.g. [13]). Analogously, the time-dependent vector field Υ is tangent to M, i.e. $\Upsilon(t,p,q) \in T_{(p,q)}M$, for all $(t,p,q) \in \mathbb{R} \times M$.

We claim that (1.3) is equivalent to the following ODE on M, which implicitly keeps track of the condition $g(x,y) = 0$:

$$\dot{\zeta} = a(t)\Psi(\zeta) + \lambda\Upsilon(t,\zeta), \tag{2.4}$$

meaning that $\zeta = (x,y)$ is a solution of (2.4), in an interval $I \subseteq \mathbb{R}$, if and only if so is (x,y) for (1.3). We prove the claim: for a given $\lambda > 0$, let (x,y) be a solution to (1.3) defined on $I \subseteq \mathbb{R}$. Differentiating the algebraic condition $g\big(x(t), y(t)\big) = 0$, one obtains

$$0 = \partial_1 g(x(t), y(t))\dot{x}(t) + \partial_2 g(x(t), y(t))\dot{y}(t),$$

whence

$$\dot{y}(t) = -[\partial_2 g(x(t), y(t))]^{-1}\partial_1 g(x(t), y(t))\left[a(t)f(x(t), y(t)) + \lambda h(t, x(t), y(t))\right],$$

with $t \in I$. Then, the solutions of (1.3) correspond to those of (2.4). Conversely, if $\zeta = (x,y)$ is a solution of (2.4) defined on an interval $I \subseteq \mathbb{R}$, then it satisfies identically $\big(x(t), y(t)\big) \in M$, which implies $g(x(t), y(t)) = 0$, and fulfills

$$\dot{\zeta}(t) = a(t)\Psi\big(\zeta(t)\big) + \lambda\Upsilon(t, \zeta(t)), \ \forall t \in \mathbb{R},$$

whose first component is (1.3).

Now, consider the unperturbed version of Equation (2.4):

$$\dot{\zeta} = a(t)\Psi(\zeta). \tag{2.5}$$

By the definition of the vector field Ψ, if f is C^1, then Ψ is C^1 too. This condition ensures that any initial value problem associated to (2.5) admits a unique solution. Then, as a consequence of the equivalence of (1.1) with (2.5), the local results on existence, uniqueness and continuous dependence of local solutions of the initial value problems translate to (1.1) from the theory of ODEs on manifolds (see, e.g. [7]). Observe that, if also h is C^1, a similar statement holds for (1.3) and (2.4).

Let $N \subseteq \mathbb{R}^n$ be a differentiable manifold and let $\Xi : \mathbb{R} \times N \to \mathbb{R}^n$ be a time-dependent tangent vector field sufficiently regular in order to guarantee the existence and uniqueness of the solutions for the initial value problems associated with the differential equation

$$\dot{\zeta} = \Xi(t,\zeta), \quad t \in \mathbb{R}. \tag{2.6}$$

Denote by

$$\begin{aligned} \mathcal{D} = \big\{(\tau,p) \in \mathbb{R} \times N :\ &\text{the solution of (2.6) which satisfies } \zeta(0) = p \\ &\text{is continuable at least up to } t = \tau\big\}. \end{aligned}$$

A well known argument based on some global continuation properties of the flows (see, e.g. [7, 8]) shows that $\mathcal{D}$ is an open set containing $\{0\} \times M$. Let $P^{\Xi} : \mathcal{D} \to N$ be the map that associates to each $(t,p) \in \mathcal{D}$ the value $\zeta(t)$ of the maximal solution ζ to (2.6) such that $\zeta(0) = p$ (i.e. $P^{\Xi}(t,p) = \zeta(t)$). Here and in the sequel, we will denote by $P^{\Xi}_{\tau}, \tau \in \mathbb{R}$, the local (Poincaré) τ-translation operator associated to the equation (2.6). It holds true that $P^{\Xi}_{\tau}(p) = P^{\Xi}(\tau,p)$, with $(\tau,p) \in \mathcal{D}$. Therefore, the domain of P^{Ξ}_{τ} is an open set formed by the points $p \in N$ for which the maximal solution of (2.6), starting from p at $t = 0$, is defined up to τ.

Remark 2.2. By the equivalence of (2.5) with (1.1), the initial value problems (2.1) and (2.2) become

$$\begin{gathered}\dot{\zeta} = a(t)\Psi(\zeta),\\ \zeta(0) = \zeta_0,\end{gathered} \tag{2.7}$$

and

$$\begin{gathered}\dot{\zeta} = \Psi(\zeta),\\ \zeta(0) = \zeta_0,\end{gathered} \tag{2.8}$$

respectively, with $\zeta_0 = (p_0, q_0) \in M$. Let J and I be the intervals on which are defined the (unique) maximal solutions of (2.7) and (2.8) respectively. As before, consider the map $\phi_a : J \to I$, $t \mapsto \phi_a(t) = \int_0^t a(s)ds$ and assume $T \in J$, so that $\phi_a(T) = T \in I$. Then, also in this case, the map ϕ_a allows us to write the solution of (2.7) in terms of the solution of SVDAEs:equivalence-on-manifold' on the interval $[0, T]$. Rephrasing all of this in terms of the local Poincaré operators $P_t^{a\Psi}, P_t^{\Psi}$, $t \in \mathbb{R}$, associated to (2.7) and (2.8), it holds true that: if $P_T^{a\Psi}(\zeta_0)$ is defined, then $P_T^{\Psi}(\zeta_0)$ is defined too and, in such a case, $P_T^{\Psi}(\zeta_0) = P_T^{a\Psi}(\zeta_0)$.

To conclude this subsection, we give some conventions that will be widely used in the sequel. Let $N \subseteq \mathbb{R}^n$ be a differentiable manifold, and $T > 0$ positive number. We denote by $C_T(N)$ the metric subspace of the Banach space $C_T(\mathbb{R}^n)$ of all the T-periodic continuous functions $\xi : \mathbb{R} \to N$. Notice that $C_T(N)$ is not complete unless N is closed in $\mathbb{R}^n$. Consider the following diagram of closed embeddings:

$$\begin{tikzcd} {[0,\infty) \times N} \arrow[r] & {[0,\infty) \times C_T(N)} \\ N \arrow[u] \arrow[r] & C_T(N) \arrow[u] \end{tikzcd} \tag{2.9}$$

which allow us to identify any space in the above diagram with its image. In particular, N will be regarded as its image in $C_T(N)$ under the embedding that associates to any $p \in N$ the map $\overline{p} \in C_T(N)$ constantly equal to p. Furthermore, we will regard N as the slice $\{0\} \times N \subseteq [0, \infty) \times N$ and, analogously, $C_T(N)$ as $\{0\} \times C_T(N)$. Thus, if Ω is a subset of $[0, \infty) \times C_T(N)$, then $\Omega \cap N$ represents the set of points of N that, regarded as constant functions, belong to Ω. Namely, we have that

$$\Omega \cap N = \{\zeta \in N : (0, \overline{p}) \in \Omega\}. \tag{2.10}$$

2.2. **The degree of the tangent vector field Ψ and some related properties.** We now give some notions about the degree of tangent vector fields on manifolds. Recall that if $w : N \to \mathbb{R}^n$ is a tangent vector field on the differentiable manifold $N \subseteq \mathbb{R}^n$ which is (Fréchet) differentiable at $p \in N$ and $w(p) = 0$, then the differential $\mathrm{d}_p w : T_pN \to \mathbb{R}^n$ maps T_pN into itself (see, e.g. [9]), so that, the determinant $\det \mathrm{d}_p w$ is defined. In the case when p is a non-degenerate zero (i.e. $\mathrm{d}_p w : T_pN \to \mathbb{R}^n$ is injective), p is an isolated zero and $\det \mathrm{d}_p w \neq 0$. Let W be an open subset of N in which we assume w admissible for the degree, that is we suppose the set $w^{-1}(0) \cap W$ is compact. Then, it is possible to associate to the pair (w, W) an integer, $\deg(w, W)$, called the degree (or characteristic) of the vector field w in W (see e.g. [4, 5]), which, roughly speaking, counts (algebraically) the zeros of w in W in the sense that when the zeros of w are all non-degenerate, then the set $w^{-1}(0) \cap W$ is finite and

$$\deg(w, W) = \sum_{q \in w^{-1}(0) \cap W} \operatorname{sign} \det \mathrm{d}_q w. \tag{2.11}$$

The concept of degree of a tangent vector field is related to the classical one of Brouwer degree (whence its name), but the former notion differs from the latter when dealing with manifolds. In particular, this notion of degree does not need the orientation of the underlying manifolds. However, when $N = \mathbb{R}^n$, the degree of a vector field $\deg(w, W)$ is essentially the well known Brouwer degree of w on W with respect to 0. The degree of a tangent vector field enjoys, for instance, the following properties: *Additivity, Excision, Homotopy invariance, Invariance under diffeomorphisms* and *Solution*. For an exhaustive exposition of this topic, we refer e.g. to [4, 5, 9].

The Excision property allows the introduction of the notion of *index* of an isolated zero of a tangent vector field. Let $q \in N$ be an isolated zero of a tangent vector field $w : N \to \mathbb{R}^n$. Clearly, $\deg(w, V)$ is well defined for any open set $V \subseteq N$ such that $V \cap w^{-1}(0) = \{q\}$. Moreover, by the Excision property, the value of $\deg(w, V)$ is constant with respect to such V's. This common value of $\deg(w, V)$ is, by definition, the index of w at q, and is denoted by $\mathrm{i}(w, q)$. Using this notation, if (w, W) is admissible, by the Additivity property we have that if all the zeros in W of w are isolated, then

$$\deg(w, W) = \sum_{q \in w^{-1}(0) \cap W} \mathrm{i}(w, q).$$

By formula (2.11) we have that if q is a non-degenerate zero of w, then $\mathrm{i}(w, q) = \operatorname{sign} \det d_q w$.

Take $U \subseteq \mathbb{R}^k \times \mathbb{R}^s \cong \mathbb{R}^n$ open and connected set. Let $g : U \to \mathbb{R}^s$ and $f : \mathbb{R} \times U \to \mathbb{R}^k$ be given maps such that f is continuous and g is C^∞ with the property that $\partial_2 g(p, q)$ is invertible for all $(p, q) \in U$. Let Ψ be the tangent vector field on $M = g^{-1}(0)$ given by (2.3a). As we will see, a key requirement for the rest of the paper is that the degree of Ψ is nonzero. Define the map $F : U \to \mathbb{R}^k \times \mathbb{R}^s$, as follows

$$F(p, q) := \big(f(p, q), g(p, q)\big). \tag{2.12}$$

The following result [13, Theorem 4.1] allows us to reduce the computation of the degree of the tangent vector field Ψ on M to that of the Brouwer degree of the map F with respect to 0, which is in principle handier. Namely, we have that

Theorem 2.3. *Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected, and let $F : U \to \mathbb{R}^k \times \mathbb{R}^s$ be given by* (2.12)*. Then, for any $V \subseteq U$ open, if either* $\deg(\Psi, M \cap V)$ *or* $\deg(F, V)$ *is well defined, so is the other, and*

$$|\deg(\Psi, M \cap V)| = |\deg(F, V)|.$$

3. Connected sets of T-periodic solutions

The focus of this section is the study of the T-periodic solutions to (1.3), where $T > 0$ is given. In particular, we dwell on the topological structure of the set of T-periodic solutions to this DAE.

Recall that $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ is open and connected, $f : U \to \mathbb{R}^k, g : U \to \mathbb{R}^s$, $a : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \times U \to \mathbb{R}^k$ are continuous. We also assume that a is T-periodic with nonzero average, that h is locally Lipschitz and T-periodic in the first variable, $f \in C^1$, and g is C^∞ with the property that $\det \partial_2 g(p,q) \neq 0$ for all $(p,q) \in U$.

In the following, we say that $(\lambda,(x,y)) \in [0,\infty) \times C_T(\mathbb{R}^k \times \mathbb{R}^s)$ is a *T-periodic pair* to (1.3), if (x,y) is a T-periodic solution of (1.3) corresponding to λ. According to the convention introduced in (2.9)-(2.10), any $(p,q) \in U$ will be identified with the element $(\overline{p},\overline{q})$ of $C_T(U)$ that is constantly equal to (p,q). Let $F : U \to \mathbb{R}^k \times \mathbb{R}^s$ be given by (2.12). Since $a(t)$ is not identically zero, a point $(p,q) \in U$ corresponds to a constant solution $(\overline{p},\overline{q})$ of (1.3), for $\lambda = 0$, if and only if $F(p,q) = (0,0)$. A T-periodic pair of this form will be called *trivial*. Thus, with this notation, the set of trivial T-periodic pairs can be written as

$$\big\{(0,(\overline{p},\overline{q})) \in [0,\infty) \times C_T(U) : F(p,q) = (0,0)\big\}.$$

Observe that, in the case $\lambda = 0$, we may have the existence of nontrivial T-periodic pairs to the system of equations (1.3).

Given $\Omega \subseteq [0,\infty) \times C_T(U)$, with $\Omega \cap U$ we denote the subset of U whose points, regarded as constant functions, lie in Ω. Namely, one has that $\Omega \cap U = \big\{(p,q) \in U : (0,(\overline{p},\overline{q})) \in \Omega\big\}$.

We will use the following result which is a direct consequence of [12, Theorem 3.3]

Theorem 3.1. *Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $\omega : N \to \mathbb{R}^k$ and $\varrho : \mathbb{R} \times N \to \mathbb{R}^k$ be two continuous tangent vector fields on the boundaryless manifold $N \subseteq \mathbb{R}^k$. Consider the following parametrized differential equation on N*

$$\dot{\zeta} = a(t)\omega(\zeta) + \lambda\varrho(t,\zeta),\ \ \lambda \geq 0. \tag{3.1}$$

Assume that ϱ and a are T-periodic, with $1/T\int_0^T a(t)dt \neq 0$. Let Σ be an open subset of $[0,\infty) \times C_T(N)$, and assume that the degree $\deg(\omega, \Sigma \cap N)$ is well-defined and nonzero. Then Σ contains a connected set of nontrivial T-periodic pairs of (3.1) *whose closure in Σ is noncompact and meets $\omega^{-1}(0) \cap \Sigma$.*

Remark 3.2. Until now, condition (1.2) has been used just to characterize some properties of the map ϕ_a introduced in Remark 2.1. However, even if it will not be shown explicitly here, this assumption has some other important implications in our method. Indeed, it is crucial for the proof of Theorem 3.1, which in turn is the basis of the main result of this section.

We state and prove the following Theorem

Theorem 3.3. *Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected. Let $g : U \to \mathbb{R}^s$, $f : U \to \mathbb{R}^k$, $a : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \times U \to \mathbb{R}^k$ be as above. Let also $F(p,q) = \big(f(p,q), g(p,q)\big)$ be defined as in* (2.12). *Given $\Omega \subseteq [0,\infty) \times C_T(U)$ open, assume that $\deg(F, \Omega \cap U)$ is well-defined and nonzero. Then, there exists a connected set of nontrivial T-periodic pairs of* (1.3) *whose closure in Ω is noncompact and meets the set $\big\{(0,(\overline{p},\overline{q})) \in \Omega : F(p,q) = (0,0)\big\}$ of the trivial T-periodic pairs of* (1.3).

Proof. In Section 2 it has been shown that the system of equations (1.3) is equivalent to an ODE of type (2.4) on $M = g^{-1}(0)$. Let Ψ and Υ be the tangent vector fields defined in (2.3). Let also $\mathcal{O}$ be the open subset of $[0,\infty) \times C_T(M)$ given by

$$\mathcal{O} = \Omega \cap \Big([0,\infty) \times C_T(M)\Big).$$

For any $Y \subseteq M$, by $\mathcal{O} \cap Y$ we mean the set of all those points of Y that, regarded as constant functions, lie in $\mathcal{O}$. Using this convention one has that $\Omega \cap Y = \mathcal{O} \cap Y$

and, in particular, $\Omega \cap M = \mathcal{O} \cap M$. This identity, together with Theorem 2.3, implies that

$$\begin{aligned}|\deg(\Psi, \mathcal{O} \cap M)| &= |\deg(\Psi, \Omega \cap M)| \\ &= |\deg(\Psi, (\Omega \cap U) \cap M)| = |\deg\big(F, \Omega \cap U\big)| \neq 0.\end{aligned}$$

Hence, the hypotheses of Theorem 3.1 are satisfied, and we thereby obtain the existence of a connected subset Λ of

$$\big\{(\lambda, (x, y)) \in \mathcal{O} : (x, y) \text{ is a nonconstant solution of } (2.4)\big\},$$

whose closure in $\mathcal{O}$ is not compact and meets $\{(0, (\overline{p}, \overline{q})) \in \mathcal{O} : \Psi(p, q) = (0, 0)\}$. Observe that this set coincides with $\big\{(0, \overline{p}, \overline{q}) \in \Omega : F(p, q) = (0, 0)\big\}$ which is the set of the trivial T-periodic pairs to (1.3). Moreover, from the equivalence of (2.4) with (1.3), we have that each $(\lambda, (x, y)) \in \Lambda$ is a nontrivial T-periodic pair to (1.3). Since M is closed in U, it follows that any relatively closed subset of $\mathcal{O}$ is relatively closed in Ω too and vice versa. Thus, the closure of Λ in $\mathcal{O}$ coincides with the closure of Λ in Ω, and hence Λ fulfills the assertion. □

Under the extra assumption that $M = g^{-1}(0)$ is closed in $\mathbb{R}^k \times \mathbb{R}^s$, we are able to retrieve some further information about the connected components of the set of T-periodic pairs to (1.3).

Lemma 3.4. *Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected with $M \subseteq U$. Assume that M is closed in $\mathbb{R}^k \times \mathbb{R}^s$. Let $\Omega \subseteq [0, \infty) \times C_T(U)$ be open, and let $\Gamma \subseteq [0, \infty) \times C_T(M)$ be a connected component of the set of T-periodic pairs to* (1.3) *that meets $\{(0, (\overline{p}, \overline{q})) \in \Omega : F(p, q) = (0, 0)\}$. Assume also that the intersection $\Gamma \cap \Omega$ is not compact. Then, Γ is either bounded or contained in Ω. Moreover, if Ω is bounded, then $\Gamma \cap \partial\Omega \neq \emptyset$.*

Proof. Since M is a closed subset of $\mathbb{R}^k \times \mathbb{R}^s$, it follows that the metric space $[0, \infty) \times C_T(M)$ is complete. Ascoli's Theorem implies that any bounded set of T-periodic pairs to (1.3) is totally bounded, which means relatively compact due to the completeness of $C_T(M)$. As a straightforward consequence, since Γ is closed, if Γ is bounded then it is also compact. Therefore, Γ cannot be both bounded and contained in Ω. The last part of the assertion follows from the fact that Γ is connected and that $\emptyset \neq \Gamma \cap \Omega \neq \Gamma$. □

In particular, we have the corollary.

Corollary 3.5. *Let a, f, h, g, U and F be as in Theorem 3.3. Assume that M is closed in $\mathbb{R}^k \times \mathbb{R}^s$. Let $\Omega \subseteq [0, \infty) \times C_T(M)$ be open and such that* $\deg(F, \Omega \cap M)$ *is defined and nonzero. Then there exists a connected component Γ of T-periodic pairs that meets $\{(0, (\overline{p}, \overline{q})) \in \Omega : F(p, q) = 0\}$, and cannot be both bounded and contained in Ω. In particular, if Ω is bounded, then $\Gamma \cap \Omega \neq \emptyset$.*

Proof. Applying Theorem 3.3 and Lemma 3.4 the thesis follows readily. □

As a consequence of Theorem 3.3 and Lemma 3.4, we now establish the following continuation result

Corollary 3.6. *Let a, f, h, g, U and F be as in Theorem 3.3. Assume that $M = g^{-1}(0)$ is closed in $\mathbb{R}^k \times \mathbb{R}^s$. Let $V \subseteq U$ be open and such that* $\deg(F, V)$ *is* *well defined and nonzero. Then, there exists a connected component Γ of T-periodic pairs to* (1.3) *that meets the set*

$$\big\{(0,(\overline{p},\overline{q}))\in[0,\infty)\times C_T(U):(p,q)\in V\cap F^{-1}(0,0)\big\}$$

and is either unbounded or meets

$$\big\{(0,(\overline{p},\overline{q}))\in[0,\infty)\times C_T(U):(p,q)\in F^{-1}(0,0)\setminus V\big\}.$$

Proof. Consider the open subset Ω of $[0,\infty)\times C_T(U)$ given by

$$\Big([0,\infty)\times C_T(U)\Big)\Big\backslash\Big\{(0,(\overline{p},\overline{q}))\in[0,\infty)\times C_T(U):(p,q)\in F^{-1}(0,0)\setminus V\Big\}.$$

Clearly, we have that $U\cap\Omega=V$ and hence $\deg(F,U\cap\Omega)\neq 0$. Thus, Theorem 3.3 implies the existence of a connected component Γ of T-periodic pairs of (1.3) that meets the set of the trivial T-periodic pairs $\{(0,\overline{p},\overline{q})\in\Omega:F(p,q)=0\}$, and whose intersection with Ω is not compact. By Lemma 3.4, if Γ is bounded, then it necessarily intersects the boundary of Ω which is given by

$$\big\{(0,(\overline{p},\overline{q}))\in[0,\infty)\times C_T(U):(p,q)\in F^{-1}(0,0)\setminus V\big\}.$$

Then, the conclusion follows. □

Now, we give some examples in order to illustrate our results.

example 3.7. We examine the scalar DAE:

$$\begin{gathered}\dot{x}=\frac{a(t)}{b(t)}x+\lambda h(t,x,y),\\ g(b(t)x,y)=0,\end{gathered}\tag{3.2}$$

where $a:\mathbb{R}\to\mathbb{R}$, $b:\mathbb{R}\to(0,\infty)$, $g:\mathbb{R}^2\to\mathbb{R}$ and $h:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions. We actually assume that $b\in C^\infty$, h is C^1, g is C^∞ with the property that $\partial_2 g(p,q)$ is nonzero for all $(p,q)\in\mathbb{R}^2$. Given $T>0$, we also require that a,b are T-periodic and, that h is T-periodic with respect to its first variable. Consider the change of variables

$$\Theta_b:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}\times\mathbb{R}^2,\quad (t,(x,y))\mapsto(t,(\mathbf{x},y)),\tag{3.3}$$

with $\mathbf{x}=b(t)x$. Differentiating we get

$$\dot{x}(t)=\frac{\dot{\mathbf{x}}(t)-\dot{b}(t)x(t)}{b(t)}.$$

Problem (3.2) can be equivalently rewritten in the form

$$\begin{gathered}\dot{\mathbf{x}}=\frac{\dot{b}(t)+a(t)}{b(t)}\mathbf{x}+\lambda b(t)h(t,\frac{\mathbf{x}}{b(t)},y),\\ g(\mathbf{x},y)=0.\end{gathered}\tag{3.4}$$

Assume that $\frac{1}{T}\int_0^T\frac{a(s)}{b(s)}ds\neq 0$. Then, one has that

$$\frac{1}{T}\int_0^T\frac{\dot{b}(s)+a(s)}{b(s)}ds=\frac{1}{T}\int_0^T\frac{\dot{b}(s)}{b(s)}ds+\frac{1}{T}\int_0^T\frac{a(s)}{b(s)}ds=\frac{1}{T}\int_0^T\frac{a(s)}{b(s)}ds\neq 0,$$

and the system of equations (3.4) is of type (1.3).

For instance, take $a(t)=|\cos(t)|$, $b(t)=2+\sin(t)$, $g(x,y)=y^5+y^3+y+x^3$, $T=2\pi$ and let $\Omega\subseteq[0,\infty)\times C_{2\pi}(\mathbb{R}^2)$ be the open set given by

$$[0,\infty)\times\big\{(x,y)\in C_{2\pi}(\mathbb{R}^2):x(t)>-1/b(t),\ \forall\, t\in\mathbb{R}\big\}.$$

It can be easily checked that $\int_0^{2\pi} a(t)/b(t)dt = 2\ln(3)$. Referring to (3.3), we take into account the induced transformation $\widehat{\Theta}_b : [0,\infty) \times C_{2\pi}(\mathbb{R}^2) \to [0,\infty) \times C_{2\pi}(\mathbb{R}^2)$ given by

$$\widehat{\Theta}_b(\lambda, (\varphi, \psi))(t) = \big(\lambda, (b(t)\varphi(t), \psi(t))\big), \quad \forall t \in \mathbb{R}. \tag{3.5}$$

Since Θ_b is continuous and invertible with continuous inverse, then $\widehat{\Theta}_b$ enjoys the same properties and the inverse $\widehat{\Theta}_b^{-1} : [0,\infty) \times C_{2\pi}(\mathbb{R}^2) \to [0,\infty) \times C_{2\pi}(\mathbb{R}^2)$ is defined by

$$\widehat{\Theta}_b^{-1}(\mu, (\zeta, \omega))(t) = \Big(\mu, \Big(\frac{\zeta(t)}{b(t)}, \omega(t)\Big)\Big), \quad \forall t \in \mathbb{R}. \tag{3.6}$$

Through the transformation (3.5), the open set Ω becomes

$$\widehat{\Omega} = [0,\infty) \times \big\{(\mathbf{x}, y) \in C_{2\pi}(\mathbb{R}^2) : \mathbf{x}(t) > -1, \forall\, t \in \mathbb{R}\big\}.$$

With respect to the Equation (3.4), the map F in (2.12) is given by $F(\mathbf{x}, y) = (\mathbf{x}, y^5 + y^3 + y + \mathbf{x}^3)$. A direct computation shows that $F^{-1}((0,0)) \cap \widehat{\Omega}$ consists of the singleton $\{(0,0)\}$ and that $\deg(F, \mathbb{R}^2 \cap \widehat{\Omega}) = 1$. Therefore, Theorem 3.3 yields a connected set $\Pi \subseteq \widehat{\Omega}$ of nontrivial 2π-periodic pairs to (3.4) emanating from the trivial 2π-periodic pair $\{(0, (\overline{0}, \overline{0}))\}$, whose closure in $\widehat{\Omega}$ is noncompact.

It is not difficult to prove that $\widehat{\Theta}_b^{-1}$ sends 2π-periodic pairs of (3.4) into 2π-periodic pairs of (3.2). Moreover, the trivial 2π-periodic pair $(0, (\overline{0}, \overline{0}))$ (which is the unique trivial 2π-periodic pair to (3.2)) is sent to itself by $\widehat{\Theta}_b^{-1}$. Thus, we can easily infer that $\Lambda := \widehat{\Theta}_b^{-1}(\Pi) \subseteq \Omega$ is a connected set of 2π-periodic pairs to (1.3), whose closure in Ω is noncompact and meets $\{(0, (\overline{0}, \overline{0}))\}$.

example 3.8. Let us now analyze a separated variables DAE coming from a system of equations describing the process of heat generation in a exothermic chemical reactor. The considered equation, which is derived from a well known model (see, e.g. [6, 10]), is affected by a T-periodic perturbation due to the presence of a T-perturbation pair (α, h), $T > 0$ given. In what follows C_0 represents the initial reactant concentration, T_0 is the initial temperature, T_c is the cooling temperature, $c = c(t)$ and $\mathsf{T} = \mathsf{T}(t)$ are concentration and temperature at time t, and by $R = R(t)$ we denote the reaction rate for unit volume. The equation is the following

$$\begin{pmatrix} \dot{c} \\ \dot{\mathsf{T}} \\ 0 \end{pmatrix} = (1+\alpha(t)) \begin{pmatrix} k_1(C_0 - c) - R, \\ k_1(\mathsf{T}_0 - \mathsf{T}) + k_2 R - k_3(\mathsf{T} - \mathsf{T}_c) \\ R - k_3 e^{-\frac{k_4 c}{\mathsf{T}}} \end{pmatrix} + \lambda \begin{pmatrix} h_1(t, c, \mathsf{T}, R) \\ h_2(t, c, \mathsf{T}, R) \\ 0 \end{pmatrix}, \tag{3.7}$$

where k_1, k_2, k_3 and k_4 are given constants. Here, $\alpha : \mathbb{R} \to \mathbb{R}$ is continuous with $\frac{1}{T}\int_0^T \alpha(t)dt = 0$. The map $h = (h_1, h_2, h_3) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ depends on time and on the state of the system, h is C^1, and both α and h are supposed to be T-periodic. Notice that, when $\lambda = 0$ and $\alpha \equiv 0$, the system of equations (3.7) describes the mentioned exothermic reactor model.

Since we want to show how Theorem 3.3 applies to (3.7), we are interested in the equation itself regardless of its physical meaning. Thus, in what follows, we assume that all the quantities involved in (3.7) are dimensionless.

Set $\mathbf{x} = (x_1, x_2) := (c, \mathsf{T})$ and $y := R$. Let $U \subseteq \mathbb{R}^3$ be the open set given by $\{(\mathbf{x}, y) \in \mathbb{R}^2 \times \mathbb{R} : x_2 > 0\}$. Consider the maps $g : U \to \mathbb{R}$ and $f : U \to \mathbb{R}^2$ given by

$$f(\mathbf{x}, y) = \big(\ A^{12} \mid A^3\ \big) \begin{pmatrix} \mathbf{x} \\ y \end{pmatrix} + B \quad \text{and} \quad g(\mathbf{x}, y) = y - k_3 e^{-\frac{k_4 x_1}{x_2}},$$

where

$$A^{12} = \begin{pmatrix} -k_1 & 0 \\ 0 & -(k_1+k_3) \end{pmatrix}, \; A^3 = \begin{pmatrix} -1 \\ k_2 \end{pmatrix} \text{ and } B = \begin{pmatrix} k_1 C_0 \\ k_1 \mathsf{T}_0 + k_3 \mathsf{T}_c \end{pmatrix}.$$

It is immediate to check that $\partial_2 g \equiv 1$. Let $F(\mathbf{x}, y) = (f(\mathbf{x}, y), g(\mathbf{x}, y))$ be as in (2.12). Using the so called "generalized Gauss algorithm", we obtain

$$\begin{aligned} \det d_{(\mathbf{x},y)} F &= \det \left(\begin{array}{c|c} A^{12} & A^3 \\ \hline \partial_1 g(\mathbf{x}, y) & 1 \end{array} \right) \\ &= \det(A^{12} - A^3 \partial_1 g(\mathbf{x}, y)) \\ &= (k_1 - \eta(\mathbf{x}))(k_1 + k_3) - k_1 k_2 \eta(\mathbf{x}) x_1 / x_2, \end{aligned}$$

where $\eta(\mathbf{x}) = \frac{k_3 k_4}{x_2} e^{\frac{-k_4 x_1}{x_2}}$. Now, for instance, take $k_1 = k_3 = 1/2, k_2 = 2, k_4 = 1$ and $C_0 > (\mathsf{T}_0 + \mathsf{T}_c)/2 > 0$. Under these hypotheses, a direct computation shows that $F^{-1}(0)$ consists exactly of a single point $(\mathbf{x}_0, y_0)$, and that $\det d_{(\mathbf{x}_0, y_0)} F \neq 0$. Hence, Theorem 3.3 applies to the considered DAE, and we can conclude that there exists a connected set Λ of nontrivial T-periodic pairs to (3.7) whose closure in $\Omega = [0, \infty) \times C_T(U)$ in noncompact, and that meets $\{(0, (\mathbf{x}_0, y_0))\}$.

As a last example, we take into account a problem which is an adapted version of a retarded equation examined in [1].

example 3.9. Let $A, E \in \mathbb{R}^{n \times n}$. Consider the following implicit differential equation

$$E\dot{\mathbf{x}} = a(t) A\mathbf{x} + \lambda C(t) S(\mathbf{x}), \quad \lambda \geq 0, \tag{3.8}$$

where $a : \mathbb{R} \to \mathbb{R}$, $S : \mathbb{R}^n \to \mathbb{R}^n$ and $C : \mathbb{R} \to \mathbb{R}^{n \times n}$ are continuous, with a and C T-periodic, $T > 0$. Assume also that C and E satisfies the following relations:

$$\ker C^T(t) = \ker E^T, \; \forall t \in \mathbb{R}, \text{ and } n > \dim \ker E^T > 0.$$

In particular, we have that $s = \operatorname{rank} E = \operatorname{rank} C(t)$ is a positive constant for all $t \in \mathbb{R}$. Under these assumptions, [1, Lemma 5.5] applies to the DAE (3.8), and so there exist orthogonal matrices $P, Q \in \mathbb{R}^{n \times n}$ that realize a singular value decomposition (SVD) for E, such that (3.8) can be equivalently rewritten as

$$PEQ^T \dot{\mathbf{x}} = a(t) PAQ^T \mathbf{x} + \lambda PC(t) Q^T QS(Q^T \mathbf{x}), \tag{3.9}$$

with $\mathbf{x} = Q^T \mathsf{x}$,

$$PEQ^T = \begin{pmatrix} \widetilde{E}_s & 0 \\ 0 & 0 \end{pmatrix}, \quad PAQ^T = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix}, \; PC(t)Q^T = \begin{pmatrix} \widetilde{C}_s(t) & 0 \\ 0 & 0 \end{pmatrix},$$

and, setting $\mathbf{p} = Q^T \mathsf{p}$, $\mathbf{p} \in \mathbb{R}^n$, we also have

$$QS(Q^T \mathsf{p}) = \begin{pmatrix} \widetilde{S}_s(Q^T \mathsf{p}) \\ \widetilde{S}_{n-r}(Q^T \mathsf{p}) \end{pmatrix},$$

where $\widetilde{E}_s \in \mathbb{R}^{s \times s}$ is a diagonal matrix with positive diagonal elements, $\widetilde{C}_s \in C(\mathbb{R}, \mathbb{R}^{s \times s})$ is nonsingular for any $t \in \mathbb{R}$, $\widetilde{S}_s \in C(\mathbb{R}^n, \mathbb{R}^s)$ $\widetilde{S}_{n-s} \in C(\mathbb{R}^n, \mathbb{R}^{n-s})$, $\widetilde{A}_{11} \in \mathbb{R}^{s \times s}$ and $\widetilde{A}_{22} \in \mathbb{R}^{n-s \times n-s}$. Decompose the Euclidean space $\mathbb{R}^n$ with respect to its orthogonal subspaces $\operatorname{im} E^T \cong \mathbb{R}^s$ and $\ker E \cong \mathbb{R}^{n-s}$, so that $\mathbb{R}^n \simeq \mathbb{R}^s \times \mathbb{R}^{n-s}$, and set $\mathbf{x} = (x, y) \in \mathbb{R}^s \times \mathbb{R}^{n-s}$. Thus (3.9) becomes

$$\begin{gathered} \widetilde{E}_s \dot{x} = a(t)\big(\widetilde{A}_{11}x + \widetilde{A}_{12}y\big) + \lambda \widetilde{C}_s(t)\widetilde{S}_s\big(x\big), \quad \lambda \geq 0, \\ \widetilde{A}_{21}x + \widetilde{A}_{22}y = 0. \end{gathered} \tag{3.10}$$

Assume that $\widetilde{A}_{22}$ is invertible. Notice that rank $\widetilde{A}_{22}$ remains the same for any other choice of matrices P and Q realizing a SVD for E, so the required condition does not depend on the chosen SVD for E. Define $F(x, y) = (\widetilde{A}_{11}x + \widetilde{A}_{12}y, \widetilde{A}_{21}x + \widetilde{A}_{22}y)$, with $(x, y) \in \mathbb{R}^s \times \mathbb{R}^{n-s}$. Therefore, Theorem 3.3 ensures the existence of a connected subset Λ of nontrivial T-periodic pairs to (3.10) whose closure in $[0, \infty) \times C_T(\mathbb{R}^s \times \mathbb{R}^{n-s})$ is noncompact and meets the set $\big\{(0, (\overline{p}, \overline{q})) \in [0, \infty) \times C_T(\mathbb{R}^s \times \mathbb{R}^{n-s}) : F(p, q) = 0\big\}$. Recasting the argument in [1, Corollary 5.7], it follows that Λ generates an unbounded connected set of nontrivial T-periodic pairs to (3.7) emanating from $\big\{(0, \overline{\mathbf{p}}) \in [0, \infty) \times C_T(\mathbb{R}^n) : A\mathbf{p} = 0\big\}$.

4. A multiplicity result

In this section we give a multiplicity result that can be inferred from Theorem 3.3 and Corollary 3.6. For the reminder of this section a, f, g, h, U, T and F will be as in Section 3. We will also suppose that $\frac{1}{T}\int_0^T a(t)\,dt = 1$, for the sake of simplicity. The approach followed here leans on the argument given in Theorem 3.3 and on a local analysis regarding the set of T-periodic solutions to (1.3).

Let (p_0, q_0) be an isolated zero of F. Then, since $\partial_2 g(p_0, q_0)$ is invertible, we can locally "decouple" (1.3). Namely, by the Implicit Function Theorem, there exist neighborhoods $V \subseteq \mathbb{R}^k$ of p_0 and $W \subseteq \mathbb{R}^s$ of q_0, and a C^1-function $\gamma : V \to \mathbb{R}^s$ such that $M \cap (V \times W)$ is the graph of γ, where $M = g^1(0)$. Thus, within $V \times W$, Equation (1.3) can be rewritten with $y = \gamma(x)$ as follows:

$$\dot{x} = a(t) f\big(x, \gamma(x)\big) + \lambda h\big(t, x, \gamma(x)\big).$$

Linearizing the above equation at (p_0, q_0), for $\lambda = 0$, we get

$$\dot{\xi} = a(t)\big[\partial_1 f(p_0, q_0) + \partial_2 f(p_0, q_0) d_{(p_0,q_0)}\gamma\big]\xi, \text{ on } \mathbb{R}^k, \tag{4.1}$$

which is a non-autonomous linear ODE in $\mathbb{R}^k$.

We will say that (p_0, q_0) is a *T-resonant zero* of F, if (4.1) admits nonzero T-periodic solutions. Notice that this definition is analogous to the one given in [13] in the context of T-periodic perturbations to autonomous semi-explicit DAEs of the form

$$\begin{gathered} \dot{x} = f(x, y) + \lambda h(t, x, y), \quad \lambda \geq 0, \\ g(x, y) = 0, \end{gathered} \tag{4.2}$$

where f, h and g are as above.

Let $\Phi(p_0, q_0)$ be the linear endomorphism of $\mathbb{R}^k$ given by

$$\partial_1 f(p_0, q_0) - \partial_2 f(p_0, q_0)[\partial_2 g(p_0, q_0)]^{-1}\partial_1 g(p_0, q_0),$$

then Equation (4.1) becomes

$$\dot{\xi} = a(t)\Phi(p_0, q_0)\xi, \quad \text{on } \mathbb{R}^k. \tag{4.3}$$

Applying to (4.2) the same linearization procedure used to obtain (4.3), we get the following autonomous ODE on $\mathbb{R}^k$

$$\dot{\xi} = \Phi(p_0, q_0)\xi. \tag{4.4}$$

The next result shows that the T-resonancy condition at $(p_0, q_0) \in F^{-1}(0)$ does not depend on the presence of the perturbation factor $a : \mathbb{R} \to \mathbb{R}$ in (1.3). Namely, it holds that

Proposition 4.1. *Let $(p_0, q_0) \in M$ be a zero of $F = (f, g)$. Then (p_0, q_0) is T-resonant for* (1.3) *if and only if it is T-resonant for* (4.2).

Proof. Let $P_\tau^{a\Phi}$ and P_τ^{Φ} be the local Poincaré τ-translation operators, $\tau \in \mathbb{R}$, associated to the equations (4.3) and (4.4) respectively.

Assume that (p_0, q_0) is a T-resonant zero for the equation (1.3), and let $\xi_0 \in \mathbb{R}^k$ be an initial point of a T-periodic solution ξ to (4.3). In such a case, $P_T^{a\Phi}(\xi_0)$ is defined and it holds true that $P_T^{a\Phi}(\xi_0) = \xi_0$. Arguing as in Remark 2.2, it follows that also $P_T^{\Phi}(\xi_0)$ is defined and that $P_T^{\Phi}(\xi_0) = P_T^{a\Phi}(\xi_0) = \xi_0$. This means that ξ_0 is an initial point of a nonzero T-periodic solution of $\dot{\xi} = \Phi(p_0, q_0)\xi$. Hence, (p_0, q_0) is T-resonant for the Equation (4.2) too. The converse implication is straightforward. $\square$

Remark 4.2. The T-resonancy condition at (p_0, q_0) can be read on the spectrum $\sigma(\Phi(p_0, q_0))$ of $\Phi(p_0, q_0)$. Indeed, the (unique) solution to the the Cauchy problem

$$\dot{\xi} = a(t)\Phi(p_0, q_0)\xi, \quad \xi(0) = \xi_0$$

is given by

$$\xi = e^{-\int_0^t a(s)ds\Phi(p_0,q_0)}\xi_0.$$

Hence, ξ is a T-periodic solution to (4.1) if and only if $\xi_0 \in \ker(I - e^{T\Phi(p_0,q_0)})$, where I is the identity on $\mathbb{R}^k$. Thus, (p_0, q_0) is T-resonant if and only if, for some $n \in \mathbb{Z}$ one has that $\frac{2n\pi i}{T} \in \sigma(\Phi(p_0, q_0))$, with i the imaginary unit. Again, using the generalized Gauss algorithm, we obtain

$$\det d_{(p_0,q_0)}F = \det\big(\partial_2 g(p_0, q_0)\big) \cdot \det\big(\Phi(p_0, q_0)\big),$$

so, if (p_0, q_0) is non-T-resonant, then it is a non-degenerate zero of F. In particular, we have that $i(F, (p_0, q_0)) \neq 0$.

We have the following lemma.

Lemma 4.3. *Suppose that a, f, g, h, U, T and F be as in Theorem 3.3. Let (p_0, q_0) be a non-T-resonant zero of F. Then*

1. *the trivial T-periodic pair $(0, (\overline{p}_0, \overline{q}_0))$ is isolated in the set of T-periodic pairs corresponding to $\lambda = 0$;*
2. *there exists a connected set of nontrivial T-periodic pairs to* (1.3) *whose closure in $[0, \infty) \times C_T(U)$ contains $(0, (\overline{p}_0, \overline{q}_0))$ and is either noncompact or intersects*

$$\{(0, (\overline{p}, \overline{q})) \in [0, \infty) \times C_T(U) : (p, q) \in F^{-1}(0, 0)\} \backslash \{(0, (\overline{p}_0, \overline{q}_0))\}.$$

The proof follows by applying Proposition 4.1 and [13, Lemma 5.5].

Here, we fix some further notation. Let Y be a metric space and X a subset of $[0, \infty) \times Y$. Given $\mu \geq 0$, we denote by X_μ the slice $\{y \in Y : (\mu, y) \in X\}$. Using the convention introduced in (2.9), Y will be identified with the subset $\{0\} \times Y \subseteq [0, \infty) \times Y$.

Let X be a subset of $[0, \infty) \times Y$. We say that a subset A of X_0 is an *ejecting set* (for X) if it is relatively open in X_0 and there exists a connected subset of X which meets A and is not included in X_0. In particular, we say that $(p_0, q_0) \in X_0$ is an ejecting point for X if $\{(p_0, q_0)\}$ is an ejecting set. Notice that (p_0, q_0) is isolated in X_0 being $\{(p_0, q_0)\}$ open in X_0.

We now give a sufficient condition for the existence of ejecting points for the set of the T-periodic pairs of (1.3).

Corollary 4.4. *Let (p_0, q_0) be a non-T-resonant zero of $F : U \to \mathbb{R}^k \times \mathbb{R}^s$. Then, (p_0, q_0) (regarded as the trivial T-periodic pair $(0, (\overline{p}_0, \overline{q}_0))$) is an ejecting point for the set of the T-periodic pairs of* (1.3).

Proof. Let X be the subset of $[0, \infty) \times C_T(U)$ of all the T-periodic pairs to (1.3). Due to Lemma 4.3, the non-T-resonant zero (p_0, q_0) results to be an isolated point of X_0, which means that it is ejecting. $\square$

We are now ready to establish the following result

Proposition 4.5. *Let a, f, g, h, U, T and F be as in Theorem 3.3. Assume also that $M = g^{-1}(0)$ is closed in $\mathbb{R}^k \times \mathbb{R}^s$. Let $\{(p_i, q_i)\}, i = 1, \dots, r$, be non-$T$-resonant zeros of F such that*

$$\deg(F, U) \neq \sum_{j=1}^{r} \mathrm{i}(F, (p_j, q_j)).$$

Suppose that (1.1) *does not admit an unbounded connected set of T-periodic solutions in $C_T(U)$. Then, there are at least $r+1$ different T-periodic solutions of* (1.3) *when $\lambda > 0$ is sufficiently small.*

The proof follows from applying Proposition 4.1 and [13, Proposition 5.7].

example 4.6. Consider the following DAE in $U = (-\frac{3}{2}, \infty) \times (-\frac{3}{2}, \infty)$

$$\begin{gathered}\dot{x} = a(t)\big(xy^2 - x^2y\big) + \lambda h(t, x, y),\\ y - x^3 + \varepsilon y^3 = 0,\end{gathered} \tag{4.5}$$

where $0 < \varepsilon < 1$ is a given constant, $a : \mathbb{R} \to (0, \infty)$ and $h : \mathbb{R} \times (\mathbb{R} \times \mathbb{R}) \to \mathbb{R}$ are continuous and T-periodic, with respect to the variable t. Actually, we require that $h \in C^1$. As usual, we assume that $\frac{1}{T}\int_0^T a(t)dt = 1$. It is immediate to check that the zero set of $F(p, q) = (pq^2 - p^2q, q - p^3 + \varepsilon q^3)$ is formed by the points $(0, 0)$, $(a_\varepsilon, a_\varepsilon)$ and $(-a_\varepsilon, -a_\varepsilon)$, where $a_\varepsilon := 1/\sqrt{1-\varepsilon}$, and that just the first of them is T-resonant. Again, it can be easily seen that, in the case $\lambda = 0$, the only possible T-periodic solutions of (4.5) are those corresponding to the zeros of F.

Consider the homotopy $H : U \times [0, 1] \to \mathbb{R} \times \mathbb{R}$, defined by

$$H(p, q; \rho) = (q^2p - p^2q, q - p^3 + (1 - \rho)\varepsilon q^3 - \rho),$$

which is admissible in the sense that the set $\{(p, q) \in U : H(p, q; \rho) = 0 \text{ for some } \rho\}$ is compact. Obviously, $H(p, q; 1) \neq 0$ for any $(p, q) \in U$, then it follows that $\deg(H(\cdot, \cdot; 1), U) = 0$. By the Homotopy Invariance property of the degree, one has that $\deg(F, U) = 0$.

Since $(a_\varepsilon, a_\varepsilon)$ and $(-a_\varepsilon, -a_\varepsilon)$ are non-T-resonant zeros of F, they are nondegenerate, and in particular it holds true that $i(F, (a_\varepsilon, a_\varepsilon)) = i(F, (-a_\varepsilon, -a_\varepsilon)) = -1$. Therefore, Proposition 4.5 ensures the existence of at least three T-periodic solutions of (4.5), for sufficiently small $\lambda > 0$.

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