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HOMOGENIZATION AND CORRECTORS FOR COMPOSITE MEDIA WITH COATED AND HIGHLY ANISOTROPIC FIBERS

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ABSTRACT. This article presents the homogenization of a quasilinear elliptic-parabolic problem in an ε-periodic medium consisting of a set of highly anisotropic fibers surrounded by coating layers, the whole being embedded in a third material having an order 1 conductivity. The conductivity along the fibers is of order 1 but the conductivities in the transverse directions and in the coatings are scaled by $\mu = o(\varepsilon)$ and ε^p, as $\varepsilon \to 0$, respectively. The heat flux are quasilinear, monotone functions of the temperature gradient. The heat capacities of the medium components are bounded but may vanish on certain subdomains, so the problem may become degenerate. By using the two-scale convergence method, we can derive the two-scale homogenized systems and prove some corrector-type results depending on the critical value $\gamma = \lim_{\varepsilon \searrow 0} \varepsilon^p/\mu$.

1. Introduction and statement of the problem

Homogenization of problems, in composite media with fibers, has been considered in [2, 5, 4, 13] and further references therein. Most of the previous works dealt with the case of the fiber-reinforced composite materials without coatings. Motivated by the study of the effects of the combination of the insulating coatings and the high anisotropy of fibers in the overall behavior of composite media, we propose here, a special class of fibrous structure exhibiting non-standard effective models. Especially, in the present work, we consider the homogenization of a quasilinear elliptic-parabolic problem in a three-phase conducting composite. One of the constituent materials corresponds to a set of fibers surrounded by a second material which works as an insulating or coated layers, and the whole is being embedded in a third material termed matrix. The fibers are considered to be highly anisotropic, with a longitudinal order 1 conductivity and a very low conductivity in the transverse directions. The conductivity of the matrix is of order 1 but becomes very small in the coatings. We shall refer to such material as a composite medium with coated and highly anisotropic fibers.

In [3], the author has dealt with the linear case. Here, we continue this investigation by studying the case where the heat flux are non-linear functions of the temperature gradient. One common peculiarity of [3] and the present work is that

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the heat capacities $c_j, j = 1, 2, 3$ are assumed to degenerate at some subdomains and even to vanish in the whole domain. Thus, our problem covers the quasilinear elliptic equation as well as the quasilinear parabolic one in a composite medium with coated and highly anisotropic fibers.

The geometry of the medium is the same as in [3]. We shall recall it and keep globally the same notations. We denote by $\widetilde{Y}$ and Y the cubes $]-\frac{1}{2}, \frac{1}{2}[^2$ and $]-\frac{1}{2}, \frac{1}{2}[^3$ respectively, thus $Y = \widetilde{Y} \times \mathrm{I}$, $\mathrm{I} =]-\frac{1}{2}, \frac{1}{2}[$. We assume that $\widetilde{Y}$is partitioned as $\widetilde{Y} = \widetilde{Y}_1 \cup \widetilde{Y}_{13} \cup \widetilde{Y}_3 \cup \widetilde{Y}_{23} \cup \widetilde{Y}_2$ where $\widetilde{Y}_1, \widetilde{Y}_2, \widetilde{Y}_3$ are three connected open subsets such that $\widetilde{Y}_1 \cap \widetilde{Y}_2 = \emptyset$, $\partial\widetilde{Y} \cap \widetilde{Y}_3 = \emptyset$ and where $\widetilde{Y}_{\alpha 3},\ \alpha = 1, 2$ is the interface between $\widetilde{Y}_\alpha$ and $\widetilde{Y}_3$; thus $\widetilde{Y}_3$ separates $\widetilde{Y}_1$ and $\widetilde{Y}_2$ (see Figure 1). For $i = 1, 2, 3$ we denote χ_i the characteristic function of $Y_i := \widetilde{Y}_i \times \mathrm{I}$ and $\theta_1, \theta_2, \theta_3$ their respective Lebesgue measures which are supposed to be of the same magnitude order. Let $\widetilde{E}_i$ the $\mathbb{Z}^2$-translates of $\widetilde{Y}_i$ (i.e., $\widetilde{E}_i := \widetilde{Y}_i + \mathbb{Z}^2$) and $\widetilde{\Gamma}_{\alpha 3}$, $\alpha = 1, 2$ the surface separating $\widetilde{E}_\alpha$ and $\widetilde{E}_3$. *We shall assume that only $\widetilde{E}_2$ is connected*. We introduce the contracted sets $\widetilde{Y}_i^\varepsilon := \varepsilon\widetilde{Y}_i$, $\widetilde{E}_i^\varepsilon := \varepsilon\widetilde{E}_i$, $i = 1, 2, 3$ and $\widetilde{\Gamma}_{\alpha 3}^\varepsilon := \varepsilon\widetilde{\Gamma}_{\alpha 3}$, $\alpha = 1, 2$, where ε is a small positive parameter. Now, let $\widetilde{\Omega}$ be a regular bounded domain in $\mathbb{R}^2$. We denote by $\widetilde{\Omega}_i^\varepsilon := \widetilde{\Omega} \cap \widetilde{E}_i^\varepsilon$, and $\widetilde{S}_{\alpha 3}^\varepsilon := \widetilde{\Omega} \cap \widetilde{\Gamma}_{\alpha 3}$. Finally, let $\Omega := \widetilde{\Omega} \times \mathrm{I}$ be the cylinder having a base $\widetilde{\Omega}$ and a height 1 and $\Omega_i^\varepsilon := \widetilde{\Omega}_i^\varepsilon \times \mathrm{I}$, $i = 1, 2, 3$.

Henceforth, $x = (\widetilde{x}, x_3)$ and $y = (\widetilde{y}, y_3)$ denote points of $\mathbb{R}^3$ and Y respectively and by $\widetilde{y}$ and $\widetilde{x}$ we denote the transverse vectors (y_1, y_2) and (x_1, x_2) respectively. We use the notation ∂_{x_i} for the partial derivative with respect to x_i. Let $T > 0$ be given, we define, then, the corresponding space-time domains $Q = (0, T) \times \Omega$ and $Q_i^\varepsilon = (0, T) \times \Omega_i^\varepsilon$, $i = 1, 2, 3$.

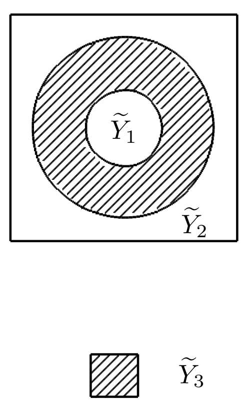

FIGURE 1. A typical basic cell $\widetilde{Y}$

Let $p > 1$ be a real number and let p' its conjugate: $p = p'(p-1)$. For $k = 1, 2, 3$, let $c_k \in L^\infty(\mathbb{R}^3)$ be the heat capacity of the k-th component. These functions are Y-periodic with respect to y with a period Y and satisfy the following assumption:

(A1) $0 \le c_k(y)$ a.e. $y \in Y$, $k = 1, 2, 3$.

The corresponding ε-periodic coefficients are defined by

$$c_k^\varepsilon(x) = c_k(\frac{x}{\varepsilon}), \quad x \in \Omega_k^\varepsilon,\ k = 1, 2, 3. \tag{1.1}$$

Concerning the heat flux, we shall suppose that they are given by three non-linear Y-periodic vectorial functions

$$\mathbb{A}_k(y,\xi):\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3,\quad k=1,2,3, \tag{1.2}$$

satisfying the following assumptions

- (A2) for all $\xi\in\mathbb{R}^3$, the function $y\mapsto\mathbb{A}_k(y,\xi)$ is measurable and Y-periodic,
- (A3) for a.e. $y\in Y$, the function $\xi\mapsto\mathbb{A}_k(y,\xi)$ is continuous,
- (A4) there exist a constant $c_0>0$ and $p>2$ such that, for all $\xi\in\mathbb{R}^3$,
$$0\le c_0|\xi|^p\le\mathbb{A}_k(y,\xi).\xi$$
- (A5) there exist a constant $c>0$ and $p>2$ such that for all $\xi\in\mathbb{R}^3$,
$$|\mathbb{A}_k(y,\xi)|\le c(1+|\xi|^{p-1}),$$
- (A6) the operators $\mathbb{A}_k$ are strictly monotone; i.e., for a.e. $y\in Y$,
$$(\mathbb{A}_k(y,\xi)-\mathbb{A}_k(y,\eta)).(\xi-\eta)>0,\quad\forall\xi\neq\eta\quad\text{in }\mathbb{R}^3.$$

To prove the corrector results, we need to assume stronger hypotheses of monotonicity:

- (A5') there exist a constant $K_1>0$ such that, for $\xi,\eta\in\mathbb{R}^3$ and a.e. $y\in Y$,
$$|\mathbb{A}_k(y,\xi)-\mathbb{A}_k(y,\eta)|\le K_1(|\xi|+|\eta|)^{p-2}|\xi-\eta|,$$
- (A6') there exist a constant $K_2>0$ such that, for $\xi,\eta\in\mathbb{R}^3$ and a.e. $y\in Y$,
$$(\mathbb{A}_k(y,\xi)-\mathbb{A}_k(y,\eta)).(\xi-\eta)\ge K_2(|\xi|+|\eta|)^{p-2}|\xi-\eta|^2.$$
- (A7) The function $\mathbb{A}_1$ is independent of the vertical coordinate and has the following form
$$\mathbb{A}_1(y,\xi):=\mathbb{A}_1(\widetilde{y},\xi)=\begin{pmatrix}\widetilde{\mathbb{A}}_1(\widetilde{y},\widetilde{\xi})\\ \mathbb{A}_{13}(\widetilde{y},\xi_3)\end{pmatrix}.$$

Obviously, the functions

$$\widetilde{\mathbb{A}}_1(\widetilde{y},\widetilde{\xi}):\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2,$$
$$\mathbb{A}_{13}(\widetilde{y},\xi_3):\ \mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$$

satisfy assumptions (A4)–(A6) by choosing $\xi=(\widetilde{\xi},0)$ and $(0,\xi_3)$ respectively.

An example of $\mathbb{A}_k$ satisfy the assumptions (A2)–(A7) is

$$\mathbb{A}_1(y,\xi)=\begin{pmatrix}|\widetilde{\xi}|^{p-2}\widetilde{\xi}\\ |\xi_3|^{p-2}\xi_3\end{pmatrix},\quad\mathbb{A}_\alpha(y,\xi)=|\xi|^{p-2}\xi,\quad\alpha=2,3;$$

i.e., the corresponding p-Laplacian operators.

Then, the diffusion through the material filling the sets E_1^ε, E_2^ε and E_3^ε is, respectively,

$$\mathbb{A}_1^\varepsilon(x,\xi):=\begin{pmatrix}\mu(\varepsilon)\widetilde{\mathbb{A}}_1^\varepsilon(x,\widetilde{\xi})\\ \mathbb{A}_{13}^\varepsilon(x,\xi_3)\end{pmatrix},\quad\mathbb{A}_2^\varepsilon(x,\xi):=\mathbb{A}_2(\frac{x}{\varepsilon},\xi),\quad\mathbb{A}_3^\varepsilon(x,\xi):=\mathbb{A}_3(\frac{x}{\varepsilon},\xi),$$

where

$$\widetilde{\mathbb{A}}_1^\varepsilon(x,\widetilde{\xi})=\widetilde{\mathbb{A}}_1(\frac{\widetilde{x}}{\varepsilon},\widetilde{\xi}),\quad\mathbb{A}_{13}^\varepsilon(x,\xi_3)=\mathbb{A}_{13}(\frac{\widetilde{x}}{\varepsilon},\xi_3).$$

The global diffusion and the heat capacity of the medium is respectively

$$\mathbb{A}^\varepsilon(x,\xi)=\sum_{k=1}^{2}\chi_k^\varepsilon(x)\mathbb{A}_k^\varepsilon(x,\xi)+\varepsilon^p\chi_3^\varepsilon(x)\mathbb{A}_3^\varepsilon(x,\xi),$$

$$c^\varepsilon(x) = \sum_{k=1}^{3} \chi_k^\varepsilon(x) c_k^\varepsilon(x).$$

Let us assume that the lateral and bottom boundaries of Ω are maintained at a fixed temperature (homogeneous Dirichlet condition), while the top boundary is insulated (homogeneous Neumann condition), and that the initial distribution of the temperature on Ω is given for every ε as

$$u_0^\varepsilon(x) = \sum_{k=1}^{3} \chi_k^\varepsilon(x) u_{0k}^\varepsilon(x).$$

Then, the evolution of the temperature $u^\varepsilon(t,x)$ is governed by the following initial boundary value problem, being in fact, a sequence of problems $(\mathcal{P}_\varepsilon)$ indexed by ε:

$$\begin{gathered} \frac{\partial}{\partial t}(c^\varepsilon(x) u^\varepsilon(t,x)) = \operatorname{div}(\mathbb{A}^\varepsilon(x, \nabla u^\varepsilon(t,x))) + f^\varepsilon(t,x), \quad x \in \Omega,\ t > 0, \\ u^\varepsilon(t,x) = 0,\ x \in \partial\Omega \cap \{-\frac{1}{2} \le x_3 < \frac{1}{2}\} =: \Gamma_{LB}, \quad t > 0, \\ \mathbb{A}^\varepsilon(x, \nabla u^\varepsilon(t,x)).n = 0, \quad x \in \partial\Omega \setminus \Gamma_{LB}, \quad t > 0, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), \quad x \in \Omega, \end{gathered} \tag{1.3}$$

where n denotes the outward normal to the boundary of Ω, the subscript L (resp. B) stands for lateral (resp. bottom) boundary and $f^\varepsilon \in L^{p'}(0,T; L^{p'}(\Omega))$ represents a given time-dependent heat source. The precise meaning of the initial condition will be done in the following section.

In the linear context, models of particular interest are developed by Mabrouk-Samadi [9], Mabrouk-Boughammoura [8] and Showalter-Visarraga [10] for the so-called highly heteregeneous medium which consits of two connected "hard" components having comparable conductivities, separated by a third "soft" material having a much lower conductivity. The common point of these works is that the three phases have only highly contrasting isotropic conductivities. These models do not display a directional dependence of the effective fields in the resulting limit problems. However, in the present model and in [3], one of the phases (the fibers) have also highly anisotropic conductivity. This "partially" highly anisotropy in the fibers leads to some kind of directional dependence on the macro and micro variables.

Mathematically, the combination of the "partially" highly anisotropy and the insulating coatings poses an interesting challenge in the homogenization process. In particular, we will see, in the case $\gamma \in \mathbb{R}_+^*$, that the resulting two-scale homogenized systems is "strongly" influenced by this combination : the effective temperature field is obtained by solving a homogenized problem in the domain Ω and an auxiliary problem in the coated fiber $\overline{Y_1} \cup Y_3$ with a non-standard boundary conditions across the interface between the fiber and the coating (see (5.2) and Remark 5.2). Hence, the main feature of the present work is to provide "rigorous" models for quasilinear heat transfer problem in fibrous composite materials taking in account the influence of the physical properties, at the micro-scale, of the coating and the fiber. In particular, we derive some new effective interface conditions which describe the interaction between the heat transfer processes of conduction in the fibers and the coatings (see (5.3) and (5.4)). Furthermore, we improve these models by some corrector-type results.

Finally, the closest work, as far as we know, to ours was done by Mabrouk [6], in which the author studied the homogenization of a nonlinear degenerate heat transfer problem in a highly heterogeneous medium. Although the mathematical framework used in [6] is closely similar, the two situations are clearly distinct in the geometry of the microstructure. Moreover, the homogenized results of [6] are recovered, here, when $\gamma := \lim_{\varepsilon\to 0} \varepsilon^p/\mu = 0$ by replacing, formally, the operator ∂_{x_3} by ∇. However, our results in the case $0 < \gamma < \infty$ can not be obtained by the physical setting considered in [6]. The corrector results are not addressed in [6], that is only weakly convergent results are proved. Yet, here we shall prove strong convergence of the gradients of temperature as well as the heat flux by adding some correctors (see Section 5). Thus, the present study is actually quite different and can be considered as an improvement of [6] and a generalization of [3] to quasilinear (monotone operators in the gradient) heat transfer problem in composite materials with coated and highly anisotropic fibers.

2. Mathematical framework

Hereafter, various spaces of functions on Ω will be used. For each $p > 1$, $L^p(\Omega)$ and $W^{1,p}(\Omega)$ are the usual Lebesgue space and Sobolev space respectively. If $\mathbb{R}$ is a Banach space, we denote $\mathbb{R}'$ its dual ; the value of $x' \in \mathbb{R}'$ at $x \in \mathbb{R}$ is denoted $x'(x)$ or sometimes $\langle x', x\rangle_{\mathbb{R}',\mathbb{R}}$. If $\mathbb{H}$ is a Hilbert space, we denote its scalar product $(.,.)_{\mathbb{H}}$, the dot denotes the usual scalar product in $\mathbb{R}^3$. If $\mathbb{R}$ is a Banach space and X is a topological one, $\mathcal{C}(X;\mathbb{R})$ is the space of continuous $\mathbb{R}$-valued functions on X with the sup-norm. For any measure space Ω, $L^p(\Omega;\mathbb{R})$ is the space of p-th power norm-summable functions on Ω with values in $\mathbb{R}$. If $\Omega = (0,T)$ is the time space, we shall often write $L^p(0,T;\mathbb{R})$. In particular, spaces of Y-periodic functions will be denoted by a subscript $\sharp$. For example, $\mathcal{C}_\sharp(Y)$ is the Banach space of functions which are defined on $\mathbb{R}^3$, continuous and Y-periodic. Similarly, $L^p_\sharp(Y)$ is the Banach space of functions in $L^p{}_{\rm loc}(\mathbb{R}^3)$ which are Y-periodic. We endow this space with the norm of $L^p(Y)$ and remark that it can be identified with the space of Y-periodic extensions to $\mathbb{R}^3$ of the functions in $L^p(Y)$. Similarly, we define the Banach space $W^{1,p}_\sharp(Y)$ with the usual norm of $W^{1,p}(Y)$.

As in [3], to have a weak formulation of the above problem we shall use the convenient mathematical model built in [6], using the functional framework, developed by Showalter for degenerate parabolic equations (see [12], Section III.6). Let us recall the precise meaning of the weak formulation of the problem we investigate. For more details see [7, 6, 3].

Let $p \geq 2$ and p' its conjugate. We define the following Banach spaces

$$V = W^{1,p}_{\Gamma_{LB}}(\Omega) := \{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \Gamma_{LB}\}, \quad \mathcal{V} = L^p(0,T;V),$$
$$V', \mathcal{V}' = L^{p'}(0,T;V')$$

be their dual spaces. For $\varepsilon > 0$, let $C^\varepsilon, A^\varepsilon : V \to V'$ be continuous operators, which are defined by the continuous bilinear forms on $V \times V$:

$$\langle C^\varepsilon u, v\rangle_{V',V} = c^\varepsilon(u,v) := \int_\Omega c^\varepsilon(x)u(x)v(x)dx,$$
$$\langle A^\varepsilon u, v\rangle_{V',V} = a^\varepsilon(u,v) := \int_\Omega \mathbb{A}^\varepsilon(x, \nabla u(x))\nabla v(x)dx.$$

Let V_c^ε be the completion of V with the semi-scalar product, defined by the form c^ε and let $V_c^{'\varepsilon}$ be its dual. Then, we have $V_c^\varepsilon = \{u : (c^\varepsilon)^{1/2}u \in L^2(\Omega)\}$ and $V_c^{\varepsilon\prime} = \{(c^\varepsilon)^{1/2}u,\ u \in L^2(\Omega)\}$. The operator C^ε admits a continuous extension from V_c^ε into $V_c^{\varepsilon\prime}$ denoted also by C^ε. Given $f^\varepsilon \in L^{p'}(0,T;L^{p'}(\Omega))$ or more generally f^ε in $\mathcal{V}'$ and w_0^ε in $V_c^{\varepsilon\prime}$, we are now able to give a weak formulation of the above initial-boundary value problem as the following abstract Cauchy problem

$$\text{Find } u \in \mathcal{V}:\ \frac{d}{dt}\mathcal{C}^\varepsilon u + \mathcal{A}^\varepsilon u = f^\varepsilon \in \mathcal{V}', \quad \mathcal{C}^\varepsilon u(0) = w_0^\varepsilon \in V_c^{\varepsilon\prime}. \tag{2.1}$$

Here, $\mathcal{A}^\varepsilon$ and $\mathcal{C}^\varepsilon$ are the realization of A^ε and C^ε as operators from $\mathcal{V}$ to $\mathcal{V}'$, that is precisely $(\mathcal{A}^\varepsilon u(t), \mathcal{C}^\varepsilon u(t)) = (A^\varepsilon(u(t)), C^\varepsilon(u(t)))$ for a.e. $t \in (0,T)$.

Let us underline that, in the abstract formulation above, we implicitly require that $\frac{d}{dt}\mathcal{C}^\varepsilon u$ belongs to $\mathcal{V}'$. This allows us to give a precise meaning to the initial condition $\mathcal{C}^\varepsilon u(0)$. Thus, given u_0^ε in V_c^ε and w_0^ε in $V_c^{\varepsilon\prime}$ related by $w_0^\varepsilon = c^\varepsilon u_0^\varepsilon$, we can express the initial condition by one of the two equivalent equalities

$$(\mathcal{C}^\varepsilon u^\varepsilon)(0) = C^\varepsilon u^\varepsilon(0) = w_0^\varepsilon \in V_c^{\varepsilon\prime} \Longleftrightarrow (c^\varepsilon)^{1/2}u^\varepsilon(0) = (c^\varepsilon)^{1/2}u_0^\varepsilon \in L^2(\Omega). \tag{2.2}$$

We define the Banach space $W_p^\varepsilon(0,T) := \{u \in \mathcal{V} : \frac{d}{dt}\mathcal{C}^\varepsilon u \in \mathcal{V}'\}$, then, the abstract Cauchy problem can, thereby, be written more explicitly as: Find u in $W_p^\varepsilon(0,T)$ such that

$$\begin{gathered}\frac{d}{dt}\mathcal{C}^\varepsilon u(t) + \mathcal{A}^\varepsilon u(t) = f^\varepsilon(t) \in V' \quad \text{for a.e. } t \in (0,T),\\ \mathcal{C}^\varepsilon u(0) = w_0^\varepsilon \quad \text{in } V_c^{\varepsilon\prime}.\end{gathered} \tag{2.3}$$

The initial condition is meaningful since u is in $W_p^\varepsilon(0,T)$ then $\mathcal{C}^\varepsilon u \in \mathcal{C}(0,T;V_c^{\varepsilon\prime})$ by [12, Proposition 6.3].

For the present study, we need to recall some equivalent variational formulations of the problem (2.3) from [6, Proposition 1.2].

Proposition 2.1. *The following statements are equivalent:*

(1) *u is the solution of (2.3).*
(2) *$u \in W_p^\varepsilon(0,T)$ and for all $v \in W_p^\varepsilon(0,T)$ with $v(T) = 0$, we have*

$$-\int_0^T \langle u(t), v'(t)\rangle_{V_c^\varepsilon} dt + \int_0^T a^\varepsilon(u(t), v(t))dt = \int_0^T f^\varepsilon(t)(v(t))dt + w_0^\varepsilon(v(0)), \tag{2.4}$$

this, by density, holds for all $v \in L^p(0,T;V)$ such that $v' \in L^{p'}(0,T;V_c^\varepsilon)$.

(3) *$u \in L^p(0,T;V)$ and for all $v \in W^{1,p}(0,T;V)$, we have*

$$\begin{aligned}&-\int_Q c^\varepsilon uv'dx\,dt + \mu(\varepsilon)\int_{Q_1^\varepsilon} \widetilde{\mathbb{A}}_1^\varepsilon(\widetilde{x}, \nabla_{\widetilde{x}}u).\nabla_{\widetilde{x}}v dx\,dt\\ &+ \int_{Q_1^\varepsilon} \mathbb{A}_{13}^\varepsilon(\widetilde{x}, \partial_{x_3}u).\partial_{x_3}v dx\,dt\\ &+ \int_{Q_2^\varepsilon} \mathbb{A}_2^\varepsilon(x, \nabla_x u).\nabla_x v\,dx\,dt + \varepsilon^p\int_{Q_3^\varepsilon} \mathbb{A}_3^\varepsilon(x, \nabla_x u).\nabla_x v\,dx\,dt\\ &= \int_Q fv\,dx\,dt - \int_\Omega c^\varepsilon u(T,x)v(T,x)dx + \int_\Omega c^\varepsilon u_0^\varepsilon v(0,x)dx.\end{aligned} \tag{2.5}$$

Remark 2.2. For each $\varepsilon > 0$, the operator $\mathcal{A}^\varepsilon : \mathcal{V} \to \mathcal{V}'$ is continuous, monotone, coercive and bounded, and the operator $\mathcal{C}^\varepsilon : \mathcal{V} \to \mathcal{V}'$ is continuous, linear, symmetric and monotone. Hence, the Cauchy problem (2.3) admits, for each $\varepsilon > 0$, a unique solution $u \in W_p^\varepsilon(0,T)$ by [12, Corollary 6.3].

Throughout this work, we shall assume that

$$u_0^\varepsilon(x) = u_0(x) \in L^p(\Omega).$$

Hereafter, let $f \in L^{p'}(0,T;L^{p'}(\Omega))$ be fixed.

Our objective is to study the behavior of the sequence $\{u^\varepsilon\}$ as $\varepsilon \to 0$ moreover, we prove a corrector results for the gradients and flux under the strong monotonicity conditions (A5') and (A6'). This will be achieved below, in particular, we will show that the limit depends on the critical value $\gamma = \lim_{\varepsilon\downarrow 0} \frac{\varepsilon^p}{\mu}$.

Our further analysis will be, as in [3], based on the method of the two-scale convergence [1, 11]. For the sake of clarity, we recall its definition.

Definition 2.3. A function $\phi(t,x,y) \in L^p(Q \times Y, \mathcal{C}_\sharp(Y))$ satisfying

$$\lim_{\varepsilon\to 0} \int_Q \phi(t,x,\frac{x}{\varepsilon})^p \,dt\,dx = \int_Q \int_Y \phi(t,x,y)^p \,dt\,dx\,dy. \tag{2.6}$$

is called admissible test function.

Definition 2.4. A sequence u^ε in $L^p(Q)$ two-scale converges to a function $u^0 \in L^p(Q\times Y)$, and we denote this $u^\varepsilon \overset{2s,p}{\to} u^0$ ($u^\varepsilon \overset{2s}{\to} u^0$ if p=2), if, for any $\phi(t,x,y) \in \mathcal{D}(Q, \mathcal{C}_\sharp(Y))$,

$$\lim_{\varepsilon\to 0} \int_Q u^\varepsilon(t,x)\phi(t,x,\frac{x}{\varepsilon}) \,dt\,dx = \int_Q \int_Y u^0(t,x,y)\phi(t,x,y) \,dt\,dx\,dy. \tag{2.7}$$

Throughout the paper, we denote by C a constant not depending on ε and whose value may vary from one line to the next. From a bounded sequence in a Lebesgue space, we can take a subsequence that converges weakly, but virtually all the subsequences converge to the same limit as the limiting equations have a unique solution, so we normally ignore to mention the term "subsequence".

3. A priori estimates

First, we recall the fundamental lemma which generalizes to the case $p > 2$ lemma 2.1. of [3], proved for $p = 2$. We shall not give the proof since it involves only minor modifications of the case $p = 2$.

Lemma 3.1. *There exists a constant C such that, for every $v \in V$, we have*

$$\|v\|^p_{L^p(\Omega)} \le C(\|\partial_{x_3} v\|^p_{L^p(\Omega_1^\varepsilon)} + \|\nabla v\|^p_{L^p(\Omega_2^\varepsilon)} + \varepsilon^p \|\nabla v\|^p_{L^p(\Omega_3^\varepsilon)}). \tag{3.1}$$

The above lemma is used for proving the following a priori estimates.

Lemma 3.2. *Let $f^\varepsilon = f$, then, here exists a constant C such that*

$$\|u^\varepsilon\|_{L^p(Q)} \le C, \tag{3.2}$$

$$(\mu^{1/p}\|\nabla_{\widetilde{x}} u^\varepsilon\|_{L^p(Q_1^\varepsilon)}, \|\partial_{x_3} u^\varepsilon\|_{L^p(Q_1^\varepsilon)}) \le C, \tag{3.3}$$

$$\Big(\|\nabla u^\varepsilon\|_{L^p(Q_2^\varepsilon)}, \varepsilon\|\nabla u^\varepsilon\|_{L^p(Q_3^\varepsilon)}\Big) \le C, \tag{3.4}$$

$$\Big(\mu^{1/p'}\|\widetilde{\mathbb{A}}_1^\varepsilon(x, \nabla_{\widetilde{x}} u^\varepsilon)\|_{L^{p'}(Q_1^\varepsilon)}, \|\mathbb{A}_{13}^\varepsilon(x, \partial_{x_3} u^\varepsilon)\|_{L^{p'}(Q_1^\varepsilon)}\Big) \le C, \tag{3.5}$$

$$(\|\mathbb{A}_2^\varepsilon(x,\nabla u^\varepsilon)\|_{L^{p'}(Q_2^\varepsilon)}, \varepsilon^{\frac{p}{p'}}\|\mathbb{A}_3^\varepsilon(x,\nabla u^\varepsilon)\|_{L^{p'}(Q_3^\varepsilon)}) \le C. \tag{3.6}$$

Moreover, if $0 < c_0 \le c_3(y)$, *then* $\|u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}$.

Proof. First, let us assume that u^ε is a solution of (1.3). Since $u^\varepsilon \in W_p^\varepsilon(0,T)$, we can choose $v = u^\varepsilon(t)$ in (2.4) and using the following identity from [12, Proposition 3.1], or [6, Proposition 1.1],

$$\frac{1}{2}\frac{d}{dt}\langle\, \mathcal{C}^\varepsilon u(t), u(t)\rangle_{V',V} = \langle \frac{d}{dt}\mathcal{C}^\varepsilon u(t), u(t)\rangle_{V',V}$$

after integration over $(0,T)$, we deduce

$$\begin{aligned} &\frac{1}{2}\int_\Omega c^\varepsilon(x)(u^\varepsilon(T,x))^2dx + \int_0^T\int_\Omega \mathbb{A}^\varepsilon(x,\nabla u^\varepsilon(s,x)).\nabla u^\varepsilon(s,x)\,dx\,ds \\ &= \int_0^T\int_\Omega f(s,x)(u^\varepsilon(s,x))\,dx\,ds + \frac{1}{2}\int_\Omega c^\varepsilon(x)(u_0)^2dx. \end{aligned} \tag{3.7}$$

Thus,

$$\begin{aligned} &\frac{1}{2}\int_\Omega c^\varepsilon(x)(u^\varepsilon(T,x))^2dx + \int_0^T\int_\Omega \mathbb{A}^\varepsilon(x,\nabla u^\varepsilon(s,x)).\nabla u^\varepsilon(s,x)\,dx\,ds \\ &\le \int_0^T\int_\Omega |f(s,x)||(u^\varepsilon(s,x))|\,dx\,ds + C. \end{aligned}$$

By Young's inequality, for all $\eta > 0$,

$$\int_0^T\int_\Omega |f(s,x)||(u^\varepsilon(s,x))|\,dx\,ds \le \frac{\eta}{p}\|f\|_{L^{p'}(Q)}^{p'} + \frac{1}{p\eta^{p-1}}\|u^\varepsilon\|_{L^p(Q)}^p.$$

Thus, using assumption (A5),

$$\begin{aligned} &\mu\int_{Q_1^\varepsilon}|\nabla_{\widetilde{x}}u^\varepsilon|^p\,dx\,dt + \int_{Q_1^\varepsilon}|\partial_{x_3}u^\varepsilon|^p\,dx\,dt \\ &+ \int_{Q_2^\varepsilon}|\nabla_x u^\varepsilon|^p\,dx\,dt + \varepsilon^p\int_{Q_3^\varepsilon}|\nabla_x u^\varepsilon)|^p\,dx\,dt \\ &\le \frac{\eta}{p}\|f\|_{L^{p'}(Q)}^{p'} + \frac{1}{p\eta^{p-1}}\|u^\varepsilon\|_{L^p(Q)}^p + C. \end{aligned}$$

Since $u^\varepsilon(t,.) \in V$, using lemma 3.1 in the right hand side, we obtain

$$\begin{aligned} &\mu\int_{Q_1^\varepsilon}|\nabla_{\widetilde{x}}u^\varepsilon|^p\,dx\,dt + \int_{Q_1^\varepsilon}|\partial_{x_3}u^\varepsilon|^p\,dx\,dt + \int_{Q_2^\varepsilon}|\nabla_x u^\varepsilon|^p\,dx\,dt + \varepsilon^p\int_{Q_3^\varepsilon}|\nabla_x u^\varepsilon)|^p\,dx\,dt \\ &\le C\frac{\eta}{p} + \frac{C}{p\eta^{p-1}}(\|\partial_{x_3}u^\varepsilon\|_{L^p(\Omega_1^\varepsilon)}^p + \|\nabla u^\varepsilon\|_{L^p(\Omega_2^\varepsilon)}^p + \varepsilon^p\|\nabla u^\varepsilon\|_{L^p(\Omega_3^\varepsilon)}^p) + C, \end{aligned}$$

we can absorb the right-hand side by choosing $\eta^{p-1} > 1$. Thus

$$\|u^\varepsilon\|_{L^p(Q)} \le C,$$
$$(\mu\|\nabla_{\widetilde{x}}u^\varepsilon\|_{L^p(Q_1^\varepsilon)}^p, \|\partial_{x_3}u^\varepsilon\|_{L^p(Q_1^\varepsilon)}, \|\nabla u^\varepsilon\|_{L^p(Q_2^\varepsilon)}, \varepsilon\|\nabla u^\varepsilon\|_{L^p(Q_3^\varepsilon)}) \le C.$$

The bounds of $\widetilde{\mathbb{A}}_1^\varepsilon$, $\mathbb{A}_{13}^\varepsilon$ and $\mathbb{A}_\alpha^\varepsilon$, $\alpha = 2,3$ are obtained using Hölder's inequality and assumption (A6). □

As a consequence of the a priori estimates mentioned above, we have the following result.

Lemma 3.3. *Let $\gamma := \lim_{\varepsilon\to 0}\frac{\varepsilon^p}{\mu}$. Assume that $\gamma < +\infty$ and $f^\varepsilon = f$. There exists*

$$u_2 \in L^p(0,T;V), \quad v_1 \in L^p\big(Q; W^{1,p}_\sharp(\widetilde{Y}_1)/\mathbb{R}\big), \quad z \in L^p(Q\times Y),$$

$$(v_2, v_3) \in \prod_{i=2}^{3} L^p(Q; W^{1,p}_\sharp(Y_i)/\mathbb{R}), \quad g_k \in L^{p'}(Q\times Y),\ u_k^* \in L^2(Q\times Y),\ k=1,2,3,$$

such that we have the following two-scale convergence holds:

$$\begin{gathered}
u^\varepsilon(t,x) \overset{2s,p}{\to} \chi_1(y)v_1(t,x,\widetilde{y}) + \chi_2(y)u_2(t,x) + \chi_3(y)v_3(t,x,y),\\
\chi_1^\varepsilon(x)(u^\varepsilon(t,x), \varepsilon\nabla_{\widetilde{x}}u^\varepsilon(x)) \overset{2s,p}{\to} \chi_1(y)(v_1(t,x,\widetilde{y}), \nabla_{\widetilde{y}}v_1(t,x,\widetilde{y})),\\
\chi_1^\varepsilon(x)\partial_{x_3}u^\varepsilon(x) \overset{2s,p}{\to} \chi_1(y)z(t,x,y), \quad \text{such that } \partial_{x_3}v_1(t,x,\widetilde{y}) = \int_{\mathrm{I}} z(t,x,y)dy_N,\\
\chi_2^\varepsilon(x)(u^\varepsilon(t,x), \nabla_x u^\varepsilon(t,x)) \overset{2s,p}{\to} \chi_2(y)(u_2(t,x), [\nabla_x u_2(t,x) + \nabla_y v_2(t,x,y)]),\\
\chi_3^\varepsilon(x)(u^\varepsilon(t,x), \varepsilon\nabla_x u^\varepsilon(t,x)) \overset{2s,p}{\to} \chi_3(y)(v_3(t,x,y), \nabla_y v_3(t,x,y)),\\
\mu^{1/p'}\chi_1^\varepsilon(x)\widetilde{\mathbb{A}}_1^\varepsilon(x, \nabla_{\widetilde{x}}u^\varepsilon(t,x)) \overset{2s,p'}{\to} \chi_1(y)\widetilde{g}_1(t,x,y),\\
\chi_1^\varepsilon(x)\mathbb{A}_{13}^\varepsilon(x, \partial_{x_3}u^\varepsilon(t,x)) \overset{2s,p'}{\to} \chi_1(y)g_{13}(t,x,y),\\
\chi_2^\varepsilon(x)\mathbb{A}_2^\varepsilon(x, \nabla_x u^\varepsilon(t,x)) \overset{2s,p'}{\to} \chi_2(y)g_2(t,x,y),\\
\varepsilon^{\frac{p}{p'}}\chi_3^\varepsilon(x)\mathbb{A}_3^\varepsilon(x, \nabla_x u^\varepsilon(t,x)) \overset{2s,p'}{\to} \chi_3(y)g_3(t,x,y),\\
\chi_k^\varepsilon(x)(c_k^\varepsilon)^{1/2}u^\varepsilon(T,x) \overset{2s}{\to} \chi_k(y)u_k^*(x,y), \quad k=1,2,3,\\
(c^\varepsilon)^{1/2}u^\varepsilon(T,x) \overset{2s}{\to} u^*(x,y) := \sum_{k=1}^{3}\chi_k(y)u_k^*(x,y).
\end{gathered}$$

Moreover, there exists a unique function $w_3 \in L^p(Q; W^{1,p}_\sharp(Y_3))$ such that

$$\begin{gathered}
v_3(t,x,y) = u_2(t,x) + w_3(t,x,y) \ \text{ in } Y_3\\
w_3(t,x,y) = v_1(t,x,\widetilde{y}) - u_2(t,x) \ \text{ on } Y_{13} := \widetilde{Y}_{13}\times \mathrm{I}\\
w_3(t,x,y) = 0 \ \text{ on } Y_{23} := \widetilde{Y}_{23}\times \mathrm{I}
\end{gathered} \tag{3.8}$$

and u^ε converges weakly in $L^p(Q)$ to the function

$$U(t,x) = (1-\theta_1)u_2(t,x) + \int_{\widetilde{Y}_1} v_1(t,x,\widetilde{y})d\widetilde{y} + \int_{Y_3} w_3(t,x,y)dy.$$

The proof of the above lemma is the same as that of [3, Lemma 2.3], we omit it.

Remark 3.4. If $\gamma = \infty$, the sequence

$$\varepsilon\chi_1^\varepsilon\nabla_{\widetilde{x}}u^\varepsilon = \frac{\varepsilon}{\mu^{1/p}}\mu^{1/p}\chi_1^\varepsilon\nabla_{\widetilde{x}}u^\varepsilon$$

is not bounded in $L^p(Q,\mathbb{R}^2)$ in general. The scaled sequence $\frac{\mu^{1/p}}{\varepsilon}\chi_1^\varepsilon u^\varepsilon$ converges strongly to zero in $L^p(Q)$ as $\varepsilon\to 0$ since $\|\chi_1^\varepsilon u^\varepsilon\|_{L^p(Q)} \le C$. Thus, hereafter, we shall consider only the most interesting cases $\gamma = 0$ and $0<\gamma<\infty$.

4. Homogenization in the case $\gamma = 0$

Since $\gamma = 0$ and $\sup_\varepsilon \Big(\mu \|\nabla u^\varepsilon\|^p_{Q_1^\varepsilon}\Big) \le C$, the function

$$\varepsilon \chi_1(\frac{x}{\varepsilon}) \nabla_{\widetilde{x}} u^\varepsilon(t,x) = \frac{\varepsilon}{\mu^{1/p}} \mu^{1/p} \chi_1(\frac{x}{\varepsilon}) \nabla_{\widetilde{x}} u^\varepsilon(t,x)$$

converges strongly to zero in $L^p(Q;\mathbb{R}^2)$. Thus $\chi_1(y)\nabla_{\widetilde{y}} v_1(t,x,\widetilde{y}) = 0$, then

$$\chi_1(y) v_1(t,x,\widetilde{y}) := u_1(t,x)$$

and especially

$$U(t,x) = \theta_1 u_1(t,x) + (1-\theta_1) u_2(t,x) + \int_{Y_3} w_3(t,x,y)dy. \tag{4.1}$$

Moreover, the sequence

$$\mu \chi_1^\varepsilon \widetilde{\mathbb{A}}_1(x, \nabla_{\widetilde{x}} u^\varepsilon) = \mu^{1/p} \mu^{1/p'} \chi_1^\varepsilon \widetilde{\mathbb{A}}_1(x, \nabla_{\widetilde{x}} u^\varepsilon) \to 0$$

strongly in $L^{p'}(Q)$.

Now, for every datum $Z \in \mathbb{R}^3$, let

$$\mathbb{A}_{13}^{\text{hom}}(Z) := \int_{\widetilde{Y_1}} \mathbb{A}_{13}(\widetilde{y}, Z) d\widetilde{y}, \tag{4.2}$$

and let $w_{2,Z}$ be the unique solution (v_2) of the following cellular problem

$$\begin{gathered} -\operatorname{div}_y\Big(\mathbb{A}_2(y, Z + \nabla_y w_{2,Z})\Big) = 0 \quad \text{in } Y_2 \\ \mathbb{A}_2(y, Z + \nabla_y w_{2,Z}).n(y) = 0 \quad \text{on } Y_{23} \\ y \mapsto w_{2,Z}(y), \quad \mathbb{A}_2(y, Z + \nabla_y w_{2,Z}).n(y)\big|_{\partial Y_2 \cap \partial Y} \quad Y-\text{periodic}, \end{gathered} \tag{4.3}$$

we define the function

$$\mathbb{A}_2^{\text{hom}}(Z) := \int_{Y_2} \mathbb{A}_2(y, Z + \nabla_y w_{2,Z}(t,x,y))dy. \tag{4.4}$$

Theorem 4.1. *The functions $(u_\alpha, w_3) \in L^p(0,T;V) \times L^p(Q; W^{1,p}_\sharp(Y_3))$, $\alpha = 1,2$ are the unique solutions of the homogenized coupled problems*

$$\begin{gathered} \widetilde{c_1} \frac{\partial u_1}{\partial t}(t,x) - \partial_{x_3}\big(\mathbb{A}_{13}^{\text{hom}}(\partial_{x_3} u_1(t,x))\big) \\ + \int_{Y_{13}} \mathbb{A}_3(y, \nabla_y w_3(t,x,y)) n_3(y) dS(y) = \theta_1 f \quad \textit{in } Q \\ \widetilde{c_2} \frac{\partial u_2}{\partial t}(t,x) - \operatorname{div}_x\big(\mathbb{A}_2^{\text{hom}}(\nabla_x u_2(t,x))\big) \\ + \int_{Y_{23}} \mathbb{A}_3(y, \nabla_y w_3(t,x,y)) n_3(y) dS(y) = \theta_2 f \quad \textit{in } Q \\ \widetilde{c_\alpha} u_\alpha(0,x) = \widetilde{c_\alpha} u_0(x) \quad \textit{in } \Omega, \quad \widetilde{c_\alpha} = \int_{Y_\alpha} c_\alpha(y)dy, \\ u_\alpha(t,x) = 0 \quad \textit{on } \Gamma_{LB} \end{gathered} \tag{4.5}$$

and

$$
\begin{gathered}
c_3(y)(\frac{\partial u_2}{\partial t}(t,x)+\frac{\partial w_3}{\partial t}(t,x,y))-\operatorname{div}_y\Big(\mathbb{A}_3(y,\nabla_y w_3(t,x,y))\Big)=f \quad \text{in } Y_3\\
w_3(t,x,y)=u_1(t,x)-u_2(t,x) \quad \text{on } Y_{13}\\
w_3(t,x,y)=0 \quad \text{on } Y_{23}\\
c_3(y)w_3(0,x,y)=c_3(y)(u_0(x)-u_2(0,x)), \quad y\in Y_3\\
y\mapsto \mathbb{A}_3(y,\nabla_y w_3(t,x,y)).n(y)\big|_{\partial Y\cap\partial Y_3} \quad Y-\text{periodic}
\end{gathered}
\tag{4.6}
$$

Remark 4.2. Let us comment on these results. These problems involve, roughly speaking, three coupled fields : two macroscopic functions (u_1,u_2) and a microscopic one w_3. Notice that, only the longitudinal heat flux in the fiber is shown to be the unique factor contributing on the effective behavior of the composite (see second term of the first equation in (4.5)). Moreover, the auxiliary problem (4.6) is defined on the surrounding coating (Y_3) of the coated fiber ($\overline{Y_1}\cup Y_3$). Besides, there is no heat flux exchange across the fiber-coating interface.

Proof of Theorem 4.1. Let $C^1_{LB}(\overline{\Omega})=\{v\in C^1(\overline{\Omega}): v=0 \text{ on } \Gamma_{LB}\}$. We shall consider test functions ψ_α,ψ,ϕ_2 defined as follows:

$$
\begin{gathered}
\psi_\alpha(t,x)\in W^{1,p}(0,T;C^1_{LB}(\overline{\Omega}))\\
\psi(t,x,y)\in W^{1,p}(0,T;C^1_{LB}(\overline{\Omega};\mathcal{C}^\infty_\sharp(Y))),\quad \psi(t,x,y)=\psi_\alpha(t,x) \quad \text{in } Y_\alpha \text{ a.e.}\\
\phi_2(t,x,y)\in\mathcal{D}(Q;\mathcal{C}^\infty_\sharp(Y))
\end{gathered}
$$

We define the function $v^\varepsilon(t,x)=\psi(t,x,\frac{x}{\varepsilon})+\varepsilon\phi_2(t,x,\frac{x}{\varepsilon})$. Then $v^\varepsilon\in W^{1,p}(0,T;V)$, hence v^ε is an allowable test function. By putting it in the formulation (2.5), using the fact that $\mu=\mu^{1/p'}\mu^{1/p}$ and letting $\varepsilon\to 0$, we obtain

$$
\begin{aligned}
&-\sum_{\alpha=1}^{2}\int_Q\int_{Y_\alpha} c_\alpha(y)u_\alpha(t,x)\psi'_\alpha(t,x)\,dt\,dx\,dy\\
&-\sum_{\alpha=1}^{2}\int_\Omega\int_{Y\alpha} c_\alpha(y)u_0(x)\psi_\alpha(0,x)\,dx\,dy\\
&+\sum_{\alpha=1}^{2}\int_\Omega\int_{Y\alpha} c_\alpha^{1/2}(y)u^*_\alpha(x,y)\psi_\alpha(T,x)\,dx\,dy\\
&-\int_Q\int_{Y_3} c_3(y)\Big[u_2(t,x)+w_3(t,x,y)\Big]\psi'(t,x,y)\,dt\,dx\,dy\\
&-\int_\Omega\int_{Y_3} c_3(y)u_0(x)\psi(0,x,y)\,dx\,dy\\
&+\int_\Omega\int_{Y_3} c_3^{1/2}(y)u^*_3(x,y)\psi(T,x,y)\,dx\,dy+\int_Q\int_{Y_1} g_{13}(t,x,y).\partial_{x_3}\psi_1(t,x)\,dt\,dx\,dy\\
&+\int_Q\int_{Y_2} g_2(t,x,y).\Big[\nabla_x\psi_2(t,x)+\nabla_y\phi_2(t,x,y)\Big]\,dt\,dx\,dy\\
&+\int_Q\int_{Y_3} g_3(t,x,y).\nabla_y\psi(t,x,y)\,dt\,dx\,dy\\
&=\sum_\alpha\int_Q\int_{Y_\alpha} f(t,x)\psi_\alpha(t,x)\,dt\,dx\,dy+\int_Q\int_{Y_3} f(t,x)\psi(t,x,y)\,dt\,dx\,dy.
\end{aligned}
\tag{4.7}
$$

(i) Take $\psi_1 = 0 = \psi_2$ and $\phi_2 = 0$. Then

$$\begin{aligned}
&-\int_Q\int_{Y_3} c_3(y)\Big[u_2(t,x)+w_3(t,x,y)\Big]\psi'(t,x,y)\,dt\,dx\,dy\\
&-\int_\Omega\int_{Y_3} c_3(y)u_0(x)\psi(0,x,y)\,dx\,dy+\int_\Omega\int_{Y_3} c_3^{1/2}(y)u_3^*(x,y)\psi(T,x,y)\,dx\,dy\\
&+\int_Q\int_{Y_3} g_3(t,x,y).\nabla_y\psi(t,x,y)\,dt\,dx\,dy\\
&=\int_{\Omega_T}\int_{Y_3} f(t,x)\psi(t,x,y)\,dt\,dx\,dy
\end{aligned}$$

for all $\psi \in W^{1,p}(0,T;C^1_{LB}(\overline{\Omega};\mathcal{C}^\infty_\sharp(Y)))$ with $\psi(t,x,.)=0$ in $Y_1\cup Y_2$. This remains true, by density, for all $\psi\in W^{1,p}_{LB}(\Omega;W^{1,p}_\sharp(Y))$, $\psi=0$ on $Y_1\cup Y_2$. For a.e. $x\in\Omega$, we have, thus, a cellular problem on Y_3: Find $w_3=w_3(.,x,.)\in L^p((0,T);W^{1,p}_\sharp(Y))$ such that

$$\begin{aligned}
&-\int_0^T\int_{Y_3} c_3(y)\Big[u_2(t,x)+w_3(t,x,y)\Big]\psi'(t,y)\,dt\,dy\\
&-\int_{Y_3} c_3(y)u_0(x)\psi(0,y)dy\\
&+\int_{Y_3} c_3^{1/2}(y)u_3^*(x,y)\psi(T,y)dy+\int_0^T\int_{Y_3} g_3(t,x,y).\nabla_y\psi(t,y)\,dt\,dy\\
&=\int_0^T\int_{Y_3} f(t,x)\psi(t,y)\,dt\,dy,
\end{aligned}\tag{4.8}$$

for all $\psi(t,y)\in W^{1,p}((0,T);W^{1,p}_\sharp(Y))$, with $\psi=0$ on $Y_1\cup Y_2$.

Integrating by parts in t and in y successively, we obtain

$$\begin{aligned}
&\int_0^T\int_{Y_3} c_3(y)\frac{\partial}{\partial t}\Big[u_2(t,x)+w_3(t,x,y)\Big]\psi(t,y)\,dt\,dy\\
&+\int_{Y_3} c_3(y)(u_0(x)-(u_2(0,x)+w_3(0,x,y)))\psi(0,y)dy\\
&-\int_{Y_3} c_3(y)(u_2(T,x)+w_3(T,x,y))\psi(T,y)dy+\int_{Y_3} c_3^{1/2}(y)u_3^*(x,y)\psi(T,y)dy\\
&-\int_0^T\int_{Y_3}\operatorname{div}_y\Big(g_3(t,x,y)\Big)\psi(t,y)\,dt\,dy\\
&+\int_0^T\int_{\partial Y\cap\partial Y_3}\Big(g_3(t,x,y)\Big).n(y)\psi(t,y)dtdS(y)\\
&=\int_0^T\int_{Y_3} f(t,x)\psi(t,y)\,dt\,dy.
\end{aligned}$$

This is the variational form of an evolution problem on Y_3 which we write in a more explicit form (x is a parameter): Find $w_3\in L^p(Q;W^{1,p}_\sharp(Y_3))$ such that $c_3(y)w_3'\in L^p(Q;(W^{1,p}_\sharp(Y_3)))$ and

$$c_3(y)\Big(\frac{\partial u_2}{\partial t}(t,x)+\frac{\partial w_3}{\partial t}(t,x,y)\Big)-\operatorname{div}_y\Big(g_3(t,x,y)\Big)=f\quad\text{in } Y_3$$

$$
\begin{aligned}
w_3(t,x,y) = u_1(t,x) - u_2(t,x) \quad &\text{on } Y_{13}\\
w_3(t,x,y) = 0 \quad &\text{on } Y_{23}\\
c_3(y)w_3(0,x,y) = c_3(y)(u_0(x) - u_2(0,x)) \quad &y \in Y_3\\
y \mapsto g_3(t,x,y).n(y)\big|_{\partial Y \cap \partial Y_3} \quad &Y-\text{periodic}
\end{aligned}
\tag{4.9}
$$

and the final condition

$$
c_3^{1/2}(y)u_3^*(x,y) = c_3(y)(u_2(T,x) + w_3(T,x,y))
$$

which is however not a part of the problem. It will be only used below to identify the functions g_{13}, g_2, g_3.

(ii) Taking now $\phi_2 = 0$ and $\psi = 0$ in $Y_3 \cup Y_1$, and using an integration by parts and the initial and final conditions satisfied by w_3, we have

$$
\begin{aligned}
&-\int_Q \int_{Y_2} c_2(y)u_2(t,x)\psi_2'(t,x)\,dt\,dx\,dy - \int_\Omega \int_{Y_2} c_2(y)u_0(x)\psi_2(0,x)\,dx\,dy\\
&+ \int_\Omega \int_{Y_2} c_2^{1/2}(y)u_2^*(x,y)\psi_2(T,x)\,dx\,dy\\
&+ \int_Q \int_{Y_2} g_2(t,x,y)\big[\nabla_x\psi_2(t,x) + \nabla_y\phi_2(t,x,y)\big]\,dt\,dx\,dy\\
&+ \int_Q \int_{Y_{23}} g_3(t,x,y).n(y)\psi_2\,dt\,dx\,dS(y)\\
&= \int_Q \int_{Y_2} f\psi_2\,dt\,dx\,dy.
\end{aligned}
\tag{4.10}
$$

Taking $\phi_2 = 0$ and ψ_2 arbitrary in $W^{1,p}(0,T;V)$, we obtain the variational form of the following initial-boundary value problem in Q:

$$
\begin{gathered}
\widetilde{c_2}\frac{\partial u_2}{\partial t}(t,x) - \operatorname{div}_x\Big(\int_{Y_2} g_2(t,x,y)dy\Big)\int_{Y_{23}} g_3(t,x,y).n(y)dS(y) = \theta_2 f \quad \text{in } Q\\
\widetilde{c_2}u_2(0,x) = \widetilde{c_2}u_0(x) \quad \text{in } \Omega, \quad \tilde{c_2} = \int_{Y_2} c_2(y)dy\\
u_2(t,x) = 0 \quad \text{on } \partial\Omega
\end{gathered}
\tag{4.11}
$$

and the final condition

$$
\widetilde{c_2}u_2(T,x) = \int_{Y_2} c^{1/2}(y)u_2^*(x,y)dy.
$$

Now, taking $\psi_2 = 0$ and ϕ_2 arbitrary in $\mathcal{D}(Q; W_\sharp^{1,p}(Y_2))$, we have

$$
\int_Q \int_{Y_2} g_2(t,x,y)\nabla_y\phi_2(t,x,y)\,dt\,dx\,dy = 0,
$$

by integration by parts in y, we obtain

$$
-\int_{Y_2} \operatorname{div}_y\Big(g_2(t,x,y)\Big)dy + \int_Q \int_{\partial Y_2} g_2(t,x,y).n(y)\phi_2(t,x,y)\,dt\,dx\,dS(y) = 0
$$

for a.e. $(t,x)\in Q$. This remains true, by density, for all $\phi_2\in L^p(Q;W^{1,p}_\sharp(Y_2))$ and gives for each $(t,x)\in Q$ the variational formulation of a cellular problem on Y_2,

$$
\begin{gathered}
-\operatorname{div}_y\Big(g_2(t,x,y)\Big)=0 \quad \text{in } Y_2\\
\Big(g_2(t,x,y)\Big).n(y)\big|_{\partial Y_2\cap\partial Y_3}=0\\
y\mapsto g_2(t,x,y).n(y)\big|_{\partial Y\cap\partial Y_3}\quad Y-\text{periodic}
\end{gathered}
\tag{4.12}
$$

Similarly, for all $\psi_1\in W^{1,p}(0,T;W^{1,p}_{LB}(\Omega))$, we obtain

$$
\begin{aligned}
&-\int_Q\int_{Y_1} c_1(y)u_1(t,x)\psi_1'(t,x)\,dt\,dx\,dy-\int_\Omega\int_{Y_1} c_1(y)u_0(x)\psi_1(0,x)\,dx\,dy\\
&+\int_\Omega\int_{Y_1} c_1^{1/2}(y)u_1^*(x,y)\psi_1(T,x)\,dx\,dy\\
&+\int_Q\int_{\widetilde{Y}_1} g_{13}(t,x,y)\partial_{x_3}\psi_1(t,x)\,dt\,dx d\widetilde{y}\\
&+\int_Q\int_{Y_{13}} g_3(t,x,y).n(y)dS(y)\\
&=\int_Q\int_{Y_1} f\psi_1\,dt\,dx\,dy,
\end{aligned}
\tag{4.13}
$$

which is the variational formulation of the following initial-boundary value problem in Q.

$$
\begin{gathered}
\widetilde{c_1}\frac{\partial u_1}{\partial t}(t,x)-\partial_{x_3}(g_{13}(t,x,y))+\int_{Y_{13}} g_3(t,x,y).n(y)dS(y)=\theta_1 f \quad \text{in } \Omega\\
\widetilde{c_1}u_1(0,x)=\widetilde{c_1}u_0(x)\quad \text{in } \Omega,\quad \tilde{c_1}=\int_{Y_1} c_1(y)dy\\
u_1(t,x)=0\quad \text{on } \partial\Omega
\end{gathered}
\tag{4.14}
$$

and the final condition

$$
\widetilde{c_1}u_1(T,x)=\int_{Y_1} c^{1/2}(y)u_1^*(x,y)dy.
$$

From this, we obtain the homogenized problem (4.5). It remains to identify g_k in terms of v_k,u_2. Before proceeding, we prove the following useful identity.

Lemma 4.3.

$$
\begin{aligned}
&\sum_{\alpha=1}^2\frac12\widetilde{c}_\alpha\int_\Omega|u_\alpha(T,x)|^2dx+\frac12\int_\Omega\int_{Y_3} c_3(y)|v_3(T,x,y)|^2dx\,dy\\
&-\frac12\sum_{k=1}^3\widetilde{c}_k\int_\Omega|u_0(x)|^2dx+\int_Q\int_{Y_1} g_{13}(t,x,y)\partial_{x_3}u_1(t,x)\,dt\,dx\,dy\\
&+\int_Q\int_{Y_2} g_2(t,x,y)\Big(\nabla_x u_2(t,x)+\nabla_y v_2(t,x,y)\Big)\,dt\,dx\,dy\\
&+\int_Q\int_{Y_3} g_3(t,x,y).\nabla_y v_3(t,x,y)\,dt\,dx\,dy\\
&=\int_Q f(t,x)U(t,x)dt\,dx.
\end{aligned}
$$

Proof. We start from the two-scale homogenized problem (4.7) and we consider the sequences

$$\psi_{k,n},\ k=1,2,3, \quad \phi_{2,n},\ \phi_{3,n}$$

such that

(1) $\psi_{\alpha,n} \to u_\alpha$ in $L^p(0,T;V)$, $\frac{\partial}{\partial t}\psi_{1,n} \to \frac{\partial}{\partial t}u_\alpha 1$ in $L^{p'}(0,T;V')$, $\alpha=1,2$,
(2) $\psi_{3,n} \to v_3$ in $L^p(0,T;W_\sharp^{1,p}(Y_3))$, $\frac{\partial}{\partial t}\psi_{3,n} \to \frac{\partial}{\partial t}v_3$ in $L^{p'}(0,T;W_\sharp^{1,p}(Y_3)')$,
(3) $\nabla_y\phi_{2,n} \to \nabla_y v_2$ in $L^p(Q\times Y_2)$, $\nabla_y\phi_{3,n} \to \nabla_y v_3$ in $L^p(Q\times Y_3)$.

Note that the smoothness of the above sequences $\psi_{k,n}$, $k=1,2,3$, $\phi_{2,n}$, $\phi_{3,n}$ implies their two-scale convergence in strong sense to the corresponding limits [1, Theorem 1.8]. Therefore, passing to the limit with respect to n and taking in account of the final conditions, we obtain

$$\begin{aligned}
&-\sum_{\alpha=1}^{2}\int_Q\int_{Y_\alpha} c_\alpha(y)u_\alpha(t,x)u'_\alpha(t,x)\,dt\,dx\,dy - \sum_{k=1}^{3}\int_\Omega\int_{Y\alpha} c_k(y)u_0(x)^2\,dx\,dy\\
&+\sum_{\alpha=1}^{2}\int_\Omega\int_{Y\alpha} c_\alpha(y)u_\alpha(T,x)^2\,dx\,dy\\
&-\int_Q\int_{Y_3} c_3(y)v_3(t,x,y)v'_3(t,x,y)\,dt\,dx\,dy + \int_\Omega\int_{Y_3} c_3(y)v_3(T,x,y)^2\,dx\,dy\\
&+\int_Q\int_{Y_1} g_{13}(t,x,y).\partial_{x_3}u_1(t,x)\,dt\,dx\,dy\\
&+\int_Q\int_{Y_2} g_2(t,x,y).\Big[\nabla_x u_2(t,x)+\nabla_y v_2(t,x,y)\Big]\,dt\,dx\,dy\\
&+\int_Q\int_{Y_3} g_3(t,x,y).\nabla_y v_3(t,x,y)\,dt\,dx\,dy\\
&=\int_Q f(t,x)U(t,x)\,dt\,dx.
\end{aligned}$$

Integrating the above equality with respect to the t variable, we obtained the states result. □

We are now equipped to identify g_{13}, g_2 and g_3.

Identification of g_{13}, g_2 and g_3. Let ϕ and Φ be in $\mathcal{C}_0^\infty(Q;\mathcal{C}_\sharp^\infty(Y))^N$ and $\mathcal{C}_0^\infty(Q;\mathcal{C}_\sharp^\infty(Y))$ respectively. For $\varepsilon>0$ and $h>0$ we define the test function

$$\begin{aligned}
\eta^\varepsilon(t,x) = \chi_1^\varepsilon(x)\begin{pmatrix}0\\ \partial_{x_3}\end{pmatrix}u_1(t,x) + \chi_2^\varepsilon(x)\nabla_x u_2(t,x)\\
+\varepsilon\nabla_x\phi(t,x,\frac{x}{\varepsilon}) + h\Phi(t,x,\frac{x}{\varepsilon}).
\end{aligned} \tag{4.15}$$

Note that η^ε and (by the continuity assumption) $\mathbb{A}_k^\varepsilon(x,\eta^\varepsilon) := \mathbb{A}_k^\varepsilon(\frac{x}{\varepsilon},\eta^\varepsilon(t,x))$, $k=1,2,3$ are admissible test functions (in $L^p(Q)$) for the two-scale convergence and

$$\begin{aligned}
\eta^\varepsilon(t,x) \overset{2s,p'}{\to} \eta(t,x,y) =: \chi_1(y)\begin{pmatrix}0\\ \partial_{x_3}\end{pmatrix}u_1(t,x) + \chi_2(y)\nabla_x u_2(t,x)\\
+\nabla_y\phi(t,x,y) + h\Phi(t,x,y).
\end{aligned}$$

The monotonicity condition (A6) yields

$$\int_Q \Big(\mathbb{A}^\varepsilon(x,\nabla_x u^\varepsilon(t,x)) - \mathbb{A}^\varepsilon(x,\eta^\varepsilon(t,x))\Big)\Big(\nabla_x u^\varepsilon(t,x) - \eta^\varepsilon(t,x)\Big)\,dt\,dx \geq 0. \quad (4.16)$$

Expanding this expression and employing (3.7) yields

$$\begin{aligned}
&\int_0^T \int_\Omega f(t,x)u^\varepsilon(t,x)\,dx\,dt - \frac{1}{2}\int_\Omega c^\varepsilon(x)u^\varepsilon(T,x)^2 dx + \frac{1}{2}\int_\Omega c^\varepsilon(x)(u_0^\varepsilon)^2 dx \\
&- \int_Q \Big(\mathbb{A}^\varepsilon(x,\nabla_x u^\varepsilon(t,x))\eta^\varepsilon(t,x) + \mathbb{A}^\varepsilon(x,\eta^\varepsilon(t,x))(\nabla_x u^\varepsilon(t,x) - \eta^\varepsilon(t,x))\Big)\,dt\,dx \geq 0.
\end{aligned}$$

Letting $\varepsilon \to 0$, the two-scale convergence of u^ε and $\chi_k^\varepsilon \mathbb{A}_k^\varepsilon$ and the continuity of $\mathbb{A}_k$ give in the limit

$$\begin{aligned}
&\int_Q f(t,x)U(t,x)\,dx\,dt - \liminf_{\varepsilon\to 0}\frac{1}{2}\int_\Omega c^\varepsilon(x)u^\varepsilon(T,x)^2 dx + \frac{1}{2}\int_\Omega\int_Y c(y)dy(u_0)^2 dx \\
&- \int_Q\int_{Y_1} g_{13}(t,x,y)\Big(\partial_{x_3}u_1(t,x) + h\Phi_N(t,x,y)\Big)\,dt\,dx\,dy \\
&- \int_Q\int_{Y_2} g_2(t,x,y)\Big(\nabla_x u_2(t,x) + \nabla_y\phi(t,x,y) + h\Phi(t,x,y)\Big)\,dt\,dx\,dy \\
&- \int_Q\int_{Y_3} g_3(t,x,y)\Big(\nabla_y\phi(t,x,y) + h\Phi(t,x,y)\Big)\,dt\,dx\,dy \qquad (4.17)\\
&+ \int_Q\int_{Y_1} \mathbb{A}_{13}(x,\eta_N(t,x,y))\Big(\partial_{x_3}u_1(t,x) + h\Phi_N(t,x,y)\Big)\,dt\,dx\,dy \\
&+ \int_Q\int_{Y_2} \mathbb{A}_2(x,\eta(t,x,y))\Big(-\nabla_x u_2(t,x) + \nabla_y\phi(t,x,y) + h\Phi(t,x,y)\Big)\,dt\,dx\,dy \\
&+ \int_Q\int_{Y_3} \mathbb{A}_3(x,\eta(t,x,y))\Big(\nabla_y\phi(t,x,y) + h\Phi(t,x,y)\Big)\,dt\,dx\,dy \geq 0.
\end{aligned}$$

Since $\mathbb{A}_k$ is continuous we may replace $\phi_\beta, \beta = 2,3$ by a sequence converging strongly in $L^p(Q;W_\sharp^{1,p}(Y_\beta)/\mathbb{R})$ to v_β; thus replacing $\eta(t,x,y)$ in (4.17) with $\nabla_x u_2 + \nabla_y v_2 + h\Phi$ and $\nabla_y v_3 + h\Phi$ successively and using Lemma 4.3, the above sum simplified to

$$\begin{aligned}
&\int_Q\int_{Y_1}\Big[\mathbb{A}_{13}\big(\widetilde{y},\partial_{x_3}u_1(t,x) + h\Phi(t,x,y)\big) - g_{13}(t,x,y)\Big]h\Phi_N(t,x,y)\,dt\,dx\,dy \\
&+ \int_Q\int_{Y_2}\Big[\mathbb{A}_2\big(x,\nabla_x u_2 + \nabla_y v_2 + h\Phi(t,x,y)\big) - g_2(t,x,y)\Big]h\Phi(t,x,y)\,dt\,dx\,dy \\
&+ \int_Q\int_{Y_3} g_3(t,x,y)\Big[\mathbb{A}_3\big(x,\nabla_y v_3 + h\Phi(t,x,y)\big) - g_3(t,x,y)\Big]h\Phi(t,x,y)\,dt\,dx\,dy \\
&\geq \frac{1}{2}\int_\Omega c^\varepsilon(x)u^\varepsilon(T,x)^2 dx + \frac{1}{2}\int_\Omega\int_Y c(y)dy(u_0)^2 dx \\
&\quad + \liminf_{\varepsilon\to 0}\Big[\frac{1}{2}\int_\Omega c^\varepsilon(x)u^\varepsilon(T,x)^2 dx + \frac{1}{2}\int_\Omega\int_Y c(y)dy(u_0)^2 dx\Big].
\end{aligned}$$

Thus, dividing by h and letting $h \to 0$ we see that for every Φ,

$$\begin{aligned} &\int_Q \int_{Y_1} \Big[\mathbb{A}_{13}\big(\widetilde{y}, \partial_{x_3} u_1(t,x) - g_{13}(t,x,y)\Big] h\Phi_N(t,x,y)\, dt\, dx\, dy \\ &+ \int_Q \int_{Y_2} \Big[\mathbb{A}_2\big(y, \nabla_x u_2 + \nabla_y v_2\big) - g_2(t,x,y)\Big] h\Phi(t,x,y)\, dt\, dx\, dy \\ &+ \int_Q \int_{Y_3} g_3(t,x,y)\Big[\mathbb{A}_3\big(y, \nabla_y v_3\big) - g_3(t,x,y)\Big] h\Phi(t,x,y)\, dt\, dx\, dy \geq 0. \end{aligned} \tag{4.18}$$

We therefore have proved the desired results, namely, that

$$\begin{gathered} g_{13}(t,x,y) = \mathbb{A}_{13}(\widetilde{y}, \partial_{x_3} u_1(t,x)) \\ g_2(t,x,y) = \mathbb{A}_2(y, \nabla_x u_2(t,x) + \nabla_y v_2(t,x,y)) \\ g_3(t,x,y) = \mathbb{A}_3(y, \nabla_y v_3(t,x,y)) \end{gathered}$$

Hence equations (4.5)-(4.6) are satisfied. To complete the proof it suffices to show that $\{u_2, v_1, v_2, v_3\}$ is the unique solution of (4.5)-(4.6). In fact, the uniqueness is a consequence of the strict monotonicity of $\mathbb{A}_k,\ k = 1,2,3$. Indeed, if $\{u_2^1, v_1^1, v_2^1, v_3^1\}$ and $\{u_2^2, v_1^2, v_2^2, v_3^2\}$ are two solutions of (4.5)-(4.6), using (4.7), by difference we obtain

$$\begin{aligned} &-\sum_{\alpha=1}^{2} \int_Q \int_{Y_\alpha} c_\alpha(y)(u_\alpha^1 - u_\alpha^2)\psi'_\alpha\, dt\, dx\, dy \\ &+ \sum_{\alpha=1}^{2} \int_\Omega \int_{Y\alpha} c_\alpha^{1/2}(y)(u_\alpha^{*\,1} - u_\alpha^{*\,2})\psi_\alpha(T,x)\, dx\, dy \\ &- \int_Q \int_{Y_3} c_3(y)(v_3^1 - v_3^2)\psi'\, dt\, dx\, dy + \int_\Omega \int_{Y_3} c_3^{1/2}(y)(v_3^{*1} - v_3^{*2})\psi\, dx\, dy \\ &+ \int_Q \int_{Y_1} (\mathbb{A}_{13}(\widetilde{y}, \partial_{x_3} u_1^1) - \mathbb{A}_{13}(\widetilde{y}, \partial_{x_3} u_1^2)).\partial_{x_3}\psi_1\, dt\, dx\, dy \\ &+ \int_Q \int_{Y_2} (\mathbb{A}_2(y, \nabla_x u_2^1 + \nabla_y v_2^1) - \mathbb{A}_2(y, \nabla_x u_2^2 + \nabla_y v_2^2)).(\nabla_x\psi_2 + \nabla_y\phi_2)\, dt\, dx\, dy \\ &+ \int_Q \int_{Y_3} (\mathbb{A}_3(y, \nabla_y v_3^1) - \mathbb{A}_3(y, \nabla_y v_3^2)).\nabla_y\psi\, dt\, dx\, dy = 0. \end{aligned}$$

In particular, for $\psi_\alpha = u_\alpha^1 - u_\alpha^2$, $\phi_2 = v_2^1 - v_2^2$ and $\psi = v_3^1 - v_3^2$, we obtain, in view of the initial and final conditions satisfied by u_α, v_3 and by the strict monotonicity of $\mathbb{A}_k,\ k = 1,2,3$:

$$\partial_{x_3}(u_1^1 - u_1^2) = 0, \quad \nabla_y(v_3^1 - v_3^2) = 0 \quad \text{in } Q \times Y_3,$$

since $(u_1^1 - u_1^2) = 0$ on $(0,T) \times \Gamma_{LB}$ and $(v_3^1 - v_3^2) = 0$ on $(0,T) \times Y_{23}$, thus $u_1^1 = u_1^2$ and $v_3^1 = v_3^2$. As a consequence,

$$\int_Q \int_{Y_2} (\mathbb{A}_2(y, \nabla_x u_2^1 + \nabla_y v_2^1) - \mathbb{A}_2(y, \nabla_x u_2^2 + \nabla_y v_2^2)).(\nabla_x\psi_2 + \nabla_y\phi_2)\, dt\, dx\, dy = 0,$$

so this problem has a unique solution in the space $W_{LB}^{1,p}(\Omega) \times L^p(\Omega; W_\sharp^{1,p}(Y_2))$ by an application of Lax-Milligram lemma, then $u_2^1 = u_2^2$ and $v_2^1 = v_2^2$.

5. Homogenization in the case $0 < \gamma < \infty$

In this case we shall proof the following result.

Theorem 5.1. *Functions* $(u_2, v_1, w_3) \in L^p(0,T; W^{1,p}_{LB}(\Omega)) \times L^p(Q; W^{1,p}_\sharp(\widetilde{Y}_1)/\mathbb{R}) \times L^p(Q; W^{1,p}_\sharp(Y_3))$ *are the unique solutions of the two-scale homogenized problems*

$$\begin{gathered}
\widetilde{c_2}\frac{\partial u_2}{\partial t} - \operatorname{div}_x(\mathbb{A}_2^{\rm hom}(\nabla_x u_2(t,x))) \\
+ \int_{Y_{23}} \mathbb{A}_3(y, \nabla_y w_3(t,x,y)).n_3(y) dS(y) = \theta_2 f \quad \text{in } Q \\
\widetilde{c_2} u_2(0,x) = \widetilde{c_2} u_0(x) \quad \text{in } \Omega, \quad \widetilde{c_2} = \int_{Y_2} c_2(y) dy \\
u_2(t,x) = 0 \quad \text{on } (0,T) \times \partial\widetilde{\Omega} \times [0,1[
\end{gathered} \tag{5.1}$$

where $\mathbb{A}_2^{\rm hom}$ *is define by* (4.4).

$$\begin{gathered}
\langle c_1\rangle_{\rm I}(\widetilde{y})\frac{\partial v_1}{\partial t}(t,x,\widetilde{y}) - \frac{1}{\gamma^{1/p}}\operatorname{div}_{\widetilde{y}}\big(\widetilde{\mathbb{A}}_1(\widetilde{y}, \nabla_{\widetilde{y}} v_1(t,x,\widetilde{y}))\big) \\
- \partial_{x_3}(\mathbb{A}_{13}(\widetilde{y}, \partial_{x_3} v_1(t,x,\widetilde{y}))) = f \quad \text{in } \widetilde{Y}_1 \\
c_3(y)\Big(\frac{\partial u_2}{\partial t}(t,x) + \frac{\partial w_3}{\partial t}(t,x,y)\Big) - \operatorname{div}_y(\mathbb{A}_3(y, \nabla_y w_3(t,x,y))) = f \quad \text{in } Y_3 \\
\widetilde{\mathbb{A}}_1(\widetilde{y}, \nabla_{\widetilde{y}} v_1).n(\widetilde{y}) = \gamma^{1/p}\langle \mathbb{A}_3(y, \nabla_y w_3).n(y)\rangle_{\rm I} \quad \text{on } \widetilde{Y}_{13} \\
w_3(t,x,y) = v_1(t,x,\widetilde{y}) - u_2(t,x) \quad \text{on } Y_{13}, \quad w_3(t,x,y) = 0 \quad \text{on } Y_{23} \\
\langle c_1\rangle_{\rm I}(\widetilde{y}) v_1 = \langle c_1\rangle_{\rm I}(\widetilde{y}) u_0, \quad \widetilde{y} \in \widetilde{Y}_1 \\
c_3(y) w_3(0,x,y) = c_3(y)(u_0(x) - u_2(0,x)), \quad y \in Y_3 \\
y \mapsto \mathbb{A}_3(y, \nabla_y w_3(t,x,y)).n(y)\big|_{\partial Y \cap \partial Y_3} \quad Y - \text{periodic}
\end{gathered} \tag{5.2}$$

where $\langle . \rangle_{\rm I}$ *denotes the integration with respect to* y_3 *over* I.

Remark 5.2. Contrary to the previous case, here the problems (5.1)-(5.2) involve a unique macroscopic function u_2 and two microscopic functions v_1, w_3. The functions v_1, w_3 are "strongly" coupled via the following non-standard boundary conditions

$$w_3(t,x,y) - v_1(t,x,\widetilde{y}) = u_2(t,x) \quad \text{on } (0,T) \times Y_{13}, \tag{5.3}$$

$$\widetilde{\mathbb{A}}_1(\widetilde{y}, \nabla_{\widetilde{y}} v_1).n(\widetilde{y}) = \gamma^{1/p}\langle \mathbb{A}_3(y, \nabla_y w_3).n(y)\rangle_{\rm I} \quad \text{on } (0,T) \times \widetilde{Y}_{13}. \tag{5.4}$$

As a consequence, the above interface conditions exhibit a remarkable temperature jump and a transverse heat flux continuity. This, might be interpreted as the combined effects of fiber coatings together with the high anisotropy of the fibers in the overall behavior of the composite. In addition, it should be noted that the auxiliary problem (5.2) is defined in the coated fiber $\overline{Y_1} \cup Y_3$ and involves both the longitudinal and the transverse thermal conductivities of the fiber.

Proof of Theorem 5.1. Let ψ_2, ψ, ϕ_2 be test functions as defined in the proof of Theorem 4.1 and let

$$\psi_1(t,x,\widetilde{y}) \in W^{1,p}(0,T; C^1_{LB}(\overline{\Omega}; \mathcal{C}^\infty_\sharp(\widetilde{Y})))$$

be a test function such that $\psi(t,x,y) = \psi_1(t,x,\widetilde{y})$ in Y_1 a.e.

As in the previous case, we take $v^\varepsilon(t,x)=\psi(t,x,\frac{x}{\varepsilon})+\varepsilon\phi_2(t,x,\frac{x}{\varepsilon})$ in (2.5) and letting $\varepsilon\to 0$, we deduce the following two-scale limit

$$
\begin{aligned}
&-\int_Q\int_{Y_1} c_1(y)v_1(t,x,\widetilde{y})\psi_1'(t,x,\widetilde{y})\,dt\,dx\,dy-\int_\Omega\int_{Y_1} c_1(y)u_0(x)\psi_1(0,x,\widetilde{y})\,dx\,dy\\
&+\int_\Omega\int_{Y_1} c_1^{1/2}(y)u_1^*(x,y)\psi_1(T,x,\widetilde{y})\,dx\,dy\\
&-\int_Q\int_{Y_2} c_2(y)u_2(t,x)\psi_2'(t,x)\,dt\,dx\,dy-\int_\Omega\int_{Y2} c_2(y)u_0(x)\psi_2(0,x)\,dx\,dy\\
&+\int_\Omega\int_{Y_2} c_2^{1/2}(y)u_2^*(x,y)\psi_2(T,x)\,dx\,dy\\
&-\int_Q\int_{Y_3} c_3(y)\Big[u_2(t,x)+w_3(t,x,y)\Big]\psi'(t,x,y)\,dt\,dx\,dy\\
&-\int_\Omega\int_{Y_3} c_3(y)u_0(x)\psi(0,x,y)\,dx\,dy\\
&+\int_\Omega\int_{Y_3} c_3^{1/2}(y)u_3^*(x,y)\psi(T,x,y)\,dx\,dy\\
&+\frac{1}{\gamma^{1/p}}\int_Q\int_{Y_1} \widetilde{g}_1(t,x,y)\nabla_{\widetilde{y}}\psi_1(t,x,\widetilde{y})\,dt\,dxdy\\
&+\int_Q\int_{Y_1} g_{13}(t,x,y)\partial_{x_3}\psi_1(t,x,\widetilde{y})\,dt\,dxdy\\
&+\int_Q\int_{Y_2} g_2(t,x,y)\Big[\nabla_x\psi_2(t,x)+\nabla_y\phi_2(t,x,y)\Big]\,dt\,dx\,dy\\
&+\int_Q\int_{Y_3} g_3(t,x,y)\nabla_y\psi(t,x,y)\,dt\,dx\,dy\\
&=\int_Q\int_{Y_1} f(t,x)\psi_1(t,x)\,dt\,dxdy+\int_Q\int_{Y_2} f(t,x)\psi_2(t,x)\,dt\,dx\,dy\\
&\quad+\int_Q\int_{Y_3} f(t,x)\psi(t,x,y)\,dt\,dx\,dy.
\end{aligned}
$$

(i) We choose $\psi_1=0=\psi_2$ and $\phi_2=0$. Then, we have the cellular problem in Y_3.

(ii) Taking $\phi_2=0$ and $\psi_2=0$ and an integration by parts with respect to x_3, we obtain

$$
\begin{aligned}
&\int_Q\int_{Y_1} g_{13}(t,x,y)\partial_{x_3}\psi_1(t,x,\widetilde{y})\,dt\,dx\,dy\\
&=\int_Q\int_{\widetilde{Y}_1}(\int_{\mathrm{I}} g_{13}(t,x,y)dy_3)\partial_{x_3}\psi_1(t,x,\widetilde{y})\,dt\,dx\,d\widetilde{y}\\
&=-\int_Q\int_{\widetilde{Y}_1}\partial_{x_3}(\int_{\mathrm{I}} g_{13}(t,x,y)dy_3)\psi_1(t,x,\widetilde{y})\,dt\,dx\,d\widetilde{y}.
\end{aligned}
$$

After integration by parts with respect to t and $\widetilde{y}$ successively we have

$$
\int_Q\int_{\widetilde{Y}_1}\langle c_1\rangle_{\mathrm{I}}(\widetilde{y})v_1'(t,x,\widetilde{y})\psi_1\,dt\,dxd\widetilde{y}-\int_\Omega\int_{\widetilde{Y}_1}\langle c_1\rangle_{\mathrm{I}}(\widetilde{y})\Big(u_0(x)-v_1(0,x,\widetilde{y})\Big)\psi_1 dx\,d\widetilde{y}
$$

$$
\begin{aligned}
&+\int_\Omega\int_{Y_1}\Big(c_1^{1/2}(y)u_1^*(x,y)-\langle c_1\rangle_{\rm I}(\widetilde{y})v_1(T,x,\widetilde{y})\Big)\psi_1(T,x,\widetilde{y})\,dx\,dy\\
&-\frac{1}{\gamma^{1/p}}\int_Q\int_{\widetilde{Y}_1}\operatorname{div}_{\widetilde{y}}\big(\widetilde{g}_1(t,x,y)\big)\psi_1 d\widetilde{y}\,dt\,dx\\
&-\frac{1}{\gamma^{1/p}}\int_Q\int_{\widetilde{Y}_{13}}\widetilde{g}_1(t,x,y).n(y)\psi_1 dS(\widetilde{y})\,dt\,dx\\
&-\int_Q\int_{\widetilde{Y}_1}\partial_{x_3}\langle g_{13}(t,x,y)\rangle_{\rm I}(\widetilde{y})\psi_1\,dt\,dx\,d\widetilde{y}\\
&+\int_Q\int_{\widetilde{Y}_{13}}\langle g_3(t,x,y).n(y)\rangle_{\rm I}(\widetilde{y})\psi_1\,dt\,dx\,dS(\widetilde{y})\\
&=\int_Q\int_{\widetilde{Y}_1} f\psi_1\,dt\,dx\,d\widetilde{y}
\end{aligned}
\tag{5.5}
$$

for all $\psi_1\in W^{1,p}\big(0,T;C^1_{LB}(\overline{\Omega};\mathcal{C}^\infty_\sharp(\widetilde{Y}_1))\big)$. Thus, we get the cell problem in $\widetilde{Y}_1$ (5.2).

(iii) Taking ($\psi_1=0$, $\phi_2=0$), then ($\psi_1=0,\psi_2=0$) we obtain the initial-boundary value problem (5.1).

It remains to identify $\widetilde{g}_1$ and g_{13}. This is done as in the preceding case. More precisely, let $\phi(t,x,y)$, $\Phi(t,x,y)$ and $\psi_1(t,x,\widetilde{y})$ be in $\mathcal{C}^\infty_0(Q;\mathcal{C}^\infty_\sharp(Y))^N$, $\mathcal{C}^\infty_0(Q;\mathcal{C}^\infty_\sharp(Y))$ and $\mathcal{C}^\infty_0(Q;\mathcal{C}^\infty_\sharp(\widetilde{Y}_1))$ respectively. For $\varepsilon>0$ and $h>0$ we define the following test function

$$
\begin{aligned}
\eta^\varepsilon(t,x)&=\chi_1^\varepsilon(x)\nabla_x^\varepsilon\psi_1(t,x,\frac{\widetilde{x}}{\varepsilon})+\chi_2^\varepsilon(x)\nabla_x u_2(t,x)\\
&\quad+\varepsilon(1-\chi_1^\varepsilon(x))\nabla_x\phi(t,x,\frac{x}{\varepsilon})+h\Phi(t,x,\frac{x}{\varepsilon}),
\end{aligned}
$$

where

$$
\nabla_x^\varepsilon=\begin{pmatrix}\varepsilon\nabla_{\widetilde{x}}\\ \partial_{x_3}\end{pmatrix}. \tag{5.6}
$$

Note that $\mathbb{A}_k^\varepsilon(x,\eta^\varepsilon):=\mathbb{A}_k^\varepsilon(\frac{x}{\varepsilon},\eta^\varepsilon(t,x))$, $k=1,2,3$ are admissible test functions (in $L^p(Q)$) for the two-scale convergence and

$$
\eta^\varepsilon(t,x)\overset{2s,p'}{\rightarrow}\eta(t,x,y),
$$

where

$$
\begin{aligned}
\eta(t,x,y)&=\chi_1(y)\begin{pmatrix}\nabla_{\widetilde{y}}\\ \partial_{x_3}\end{pmatrix}\psi_1(t,x,\widetilde{y})+\chi_2(y)\nabla_x u_2(t,x)\\
&\quad+(1-\chi_1(y))\nabla_y\phi(t,x,y)+h\Phi(t,x,y).
\end{aligned}
$$

Using monotonicity condition (A6) and letting $\varepsilon\to 0$, exactly as in the previous case, we obtain

$$
\begin{aligned}
&\int_Q f(t,x)U(t,x)\,dx\,dt-\liminf_{\varepsilon\to0}\frac12\int_\Omega c^\varepsilon(x)u^\varepsilon(T,x)^2dx+\frac12\int_\Omega\int_Y c(y)dy(u_0)^2dx\\
&-\frac{1}{\gamma^{1/p}}\int_Q\int_{Y_1}\widetilde{g}_1(t,x,y)\Big(\nabla_{\widetilde{y}}\psi_1(t,x,\widetilde{y})+h\widetilde{\Phi}(t,x,y)\Big)\,dt\,dx\,dy\\
&-\int_Q\int_{Y_1}g_{13}(t,x,y)\Big(\partial_{x_3}\psi_1(t,x,\widetilde{y})+h\phi_3(t,x,y)\Big)\,dt\,dx\,dy\\
&-\int_Q\int_{Y_2}g_2(t,x,y)\Big(\nabla_x u_2(t,x)+\nabla_y\phi(t,x,y)+h\Phi(t,x,y)\Big)\,dt\,dx\,dy
\end{aligned}
$$

$$
\begin{aligned}
&- \int_Q \int_{Y_3} g_3(t,x,y)\Big(\nabla_y \phi(t,x,y) + h\Phi(t,x,y)\Big)\,dt\,dx\,dy \\
&+ \frac{1}{\gamma^{1/p}} \int_Q \int_{Y_1} \widetilde{\mathbb{A}}_1(x,\eta(t,x,y))\Big(\nabla_{\widetilde{y}} \psi_1(t,x,\widetilde{y}) + h\widetilde{\Phi}(t,x,y)\Big)\,dt\,dx\,dy \\
&+ \int_Q \int_{Y_1} \mathbb{A}_{13}(x,\eta(t,x,y))\Big(-\partial_{x_3} \psi_1(t,x,\widetilde{y}) + h\phi_3(t,x,y)\Big)\,dt\,dx\,dy \\
&+ \int_Q \int_{Y_2} \mathbb{A}_2(x,\eta(t,x,y))\Big(-\nabla_x u_2(t,x) + \nabla_y \phi(t,x,y) + h\Phi(t,x,y)\Big)\,dt\,dx\,dy \\
&+ \int_Q \int_{Y_3} \mathbb{A}_3(x,\eta(t,x,y))\Big(\nabla_y \phi(t,x,y) + h\Phi(t,x,y)\Big)\,dt\,dx\,dy \geq 0.
\end{aligned}
$$

Now, we may replace ψ_1 and $\phi_\beta, \beta = 2,3$ by a sequence converging strongly in $L^p(Q; W^{1,p}_\sharp(\widetilde{Y}_1)/\mathbb{R})$ and $L^p(Q; W^{1,p}_\sharp(Y_\beta)/\mathbb{R})$ to v_1 and v_β respectively, thus replacing $\eta(t,x,y)$ in (4.17) with $\begin{pmatrix} \nabla_{\widetilde{y}} v_1 \\ \partial_{x_3} v_1 \end{pmatrix}$, $\nabla_x u_2 + \nabla_y v_2 + h\Phi$ and $\nabla_y v_3 + h\Phi$ successively. Using the conservation of energy given by the lemma 4.3 adapted to the present case, the above sum simplified to

$$
\begin{aligned}
&\frac{1}{\gamma^{1/p}} \int_Q \int_{Y_1} \Big[\widetilde{\mathbb{A}}_1\big(\widetilde{y}, \nabla_{\widetilde{y}} v_1(t,x,\widetilde{y}) + h\widetilde{\Phi}(t,x,y)\big) - \widetilde{g}_1(t,x,y)\Big] h\widetilde{\Phi}(t,x,y)\,dt\,dx\,dy \\
&+ \int_Q \int_{Y_1} \Big[\mathbb{A}_{13}\big(\widetilde{y}, \partial_{x_3} v_1(t,x,\widetilde{y}) + h\phi_3(t,x,y)\big) - g_{13}(t,x,y)\Big] h\phi_3(t,x,y)\,dt\,dx\,dy \\
&+ \int_Q \int_{Y_2} \Big[\mathbb{A}_2\big(x, \nabla_x u_2 + \nabla_y v_2 + h\Phi(t,x,y)\big) - g_2(t,x,y)\Big] h\Phi(t,x,y)\,dt\,dx\,dy \\
&+ \int_Q \int_{Y_3} g_3(t,x,y)\Big[\mathbb{A}_3\big(x, \nabla_y v_3 + h\Phi(t,x,y)\big) - g_3(t,x,y)\Big] h\Phi(t,x,y)\,dt\,dx\,dy \\
&\geq \frac{1}{2}\int_\Omega c^\varepsilon(x) u^\varepsilon(T,x)^2 dx + \frac{1}{2}\int_\Omega \int_Y c(y)dy(u_0)^2 dx \\
&\quad + \liminf_{\varepsilon\to 0}\Big[\frac{1}{2}\int_\Omega c^\varepsilon(x) u^\varepsilon(T,x)^2 dx + \frac{1}{2}\int_\Omega \int_Y c(y)dy(u_0)^2 dx\Big].
\end{aligned}
$$

Thus dividing by h and letting $h \to 0$ we see that for every Φ,

$$
\begin{aligned}
&\frac{1}{\gamma^{1/p}} \int_Q \int_{Y_1} \Big[\widetilde{\mathbb{A}}_1\big(\widetilde{y}, \nabla_{\widetilde{y}} v_1(t,x,\widetilde{y})\big) - \widetilde{g}_1(t,x,y)\Big] h\widetilde{\Phi}(t,x,y)\,dt\,dx\,dy \\
&+ \int_Q \int_{Y_1} \Big[\mathbb{A}_{13}\big(\widetilde{y}, \partial_{x_3} v_1(t,x,\widetilde{y})\big) - g_{13}(t,x,y)\Big] h\phi_3(t,x,y)\,dt\,dx\,dy \\
&+ \int_Q \int_{Y_2} \Big[\mathbb{A}_2\big(y, \nabla_x u_2 + \nabla_y v_2\big) - g_2(t,x,y)\Big] h\Phi(t,x,y)\,dt\,dx\,dy \\
&+ \int_Q \int_{Y_3} g_3(t,x,y)\Big[\mathbb{A}_3\big(y, \nabla_y v_3\big) - g_3(t,x,y)\Big] h\Phi(t,x,y)\,dt\,dx\,dy \geq 0.
\end{aligned}
$$

Thus,

$$
\begin{gathered}
\langle \widetilde{g}_1(t,x,y)\rangle_{\mathrm{I}} = \widetilde{\mathbb{A}}_1(\widetilde{y}, \nabla_{\widetilde{y}} v_1(t,x,\widetilde{y})), \\
\langle g_{13}(t,x,y)\rangle_{\mathrm{I}} = \mathbb{A}_{13}(\widetilde{y}, \partial_{x_3} v_1(t,x,\widetilde{y})).
\end{gathered}
$$

Therefore we have proved the desired results. Now, to complete the proof of Theorem 5.1 we shall show the uniqueness of the solution following exactly the same lines as in the proof of Theorem 4.1.

6. Corrector results

Now, we prove corrector results for the gradients of temperature and the corresponding flux under the stronger hypotheses (A5')–(A6') of monotonicity. Let u^ε be the solution of the problem (1.3). Let v_1, u_2, v_2, v_3 be as in Theorem 4.1 (when $\gamma = 0$ we recall that $v_1(t,x,\widetilde{y}) = u_1(t,x)$). We define the sequences of functions

$$\begin{gathered}
\xi_1(t,x,y) := \chi_1(y)\nabla_{\widetilde{y},x_3} v_1(t,x,\widetilde{y}),\\
\xi_2(t,x,y) := \chi_2(y)(\nabla_x u_2(t,x) + \nabla_y v_2(t,x,y)),\\
\xi_3(t,x,y) := \chi_3(y)\nabla_y v_3(t,x,y),\\
\xi_k^\varepsilon(t,x) := \chi_k^\varepsilon(x)\xi_k^\varepsilon(t,x,\frac{x}{\varepsilon}),\ k = 1,2,3,\\
\mathbb{B}_1^\varepsilon(x,\xi) := \chi_1^\varepsilon(x)\begin{pmatrix} \frac{\mu}{\varepsilon}\widetilde{\mathbb{A}}_1^\varepsilon(\widetilde{x},\widetilde{\xi}) \\ \mathbb{A}_{13}^\varepsilon(\widetilde{x},\xi_3)\end{pmatrix},\\
\mathbb{B}_2^\varepsilon(x,\xi) := \chi_2^\varepsilon(x)\mathbb{A}_2^\varepsilon(\widetilde{x},\widetilde{\xi}),\\
\mathbb{B}_3^\varepsilon(x,\xi) := \chi_3^\varepsilon(x)\varepsilon^{p-1}\mathbb{A}_3^\varepsilon(\widetilde{x},\widetilde{\xi}),
\end{gathered} \tag{6.1}$$

where $\nabla_{\widetilde{y},x_3} = \begin{pmatrix}\nabla_{\widetilde{y}} \\ \partial_{x_3}\end{pmatrix}$. Note that $\mathbb{B}_k^\varepsilon$ satisfies the strong monotonicity condition (A5'), since $\frac{\mu}{\varepsilon} = \frac{\mu^{1/p}}{\varepsilon}\mu^{1/p'} \to 0$; thus, for example, we have

$$\begin{aligned}
|\mathbb{B}_3^\varepsilon(x,\xi) - \mathbb{B}_3^\varepsilon(x,\eta)| &= |\varepsilon^{p-1}\mathbb{A}_3^\varepsilon(x,\xi) - \varepsilon^{p-1}\mathbb{A}_3^\varepsilon(x,\eta)|\\
&\le K_1\varepsilon^{p-1}(|\xi| + |\eta|)^{p-2}|\xi - \eta|\\
&\le K_1(|\xi| + |\eta|)^{p-2}|\xi - \eta|.
\end{aligned} \tag{6.2}$$

In a similar manner, we get the same inequality for $\mathbb{B}_1^\varepsilon$ and $\mathbb{B}_2^\varepsilon$.

Theorem 6.1. *If the functions, $\nabla_{\widetilde{y}} v_1$, $\nabla_y v_2$ and $\nabla_y v_3$ are admissible (cf. Definition 2.6), then*

$$\limsup_{\varepsilon\searrow 0}\|\chi_1^\varepsilon(\nabla_x^\varepsilon u^\varepsilon - \xi_1^\varepsilon)\|_{L^p(Q)} = 0, \tag{6.3}$$

$$\limsup_{\varepsilon\searrow 0}\|\chi_2^\varepsilon(\nabla_x u - \xi_2^\varepsilon)\|_{L^p(Q)} = 0, \tag{6.4}$$

$$\limsup_{\varepsilon\searrow 0}\|\chi_3^\varepsilon(\varepsilon\nabla_x u^\varepsilon - \xi_3^\varepsilon)\|_{L^p(Q)} = 0. \tag{6.5}$$

$$\limsup_{\varepsilon\searrow 0}\|\chi_k^\varepsilon(\mathbb{B}_k^\varepsilon(x,\nabla_x u^\varepsilon) - \mathbb{B}_k^\varepsilon(x,\xi_k^\varepsilon))\|_{L^{p'}(Q)} = 0, \quad k = 1,2,3. \tag{6.6}$$

Where ∇^ε is defined by (5.6).

Let us mention that the convergence (6.3)-(6.6) means that, under hypotheses (A5')-(A6') and $\nabla_{\widetilde{y}} v_1$, $\nabla_y v_2$ and $\nabla_y v_3$ are admissible, the oscillations of the sequences, in the above, are all contained in the corresponding two-scale limits. Moreover, the proof of this theorem is motivated by the approach based on the two-scale convergence.

Now, let us introduce some more notation, functions and quantities which we will use hereafter. We will use M to denote a generic constant which does not depend on ε, but probably on p, K_1, K_2, c_0, c and the $L^{p'}$ (resp. L^p) norm of the data f (resp. u_0). Let $\kappa \in]0,1[$ be a constant and $\psi_k(t,x,y)$, $k=1,2,3$, be admissible test functions such that

$$\|\nabla_{\widetilde{y},x_3} v_1 - \psi_1\|^p_{L^p(Q\times Y_1)} + \sum_{\alpha=2}^{3} \|\nabla_y v_\alpha - \psi_\alpha\|^p_{L^p(Q\times Y_\alpha)} \le \kappa. \tag{6.7}$$

Define the following functions:

$$\begin{gathered}\eta_1^\varepsilon(t,x) := \chi_1^\varepsilon(x)(\nabla_x^\varepsilon u^\varepsilon(t,x) + \psi_1(t,x,\frac{x}{\varepsilon})),\\ \eta_2^\varepsilon(t,x) := \chi_2^\varepsilon(x)(\nabla_x u^\varepsilon(t,x) + \psi_2(t,x,\frac{x}{\varepsilon})),\\ \eta_3^\varepsilon(t,x) := \chi_3^\varepsilon(x)\psi_3(t,x,\frac{x}{\varepsilon}).\end{gathered} \tag{6.8}$$

Note that the functions η_k^ε and $\mathbb{B}_k^\varepsilon(x,\eta_k^\varepsilon)$, $k=1,2,3$ arise from admissible test functions and we have the following two-scale convergence (cf. Lemma 3.3):

$$\begin{gathered}\eta_1^\varepsilon(t,x) \overset{2s,p}{\to} \eta_1(t,x,y) := \chi_1(y)(\nabla_{\widetilde{y},x_3} v_1(t,x,y) + \psi_1(t,x,y)),\\ \eta_2^\varepsilon(t,x) \overset{2s,p}{\to} \eta_2(t,x,y) := \chi_2(y)(\nabla_x u_2(t,x) + \psi_2(t,x,y)),\\ \eta_3^\varepsilon(t,x) \overset{2s,p}{\to} \eta_3(t,x,y) := \chi_3(y)\psi_3(t,x,y),\\ \mathbb{B}_k^\varepsilon(x,\eta_k^\varepsilon) \overset{2s,p'}{\to} \chi_k(y)\mathbb{B}_k(y,\eta_k(t,x,y)),\ k=1,2,3,\end{gathered}$$

where $\mathbb{B}_1(y,\eta) = \begin{pmatrix}\nu\widetilde{\mathbb{A}}_1(\widetilde{y},\widetilde{\eta})\\ \mathbb{A}_{13}(\widetilde{y},\eta_{1N})\end{pmatrix}$, ($\nu=0$ if $\gamma=0$ and $\nu=\frac{1}{\gamma^{1/p}}$ else), $\mathbb{B}_2=\mathbb{A}_2$ and $\mathbb{B}_3=\mathbb{A}_3$.

Lemma 6.2.

$$\sum_{k=1}^{3} \|\xi_k\|^p_{L^p(Q\times Y_k)} \le \frac{c}{c_0}(\|f\|^{p'}_{L^{p'}(Q)} + \|u_0\|^p_{L^p(\Omega)}).$$

The proof of the above lemma follows from the identity in Lemma 4.3 and assumption (A5).

Lemma 6.3. *Let ξ_k, η_k, $\mathbb{B}_k^\varepsilon$, ξ_k^ε, η_k^ε, $\mathbb{B}_k$, $k=1,2,3$ be functions as defined above. Then*

$$\begin{gathered}\limsup_{\varepsilon\searrow 0}\int_{Q_1^\varepsilon}(\mathbb{B}_1^\varepsilon(x,\nabla_x u^\varepsilon) - \mathbb{B}_1^\varepsilon(x,\eta_1^\varepsilon)).(\nabla^\varepsilon u^\varepsilon - \eta_1)\,dx\,dt \le \mathbb{E},\\ \limsup_{\varepsilon\searrow 0}\int_{Q_2^\varepsilon}(\mathbb{B}_2^\varepsilon(x,\nabla_x u^\varepsilon) - \mathbb{B}_2^\varepsilon(x,\eta_2^\varepsilon)).(\nabla u^\varepsilon - \eta_2)\,dx\,dt \le \mathbb{E},\\ \limsup_{\varepsilon\searrow 0}\int_{Q_3^\varepsilon}(\mathbb{B}_3^\varepsilon(x,\nabla_x u^\varepsilon) - \mathbb{B}_3^\varepsilon(x,\eta_3^\varepsilon)).(\varepsilon\nabla u^\varepsilon - \eta_3)\,dx\,dt \le \mathbb{E},\end{gathered}$$

where

$$\mathbb{E} := \sum_{k=1}^{3}\int_{Q\times Y_k}(\mathbb{B}_k(y,\xi_k) - \mathbb{B}_k(y,\eta_k)).(\xi_k - \eta_k)dy\,dx\,dt.$$

Proof. Firstly, we denote the integrals appearing in the left-side of the above inequalities by $\mathbb{E}_1^\varepsilon$, $\mathbb{E}_2^\varepsilon$ and $\mathbb{E}_3^\varepsilon$ respectively. Secondly, we put

$$\mathbb{D}^\varepsilon(T) = \int_\Omega c^\varepsilon(x) u^\varepsilon(T,x)^2 dx, \quad \mathbb{D}^\varepsilon(0) = \int_\Omega c^\varepsilon(x) u_0(x)^2 dx,$$

$$\mathbb{D}^0(T) = \sum_{\alpha=1}^2 \widetilde{c}_\alpha \int_\Omega |u_\alpha(T,x)|^2 dx + \int_\Omega \int_{Y_3} c_3(y) |v_3(T,x,y)|^2 \,dx\,dy$$

$$\mathbb{D}^0(0) = \sum_{k=1}^3 \widetilde{c}_k \int_\Omega |u_0(x)|^2 dx.$$

Then, for $k = 1,2,3$, using (3.7), we obtain

$$\begin{aligned}
\mathbb{E}_k^\varepsilon \le \sum_{j=1}^3 \mathbb{E}_j^\varepsilon &= \int_Q f(t,x) u^\varepsilon(t,x)\,dt\,dx + \frac{1}{2}\mathbb{D}^\varepsilon(0) - \frac{1}{2}\mathbb{D}^\varepsilon(T) \\
&- \sum_{j=1}^3 \int_{Q_3^\varepsilon} \mathbb{B}_j^\varepsilon(x, \nabla_x u^\varepsilon).\eta_j^\varepsilon \,dt\,dx - \int_{Q_1^\varepsilon} \mathbb{B}_1^\varepsilon(x,\eta_1^\varepsilon).(\nabla_x^\varepsilon u^\varepsilon - \eta_1^\varepsilon)\,dt\,dx \\
&- \int_{Q_2^\varepsilon} \mathbb{B}_2^\varepsilon(x,\eta_2^\varepsilon).(\nabla_x u^\varepsilon - \eta_2^\varepsilon)\,dt\,dx - \int_{Q_3^\varepsilon} \mathbb{B}_3^\varepsilon(x,\eta_3^\varepsilon).(\varepsilon\nabla_x u^\varepsilon - \eta_3^\varepsilon)\,dt\,dx.
\end{aligned}$$

Now, using the two-scale convergence, we deduce

$$\begin{aligned}
\limsup_{\varepsilon \searrow 0} \sum_{j=1}^3 \mathbb{E}_j^\varepsilon \le \int_{Q\times Y} & f(t,x) U(t,x)\,dt\,dx\,dy + \frac{1}{2}\mathbb{D}^0(0) - \liminf_{\varepsilon\searrow 0} \frac{1}{2}\mathbb{D}^\varepsilon(T) \\
&- \sum_{j=1}^3 \int_{Q\times Y_j} \mathbb{B}_j(y,\xi_j).\eta_j \,dt\,dx\,dy \\
&- \sum_{j=1}^3 \int_{Q\times Y_j} \mathbb{B}_j(y,\eta_j).(\xi_j - \eta_j)\,dt\,dx\,dy.
\end{aligned}$$

The right-hand side can be written as

$$\begin{aligned}
&\int_{Q\times Y} f(t,x) U(t,x)\,dt\,dx\,dy + \frac{1}{2}\mathbb{D}^0(0) - \liminf_{\varepsilon\searrow 0} \frac{1}{2}\mathbb{D}^\varepsilon(T) \\
&- \sum_{j=1}^3 \int_{Q\times Y_j} \mathbb{B}_j(y,\xi_j).\xi_j \,dt\,dx\,dy \\
&+ \sum_{j=1}^3 \int_{Q\times Y_j} (\mathbb{B}_j(y,\xi_j) - \mathbb{B}_j(y,\eta_j)).(\xi_j - \eta_j)\,dt\,dx\,dy.
\end{aligned}$$

From the energy identity (cf. Lemma 4.3) we obtain

$$\begin{aligned}
&\sum_{j=1}^3 \int_{Q\times Y_j} \mathbb{B}_j(y,\xi_j).\xi_j \,dt\,dx\,dy \\
&= \int_{Q\times Y} f(t,x) U(t,x)\,dt\,dx\,dy + \frac{1}{2}\mathbb{D}^0(0) + \frac{1}{2}\mathbb{D}^0(T)
\end{aligned}$$

$$\geq \int_{Q\times Y} f(t,x)U(t,x)\,dt\,dx\,dy + \frac{1}{2}\mathbb{D}^0(0) - \liminf_{\varepsilon\searrow 0}\frac{1}{2}\mathbb{D}^\varepsilon(T).$$

This completes the proof. □

Lemma 6.4. *Let ξ_k, η_k, κ be as defined above. Then*

$$\mathbb{E} := \sum_{k=1}^{3}\int_{Q\times Y_k} (\mathbb{B}_k(y,\xi_k) - \mathbb{B}_k(y,\eta_k).(\xi_k-\eta_k)dy\,dx\,dt \leq M\kappa^{2/p}.$$

Proof. Using (A5') and Hölder's inequality,

$$\begin{aligned}
\mathbb{E} &\leq \sum_{k=1}^{3}\int_{Q\times Y_k} (\mathbb{B}_k(y,\xi_k) - \mathbb{B}_k(y,\eta_k)(\xi_k-\eta_k)dy\,dx\,dt\\
&\leq K_1\sum_{k=1}^{3}\int_{Q\times Y_k} (|\xi_k|+|\eta_k|)^{p-2}|\xi_k-\eta_k|^2 dy\,dx\,dt\\
&\leq K_1\sum_{k=1}^{3}\Big(\int_{Q\times Y_k} (|\xi_k|+|\eta_k|)^{p} dy\,dx\,dt\Big)^{\frac{p-2}{p}}\|\xi_k-\eta_k\|_{L^p}^2\\
&\leq K_1\sum_{k=1}^{3}(\|\xi_k\|_{L^p}+\|\eta_k\|_{L^p})^{p-2}\|\xi_k-\eta_k\|_{L^p}^2\\
&\leq K_1\Big(\sum_{k=1}^{3}(\|\xi_k\|_{L^p}+\|\eta_k\|_{L^p})^{p}\Big)^{\frac{p-2}{p}}\Big(\sum_{k=1}^{3}(\|\xi_k-\eta_k\|_{L^p}^p)\Big)^{2/p}\\
&\leq K_1\Big(\sum_{k=1}^{3}(2\|\xi_k\|_{L^p}+\|\xi_k-\eta_k\|_{L^p})^{p}\Big)^{\frac{p-2}{p}}\Big(\sum_{k=1}^{3}(\|\xi_k-\eta_k\|_{L^p}^p)\Big)^{2/p}\\
&\leq K_1\Big(\sum_{k=1}^{3}2^p(2^p\|\xi_k\|_{L^p}^p+\|\xi_k-\eta_k\|_{L^p}^p)\Big)^{\frac{p-2}{p}}\Big(\sum_{k=1}^{3}(\|\xi_k-\eta_k\|_{L^p}^p)\Big)^{2/p}.
\end{aligned}$$

By the estimate proved in Lemma 6.2, we deduce the result. □

Now, we prove some preliminary corrector results.

Theorem 6.5. *Under the same assumption as in Theorem 6.1, we have:*

$$\limsup_{\varepsilon\searrow 0}\|\chi_1^\varepsilon\nabla_x^\varepsilon u^\varepsilon - \eta_1^\varepsilon\|_{L^p(Q)} \leq M\kappa^{2/p}, \tag{6.9}$$

$$\limsup_{\varepsilon\searrow 0}\|\chi_2^\varepsilon\nabla_x u^\varepsilon - \eta_2^\varepsilon\|_{L^p(Q)} \leq M\kappa^{2/p}, \tag{6.10}$$

$$\limsup_{\varepsilon\searrow 0}\|\chi_3^\varepsilon\varepsilon\nabla_x u^\varepsilon - \eta_3^\varepsilon\|_{L^p(Q)} \leq M\kappa^{2/p}. \tag{6.11}$$

$$\limsup_{\varepsilon\searrow 0}\|\mathbb{B}_k^\varepsilon(x,\nabla_x u^\varepsilon) - \mathbb{B}_k^\varepsilon(x,\eta_k^\varepsilon)\|_{L^{p'}(Q)} \leq M\kappa^{2/(p-1)}, \quad k=1,2,3. \tag{6.12}$$

Proof. From (6.2), we obtain

$$|\chi_1^\varepsilon\nabla_x^\varepsilon u^\varepsilon(t,x) - \eta_1^\varepsilon(t,x)|^p \leq \frac{1}{K_2}(\mathbb{B}_1^\varepsilon(x,\nabla_x^\varepsilon u^\varepsilon) - \mathbb{B}_1^\varepsilon(x,\eta_1^\varepsilon)).(\nabla_x^\varepsilon u^\varepsilon - \eta_1^\varepsilon),$$

$$|\chi_2^\varepsilon\nabla_x u^\varepsilon(t,x) - \eta_2^\varepsilon(t,x)|^p \leq \frac{1}{K_2}(\mathbb{B}_2^\varepsilon(x,\nabla_x u^\varepsilon) - \mathbb{B}_3^\varepsilon(x,\eta_3^\varepsilon)).(\nabla_x u^\varepsilon - \eta_3^\varepsilon),$$

$$|\chi_3^\varepsilon \varepsilon \nabla_x u^\varepsilon(t,x) - \eta_3^\varepsilon(t,x)|^p \le \frac{1}{K_2}(\mathbb{B}_3^\varepsilon(x,\nabla_x u^\varepsilon) - \mathbb{B}_3^\varepsilon(x,\eta_3^\varepsilon)).(\varepsilon\nabla_x u^\varepsilon - \eta_3^\varepsilon).$$

Therefore,

$$\|\chi_1^\varepsilon \nabla_x u^\varepsilon - \eta_1^\varepsilon\|^p_{L^p(Q_1^\varepsilon)} \le \frac{1}{K_2}\int_{Q_1^\varepsilon}(\mathbb{B}_1^\varepsilon(x,\nabla_x^\varepsilon u^\varepsilon) - \mathbb{B}_1^\varepsilon(x,\eta_1^\varepsilon)).(\nabla_x^\varepsilon u^\varepsilon - \eta_1^\varepsilon)\,dt\,dx,$$
$$\|\chi_2^\varepsilon \nabla_x u^\varepsilon - \eta_2^\varepsilon\|^p_{L^p(Q_2^\varepsilon)} \le \frac{1}{K_2}\int_{Q_2^\varepsilon}(\mathbb{B}_2^\varepsilon(x,\nabla_x u^\varepsilon) - \mathbb{B}_2^\varepsilon(x,\eta_2^\varepsilon)).(\nabla_x u^\varepsilon - \eta_2^\varepsilon)\,dt\,dx,$$
$$\|\chi_3^\varepsilon \varepsilon\nabla_x u^\varepsilon - \eta_3^\varepsilon\|^p_{L^p(Q_3^\varepsilon)} \le \frac{1}{K_2}\int_{Q_3^\varepsilon}(\mathbb{B}_3^\varepsilon(x,\varepsilon\nabla_x u^\varepsilon) - \mathbb{B}_3^\varepsilon(x,\eta_3^\varepsilon)).(\varepsilon\nabla_x u^\varepsilon - \eta_3^\varepsilon)\,dt\,dx.$$

Now, let

$$\mathbb{G}^\varepsilon := \|\chi_1^\varepsilon \nabla_x^\varepsilon u^\varepsilon - \eta_1^\varepsilon\|^p_{L^p(Q_1^\varepsilon)} + \|\chi_2^\varepsilon \nabla_x u^\varepsilon - \eta_2^\varepsilon\|^p_{L^p(Q_2^\varepsilon)} + \|\chi_3^\varepsilon \varepsilon\nabla_x u^\varepsilon - \eta_3^\varepsilon\|^p_{L^p(Q_3^\varepsilon)},$$

and

$$\begin{aligned}\mathbb{F}^\varepsilon := &\int_{Q_1^\varepsilon}(\mathbb{B}_1^\varepsilon(x,\nabla_x^\varepsilon u^\varepsilon) - \mathbb{B}_1^\varepsilon(x,\eta_1^\varepsilon)).(\nabla_x^\varepsilon u^\varepsilon - \eta_1^\varepsilon)\,dt\,dx\\ &+ \int_{Q_2^\varepsilon}(\mathbb{B}_2^\varepsilon(x,\nabla_x u^\varepsilon) - \mathbb{B}_2^\varepsilon(x,\eta_2^\varepsilon)).(\nabla_x u^\varepsilon - \eta_2^\varepsilon)\,dt\,dx\\ &+ \int_{Q_3^\varepsilon}(\mathbb{B}_3^\varepsilon(x,\varepsilon\nabla_x u^\varepsilon) - \mathbb{B}_3^\varepsilon(x,\eta_3^\varepsilon)).(\nabla_x u^\varepsilon - \eta_3^\varepsilon)\,dt\,dx.\end{aligned}$$

From th above estimates, we have $\mathbb{G}^\varepsilon \le \mathbb{F}^\varepsilon/c_0$. Therefore, passing to the limit-sup as $\varepsilon \to 0$ and using Lemmas 6.3 and 6.4, we obtain

$$\limsup_{\varepsilon\searrow 0} \mathbb{G}^\varepsilon \le M\kappa^{2/p}.$$

This concludes the proof of (6.9)-(6.11). To achieve the proof, we will only prove the estimate (6.12) for $k = 1$, the others are proved in a similar manner. Let $q = p' = \frac{p}{p-1}$, then by (6.2) we have

$$\begin{aligned}&\int_{Q_1^\varepsilon}|\mathbb{B}_1^\varepsilon(x,\nabla_x^\varepsilon u^\varepsilon) - \mathbb{B}_1^\varepsilon(x,\eta_1^\varepsilon)|^q\,dt\,dx\\ &\le K_1\int_{Q_1^\varepsilon}(|\nabla_x^\varepsilon u^\varepsilon| + |\eta_1^\varepsilon|)^{(p-2)q}|\nabla_x^\varepsilon u^\varepsilon - \eta_1^\varepsilon|^q\,dt\,dx.\end{aligned}$$

Since

$$\frac{1}{p-1} + \frac{p-2}{p-1} = 1,$$

by Hölder's inequality,

$$\begin{aligned}&\int_{Q_1^\varepsilon}|\mathbb{B}_1^\varepsilon(x,\nabla_x^\varepsilon u^\varepsilon) - \mathbb{B}_1^\varepsilon(x,\eta_1^\varepsilon)|^q\,dt\,dx\\ &\le K_1\Big(\int_{Q_1^\varepsilon}(|\nabla_x^\varepsilon u^\varepsilon| + |\eta_1^\varepsilon|)^p\,dt\,dx\Big)^{\frac{p-2}{p-1}}\Big(\int_{Q_1^\varepsilon}|\nabla_x^\varepsilon u^\varepsilon - \eta_1^\varepsilon|^p\,dt\,dx\Big)^{\frac{1}{p-1}}\\ &\le M\|\chi_1^\varepsilon(\nabla_x^\varepsilon u^\varepsilon - \eta_1^\varepsilon)\|^q_{L^p(Q)}.\end{aligned}$$

Now, using (6.9), we obtain the desired estimate. □

Proof of Theorem 6.1. Since the functions $\nabla_y v_2$ and $\nabla_y v_3$ are assumed to be admissible test functions, we can choose $\psi_1 = \nabla_y v_2$ and $\psi_2 = \nabla_y v_3$. Therefore, κ can be taken arbitrarily small and thus, Theorem 6.1 follows from Theorem 6.5. □

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References

[1] G. Allaire; *Homogenization and two-scale convergence*, SIAM J. Math. Anal., 23 (6) (1992) 1482-1518.

[2] M. Bellieud and G. Bouchitté; *Homogenization of a soft elastic material reinforced by fibers*, Asymptotic Analysis, 32 (2002) 153-183.

[3] A. Boughammoura; *Homogenization of a degenerate parabolic problem in a highly heterogeneous with highly anisotropic fibers*, Math. Comp. Modelling, 49 (2009) 66-79.

[4] M. Briane; *Non-local Homogenization in some weakly connected domains*, Ricerche Mat., 47 (1987) 51-94.

[5] D. Caillerie, B. Dinari; *A perturbation problem with two small parameters in the framework of the heat conduction of a fiber reinforced body*, Partial Differential equations, 19 (1987) 59-78.

[6] M. Mabrouk; *Homogenization of a non-linear degenerate parabolic problem in a highly heterogeneous periodic medium*, Math. Meth. Appl. Sci., 28 (2005) 1141-1171.

[7] M. Mabrouk, A. Boughammoura; *Homogenization of a degenerate parabolic problem in a highly heterogeneous thin structure*, Appl. Math. Modelling, 32 (2008) 2731-2752.

[8] M. Mabrouk, A. Boughammoura; *Homogenization of a degenerate parabolic problem in a highly heteregeneous medium*, C. R. Mecanique, 331 (2003) 415-421.

[9] M. Mabrouk, H. Samadi; *Homogenization of a heat transfer problem in a highly heterogeneous periodic medium*, Internat. J. Engrg. Sci., 40 (2002) 1233-1250.

[10] R. E. Schowalter, D. Visarraga; *Double-Diffusion Models from a Highly-Heterogeneous Medium*, Jour. Math. Anal. Appl., 295 (2004), 191-210.

[11] G. Nguetseng; *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal., 20 (1989) 608-623.

[12] R. E. Schowalter; *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Mathematical Surveys and Monographs, AMS, Providence, 49 (1997).

[13] A. Sili; *Homogenization of a non linear monotone problem in an anisotropic medium*, Math. Mod. Meth. Appl. Sci., 14 (2004) 329-353.

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