GLOBAL CONTINUUM AND MULTIPLE POSITIVE SOLUTIONS TO A P-LAPLACIAN BOUNDARY-VALUE PROBLEM

CHAN-GYUN KIM, JUNPING SHI

ABSTRACT. A p-Laplacian boundary-value problem with positive nonlinearity is considered. The existence of a continuum of positive solutions emanating from $(\lambda, u) = (0, 0)$ is shown, and it can be extended to $\lambda = \infty$. Under an additional condition on the nonlinearity, it is shown that the positive solution is unique for any $\lambda > 0$; thus the continuum $\mathcal{C}$ is indeed a continuous curve globally defined for all $\lambda > 0$. In addition, by the upper and lower solutions method, existence of three positive solutions is established under some conditions on the nonlinearity.

1. INTRODUCTION

Consider a boundary-value problem

$$(w(t)\varphi_p(u'(t)))' + \lambda h(t)f(u(t)) = 0, \quad t \in (0, 1),$$
$$u(0) = u(1) = 0, \tag{1.1}$$

where $\varphi_p(x) := |x|^{p-2}x$, $p > 1$ and λ is a nonnegative parameter. For the functions in (1.1), throughout the paper, we assume that the following hypotheses hold:

(H1) $h \in C([0, 1], (0, \infty))$;

(W1) $w \in C([0, 1], [0, \infty))$, $w(t) > 0$ for $t \in (0, 1)$ and $\varphi_p^{-1}(1/w) \in L^1(0, 1)$;

(F1) $f \in C([0, \infty), (0, \infty))$, and $\lim_{u \to \infty} \frac{f(u)}{\varphi_p(u)} = 0$.

Our main result is that under the above conditions for the weight functions w, h, and the nonlinearity f, a continuum $\mathcal{C}$ of positive solutions of (1.1) emanates from $(\lambda, u) = (0, 0)$, and $\mathcal{C}$ can be extended to $\lambda = \infty$. Thus we establish the existence of at least one positive solution of (1.1) for any $\lambda > 0$. Under an additional condition

(F2) $\frac{f(u)}{\varphi_p(u)}$ is strictly decreasing on $(0, \infty)$,

it is shown that the positive solution of (1.1) is unique for any $\lambda > 0$, thus the continuum $\mathcal{C}$ is indeed a continuous curve globally defined for all $\lambda > 0$ in this case. On the other hand, we show that under some different conditions for f, problem (1.1) has at least three positive solutions, which indicates that the bifurcation diagram for (1.1) cannot be a monotone curve in this case. These results demonstrate the rich structure of the solution set of (1.1).

Problem (1.1) or similar equations arises from mathematical models of chemical catalysis [2, 9, 28], combustion theory [27, 30], and ecological modeling [15, 20]. Common to this type of models is that the equation may possess multiple positive steady state solutions, and the bifurcation diagram of the positive steady states may have multiple turning points (saddle-node bifurcation points). For the reaction-diffusion case ($p = 2$), many results on multiple positive solutions and S-shaped bifurcation diagrams have been obtained, see for example [1, 2, 3, 9, 10, 11]. Moreover for some cases, careful analysis can lead to exact multiplicity of solutions and exact S-shaped bifurcation diagrams, see for example, [4, 17, 18, 26, 27, 28].

The p-Laplacian equation (1.1) is considerably harder than the corresponding reaction-diffusion equation (which is the special case of $p = 2$). The mathematical difficulty comes from the nonlinearity and degeneracy of the differential operator and the failure of linearization and continuation technique in many places. While in general fewer existence and multiplicity results have been proved for p-Laplacian equations, we mention the nice work in [5, 7, 8, 22] as examples, and a nice survey of earlier work was given in [25]. For one-dimensional case, more precise results can be established as several exact multiplicity of solutions and exact S-shaped bifurcation diagrams have been shown in [13, 29] recently for f satisfying (F1), and similar uniqueness result when f also satisfies (F2) has been also proved in [24]. But there are very few results for the existence of three positive solutions when the problem is spatially non-homogenous as the one we consider here, and the nature of global continuum of the positive solutions has not been exploited before either. In a recent work [16] by the first author, an existence result for the three positive solutions was proved for nondecreasing nonlinearity f with p-sublinear at infinity, and in this paper we consider non-monotone nonlinearity f with a positive falling zero as well as p-sublinear at infinity.

The rest of this article is organized as follows. In Section 2, we introduce some preliminary results such as definitions and lemmas. In Section 3, our main results are given and also an example to illustrate our result is presented.

2. Preliminaries

First we recall the general framework for the solutions of (1.1). Consider the Banach space
$$C_w[0, 1] = \{u \in C[0, 1] \cap C^1(0, 1) : w\varphi_p(u') \in C[0, 1]\}$$
with norm
$$\|u\|_w = \|u\|_\infty + \|w^{1/(p-1)}u'\|_\infty,$$
where $\|v\|_\infty = \max_{t \in [0,1]} |v(t)|$. Note that if $w(t) > 0$ for $t \in [0, 1]$, then $(C_w[0, 1], \| \cdot \|_w)$ is equivalent to $(C^1[0, 1], \| \cdot \|_1)$. Here $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$. We set $\mathcal{K}_w = \{u \in C_w[0, 1] : u(t) \geq 0, \ t \in [0, 1]\}$. Then $\mathcal{K}_w$ is an ordered cone in $C_w[0, 1]$, and let $L^1_+(0, 1)$ denote the set of nonnegative Lebesgue-integrable functions on $[0, 1]$.

To define the solution operator to (1.1), let us consider the p-Laplacian problem
$$(w(t)\varphi_p(u'(t)))' + K(t) = 0, \ a.e. \ in \ (0, 1),$$
$$u(0) = u(1) = 0, \tag{2.1}$$
where $K \in L^1_+(0, 1)$.

By a solution u of problem (2.1), we understand a function $u \in C_w[0, 1]$ with $w\varphi_p(u') \in AC[0, 1]$ which satisfies (2.1). We recall the following lemmas from [16].

Lemma 2.1 ([16]). *Assume* (W1) *is satisfied, and let u be the solution of* (2.1) *with $K > 0$, a.e. in $[0, 1]$. Then there exists a unique point $A_u \in (0, 1)$ such that $(w^{1/(p-1)}u')(t) > 0$, $t \in [0, A_u)$, $u'(A_u) = 0$ and $(w^{1/(p-1)}u')(t) < 0$, $t \in (A_u, 1]$.*

Lemma 2.2 ([16]). *Assume* (W1) *is satisfied, and let B and C be positive constants such that $B < C$. Then there exists $\epsilon = \epsilon(C/B) > 0$ such that $A_u \in (2\epsilon, 1 - 2\epsilon)$ for all possible solutions u of* (2.1) *with $K \in K_B^C$, where*

$$K_B^C = \{K \in L^1(0, 1) : K \geq B, \int_0^1 K(s)ds \leq C\}$$

and A_u is the unique point in Lemma 2.1 such that $u'(A_u) = 0$.

Remark 2.3. If w and K are symmetric with respect to $1/2$, one can easily see that the unique solution u_K of (2.1) is symmetric with respect to $1/2$. Hence $A_{u_K} = 1/2$ in this case.

Next we show that (2.1) and (1.1) can be rewritten into some integral forms. Problem (2.1) can be equivalently written as

$$u(t) = G_p(K)(t) := \int_0^t \varphi_p^{-1}\Big[\frac{1}{w(s)}\Big(c(K) + \int_s^1 K(\tau)d\tau\Big)\Big]ds, \quad t \in [0, 1],$$

where $c : L^1(0, 1) \to \mathbb{R}$ is a mapping satisfying

$$\int_0^1 \varphi_p^{-1}\Big[\frac{1}{w(s)}\Big(c(K) + \int_s^1 K(\tau)d\tau\Big)\Big]ds = 0.$$

It can be proved that the mapping c is continuous and it maps bounded sets in $L^1(0, 1)$ into bounded sets in $\mathbb{R}$, and the mapping $G_p : L_+^1(0, 1) \to \mathcal{K}_w$ is continuous and it maps equi-integrable sets of $L_+^1(0, 1)$ into relatively compact sets of $\mathcal{K}_w$ by similar arguments as in the previous result (e.g., see [6, 12, 23]). Define $H : [0, \infty) \times \mathcal{K}_w \to L_+^1(0, 1)$ by $H(\lambda, u)(t) = \lambda h(t)f(u(t))$. Then it is well known that H is a continuous operator which maps bounded sets of $[0, \infty) \times \mathcal{K}_w$ into equi-integrable sets of $L_+^1(0, 1)$. Thus $T = G_p \circ H : [0, \infty) \times \mathcal{K}_w \to \mathcal{K}_w$ is completely continuous. Furthermore, (1.1) has a positive solution u if and only if $T(\lambda, \cdot)$ has a fixed point u in $\mathcal{K}_w \setminus \{0\}$ for $\lambda > 0$.

Now we recall a well-known theorem for the existence of a global continuum of solutions to $T(\lambda, u) = 0$:

Theorem 2.4 ([31, Corollary 14.12]). *Let X be a Banach space with $X \neq \{0\}$ and let $\mathcal{K}$ be an ordered cone in X. Consider*

$$x = T(\mu, x), \tag{2.2}$$

where $\mu \in [0, \infty)$ and $x \in \mathcal{K}$. If $T : [0, \infty) \times \mathcal{K} \to \mathcal{K}$ is completely continuous and $T(0, x) = 0$ for all $x \in \mathcal{K}$. Then the solution component $\mathcal{C}$ of (2.2) *in $[0, \infty) \times \mathcal{K}$ which contains $(0, 0)$ is unbounded.*

Since $T(0, u) = 0$ and $T(\lambda, 0) \neq 0$ if $\lambda > 0$, by Theorem 2.4, we obtain the following proposition.

Proposition 2.5. *Assume* (H1), (W1), (F1) *are satisfied. Then there exists an unbounded continuum $\mathcal{C}$ of positive solutions for* (1.1) *emanating from $(0, 0)$ in $[0, \infty) \times \mathcal{K}_w$.*

Next we review the notion of order in $C_w[0,1]$, and also the upper and lower solutions of (1.1), which was introduced in [16].

Definition 2.6. Given functions $u, v, w : [0,1] \to \mathbb{R}$, we say that

(1) $u \le v$ if for all $t \in [0,1]$, $u(t) \le v(t)$; and
(2) $u \in [v,w]$ if $v \le u \le w$.

Definition 2.7. Let $u, v \in C_w[0,1]$. We say that $u \prec v$ if and only if the following three conditions hold:

(i) for all $t \in (0,1)$, $u(t) < v(t)$,
(ii) either $u(0) < v(0)$ or $(w^{1/(p-1)}u')(0) < (w^{1/(p-1)}v')(0)$,
(iii) either $u(1) < v(1)$ or $(w^{1/(p-1)}u')(1) > (w^{1/(p-1)}v')(1)$.

Definition 2.8. A function $\alpha \in C_w[0,1]$ with $w\varphi_p(\alpha')$ absolutely continuous is called a lower solution of (1.1) if

(i) for a.e. $t \in (0,1)$, $(w(t)\varphi_p(\alpha'(t)))' + \lambda h(t)f(\alpha(t)) \ge 0$,
(ii) $\alpha(0) \le 0$ and $\alpha(1) \le 0$.

In the same way we define an upper solution of (1.1) by reversing the above inequalities.

Definition 2.9. A function $\alpha \in C_w[0,1]$ with $w\varphi_p(\alpha')$ absolutely continuous is called a strict lower solution of (1.1) if it is not a solution, $\alpha(0) \le 0$, $\alpha(1) \le 0$ and

(i) for any $t_0 \in (0,1)$, there exist an open interval $I_0 \subseteq (0,1)$ and $\epsilon_0 > 0$ such that $t_0 \in I_0$ and for a.e. $t \in I_0$, for all $u \in [\alpha(t), \alpha(t) + \epsilon_0]$, one has $(w(t)\varphi_p(\alpha'(t)))' + \lambda h(t)f(u) \ge 0$,
(ii) either $\alpha(0) < 0$ (respectively, $\alpha(1) < 0$) or there exist $\delta_0 > 0$ and $c_0 \in C_w[0,1]$ with $c_0 \succ 0$ such that if $I_0 = [0, \delta_0)$ (respectively, $I_0 = (1 - \delta_0, 1]$), then for a.e. $t \in I_0$, for all $u \in [\alpha(t), \alpha(t) + c_0(t)]$, one has $(w(t)\varphi_p(\alpha'(t)))' + \lambda h(t)f(u) \ge 0$.

In a similar way, a function $\beta \in C_w[0,1]$ with $w\varphi_p(\beta')$ absolutely continuous is a strict upper solution of (1.1) if it is not a solution, $\beta(0) \ge 0$, $\beta(1) \ge 0$ and

(i) for any $t_0 \in (0,1)$, there exist an open interval $I_0 \subseteq (0,1)$ and $\epsilon_0 > 0$ such that $t_0 \in I_0$ and, for a.e. $t \in I_0$, for all $u \in [\beta(t) - \epsilon_0, \beta(t)]$, one has $(w(t)\varphi_p(\beta'(t)))' + \lambda h(t)f(u) \le 0$,
(ii) either $\beta(0) > 0$ (respectively, $\beta(1) > 0$) or there exist $\delta_0 > 0$ and $c_0 \in C_w[0,1]$ with $c_0 \succ 0$ such that if $I_0 = [0, \delta)$ (respectively, $I_0 = (1 - \delta_0, 1]$), then for a.e. $t \in I_0$, for all $u \in [\beta(t) - c_0(t), \beta(t)]$, one has $(w(t)\varphi_p(\beta'(t)))' + \lambda h(t)f(u) \le 0$.

With these definition, the following existence results were proved in [16].

Theorem 2.10 ([16]). *Assume that α and β are lower and upper solutions of (1.1) respectively such that $\alpha \le \beta$. Then (1.1) has at least one solution u such that $\alpha \le u \le \beta$. Moreover, if α and β are strict, then $\alpha \prec u \prec \beta$.*

Theorem 2.11 ([16]). *Assume that α_1 and β_2 are lower and upper solutions of (1.1) respectively and α_2 and β_1 are strict lower and upper solutions of (1.1) respectively such that*

$$\alpha_1 \le \beta_1 \le \beta_2, \ \alpha_1 \le \alpha_2 \le \beta_2,$$

and there exists $t_0 \in [0,1]$ with $\beta_1(t_0) < \alpha_2(t_0)$. Then there exist three solutions of (1.1) such that

$$\alpha_1 \le u_1 \prec \beta_1, \quad \alpha_2 \prec u_2 \le \beta_2,$$
$$u_3 \in [\alpha_1, \beta_2] \setminus ([\alpha_1, \beta_1] \cup [\alpha_2, \beta_2]).$$

Finally we introduce the generalized Picone identity due to Jaroš and Kusano [14, 19]. Let us consider the following differential operators:

$$l_p[y] := (r\varphi_p(y'))' + q(t)\varphi_p(y),$$
$$L_p[z] := (R\varphi_p(z'))' + Q(t)\varphi_p(z).$$

Lemma 2.12 ([19, p. 382]). *Let r, q, R and Q be real-valued continuous functions on an interval I. If y and z are any functions such that $y, z, r\varphi_p(y')$ and $R\varphi_p(z')$ are differentiable on I and $z(t) \ne 0$ for $t \in I$, then, for $t \in I$, we have*

$$\frac{d}{dt}\Big(y\Big[r\varphi_p(y') - \varphi_p\big(\frac{y}{z}\big)R\varphi_p(z')\Big]\Big)$$
$$= (r - R)|y'|^p + (Q - q)|y|^p + R\Big[|y'|^p + (p-1)\big|\frac{yz'}{z}\big|^p - p\varphi_p(y)y'\varphi_p\big(\frac{z'}{z}\big)\Big] \quad (2.3)$$
$$+ yl_p[y] - \frac{|y|^p}{\varphi_p(z)}L_p[z].$$

Remark 2.13. By Young's inequality, we obtain

$$|y'|^p + (p-1)\big|\frac{yz'}{z}\big|^p - p\varphi_p(y)y'\varphi_p\big(\frac{z'}{z}\big) \ge 0,$$

and the equality holds if and only if $y' = yz'/z$ on I.

3. Main result

Let $\mathcal{S}$ be the set of positive solutions of (1.1) in $(0, \infty) \times \mathcal{K}_w$. For the sake of convenience, we use the following notation

$$h_0 := \min_{t \in [0,1]} h(t), \quad h^0 := \max_{t \in [0,1]} h(t),$$
$$\overline{w} := \int_0^1 \varphi_p^{-1}\big(\frac{1}{w(s)}\big)ds.$$

Lemma 3.1. *Assume (H1), (W1), (F1) are satisfied, and let $\{(\lambda_n, u_n)\}_{n=1}^\infty$ be a sequence in $\mathcal{S}$ such that $\lambda_n \to \infty$ as $n \to \infty$. Then $\|u_n\|_\infty \to \infty$ as $n \to \infty$.*

Proof. Assume on the contrary that $\lambda_n \to \infty$ as $n \to \infty$, but there exists $M > 0$ such that $\|u_n\|_\infty \le M$ for all n. Then from (F1) there exists $\delta > 0$ such that $f(u_n(t)) \ge 4\delta$, $t \in [0,1]$. For each n, the function u_n attains its maximum at the unique point $x_n \in (0,1)$ and $u_n'(x_n) = 0$. Suppose that $x_n \ge 1/2$ (the case $x_n < 1/2$ is similar). Then, for $t \in (0, 1/4)$, one has

$$u_n'(t) = \varphi_p^{-1}\Big(\frac{1}{w(t)}\int_t^{x_n} \lambda_n h(s)f(u_n(s))ds\Big)$$
$$\ge \varphi_p^{-1}\Big(\frac{1}{w(t)}\int_{1/4}^{1/2} \lambda_n h(s)f(u_n(s))ds\Big)$$
$$\ge \varphi_p^{-1}(\lambda_n \delta h_0)\varphi_p^{-1}\big(\frac{1}{w(t)}\big),$$

and hence,
$$u_n(\tfrac{1}{4}) \geq \varphi_p^{-1}(\lambda_n \delta h_0) \int_0^{1/4} \varphi_p^{-1}\big(\frac{1}{w(s)}\big) ds.$$

Since $\lambda_n \to \infty$ as $n \to \infty$, $u_n(\tfrac{1}{4}) \to \infty$ as $n \to \infty$. This contradicts the fact that $\|u_n\|_\infty \leq M$ for all n. □

Lemma 3.2. *Assume* (H1), (W1), (F1) *are satisfied, and let* $\{(\lambda_n, u_n)\}_{n=1}^\infty$ *be a sequence in* $\mathcal{S}$ *such that* $\|u_n\|_w \to \infty$ *as* $n \to \infty$. *Then* $\lambda_n \to \infty$ *as* $n \to \infty$.

Proof. Assume on the contrary that $\|u_n\|_w \to \infty$ as $n \to \infty$, but there exists $L > 0$ such that $\lambda_n \leq L$ for all n. Then one can easily see that $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Put
$$\alpha = \frac{1}{2Lh^0\overline{w}^{p-1}}.$$

By (F1), there exists $N_\alpha > 0$ such that for all $u > N_\alpha$, $f(u) < \alpha u^{p-1}$. Let $M_\alpha = \max_{0 \leq u \leq N_\alpha} f(u)$, $A_n := \{t \in [0,1] : u_n(t) \leq N_\alpha\}$ and $B_n := \{t \in [0,1] : u_n(t) > N_\alpha\}$. Then $f(u_n(t)) \leq M_\alpha$, $t \in A_n$ and $f(u_n(t)) \leq \alpha u_n(t)^{p-1}$, $t \in B_n$. Put $u_n(x_n) = \|u_n\|_\infty$. Then

$$\begin{aligned}
u_n(x_n) &= \int_0^{x_n} \varphi_p^{-1}\Big(\frac{1}{w(s)} \int_s^{x_n} \lambda_n h(\tau) f(u_n(\tau)) d\tau\Big) ds \\
&\leq \varphi_p^{-1}(\lambda_n h^0) \int_0^1 \varphi_p^{-1}\Big(\frac{1}{w(s)}\Big[\int_{A_n} f(u_n(\tau)) d\tau + \int_{B_n} f(u_n(\tau)) d\tau\Big]\Big) ds \\
&\leq \varphi_p^{-1}(\lambda_n h^0) \int_0^1 \varphi_p^{-1}\Big(\frac{1}{w(s)}\Big[M_\alpha + \alpha \int_{B_n} u_n(\tau)^{p-1} d\tau\Big]\Big) ds.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{\varphi_p^{-1}(\lambda_n h^0)} &\leq \int_0^1 \varphi_p^{-1}\Big(\frac{1}{w(s)}\Big[\frac{M_\alpha}{\|u_n\|_\infty^{p-1}} + \alpha \int_{B_n} \frac{u_n(\tau)^{p-1}}{\|u_n\|_\infty^{p-1}} d\tau\Big]\Big) ds \\
&\leq \int_0^1 \varphi_p^{-1}\Big(\frac{1}{w(s)}\Big[\frac{M_\alpha}{\|u_n\|_\infty^{p-1}} + \alpha\Big]\Big) ds.
\end{aligned}$$

Letting $n \to \infty$, by Lebesgue's dominated convergence theorem,
$$\frac{1}{\varphi_p^{-1}(Lh^0)} \leq \varphi_p^{-1}(\alpha)\overline{w},$$

which contradicts the choice of α. □

The following theorem can be easily obtained in view of Proposition 2.5, Lemma 3.1 and Lemma 3.2.

Theorem 3.3. *Assume* (H1), (W1), (F1) *are satisfied. Then there exists an unbounded continuum* $\mathcal{C}$ *of positive solutions for* (1.1) *emanating from* $(0,0)$ *in* $[0,\infty) \times \mathcal{K}_w$ *such that*

(i) *for each* $\lambda > 0$, *there exists a positive solution* u_λ *of* (1.1) *such that* $(\lambda, u_\lambda) \in \mathcal{C}$ *and*

(ii) *for* $(\lambda, u_\lambda) \in \mathcal{S}$, $\lambda \to \infty$ *if and only if* $\|u_\lambda\|_w \to \infty$.

Let v be the unique solution of
$$\begin{aligned}
(w(t)\varphi_p(v'(t)))' + h(t) &= 0, \ t \in (0,1), \\
v(0) = v(1) &= 0.
\end{aligned}$$

Then $v \in C_w[0, 1]$ and $v(t) > 0$, $t \in (0, 1)$.

Note that condition (F1) implies that f also satisfies the following condition (e.g., see [21, Lemma 4.1])

(F1*) Let $f^*(u) = \max_{0 \leq s \leq u} f(s)$, then $\lim_{u \to \infty} \frac{f^*(u)}{\varphi_p(u)} = 0$.

Now we give a uniqueness result under the additional condition (F2).

Theorem 3.4. *Assume* (H1), (W1),(F1), (F2) *are satisfied. Then* $\mathcal{S} = \mathcal{C}$ *and* $\mathcal{C}$ *is the solution curve of positive solutions for* (1.1) *such that*

(i) *for each* $\lambda > 0$, *there exists the unique positive solution* $u(\lambda)$ *of* (1.1) *such that* $(\lambda, u(\lambda)) \in \mathcal{C}$,

(ii) $\lambda \to \infty$ *if and only if* $\|u(\lambda)\|_w \to \infty$ *and*

(iii) *for any* $0 < \lambda_a < \lambda_b$, $u(\lambda_a) \prec u(\lambda_b)$.

Proof. First, we prove that (1.1) has at most one positive solution for each $\lambda > 0$. Assume on the contrary that there exists $\lambda > 0$ such that (λ, u_1) and (λ, u_2) are two distinct positive solutions of (1.1). Without loss of generality, we may assume that there exists an interval $(a, b)(\subseteq (0, 1))$ such that $u_1(t) > u_2(t)$, $t \in (a, b)$, $u_1(a) = u_2(a)$ and $u_1(b) = u_2(b)$. Then we have four cases: (1) $a > 0$, $b = 1$; (2) $a = 0$, $b = 1$; (3) $a = 0$, $b < 1$; (4) $a > 0$, $b < 1$. We only prove the case (1) since other cases are similar. In this case, we know that $u_1'(a) \geq u_2'(a)$ and $(w^{1/(p-1)}u_1')(1) \leq (w^{1/(p-1)}u_2')(1) < 0$. Then, by L'Hospital's rule,

$$\lim_{t \to 1^-} \frac{u_1(t)}{u_2(t)} = \frac{(w^{1/(p-1)}u_1')(1)}{(w^{1/(p-1)}u_2')(1)}. \tag{3.1}$$

Taking $r(t) = R(t) = w(t)$, $y(t) = u_1(t)$, $z(t) = u_2(t)$, $q(t) = \frac{\lambda h(t) f(u_1(t))}{\varphi_p(u_1(t))}$ and $Q(t) = \frac{\lambda h(t) f(u_2(t))}{\varphi_p(u_2(t))}$ in Lemma 2.12 and integrating (2.3) from a to 1, by (3.1) and Remark 2.13,

$$- u_1(a)[(w\varphi_p(u_1'))(a) - (w\varphi_p(u_2'))(a)]$$
$$\geq \lambda \int_a^1 h(s) \Big[\frac{f(u_2(s))}{\varphi_p(u_2(s))} - \frac{f(u_1(s))}{\varphi_p(u_1(s))} \Big] |u_1(s)|^p ds.$$

By the fact that $u_1'(a) > u_2'(a)$,

$$\lambda \int_a^1 h(s) \Big[\frac{f(u_2(s))}{\varphi_p(u_2(s))} - \frac{f(u_1(s))}{\varphi_p(u_1(s))} \Big] |u_1(s)|^p ds \leq 0.$$

On the other hand, since $u_1(t) > u_2(t)$, $t \in (a, 1)$, by (F_2),

$$\lambda \int_a^1 h(s) \Big[\frac{f(u_2(s))}{\varphi_p(u_2(s))} - \frac{f(u_1(s))}{\varphi_p(u_1(s))} \Big] |u_1(s)|^p ds > 0.$$

This is a contradiction and thus (1.1) has a unique positive solution $u(\lambda)$ for each $\lambda > 0$.

For $0 < \lambda_a < \lambda_b$, it can be easily see that $u(\lambda_a)$ is a strict lower solution of (1.1) with $\lambda = \lambda_b$. Put $\beta_1 = \lambda_b C_b \frac{v}{\|v\|_\infty}$, where C_b is the constant satisfying

$$\frac{f^*(\lambda_b C_b)}{(\lambda_b C_b)^{p-1}} < \frac{1}{\lambda_b \|v\|_\infty^{p-1}}$$

and $u(\lambda_a) \leq \beta_1$. Note that it is possible to choose such a constant C_b in view of (F_1^*). Then, for $t \in (0, 1)$,

$$
\begin{aligned}
-(w(t)\varphi_p(\beta_1'(t)))' &= \Big(\lambda_b C_b \frac{1}{\|v\|_\infty}\Big)^{p-1} h(t) \\
&> \lambda_b h(t) f^*(\lambda_b C_b) \\
&\geq \lambda_b h(t) f^*(\beta_1(t)) \\
&\geq \lambda_b h(t) f(\beta_1(t)),
\end{aligned}
$$

which implies that β_1 is a strict upper solution of (1.1) with $\lambda = \lambda_b$. By Theorem 2.10, there exists a positive solution u_b of (1.1), with $\lambda = \lambda_b$, such that $u(\lambda_a) \prec u_b$, and u_b must be same as $u(\lambda_b)$ since (1.1) has at most one positive solution for each $\lambda > 0$. Thus the proof is complete by Theorem 3.3. □

For the rest of this article, we assume that f satisfies

(F3) There exist $m, M > 0$ such that f is nondecreasing on (m, M).

Define $\bar{f}$ as

$$
\bar{f}(u) = \begin{cases} \hat{f}(u), & u < m, \\ f(u), & u \geq m. \end{cases}
$$

Here, $\hat{f}$ is defined so that $\bar{f}$ is nondecreasing on $[0, M]$, $0 < \bar{f}(0) < f(m)$, $\bar{f} \leq f$ and $\bar{f}$ is continuous on $[0, \infty)$.

For $0 < a < b$ and $b \in [m, M]$, let $\epsilon_b = \epsilon(C/B)$ with $B = \bar{f}(0)$, $C = f(b)$ in Lemma 2.2 and

$$
C(a, b) = \max \Big\{ \frac{\varphi_p(b) f^*(a)\varphi_p(\|v\|_\infty)}{\varphi_p(a) f(b) h_0 \epsilon_b \varphi_p(C_{\epsilon_b})}, \frac{\varphi_p(b)\varphi_p(\overline{w})}{\varphi_p(M)\epsilon_b \varphi_p(C_{\epsilon_b})} \Big\}
$$

where

$$
C_{\epsilon_b} = \min \Big\{ \int_0^{\epsilon_b} \varphi_p^{-1}\Big(\frac{1}{w(s)}\Big) ds, \int_{1-\epsilon_b}^1 \varphi_p^{-1}\Big(\frac{1}{w(s)}\Big) ds \Big\}.
$$

Now we give the following existence result of three positive solutions to (1.1).

Theorem 3.5. *Assume (H1), (W1), (F1), (F3) are satisfied, and there exist positive constants a, b such that $a < b$, $b \in [m, M]$ and $C(a, b) < 1$. Then, for all $\lambda \in (\lambda_1, \lambda_2)$, (1.1) has three distinct positive solutions. Here,*

$$
\lambda_1 := \frac{\varphi_p(b)}{h_0 \epsilon_b f(b)\varphi_p(C_{\epsilon_b})},
$$

$$
\lambda_2 := \min \Big\{ \varphi_p\Big(\frac{a}{\|v\|_\infty}\Big)\frac{1}{f^*(a)}, \varphi_p\Big(\frac{M}{\overline{w}}\Big)\frac{1}{h_0 f(b)} \Big\}.
$$

Proof. Let λ be fixed with $\lambda_1 < \lambda < \lambda_2$. Firstly, put $\alpha_1 \equiv 0$. Clearly α_1 is a lower solution of (1.1). Secondly, put $\beta_1 = \frac{a}{\|v\|_\infty} v$. Then for $t \in (0, 1)$,

$$
\begin{aligned}
-(w(t)\varphi_p(\beta_1'(t)))' &= \varphi_p\Big(\frac{a}{\|v\|_\infty}\Big) h(t) \\
&> \lambda h(t) f^*(a) \\
&\geq \lambda h(t) f^*(\beta_1(t)) \\
&\geq \lambda h(t) f(\beta_1(t)).
\end{aligned}
$$

Thus β_1 is a strict upper solution of (1.1).

Thirdly, let α_2 be the unique solution of

$$(w(t)\varphi_p(\alpha_2'(t)))' + \lambda^* h_0 \bar{f}(\rho(t)) = 0, \ t \in (0,1),$$
$$\alpha_2(0) = \alpha_2(1) = 0, \tag{3.2}$$

where $\lambda^* \in (\lambda_1, \lambda)$ and $\rho(t)$ is defined as

$$\rho(t) = \begin{cases} b\big[\int_0^{\epsilon_b} \varphi_p^{-1}\big(\frac{1}{w(s)}\big)ds\big]^{-1} \int_0^t \varphi_p^{-1}\big(\frac{1}{w(s)}\big)ds, & t \in [0, \epsilon_b], \\ b, & t \in (\epsilon_b, 1-\epsilon_b), \\ b\big[\int_{1-\epsilon_b}^1 \varphi_p^{-1}\big(\frac{1}{w(s)}\big)ds\big]^{-1} \int_t^1 \varphi_p^{-1}\big(\frac{1}{w(s)}\big)ds, & t \in [1-\epsilon_b, 1]. \end{cases}$$

Since $\lambda^* < \lambda_2$, one has

$$\|\alpha_2\|_\infty = \alpha_2(A_{\alpha_2})$$
$$= \int_0^{A_{\alpha_2}} \varphi_p^{-1}\Big(\frac{1}{w(s)} \int_s^{A_{\alpha_2}} \lambda^* h_0 \bar{f}(\rho(\tau))d\tau\Big)ds$$
$$\leq \varphi_p^{-1}(\lambda^* h_0 \bar{f}(b))\overline{w} = \varphi_p^{-1}(\lambda^* h_0 f(b))\overline{w} < M.$$

Note that $\alpha_2 = \varphi_p^{-1}(\lambda^* h_0)l$, where l is the unique solution of

$$(w(t)\varphi_p(l'(t)))' + \bar{f}(\rho(t)) = 0, \ t \in (0,1),$$
$$l(0) = l(1) = 0.$$

Since $\|\rho\|_\infty = b$, by Lemma 2.2, $A_{\alpha_2} = A_l \in (2\epsilon_b, 1-2\epsilon_b)$. For $t \in [0, \epsilon_b]$,

$$w(t)\varphi_p(\rho'(t)) = \varphi_p\Big(b\big[\int_0^{\epsilon_b} \varphi_p^{-1}\big(\frac{1}{w(s)}\big)ds\big]^{-1}\Big) \leq \varphi_p(b/C_{\epsilon_b}) \tag{3.3}$$

and for $t \in [1-\epsilon_b, 1]$,

$$-w(t)\varphi_p(\rho'(t)) = \varphi_p\Big(b\big[\int_{1-\epsilon_b}^1 \varphi_p^{-1}\big(\frac{1}{w(s)}\big)ds\big]^{-1}\Big) \leq \varphi_p(b/C_{\epsilon_b}). \tag{3.4}$$

For $t \in [0, \epsilon_b]$, integrating (3.2) from t to A_{α_2}, one has

$$w(t)\varphi_p(\alpha_2'(t)) = \lambda^* h_0 \int_t^{A_{\alpha_2}} \bar{f}(\rho(s))ds$$
$$\geq \lambda^* h_0 \int_{\epsilon_b}^{2\epsilon_b} \bar{f}(\rho(s))ds$$
$$= \lambda^* h_0 \int_{\epsilon_b}^{2\epsilon_b} \bar{f}(b)ds$$
$$= \lambda^* h_0 \epsilon_b \bar{f}(b) = \lambda^* h_0 \epsilon_b f(b)$$
$$> \varphi_p\Big(\frac{b}{C_{\epsilon_b}}\Big).$$

Thus by (3.3), for $t \in [0, \epsilon_b]$,

$$w(t)\varphi_p(\alpha_2'(t)) > w(t)\varphi_p(\rho'(t)),$$

which implies $\alpha_2(t) \geq \rho(t), \quad t \in [0, \epsilon_b]$. Similarly, by (3.4),

$$\alpha_2(t) \geq \rho(t), \quad t \in [1-\epsilon_b, 1].$$

By the facts that $A_{\alpha_2} \in (2\epsilon_b, 1 - 2\epsilon_b)$ and $\rho(t) = b$, $t \in [\epsilon_b, 1 - \epsilon_b]$, we have $\alpha_2 \geq \rho$, which implies that, for $t \in (0, 1)$,

$$-(w(t)\varphi_p(\alpha_2'(t)))' = \lambda^* h_0 \bar{f}(\rho(t)) \leq \lambda^* h_0 \bar{f}(\alpha_2(t)) < \lambda h(t) f(\alpha_2(t)),$$

and thus α_2 is a strict lower solution of (1.1). Since $\|\alpha_2\|_\infty \geq \|\rho\|_\infty = b > a = \|\beta_1\|_\infty$, there exists $t_0 \in (0, 1)$ such that $\alpha_2(t_0) > \beta_1(t_0)$.

Finally, put $\beta_2 = \lambda C_\lambda \frac{v}{\|v\|_\infty}$, where C_λ is the constant satisfying

$$\frac{f^*(\lambda C_\lambda)}{(\lambda C_\lambda)^{p-1}} < \frac{1}{\lambda \|v\|_\infty^{p-1}}, \quad \beta_1 \leq \beta_2, \quad \alpha_2 \leq \beta_2.$$

Then, for $t \in (0, 1)$, we have

$$\begin{aligned}
-(w(t)\varphi_p(\beta_2'(t)))' &= \left(\lambda C_\lambda \frac{1}{\|v\|_\infty}\right)^{p-1} h(t) \\
&> \lambda h(t) f^*(\lambda C_\lambda) \\
&\geq \lambda h(t) f^*(\beta_2(t)) \\
&\geq \lambda h(t) f(\beta_2(t)),
\end{aligned}$$

and β_2 is an upper solution of (1.1). Thus, by Theorem 2.11, (1.1) has three positive solutions for all $\lambda \in (\lambda_1, \lambda_2)$. $\qquad\square$

In the results so far, we assume that f is positive for all $u \geq 0$ as it satisfies (F1). If we assume that f has a positive falling zero instead of (F1); i.e., f satisfies

(F1') $f \in C([0, \infty), \mathbb{R})$ and there exists $k > 0$ such that $(k - u)f(u) > 0$ for $u \neq k$,

then we can obtain results similar to Theorem 3.3, Theorem 3.4 and Theorem 3.5 as follows.

Theorem 3.6. *Assume* (H1), (W1), (F1') *are satisfied. Then there exists an unbounded continuum $\mathcal{C}$ of positive solutions for* (1.1) *emanating from* $(0, 0)$ *in* $[0, \infty) \times \mathcal{K}_w$ *such that for each $\lambda > 0$, there exists a positive solution u_λ of* (1.1) *such that $(\lambda, u_\lambda) \in \mathcal{C}$, and for each $\lambda > 0$, $\|u_\lambda\|_\infty \leq k$.*

Theorem 3.7. *Assume* (H1), (W1), (F1') *and*

(F2') $\frac{f(u)}{\varphi_p(u)}$ *is strictly decreasing on* $(0, k)$

are satisfied. Then $\mathcal{S} = \mathcal{C}$ and $\mathcal{C}$ is the solution curve of positive solutions for (1.1) *such that*

(i) *for each $\lambda > 0$, there exists the unique positive solution $u(\lambda)$ of* (1.1) *such that $(\lambda, u(\lambda)) \in \mathcal{C}$ and*

(ii) *for any $0 < \lambda_a < \lambda_b$, $u(\lambda_a) \prec u(\lambda_b)$.*

Theorem 3.8. *Assume* (H1), (W1), (F1'), (F3) *are satisfied, and there exist positive constants a, b such that $a < b$, $b \in [m, M]$, $M < k$ and $C(a, b) < 1$. Then, for all $\lambda \in (\lambda_1, \lambda_2)$,* (1.1) *has three positive solutions. Here, λ_1 and λ_2 are the same constants in Theorem 3.5.*

Finally we give an example to illustrate Theorem 3.5 or Theorem 3.8.

Example 3.9. In problem (1.1), put $w(t) = t^\theta(1 - t)^\theta$, $0 \leq \theta < p - 1$ and

$$f(u) = \begin{cases} f_1(u), & 0 \leq u \leq 1, \\ \exp[\frac{\alpha u}{\alpha + u}], & 1 < u < M, \\ f_2(u), & u \geq M. \end{cases}$$

Here, $\alpha > 0$ and $M > 1$ are constants which will be chosen later, and f_1 and f_2 are the functions such that $f_1(u) \leq \exp[\alpha/(\alpha + 1)]$ for $u \in [0, 1]$ and f satisfies (F1) or (F_1'). Clearly, (W1) and (F3) are satisfied for $m = 1$ and $M > 1$.

Note that, in the proof of Theorem 3.5, ϵ_b is just needed for verifying that $A_{\alpha_2} \in (2\epsilon_b, 1 - 2\epsilon_b)$. Since w is symmetric with respect to $1/2$, so ρ is also symmetric with respect to $1/2$. Then $A_{\alpha_2} = 1/2$ (see Remark 2.3), and we can take $\epsilon_b = 1/5$. For $a = 1$ and $b = \alpha$,

$$\frac{\varphi_p(b) f^*(a)}{\varphi_p(a) f(b)} = \frac{\varphi_p(\alpha) f(1)}{\varphi_p(1) f(\alpha)} = \alpha^{p-1} \exp\Big[\frac{\alpha}{\alpha + 1} - \frac{\alpha}{2}\Big].$$

For any $h \in C([0, 1], (0, \infty))$, there exists $\alpha = \alpha(h) > 0$ such that

$$\alpha^{p-1} \exp\Big[\frac{\alpha}{\alpha + 1} - \frac{\alpha}{2}\Big] < \frac{h_0 \epsilon_b \varphi_p(C_{\epsilon_b})}{\varphi_p(\|v\|_\infty)},$$

and

$$\frac{\varphi_p(b) f^*(a) \varphi_p(\|v\|_\infty)}{\varphi_p(a) f(b) h_0 \epsilon_b \varphi_p(C_{\epsilon_b})} < 1.$$

Furthermore, we can choose $M = M(\alpha)$ such that

$$\varphi_p\Big(\frac{b}{M}\Big) = \varphi_p\Big(\frac{\alpha}{M}\Big) < \frac{\epsilon_b \varphi_p(C_{\epsilon_b})}{\varphi_p(\overline{w})}.$$

Then $C(1, \alpha) < 1$ and thus (1.1) has at least three positive solutions for a certain range of λ by Theorem 3.5 or Theorem 3.8.

References

[1] K. J. Brown, M. M. A. Ibrahim, R. Shivaji; *S-shaped bifurcation curves*, Nonlinear Anal. 5 (5) (1981) 475–486.

[2] E. N. Dancer; *On the structure of solutions of an equation in catalysis theory when a parameter is large*, J. Differential Equations 37 (3) (1980) 404–437.

[3] P. Drábek, S. B. Robinson; *Multiple positive solutions for elliptic boundary value problems*, Rocky Mountain J. Math. 36 (1) (2006) 97–113.

[4] Y. Du, Y. Lou; *Proof of a conjecture for the perturbed Gelfand equation from combustion theory*, J. Differential Equations 173 (2) (2001) 213–230.

[5] X. Garaizar; *Existence of positive radial solutions for semilinear elliptic equations in the annulus*, J. Differential Equations 70 (1) (1987) 69–92.

[6] M. García-Huidobro, R. Manásevich, F. Zanolin; *A Fredholm-like result for strongly nonlinear second order ODEs*, J. Differential Equations 114 (1) (1994) 132–167.

[7] D. D. Hai; *On singular Sturm-Liouville boundary-value problems*, Proc. Roy. Soc. Edinburgh Sect. A 140 (1) (2010) 49–63.

[8] D. D. Hai, K. Schmitt, R. Shivaji; *Positive solutions of quasilinear boundary value problems*, J. Math. Anal. Appl. 217 (2) (1998) 672–686.

[9] S. P. Hastings, J. B. McLeod; *The number of solutions to an equation from catalysis*, Proc. Roy. Soc. Edinburgh Sect. A 101 (1-2) (1985) 15–30.

[10] J. Henderson, H. B. Thompson; *Existence of multiple solutions for second order boundary value problems*, J. Differential Equations 166 (2) (2000) 443–454.

[11] J. Henderson, H. B. Thompson; *Multiple symmetric positive solutions for a second order boundary value problem*, Proc. Amer. Math. Soc. 128 (8) (2000) 2373–2379.

[12] Y. X. Huang, G. Metzen; *The existence of solutions to a class of semilinear differential equations*, Differential Integral Equations 8 (2) (1995) 429–452.

[13] K. C. Hung, S. H. Wang; *A theorem on S-shaped bifurcation curve for a positone problem with convex-concave nonlinearity and its applications to the perturbed Gelfand problem*, J. Differential Equations 251 (2) (2011) 223–237.

[14] J. Jaroš, T. Kusano; *A Picone type identity for second order half-linear differential equations*, Acta Math. Univ. Comenian. (N.S.) 68 (1) (1999) 137–151.

[15] J. Jiang, J. Shi; *Bistability dynamics in some structured ecological models*, in: R. S. Cantrell, C. Cosner, S. Ruan (eds.), Spatial Ecology, Chapman & Hall/CRC Press, Boca Raton, 2009.

[16] C. G. Kim; *The three-solutions theorem for p-laplacian boundary value problems*, Nonlinear Anal. 75 (2) (2012) 924–931.

[17] P. Korman, Y. Li; *On the exactness of an S-shaped bifurcation curve*, Proc. Amer. Math. Soc. 127 (4) (1999) 1011–1020.

[18] P. Korman, J. Shi; *Instability and exact multiplicity of solutions of semilinear equations*, in: Proceedings of the Conference on Nonlinear Differential Equations (Coral Gables, FL, 1999), Electron. J. Differ. Equ. Conf. 5, 2000.

[19] T. Kusano, J. Jaroš, N. Yoshida; *A Picone-type identity and Sturmian comparison and oscillation theorems for a class of half-linear partial differential equations of second order*, Nonlinear Anal. 40 (1-8) (2000) 381–395.

[20] E. K. Lee, S. Sasi, R. Shivaji; *S-shaped bifurcation curves in ecosystems*, J. Math. Anal. Appl. 381 (2) (2011) 732–741.

[21] Y. H. Lee, I. Sim; *Existence results of sign-changing solutions for singular one-dimensional p-Laplacian problems*, Nonlinear Anal. 68 (5) (2008) 1195–1209.

[22] S. S. Lin; *Positive radial solutions and nonradial bifurcation for semilinear elliptic equations in annular domains*, J. Differential Equations 86 (2) (1990) 367–391.

[23] R. Manásevich, J. Mawhin; *Periodic solutions for nonlinear systems with p-Laplacian-like operators*, J. Differential Equations 145 (2) (1998) 367–393.

[24] B. P. Rynne; *A global curve of stable, positive solutions for a p-Laplacian problem*, Electron. J. Differential Equations (2010) No. 58, 1-12.

[25] P. Takáč; *Degenerate elliptic equations in ordered Banach spaces and applications*, in: Nonlinear differential equations (Chvalatice, 1998), vol. 404 of Chapman & Hall/CRC Res. Notes Math., Chapman & Hall/CRC, Boca Raton, FL, 1999, pp. 111–196.

[26] S. H. Wang; *On S-shaped bifurcation curves*, Nonlinear Anal. 22 (12) (1994) 1475–1485.

[27] S. H. Wang; *Rigorous analysis and estimates of S-shaped bifurcation curves in a combustion problem with general Arrhenius reaction-rate laws*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 454 (1972) (1998) 1031–1048.

[28] S .H. Wang, F. P. Lee; *Bifurcation of an equation from catalysis theory*, Nonlinear Anal. 23 (9) (1994) 1167–1187.

[29] S. H. Wang, T. S. Yeh; *Exact multiplicity of solutions and S-shaped bifurcation curves for the p-Laplacian perturbed Gelfand problem in one space variable*, J. Math. Anal. Appl. 342 (2) (2008) 1175–1191.

[30] F. A. Williams; *Combustion theory*, Westview Press; Second edition, 1994.

[31] E. Zeidler; *Nonlinear functional analysis and its applications I*, Springer-Verlag, New York, 1986.

Chan-Gyun Kim

Department of Mathematics, College of William and Mary, Williamsburg, Virginia 23187-8795, USA
   E-mail address: cgkim75@gmail.com

Junping Shi

Department of Mathematics, College of William and Mary, Williamsburg, Virginia 23187-8795, USA.

School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, China
   E-mail address: shij@math.wm.edu