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SUMS OF ZEROS OF SOLUTIONS TO SECOND ORDER ODE WITH NON-POLYNOMIAL COEFFICIENTS

MICHAEL I. GIL'

Abstract. We consider the equation $y'' = F(z)y$ $(z \in \mathbb{C})$ with an entire function F satisfying the condition

$$|F(z)| \le A \exp\big(\frac{|z|^\rho}{\rho}\big) \quad (\rho \ge 1,\ A = \text{const} > 0).$$

Let $z_k(y)$, $k = 1, 2, \dots$ be the zeros of a solution $y(z)$ to the above equation. Bounds for the sums

$$\sum_{k=1}^{j} \frac{1}{|z_k(y)|} \quad (j = 1, 2, \dots)$$

are established. Some applications of these bounds are also considered.

1. Introduction and statement of the main result

In the present article, we consider linear differential equations with non-polynomial coefficients in the complex domain. The literature devoted to the zeros of solutions of such equations is very rich. Here the main tool is the Nevanlinna theory. An excellent exposition of the Nevanlinna theory and its applications to differential equations is given in the book [11]. In that book, in particular, the well-known results of Bank, Brűggemann, Hellerstein, Rossi, Huang and other mathematicians are featured. In connection with recent results see the very interesting papers [2]-[5], [8], [12]-[17]. In particular, in the paper [15], the authors study the convergence of the zeros of a non-trivial (entire) solution to the linear differential equation

$$f'' + \big\{Q_1(z)e^{P_1(z)} + Q_2(z)e^{P_2(z)} + Q_3(z)e^{P_3(z)}\big\}f = 0$$

where P_j are polynomials of degree $n \ge 1$ and Q_j $(Q_j \not\equiv 0)$ are entire functions of order less than n $(j = 1, 2, 3)$. The remarkable results on the zeros of a wide class of ordinary differential equations with polynomial coefficients whose solutions are classical orthogonal polynomials was established by Anghel [1]. Besides, he had derived the important results connected with the equations of mathematical physics.

Certainly we could not survey the whole subject here and we refer the reader to the above listed publications and references given therein.

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In the above cited works mainly the asymptotic distributions of zeros and counting functions of zeros are investigated. At the same time, bounds for the zeros of solutions are very important in various applications. But to the best of our knowledge, they have been investigated considerably less than the asymptotic distributions. In the paper [10], bounds for the sums of the zeros of solutions are established for the second order equations with polynomial coefficients. In this paper, we obtain such bounds in the case of non-polynomial coefficients. Besides, below we estimate the zero free domains. That estimation supplements the well-known results of Eloe and Henderson [7] on the positivity of solutions for higher order ordinary differential equations, since the positivity of solutions implies the absence of zeros. Note that, the proof of the main result of the present paper is considerably different from the proof of the paper [10].

Consider the equation

$$\frac{d^2y(z)}{dz^2} = F(z)y(z) \tag{1.1}$$

with an entire function F, satisfying the condition

$$|F(z)| \le A\exp\big(\frac{|z|^\rho}{\rho}\big) \quad (\rho \ge 1;\ A = \text{const} > 0;\ z \in \mathbb{C}). \tag{1.2}$$

In Section 3 below we check that to inequality (1.2) can be reduced the formally more general inequality

$$|F(z)| \le A\exp[B|z|^\rho] \quad (B = \text{const} > 0) \tag{1.3}$$

by the substitution

$$z = \frac{w}{(\rho B)^{1/\rho}} \tag{1.4}$$

into (1.1). Everywhere below, $y(z)$ is a solution of (1.1) with $y(0) = 1$. Enumerate the zeros $z_k(y)$ of $y(z)$, with multiplicities taken into account, in order of increasing modulus: $|z_k(y)| \le |z_{k+1}(y)|$ $(k = 1, 2, \dots)$. Put

$$v_n = n\,\frac{e^{1+2/\rho}}{\rho}.$$

Theorem 1.1. *Let $y(z)$ be a solution of* (1.1) *with $y(0) = 1$ and let condition* (1.2) *hold. Then*

$$\sum_{k=1}^{j} \frac{1}{|z_k(y)|} < \theta_0 + \zeta_0 \sum_{k=1}^{j} \frac{1}{\sqrt[\rho]{\ln v_k}} \quad (j = 1, 2, \dots),$$

where θ_0 and ζ_0 are positive constants defined by

$$\theta_0 = 2\sqrt{\frac{e}{3}}(1 + |y'(0)|)\exp[A^2e^{2/\rho}] \quad \textit{and} \quad \zeta_0 = 2e^{1/2}(3/\rho)^{1/\rho}.$$

The proof of this theorem is presented in the next section. Let us point some corollaries of Theorem 1.1.

Denote by $\nu(y, r)$ $(r > 0)$ the counting function of the zeros of y in $|z| \le r$. Theorem 1.1 implies

Corollary 1.2. *With the notation*

$$\eta_j(y) := \frac{j}{\theta_0 + \zeta_0 \sum_{k=1}^{j} \frac{1}{\sqrt[\rho]{\ln v_k}}} \quad (j = 1, 2, \dots),$$

the inequality $|z_j(y)| > \eta_j(y)$ *holds and thus* $\nu(u,r) \le j-1$ *for any* $r \le \eta_j(y)$ $(j \ge 2)$.

Furthermore, put

$$\vartheta_1 = \theta_0 + \frac{\zeta_0}{\ln^{1/\rho} v_1}, \quad \vartheta_k = \frac{\zeta_0}{\ln^{1/\rho} v_k} \quad (k = 2,3,\dots).$$

Theorem 1.1 and [9, Lemma 1.2.1] yield the following result.

Corollary 1.3. *Under the hypothesis of Theorem 1.1, let* $\phi(t)$ $(0 \le t < \infty)$ *be a continuous convex scalar-valued function, such that* $\phi(0) = 0$. *Then*

$$\sum_{k=1}^{j} \phi(|z_k(y)|^{-1}) \le \sum_{k=1}^{j} \phi(\vartheta_k) \quad (j = 1,2,\dots).$$

In particular, take

$$\phi(t) = t^{\rho+1} \exp[-\frac{\zeta_0^\rho}{t^\rho}].$$

Then

$$\phi(\vartheta_k) = \frac{\zeta_0^{\rho+1}}{\ln^{1+1/\rho} v_k} \exp[-\ln v_k] \le \frac{\text{const}}{(\ln k)^{1+1/\rho}\, k} \quad (k > 1).$$

By the previous corollary we get the following result.

Corollary 1.4. *Under the hypothesis of Theorem 1.1, we have*

$$\sum_{k=1}^{\infty} \frac{1}{|z_k(y)|^{\rho+1}} e^{-(\zeta_0 |z_k(y)|)^\rho} < \infty.$$

In addition, in light of Theorem 1.1 and [9, Lemma 1.2.2] we obtain our next result.

Corollary 1.5. *Let* $\Phi(t_1, t_2, \dots, t_j)$ *be a function with an integer* j *defined on the domain*

$$0 < t_j \le t_{j-1} \dots \le t_2 \le t_1 < \infty$$

and satisfying the condition

$$\frac{\partial \Phi}{\partial t_1} > \frac{\partial \Phi}{\partial t_2} > \dots > \frac{\partial \Phi}{\partial t_j} > 0 \quad \textit{for } t_1 > t_2 > \dots > t_j > 0.$$

Then

$$\Phi\Big(\frac{1}{|z_1(y)|}, \dots, \frac{1}{|z_j(y)|}\Big) \le \Phi(\vartheta_1, \dots, \vartheta_j).$$

In particular, let $\{d_k\}_{k=1}^\infty$ be a decreasing sequence of positive numbers. Then the previous corollary yields the inequality

$$\sum_{k=1}^{j} \frac{d_k}{|z_k(y)|} \le \sum_{k=1}^{j} d_k \vartheta_k \quad (j = 1,2,\dots).$$

2. Proof of Theorem 1.1

Consider the entire function

$$f(z)=\sum_{k-0}^{\infty} c_k z^k \quad (c_0=1) \tag{2.1}$$

satisfying the condition

$$|f(z)|\le (1+qr)\exp[A\exp(r^\rho/\rho)] \quad (q=\text{const}>0; z\in\mathbb{C},\ r=|z|). \tag{2.2}$$

Put

$$C=(1+q)\exp[\frac{1}{2}e^{2/\rho}A^2].$$

Lemma 2.1. *Let condition* (2.2) *hold. Then the Taylor coefficients of f are subjected to the inequality*

$$|c_n|\le Ce^{n/2}\Big(\frac{3}{\rho\ln v_n}\Big)^{(n-1)/\rho}.$$

Proof. Let

$$M_f(r)=\max_{|z|=r}|f(z)|.$$

By the well-known inequality for the coefficients of a power series,

$$|c_n|\le \frac{M_f(r)}{r^n} \quad (r>0).$$

Take into account that

$$ab\le \frac{a^2}{4c}+b^2c \quad (a,b,c\text{positive constants}).$$

Then for a constant $\mu>0$,

$$A\exp(r^\rho/\rho)\le \frac{A^2}{4\mu}+\mu\exp(2r^\rho/\rho).$$

Due to (2.2),

$$|c_n|\le (1+qr)\exp[\frac{A^2}{4\mu}]h(r) \text{ where } h(r):=\frac{\exp[\mu e^{2r^\rho/\rho}]}{r^n} \quad (n=1,2,\dots). \tag{2.3}$$

Let us use the usual method for finding extrema. Clearly,

$$r^{2n}h'(r)=e^{\mu e^{2r^\rho/\rho}}[2\mu e^{2r^\rho/\rho}r^{n+\rho-1}-nr^{n-1}].$$

Thus the zero $r_0=r_0(n)$ of $h'(r)$ is defined by

$$2\mu r_0^\rho e^{2r_0^\rho/\rho}=n. \tag{2.4}$$

Take

$$\mu=\frac{1}{2}e^{-2/\rho}.$$

Then

$$r_0^\rho e^{2(r_0^\rho-1)/\rho}=n. \tag{2.5}$$

So for $n\ge 1$ we have $r_0\ge 1$. Hence by (2.4),

$$\mu e^{2r_0^\rho/\rho}\le n/2. \tag{2.6}$$

Since $x\le e^{x-1}$ $(x>0)$, by (2.5) we have $e^{3r_0^\rho/\rho}\ge ne^{1+2/\rho}/\rho=v_n$, and therefore,

$$r_0\ge (\frac{\rho}{3}\ln v_n)^{1/\rho}. \tag{2.7}$$

Since $r_0 \geq 1$, we obtain $\frac{1+qr_0}{r_0} \leq 1+q$. Now (2.3) and (2.6) imply

$$|c_n| \leq \exp[\frac{A^2}{4\mu}](1+q)\frac{e^{n/2}}{r_0^{n-1}}.$$

Hence, (2.7) proves the lemma. □

Put $D = \sqrt{e}C$, and

$$\tau_n = 2e^{1/2}\left(\frac{3}{\rho \ln v_n}\right)^{1/\rho}.$$

Then according to Lemma 2.1,

$$|c_n| \leq D\frac{\tau_n^{n-1}}{2^{n-1}}.$$

Denote $\psi_1 = 1, \psi_n = \tau_n^{n-1}$ $(n > 1)$; $a_n = c_n/\psi_n$, and $m_{n+1} = \psi_{n+1}/\psi_n$. As it is proved in [9, Theorem 5.1.1], the inequality

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|} \leq \theta(f) + \sum_{k=1}^{j} m_{k+1}$$

is valid, where $z_k(f)$ are the zeros of $f(z)$, with multiplicities taken into account, enumerated in order of increasing modulus, and

$$\theta(f) := \Big[\sum_{k=1}^{\infty} |a_k|^2\Big]^{1/2}.$$

But $|a_1| = |c_1| \leq D$; $|a_n| \leq D/2^{n-1}, n \geq 2$. Moreover, since $\tau_{n+1} \leq \tau_n$, we get $m_{n+1} \leq \tau_n$ $(n = 1, 2, \dots)$, and $\theta(f) \leq \tilde{\theta}_0$, where

$$\tilde{\theta}_0^2 = D^2 \sum_{k=0}^{\infty} \frac{1}{4^k} = 4D^2/3.$$

We thus have proved the following result.

Lemma 2.2. *Let an entire function f satisfy condition* (2.2)*. Then*

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|} \leq \tilde{\theta}_0 + \sum_{k=1}^{j} \tau_n = \tilde{\theta}_0 + \zeta_0 \sum_{k=1}^{j} \frac{1}{\ln^{1/\rho} v_k} \quad (j = 1, 2, \dots).$$

Lemma 2.3. *A solution y of* (1.1) *with the conditions* (1.2) *and $y(0) = 1$ is an entire function satisfying the inequality*

$$|y(z)| \leq (1 + |y'(0)|r)\exp[Ae^{r^\rho/\rho}] \quad (z \in \mathbb{C}).$$

Proof. From (1.1) for a $z = re^{it}$ with a fixed argument t we have $e^{-2it}d^2y(z)/dr^2 = F(z)y(z)$. Hence, putting $q = |y'(0)|$, $g(r) = |y(re^{it})|$, and taking into account (1.2) we obtain

$$g(r) \leq 1 + qr + \int_0^r (r-s)|F(se^{it})|g(s)ds \leq 1 + qr + A\int_0^r (r-s)\exp[s^\rho/\rho]g(s)ds.$$

By [6, Lemma III.2.1] we have $g(r) \leq m(r)$, where $m(r)$ is a solution of the equation

$$m(r) = 1 + qr + A\int_0^r (r-s)\exp[s^\rho/\rho]m(s)ds.$$

However,
$$\int_0^r (r-s)f(s)ds = \int_0^r \int_0^s f(\tau)d\tau\ ds$$
for any integrable function f. Thus,
$$m(r) = 1 + qr + A\int_0^r \int_0^s \exp[\tau^\rho/\rho]m(\tau)d\tau\ ds.$$
Clearly the derivative of m is positive. So
$$m(r) \le 1 + qr + A\int_0^r m(\tau)\int_0^\tau \exp[s^\rho/\rho]ds\ d\tau.$$
But for $r \le 1$,
$$\int_0^r \exp[s^\rho/\rho]ds \le \exp[r^\rho/\rho],$$
and for an $r \ge 1$,
$$\int_0^r e^{s^\rho/\rho}ds \le \int_0^1 e^{s^\rho/\rho}ds + \int_1^r s^{\rho-1}e^{s^\rho/\rho}ds \le e^{1/\rho} + (e^{r^\rho/\rho} - e^{1/\rho}) = e^{r^\rho/\rho}.$$
Thus,
$$\int_0^r \exp[s^\rho/\rho]ds \le \exp[r^\rho/\rho]. \tag{2.8}$$
Consequently, $m(r) \le 1 + qr + A\int_0^r m(s)\exp[s^\rho/\rho]ds$. By the Gronwall inequality,
$$m(r) \le (1+qr)\exp\big[A\int_0^r e^{s^\rho/\rho}ds\big].$$
Now (2.8) implies the required result. □

Then the assertion of Theorem 1.1 follows from Lemmas 2.2 and 2.3.

3. Example

In this section we consider an example that illustrates Theorem 1.1. First substitute (1.4) into (1.1). Then we arrive at the equation
$$\frac{d^2x(w)}{dw^2} = F_1(w), \quad \text{where } F_1(w) = \frac{1}{(\rho B)^{2/\rho}}F\big(\frac{w}{(\rho B)^{1/\rho}}\big)$$
and $x(w) = y(w/(\rho B)^{1/\rho})$. If condition (1.3) holds, then
$$|F_1(w)| \le A_1 \exp\big(\frac{|w|^\rho}{\rho}\big),$$
where $A_1 = A/(\rho B)^{2/\rho}$. By Theorem 1.1,
$$\sum_{k=1}^j \frac{1}{|z_k(x)|} < \theta_1 + \zeta_0 \sum_{k=1}^j \frac{1}{\ln^{1/\rho} v_k} \quad (j = 1, 2, \dots), \tag{3.1}$$
where
$$\theta_1 = 2\sqrt{\frac{e}{3}}\Big(1 + |\frac{dx(0)}{dw}|\Big)\exp[A_1^2 e^{2/\rho}] = 2\sqrt{\frac{e}{3}}\Big(1 + \frac{1}{(B\rho)^{1/\rho}}|\frac{dy(0)}{dz}|\Big)\exp[A_1^2 e^{2/\rho}].$$

But $z_k(y) = z_k(x)(\rho B)^{1/\rho}$. Now (3.1) implies

$$\sum_{k=1}^{j} \frac{1}{|z_k(y)|} < (\rho B)^{1/\rho}\Big[\theta_1 + \zeta_0 \sum_{k=1}^{j} \frac{1}{\ln^{1/\rho} v_k}\Big] \quad (j = 1, 2, \dots). \tag{3.2}$$

So the following result is holds.

Corollary 3.1. *Let $y(z)$ be a solution of (1.1) with $y(0) = 1$ and condition (1.3) hold. Then inequality (3.2) is valid.*

Furthermore, it can be directly checked that the function

$$y(z) = ce^{-z/2}\sin(e^z) \tag{3.3}$$

with $c = 1/\sin(1)$ is a solution of the equation

$$y''(z) = (\frac{1}{4} - e^{2z})y(z) \tag{3.4}$$

Besides, $y(0) = 1$. Clearly, the zeros of y are $\ln \pi k$ $(k = 0, \pm 1, \pm 2, \dots)$. Hence, for a sufficiently large j we have

$$\sum_{k=1}^{2j} \frac{1}{|z_k(y)|} = \sum_{k=1}^{j} \big[\frac{1}{|z_{2k-1}(y)|} + \frac{1}{|z_{2k}(y)|}\big] = \sum_{k=1}^{j} \big[\frac{1}{\ln \pi k} + \frac{1}{|\ln(-\pi k)|}\big]. \tag{3.5}$$

On the other hand, due to (3.4),$F(z) = \frac{1}{4} - e^{2z}$ and therefore, $|F(z)| \le (1 + \frac{1}{4})e^{2|z|}$. By (3.2) with $B = 2, A = 1 + 1/4$, we have

$$\sum_{k=1}^{j} \frac{1}{|z_k(y)|} < 2\theta_1 + 2\zeta_0 \sum_{k=1}^{j} \frac{1}{\ln(ke^3)} \quad (j = 1, 2, \dots).$$

This result is rather close to (3.5).

Note that, if $F(z)$ is of infinite order, then the problem considered in this paper is much more complicated.

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Michael I. Gil'

Department of Mathematics, Ben Gurion University of the Negev, P.0. Box 653, Beer-Sheva 84105, Israel

E-mail address: gilmi@bezeqint.net