Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 115, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS FOR HARDY-SOBOLEV-MAZ'YA SYSTEMS

JIAN WANG, XIN WEI

Abstract. The main goal of this article is to investigate the existence of solutions for the Hardy-Sobolev-Maz'ya system

$$
\begin{gathered}
-\Delta u-\lambda \frac{u}{|y|^2}=\frac{|v|^{p_t-1}}{|y|^t} v, \quad \text{in } \Omega, \\
-\Delta v-\lambda \frac{v}{|y|^2}=\frac{|u|^{p_s-1}}{|y|^s} u, \quad \text{in } \Omega, \\
u=v=0, \quad \text{on } \partial \Omega
\end{gathered}
$$

where $0 \in \Omega$ which is a bounded, open and smooth subset of $\mathbb{R}^k \times \mathbb{R}^{N-k}$, $2 \leq k<N$. The non-existence of classical positive solutions is obtained by a variational identity and the existence result by a linking theorem.

1. Introduction

In this article, we are concerned with the existence of nontrivial solutions for Hardy-Sobolev-Maz'ya system

$$
\begin{gathered}
-\Delta u-\lambda \frac{u}{|y|^2}=\frac{|v|^{p_t-1}}{|y|^t} v, \quad \text{in } \Omega, \\
-\Delta v-\lambda \frac{v}{|y|^2}=\frac{|u|^{p_s-1}}{|y|^s} u, \quad \text{in } \Omega, \\
u=v=0, \quad \text{on } \partial \Omega,
\end{gathered} \tag{1.1}
$$

where $0 \in \Omega$ which is an open, bounded and smooth domain of $\mathbb{R}^N=\mathbb{R}^k \times \mathbb{R}^{N-k}$ with $2 \leq k<N$. A point $x \in \mathbb{R}^N$ is denoted as $x=(y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. We also give some assumptions for the parameters: $0 \leq \lambda<\frac{(k-2)^2}{4}$ when $k>2$, $\lambda=0$ when $k=2$, $0 \leq t, s<2$ and $p_t, p_s>1$.

The Hardy-Sobolev-Maz'ya elliptic equation:

$$
\begin{gathered}
-\Delta u-\lambda \frac{u}{|y|^2}=\frac{|u|^{p_t-1}}{|y|^t} u, \quad \text{in } \Omega, \\
u=0, \quad \text{on } \partial \Omega,
\end{gathered} \tag{1.2}
$$

2000 *Mathematics Subject Classification.* 35J47, 35J50, 35J57, 58E05.
Key words and phrases. Variational identity; (PS) condition; linking theorem; Hardy-Sobolev-Maz'ya inequality.

Submitted December 26, 2011. Published July 5, 2012.

comes from an astrophysics model with $\Omega = \mathbb{R}^3$, $\lambda = 0$, $t = 1$, (see [2] for details). The existence and regularity of the solution for problem (1.2) in bounded domain have been studied in [3] in subcritical case, that is, $p_t+1 < \frac{2N-2t}{N-2} := 2^*(t)$, and non-existence in super critical was obtained by Pohozaěv identity. It is interesting to investigate with what restrictions on p_t, p_s for existence solutions for system (1.1) by more general variational identity, see [18] for details. The natural functional corresponding to system (1.1) is

$$I_0(u,v) = \int_\Omega \Big(\nabla u\cdot\nabla v - \lambda\frac{uv}{|y|^2} - \frac{1}{p_t+1}\frac{|v|^{p_t+1}}{|y|^t} - \frac{1}{p_s+1}\frac{|u|^{p_s+1}}{|y|^s}\Big)dx$$

in the space $H_0^1(\Omega)\times H_0^1(\Omega)$ with natural exponent region:

$$p_t + 1 < 2^*(t) \quad \text{and} \quad \mathrm{p_s + 1 < 2^*(s)}. \tag{1.3}$$

The quadratic part of the functional I_0, that is, $\int_\Omega(\nabla u\cdot\nabla v - \lambda\frac{uv}{|y|^2})dx$, is positive on the infinite dimensional subspace $\{(u,u)\in H_0^1(\Omega)\times H_0^1(\Omega)\}$ and negative on the infinite dimensional subspace $\{(u,-u)\in H_0^1(\Omega)\times H_0^1(\Omega)\}$. The system is then called strongly indefinite.

There were a significant amount of research on strongly indefinite elliptic systems, see [6, 18, 10, 17]. In particular, in [6, 13], the authors did the existence solutions for the strongly indefinite elliptic systems with the weights. They extended the restriction of the exponent by destroying symmetry of the regularity of solution pair, then obtained the existence results by the linking type theorem. Inspired works in [18, 13, 3, 6], we study that the existence of infinitely many solutions for system (1.1) with

$$\frac{1}{p_t+1}(1-\frac{t}{N}) + \frac{1}{p_s+1}(1-\frac{s}{N}) > \frac{N-2}{N}.$$

which contains natural exponent region (1.3). It could happen that the exponent p_t or p_s is supercritical in the sense that

$$p_t + 1 > 2^*(t) \quad \text{or} \quad p_s + 1 > 2^*(s),$$

where the critical exponent $2^*(s)$ is from the the imbedding from Sobolev space $H_0^1(\Omega)$ to

$$L_s^{p_s+1}(\Omega) = \{u : \int_\Omega \frac{|u|^{p_s+1}}{|y|^s}dx < +\infty\},$$

which is compact if $2 \le p_s + 1 < 2^*(s)$. The main point to solve the problem is to destroy the symmetry between u and v by distributing more regularity of u than that of v if $p_s \ge p_t$. To this end, we define $A^r := (-\Delta - \frac{\lambda}{|y|^2})^{r/2}$, which a positive operator in a fractional Sobolev space $E^r(\Omega) := D(A^r)$. Then it is available to define the functional associated with system (1.1),

$$I(u,v) = \int_\Omega \Big(A^r u A^{2-r} v - \frac{1}{p_t+1}\frac{|v|^{p_t+1}}{|y|^t} - \frac{1}{p_s+1}\frac{|u|^{p_s+1}}{|y|^s}\Big)dx, \tag{1.4}$$

in the fractional Sobolev space $E(\Omega) = E^r(\Omega)\times E^{2-r}(\Omega)$. The functional I has critical points by using linking type theorem (see [12]) in fractional Sobolev spaces $E(\Omega)$. We have then the following existence results.

Theorem 1.1. *Assume that* $0 \le \lambda < \frac{(k-2)^2}{4}$ *if* $k > 2$, $\lambda = 0$ *if* $k = 2$, $0 \le t, s < 2$ *and* $p_t, p_s > 1$ *satisfying*

$$\frac{1}{p_t+1}(1-\frac{t}{N}) + \frac{1}{p_s+1}(1-\frac{s}{N}) > \frac{N-2}{N}, \tag{1.5}$$

then there are infinitely many solutions of system (1.1).

Moreover, we suppose that Ω *is star-sharped with respect to the origin and assumption* (1.5) *fails. Then system* (1.1) *does not have classical positive solution.*

Remark 1.2. Under the assumption (1.5), we do not obtain a positive strong solution of (1.1). Since it lacks regularity results for (1.2) from weak sense to classical sense, then we can't use Maximum Principle. For the regularity results, see [3] for details.

We observe that there are not just one singular point for weight functions in system (1.1), but a manifold $\{(0, z) \in \Omega\}$ with dimension $N - k$. We would like to emphasize that the restriction hyperbola (1.5) does not depend on the dimension number k, one reason for which is the critical exponent of imbedding from $E^r(\Omega) \hookrightarrow L_s^{p_s+1}(\Omega)$ independent of k. To be more precise, $2^*(s)$ defined before is equal to the critical exponent of the imbedding

$$H_0^1(\Omega) \hookrightarrow L^p(\Omega, \frac{1}{|x|^s}) := \{u : \int_\Omega \frac{|u|^p}{|x|^s} dx < +\infty\}.$$

This paper is organized as follows. Section §2 is devoted to study the compact imbedding from fractional Sobolev spaces to weighted spaces. In Section §3 we prove the existence of infinitely many solutions of (1.1). Finally, we do the nonexistence result in Theorem 1.1 by variational identity in Section §4.

2. Compactness of fractional Sobolev space

To destroy the symmetry of regularities between u and v, it is necessary to establish compact imbedding from fractional Sobolev spaces to the weighted spaces

$$L_s^{p_s+1}(\Omega) = \{u : \int_\Omega \frac{|u(x)|^{p_s+1}}{|y|^s} dx < +\infty, x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}\}.$$

Firstly, we introduce interpolation theorem (see [13, 15]). A pair E_0, E_1 of Banach spaces is called an interpolation pair, if E_0 and E_1 are continuously imbedded in some separated topological linear spaces $\mathcal{B}$. Let $A_0,\ A_1$ and E_0, E_1 be interpolation pairs, A_θ and E_θ are called interpolation spaces of exponent $\theta (0 < \theta < 1)$, with respect to $A_0,\ A_1$ and $E_0,\ E_1$ if we have the topological inclusions

$$A_0 \cap A_1 \subset A_\theta \subset A_0 + A_1, \quad E_0 \cap E_1 \subset E_\theta \subset E_0 + E_1,$$

and if each linear mapping T from a separated topological linear space $\mathcal{A}$ into $\mathcal{B}$, which maps A_i continuously into E_i $(i = 0, 1)$ and maps A_θ continuously into E_θ in such a way that

$$M \le M_0^{1-\theta} M_1^\theta,$$

where M denotes the norm of $T : A_\theta \to E_\theta$ and M_i the norm of $T : A_i \to E_i (i = 0, 1)$.

Let E_0, E_1 be the interpolation pairs. It requires the following condition [13, 15]

(H1) For each compact set $\mathcal{K} \in E_0$ there exist a constant $C > 0$ and $\mathcal{D}$ of linear operators $P : \mathcal{B} \to \mathcal{B}$, which map E_i into $E_0 \cap E_1 (i = 0, 1)$ and such that
$$\|P\|_{L(E_i,E_i)} \leq C \quad (i = 0, 1). \tag{2.1}$$
Furthermore, we suppose that to each $\epsilon > 0$ we can find a $P \in \mathcal{D}$ such that
$$\|Px - x\|_{E_0} < \epsilon \tag{2.2}$$
for all $x \in \mathcal{K}$.

Stronger hypothesis of (H1) is the following.

(H2) There exist a constant $C > 0$ and a set $\mathcal{D}$ of linear operators $P : \mathcal{B} \to \mathcal{B}$ with $P(E_i) \subset E_0 \cap E_1 (i = 0, 1)$, such that (2.1) is satisfied and every $\epsilon > 0$ and every finite set $x_1, \dots, x_m$ in E_0 we can find a $P \in \mathcal{D}$ so that
$$\|Px_k - x_k\|_{E_0} \leq \epsilon \quad (k = 1, \dots, m). \tag{2.3}$$

Lemma 2.1 ([15])**.** *Let A_0, A_1 and E_0, E_1 be interpolation pairs and suppose that A_θ and E_θ are interpolation spaces of exponent $\theta (0 < \theta < 1)$ with respect to these pairs. Suppose further that $A_\theta \subset \overline{A_\theta}$ and E_0, E_1 satisfy* (H1)*. Then, if $T : A_0 \to E_0$ is compact and $T : A_1 \to E_1$ is bounded, it follows that $T : A_\theta \to E_\theta$ is compact.*

To establish suitable interpolation pairs, we define the fractional Sobolev space $E^r(\Omega) := D((-\Delta - \frac{\lambda}{|y|^2})^{r/2})$ with $0 \leq r \leq 2$ which is a Hilbert space endowed with the norm $\|u\|_{E^r}^2 = \int_\Omega |A^r u|^2 dx$, induced by the inner product
$$\langle u, v\rangle_{E^r} = \int_\Omega A^r u A^r v dx,$$
where $A^r = (-\Delta - \frac{\lambda}{|y|^2})^{r/2}$.

Now we assume $0 \leq r \leq 2$, and define the interpolation spaces
$$E^r(\Omega) = [H^2(\Omega) \cap H_0^1(\Omega), L^2(\Omega)]_{1-r}.$$
In fact,
$$-\Delta - \frac{\lambda}{|y|^2} : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$$
and $D(-\Delta - \frac{\lambda}{|y|^2}) = D(-\Delta)$. We have the following spaces: $E^s = H^s(\Omega)$ if $0 \leq s < 1/2$; $E^s \subset H^s(\Omega)$ if $s = 1/2$; $E^s = \{u \in H^s(\Omega) : u(x) = 0, x \in \partial\Omega\}$ if $1/2 < s \leq 2$, $s \neq 3/2$; and $E^s \subset \{u \in H^s(\Omega) : u(x) = 0,\ x \in \partial\Omega\}$ if $s = 3/2$. See [12] for details.

Before using Lemma 2.1 to obtain the imbedding $E^r(\Omega) \hookrightarrow L_s^{p_s+1}(\Omega)$, we first prove some basic property of interpolation pair $L_t^{p_t+1}(\Omega)$ and $L^2(\Omega)$ as follows.

Proposition 2.2. *The interpolation pair $L_t^{p_t+1}(\Omega), L^2(\Omega)$ satisfies the condition* (H2).

Proof. Let $f_1, \dots, f_m$ be given functions in $L_t^{p_t+1}(\Omega)$ and we know $L_t^{p_t+1}(\Omega) \hookrightarrow L^2(\Omega)$.

Suppose $\epsilon > 0$ is a given number. As the set $\mathcal{E}$ of all bounded measurable functions with compact support is dense in $L^2(\Omega)$, and then in $L_t^{p_t+1}(\Omega)$, we may assume that $f_j \in \mathcal{E} (j = 1, \dots, m)$. Let K be a compact set in Ω, outside of which all f_j vanish, and choose $\eta > 0$ such that $\eta \max(1, \mu(K)) < \epsilon$, where $\mu(K)$ is the Lebesgue measure of K. We may construct finite cubes $\{K_n := K_n^y \times K_n^z \subset$

$\mathbb{R}^k \times \mathbb{R}^{N-k}\}$ with $\mu(K_n) > 0$ and K_0 of measure zero such that $\sup_{x,x' \in K_n} |f_j(x) - f_j(x')| < \eta$ $(j = 1, \dots, m)$ and the union of K_n including $n = 0$ covers K.

Let φ_n $(n = 1, 2, \dots)$ denote the characterisitic function of K_n and set

$$Pf := \sum_{n=1} (\mu(K_n)^{-1} \int_\Omega f \varphi_n dx) \varphi_n, \quad \text{for all } f \in L_t^{p_t+1}(\Omega).$$

We claim that (2.1) holds for operator P. Indeed, by Hölder's inequality,

$$\begin{aligned}
&\int_\Omega \frac{|Pf|^{p_t+1}}{|y|^t} dx \\
&= \sum_{n=1} \Big(\mu(K_n)^{-1} \int_\Omega f \varphi_n dx \Big)^{p_t+1} \int_{K_n} \frac{\varphi_n^{p_t+1}}{|y|^t} dx \\
&\le \sum_{n=1} \Big[\mu(K_n)^{-1} \Big(\int_{K_n} \frac{|f|^{p_t+1}}{|y|^t} dx \Big)^{\frac{1}{p_t+1}} \Big(\int_{K_n} |y|^{\frac{t}{p_t}} dx \Big)^{\frac{p_t}{p_t+1}} \Big]^{p_t+1} \int_{K_n} \frac{1}{|y|^t} dx \\
&\le \sum_{n=1} \Big[\mu(K_n)^{-1} \int_{K_n} |y|^t dx \Big] \Big[\mu(K_n)^{-1} \int_{K_n} \frac{1}{|y|^t} dx \Big] \int_{K_n} \frac{|f|^{p_t+1}}{|y|^t} dx \\
&= \sum_{n=1} \Big[\mu(K_n^y)^{-1} \int_{K_n^y} |y|^t dy \Big] \Big[\mu(K_n^y)^{-1} \int_{K_n^y} \frac{1}{|y|^t} dy \Big] \int_{K_n} \frac{|f|^{p_t+1}}{|y|^t} dx
\end{aligned}$$

The above equality uses $\mu(K_n) = \mu(K_n^y)\mu(K_n^z)$ and $\int_{K_n} |y|^t dx = \mu(K_n^z) \int_{K_n^y} |y|^t dy$.

So we need only prove

$$\Big[\mu(K_n^y)^{-1} \int_{K_n^y} |y|^t dy \Big] \Big[\mu(K_n^y)^{-1} \int_{K_n^y} \frac{1}{|y|^t} dy \Big] \le C. \tag{2.4}$$

where $C > 0$ is independent of n.

In fact, for $\mu(K_n^y) > 0$, there is $\delta_n > 0$ such that $\mu(K_n^y) = \mu(B_{\delta_n}(0))$, where $B_{\delta_n}(0) \subset \mathbb{R}^k$. Since K_n^y is cube, if $K_n^y \cap B_{\delta_n}(0) \ne \emptyset$, then

$$\int_{K_n^y \cap B_{\delta_n}} |y|^t dy \le \delta_n^t \int_{B_{\delta_n}} dy = \delta_n^t \mu(K_n^y)$$

and

$$\int_{K_n^y \cap B_{\delta_n}^c} |y|^t dy \le (c\delta_n)^t \int_{K_n^y \cap B_{\delta_n}^c} dy \le c^t \delta_n^t \mu(K_n^y),$$

where $c := \sqrt{2}\mu(B_1)^{1/k} + 1$ with B_1 being unit ball of $\mathbb{R}^k$, which imply that

$$\int_{K_n^y} |y|^t dy \le (c^t + 1) \delta_n^t \mu(K_n^y).$$

On the other side, there is $C > 0$ independent of n such that

$$\int_{K_n^y \cap B_{\delta_n}} |y|^{-t} dy \le \int_{B_{\delta_n}} |y|^{-t} dy = \frac{C}{k-t} \delta_n^{-t} \mu(K_n^y)$$

and

$$\int_{K_n^y \cap B_{\delta_n}^c} |y|^{-t} dy \le \delta_n^{-t} \int_{K_n^y \cap B_{\delta_n}^c} dy \le \delta_n^{-t} \mu(K_n^y),$$

which imply that

$$\int_{K_n^y} |y|^{-t} dy \le \frac{C + k - t}{k - t} \delta_n^{-t} \mu(K_n^y).$$

Then we have

$$\Big[\mu(K_n^y)^{-1}\int_{K_n^y}|y|^t dy\Big]\Big[\mu(K_n^y)^{-1}\int_{K_n^y}\frac{1}{|y|^t}dy\Big]\le C. \tag{2.5}$$

for some $C>0$ independent of n.

If $K_n^y\cap B_{\delta_n}(0)=\emptyset$, then we have $r_n:=dist(0,K_n^y)\ge\delta_n$ and

$$\int_{K_n^y}|y|^t dy\le(r_n+c\delta_n)^t\mu(K_n^y),\ \int_{K_n^y}|y|^{-t}dy\le(r_n)^{-t}\mu(K_n^y),$$

which imply

$$[\mu(K_n^y)^{-1}\int_{K_n^y}|y|^t dy][\mu(K_n^y)^{-1}\int_{K_n^y}\frac{1}{|y|^t}dy]\le(\frac{r_n+c\delta_n}{r_n})^t\le(1+c)^t. \tag{2.6}$$

Then (2.4) follows from (2.5) and (2.6). Thus

$$\int_\Omega\frac{|Pf|^{p_t+1}}{|y|^t}dx\le C\sum_{n=1}\int_{K_n}\frac{|f|^{p_t+1}}{|y|^t}dx=C\int_\Omega\frac{|f|^{p_t+1}}{|y|^t}dx;$$

that is,

$$\|Pf\|_{L_t^{p_t+1}(\Omega)}\le C\|f\|_{L_t^{p_t+1}(\Omega)}.$$

Especially, setting $t=0$ and $p_t=1$, we have (2.1). Thus the claim follows. Next, we verify (2.3). Indeed,

$$\begin{aligned}Pf_j(x)-f_j(x)&=\sum_{n=1}[\mu(K_n)^{-1}\int_\Omega f_j(x')\varphi_n(x')dx']\varphi_n(x)-f_j(x)\\&=\sum_{n=1}[\mu(K_n)^{-1}\int_\Omega(f_j(x')-f_j(x))\varphi_n(x')dx']\varphi_n(x)\end{aligned}$$

and

$$|\mu(K_n)^{-1}\int_\Omega(f_j(x')-f_j(x))\varphi_n(x')dx'|\le\eta,\ x\in K_n.$$

It follows that

$$\begin{aligned}\|Pf_j-f_j\|_{L_t^{p_t+1}(\Omega)}&=\|Pf_j-f_j\|_{L_t^{p_t+1}(K)}\\&\le\sum_{n=1}[\mu(K_n)^{-1}\int_\Omega(f_j(x')-f_j(x))\varphi_n(x')dx']\varphi_n(x)\\&\le\eta\sum_{n=1}\mu(K_n)=\eta\mu(K)<\epsilon;\end{aligned}$$

i.e., (2.3) holds. The proof is complete. $\square$

Now we give the general imbedding theorem by the interpolation Lemma 2.1.

Theorem 2.3. *The imbedding $E^r(\Omega)\hookrightarrow L_s^{p_s+1}(\Omega)$ is is compact if $2\le p_s+1<\frac{2N-2s}{N-2r}$.*

Proof. We define the interpolation space,

$$L_s^q(\Omega)=[L_t^{p_t+1}(\Omega),L^2(\Omega)]_{1-r}.$$

We claim next that $2\le q\le\frac{2N-2s}{N-2r}$. In fact, for any $u\in L_t^{p_t+1}(\Omega)$, by using Hölder's inequality, one obtains

$$\int_\Omega\frac{|u|^q}{|y|^s}dx=\int_\Omega\frac{|u|^{2\gamma+(p_t+1)(1-\gamma)}}{|y|^s}dx\le\Big(\int_\Omega|u|^2dx\Big)^\gamma\Big(\int_\Omega\frac{|u|^{p_t+1}}{|y|^{\frac{s}{1-\gamma}}}dx\Big)^{1-\gamma},$$

where $\gamma = \frac{p_t+1-q}{p_t-1} \in (0,1)$.

Let $\theta = \frac{2\gamma}{q}$, then $\frac{(p_t+1)(1-\gamma)}{q} = 1-\theta$ and

$$\Big(\int_\Omega \frac{|u|^q}{|y|^s}dx\Big)^{\frac{1}{q}} \le \|u\|^\theta_{L^2(\Omega)}\Big(\int_\Omega \frac{|u|^{p_t+1}}{|y|^{\frac{s}{1-\gamma}}}dx\Big)^{\frac{1-\theta}{p_t+1}} = \|u\|^\theta_{L^2(\Omega)}\Big(\int_\Omega \frac{|u|^{p_t+1}}{|y|^{\frac{2s}{2-q(1-r)}}}dx\Big)^{\frac{1-\theta}{p_t+1}},$$

where $r = 1-\theta$. The critical exponent of $\int_\Omega \frac{|u|^{p_t+1}}{|y|^{\frac{2s}{2-q(1-r)}}}dx$ is $(p_t+1)^*(r) = \frac{2(N-\frac{2s}{2-q(1-r)})}{N-2}$. Requiring $p_t+1 \le (p_t+1)^*(r)$, we obtain

$$2 \le q \le \frac{2N-2s}{N-2r}.$$

Hence, the claim is true.

By Proposition 2.2, we know that interpolation pair $L^{p_t+1}_t(\Omega)$, $L^2(\Omega)$ has property (H2). And the imbedding

$$H^1_0(\Omega) \hookrightarrow L^{p_s+1}_s(\Omega)$$

is compact if $2 \le p_s+1 < 2^*(s)$. Then by Lemma 2.1, we obtain the results. $\square$

Similarly, we have $E^{2-r}(\Omega) \hookrightarrow L^{p_t+1}_t(\Omega)$, if $2 \le \mathrm{p_t}+1 \le \frac{\mathrm{2N-2t}}{\mathrm{N+2r-4}}$. Hence we obtain the following theorem.

Theorem 2.4. *The imbedding of the fractional Sobolev space*

$$E(\Omega) = E^r(\Omega) \times E^{2-r}(\Omega) \hookrightarrow L^{p_s+1}_s(\Omega) \times L^{p_t+1}_t(\Omega)$$

is compact, where $2 \le p_s+1 < \frac{2N-2s}{N-2r}$, $2 \le p_t+1 < \frac{2N-2t}{N+2r-4}$ *and* $0<r<2$.

Remark 2.5. If $2 \le p_s+1 < \frac{2N-2s}{N-2r}, 2 \le p_t+1 < \frac{2N-2t}{N+2r-4}$, where $0<r<2$, then (1.5) holds. Conversely, for $p_s, p_t > 1$ satisfying (1.5), then there exists $r \in (0,2)$ such that $2 \le p_s+1 < \frac{2N-2s}{N-2r}, 2 \le p_t+1 < \frac{2N-2t}{N+2r-4}$.

Lemma 2.6. *Suppose that Ω is an open, smooth and bounded domain and $\lambda \in [0,(k-2)^2/4)$. Then there exists a sequence eigenvalues $(\mu_n)_n$ and corresponding eigenfunctions $(\varphi_n)_n$ of*

$$\begin{gathered} -\Delta u - \lambda \frac{u}{|y|^2} = \mu u, \\ u \in H^1_0(\Omega) \end{gathered} \tag{2.7}$$

such that $0 < \mu_1 < \mu_2 \le \dots \le \mu_n \dots \to +\infty$ as $n \to +\infty$, $\|\varphi_n\|_{H^1_0(\Omega)} = \mu_n\|\varphi_n\|_{L^2(\Omega)}$, where $\|\varphi_n\|^2_{H^1_0(\Omega)} = \int_\Omega(|\nabla\varphi_n|^2 - \lambda\frac{\varphi_n^2}{|y|^2})dx$.

Proof. Since $\lambda \in [0,(k-2)^2/4)$, then the norm $(\int_\Omega |\nabla u|^2dx)^{1/2}$ is equivalent to $[\int_\Omega(|\nabla u|^2 - \lambda\frac{u^2}{|y|^2})dx]^2$ in Hilbert space $H^1_0(\Omega)$. We observe that the operator $S = (-\Delta - \frac{\lambda}{|y|^2})^{-1}$ is symmetric and compact, following the the proceeding the proof of Theorem 2 in Chapter§6.5 in [7], we will have the results. $\square$

We end this section with the fact that $H^1_0(\Omega) = \overline{\mathrm{span}_{n\in\mathbb{N}}\{\varphi_n\}}$ and the space E^r could be expressed by

$$E^r = D(A^r) = \big\{u = \sum_{n=1}^{+\infty} a_n\varphi_n \in L^2(\Omega) : \sum_{n=1}^{+\infty}\mu_n^r a_n^2 < +\infty\big\}.$$

3. Existence of infinitely many solutions of (1.1)

In this section, we do the existence of infinitely many solutions of (1.1). We first recall one type of linking theorem in [9] (see also [6]) that provides us with infinitely many critical points of I. We split Hilbert space $E = X \oplus Y$ where X and Y are both infinite dimensional subspaces. Assume there exists a sequence of finite dimensional subspaces $X_n \subset X$, $Y_n \subset Y$, $E_n = X_n \oplus Y_n$ such that $\overline{\cup_{n=1}^{\infty} E_n} = E$. Let $T : E \to E$ be a linear bounded invertible operator.

We say that the functional I satisfies the $(PS)^*$ condition with respect to E_n, if any sequence $\{\mathbf{u}_j\} \subset E_{n_j}$ with $n_j \to \infty$ as $j \to +\infty$, such that

$$|I(\mathbf{u}_j)| \to c \quad \text{and} \quad I|'_{E_{n_j}}(\mathbf{u}_j) \to 0$$

possesses a subsequence converging in E.

Let $S_\rho = \{y \in Y, \|y\|_E = \rho\}$, fix $y_1 \in Y$ with $\|y_1\|_E = 1$ and subspaces Z_1, Z_2 such that

$$X \oplus \operatorname{span}\{y_1\} = Z_1 \oplus Z_2 \quad \text{and} \quad y_1 \in Z_2.$$

We next define, for $M, \sigma > 0$,

$$D = D_{M,\sigma} = \{x_1 + x_2 \in Z_1 \oplus Z_2, \|x_1\|_E \le M, \|x_2\|_E \le \sigma\}.$$

The following linking theorem is used to prove the existence result for system (1.1).

Theorem 3.1 ([9])**.** *Suppose that $I \in C^1(E, \mathbb{R})$ be an even functional. We assume that:*

- (L1) *I satisfies $(PS)^*$ condition with respect to E_n,*
- (L2) *$T : E_n \to E_n$, for n large, and $\sigma, \rho > 0$ satisfy $\sigma\|Ty_1\|_E > \rho$,*
- (L3) *There are constants $\alpha \le \beta$ such that*

$$\inf_{S_\rho \cap E_n} I \ge \alpha, \quad \sup_{T(\partial D \cap E_n)} I < \alpha \quad \text{and} \quad \sup_{T(D \cap E_n)} I \le \beta$$

for all n large.

Then I has a critical value $c \in [\alpha, \beta]$.

To apply Theorem 3.1 for solving our problem, we recall that the functional I is defined in (1.4) in $E(\Omega)$, which is a product Hilbert space defined by

$$E(\Omega) = E^r(\Omega) \times E^{2-r}(\Omega), \quad 0 < r < 2,$$

with the norm

$$\|\mathbf{u}\|_E^2 = \int_\Omega |A^r u|^2 + |A^{2-r} v|^2 dx = \|u\|_{E^r}^2 + \|v\|_{E^{2-r}}^2, \quad \mathbf{u} = (u, v) \in E(\Omega),$$

which is induced by inner product

$$\langle \mathbf{u}, \mathbf{w} \rangle_E = \langle u, \varphi \rangle_{E^r} + \langle v, \psi \rangle_{E^{2-r}},$$

where $\mathbf{u} = (u, v) \in E(\Omega)$ and $\mathbf{w} = (\varphi, \psi) \in E(\Omega)$.

We define $E^+ := \{(u, A^{r-2}A^r u) |\ u \in E^r\}$ and $E^- := \{(u, -A^{r-2}A^r u) |\ u \in E^r\}$, then E has orthogonal decomposition:

$$E(\Omega) = E^+ \oplus E^- = \{\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-, \mathbf{u}^\pm \in E^\pm\}.$$

Let

$$E_n = \operatorname{span}\{\varphi_1, \dots, \varphi_n\} \times \operatorname{span}\{\varphi_1, \dots, \varphi_n\}.$$

We first prove that I satisfies the $(PS)^*$ condition with respect to E_n.

Lemma 3.2. *The functional I satisfies $(PS)^*$ condition with respect to E_n.*

Proof. Suppose $\{\mathbf{u}_j\} \subset E_{n_j}$ be a sequence such that
$$I(\mathbf{u}_j) \to c, \quad I|'_{E_{n_j}}(\mathbf{u}_j) \to 0.$$
We claim that $(\mathbf{u}_j)$ is bounded in E. Taking $\mathbf{w} = \mathbf{u}_j$, we obtain for a sequence positive numbers $\epsilon_j \to 0$ as $j \to +\infty$
$$\begin{aligned} C + \varepsilon_j \|\mathbf{u}_j\|_E &\geq I(\mathbf{u}_j) - \frac{1}{2}\langle I'(\mathbf{u}_j), \mathbf{u}_j\rangle \\ &= \int_\Omega \big[(\frac{1}{2} - \frac{1}{p_t+1})\frac{|v_j|^{p_t+1}}{|y|^t} + (\frac{1}{2} - \frac{1}{p_s+1})\frac{|u_j|^{p_s+1}}{|y|^s}\big]dx. \end{aligned}$$
Then we have
$$\int_\Omega \frac{|v_j|^{p_t+1}}{|y|^t}dx + \int_\Omega \frac{|u_j|^{p_s+1}}{|y|^s}dx \leq C + \varepsilon_j\|\mathbf{u}_j\|_E.$$
Noting $\mathbf{u}_j = \mathbf{u}_j^+ + \mathbf{u}_j^-$, $\mathbf{u}_j^\pm = (u_j^\pm, v_j^\pm)$, we have
$$\|\mathbf{u}_j^\pm\|_E^2 - \varepsilon_j\|\mathbf{u}_j^\pm\|_E \leq |\int_\Omega (\frac{|v_j|^{p_t-1}}{|y|^t}v_j v_j^\pm + \frac{|u_j|^{p_s-1}}{|y|^s}u_j u_j^\pm)dx|.$$
By Hölder's inequality,
$$\begin{aligned} |\int_\Omega \frac{|v_j|^{p_t-1}}{|y|^t}v_j v_j^\pm dx| &\leq (\int_\Omega \frac{|v_j|^{p_t+1}}{|y|^t}dx)^{\frac{p_t}{p_t+1}}(\int_\Omega \frac{|v_j^\pm|^{p_t+1}}{|y|^t}dx)^{\frac{1}{p_t+1}} \\ &\leq (\int_\Omega \frac{|v_j|^{p_t+1}}{|y|^t}dx)^{\frac{p_t}{p_t+1}}\|v_j^\pm\|_{E^{2-r}} \\ &\leq (\int_\Omega \frac{|v_j|^{p_t+1}}{|y|^t}dx)^{\frac{p_t}{p_t+1}}\|\mathbf{u}_j^\pm\|_E. \end{aligned}$$
Similarly, we have
$$\begin{aligned} |\int_\Omega \frac{|u_j|^{p_s-1}}{|y|^s}u_j u_j^\pm dx| &\leq (\int_\Omega \frac{|u_j|^{p_s+1}}{|y|^s}dx)^{\frac{p_s}{p_s+1}}\|u_j^\pm\|_{E^r} \\ &\leq (\int_\Omega \frac{|u_j|^{p_s+1}}{|y|^s}dx)^{\frac{p_s}{p_s+1}}\|\mathbf{u}_j^\pm\|_E. \end{aligned}$$
Then we obtain
$$\begin{aligned} \|\mathbf{u}_j^\pm\|_E - \varepsilon_j &\leq (\int_\Omega \frac{|v_j|^{p_t+1}}{|y|^t}dx)^{\frac{p_t}{p_t+1}} + (\int_\Omega \frac{|u_j|^{p_s+1}}{|y|^s}dx)^{\frac{p_s}{p_s+1}} \\ &\leq (C + \varepsilon_j\|\mathbf{u}_j\|_E)^{\frac{p_t}{p_t+1}} + (C + \varepsilon_j\|\mathbf{u}_j\|_E)^{\frac{p_s}{p_s+1}}, \end{aligned}$$
which yields $\|\mathbf{u}_j\|_E \leq C$ uniformly in j.

By Theorem 2.3, u_j and v_j have subsequences which converge strongly in $L_s^{p_s+1}$ and $L_t^{p_t+1}$, respectively. Then we obtain $\mathbf{u}_j = (u_j, v_j)$ possesses a subsequence converging in a standard way. Hence I satisfies $(PS)^*$ condition with respect to E_n. □

Now fix j, and we split E_n into $X_n \oplus Y_n$, where $X_n = (E_1^- \oplus \dots \oplus E_n^-) \oplus (E_1^+ \oplus \dots \oplus E_{j-1}^+)$ and $Y_n = E_j^+ \oplus \dots \oplus E_n^+$ with $E_i^+ = \operatorname{span}\{(\varphi_i, A^{r-2}A^r\varphi_i)\}$, $E_i^- = \operatorname{span}\{(\varphi_i, -A^{r-2}A^r\varphi_i)\}$. Next we define, for $\mathbf{u} = (u,v) \in E$
$$T_\sigma(\mathbf{u}) = (\sigma^{\mu-1}u, \sigma^{\nu-1}v), \tag{3.1}$$

where $\mu,\nu > 1$ will be chosen latter. By (3.1), (L2) holds for T and $y_1 = (\varphi_j, A^{r-2}A^r\varphi_j)$.

In what follows, we prove (L3) under our assumptions above.

Lemma 3.3. *There exist $\alpha_j > 0$ and $\rho_j > 0$ independent of n such that for all $n \geq j$*

$$\inf_{S_{\rho_j}\cap Y_n} I \geq \alpha_j,$$

where $Y = E_j^+ \oplus\cdots\oplus E_n^+ \oplus\dots$ and $S_{\rho_j} = \{y \in Y, \|y\|_E = \rho_j\}$. Moreover, $\alpha_j \to +\infty$ as $j \to +\infty$.

Proof. For $\mathbf{u} = (u, v) \in Y$, we have

$$\|u\|_{E^r}^2 \geq \mu_j^{\min\{r,2-r\}}\|u\|_{L^2}^2 \quad \text{and} \quad \|v\|_{E^{2-r}}^2 \geq \mu_j^{\min\{r,2-r\}}\|v\|_{L^2}^2.$$

By Theorem 2.3 and Hölder inequality, we have that

$$\|u\|_{L_s^{p_s+1}}^{p_s+1} \leq \|u\|_{L^2}^{2\kappa}\|u\|^{p_s+1-2\kappa}_{L^{\frac{p_s+1-2\kappa}{1-\kappa}}_{\frac{s}{1-\kappa}}} \leq \frac{C}{\mu_j^{\min\{r,2-r\}\kappa}}\|u\|_{E^r}^{p_s+1} \leq \frac{C}{\mu_j^{\min\{r,2-r\}\kappa}}\|\mathbf{u}\|_E^{p_s+1}$$

and

$$\|v\|_{L_t^{p_t+1}} \leq \frac{C}{\mu_j^{\min\{r,2-r\}\bar\kappa}}\|v\|_{E^{2-r}}^{p_t+1} \leq \frac{C}{\mu_j^{\min\{r,2-r\}\bar\kappa}}\|\mathbf{u}\|_E^{p_t+1}$$

for some constants $\kappa, \bar\kappa \in (0,1)$ such that

$$E^r \hookrightarrow L^{\frac{p_s+1-2\kappa}{1-\kappa}}_{\frac{s}{1-\kappa}}(\Omega) \quad \text{and} \quad E^{2-r} \hookrightarrow L^{\frac{p_t+1-2\bar\kappa}{1-\bar\kappa}}_{\frac{t}{1-\bar\kappa}}(\Omega)$$

are continuous, and $C > 0$ independent of n. Then we have that for $\mathbf{u} = (u, v) \in Y$,

$$\begin{aligned} I(\mathbf{u}) &= \int_\Omega (|A^r u|^2 - \frac{1}{p_t+1}\frac{|v|^{p_t+1}}{|y|^t} - \frac{1}{p_s+1}\frac{|u|^{p_s+1}}{|y|^s})dx \\ &\geq \frac{1}{2}\|\mathbf{u}\|_E^2 - \frac{C}{\mu_j^{\min\{r,2-r\}\min\{\kappa,\bar\kappa\}}}(\|\mathbf{u}\|_E^{p_s+1} + \|\mathbf{u}\|_E^{p_t+1}). \end{aligned}$$

By choosing $2\rho_j^{\max\{p_s+1,p_t+1\}} = \mu_j^{\min\{r,2-r\}\min\{\kappa,\bar\kappa\}}$, we have for $\mathbf{u} \in S_{\rho_j} \cap Y_n$

$$I(\mathbf{u}) \geq \frac{1}{2}\rho_j^2 - C =: \alpha_j,$$

and we finished the proof. $\square$

Lemma 3.4. *There exist $\beta_j \geq \alpha_j$, $M_j > 0$ and $\sigma_j > \rho_j$ independent of n such that for all $n \geq j$*

$$\sup_{T_{\sigma_j}(\partial D\cap E_n)} I < \alpha_j \quad \textit{and} \quad \sup_{T_{\sigma_j}(D\cap E_n)} I \leq \beta_j,$$

where

$$D = \{\mathbf{u} \in E^- \oplus E_1^+ \oplus \cdots \oplus E_j^+, \|\mathbf{u}^-\|_E \leq M_j, \|\mathbf{u}^+\|_E \leq \sigma_j\}.$$

Proof. Let $\mathbf{z} = T_{\sigma_j}(\mathbf{u})$ with $\mathbf{u} \in D$. Then we can write

$$\mathbf{z} = (\sigma_j^{\mu-1}u^+, \sigma_j^{\nu-1}A^{r-2}A^r u^+) + (\sigma_j^{\mu-1}u^-, -\sigma_j^{\nu-1}A^{r-2}A^r u^-),$$

where $\mu,\nu > 1$ will be chosen latter, u^+ and u^- can be written as

$$u^+ = \sum_{i=1}^j \theta_i\varphi_i \quad \text{and} \quad u^- = \sum_{i=1}^j \gamma_i\varphi_i + \tilde u^-$$

where $\tilde{u}^-$ is orthogonal to φ_i, $i = 1, \dots, j$ in $L^2(\Omega)$. Using Holder's inequality and the equivalence of all the norms in finite dimensional space, we get

$$\sum_{i=1}^{j} \mu_i^{2r-2}(\theta_i^2 + \theta_i\gamma_i) = \langle u^+ + u^-, A^{r-2}A^r u^+\rangle \le C_j\|u^+ + u^-\|_{L_s^{p_s+1}}\|u^+\|_{L^2} \quad (3.2)$$

and

$$\sum_{i=1}^{j} \mu_i^{2r-2}(\theta_i^2 - \theta_i\gamma_i) = \langle v^+ + v^-, A^{r-2}A^r v^+\rangle \le C_j\|v^+ + v^-\|_{L_t^{p_t+1}}\|u^+\|_{L^2}. \quad (3.3)$$

If $\sum_{i=1}^{j} \theta_i\gamma_i \ge 0$, then (3.2) implies

$$\|u^+\|_{L^2} \le C_j\|u^+ + u^-\|_{L_s^{p_s+1}} = C_j\|u\|_{L_s^{p_s+1}},$$

otherwise, (3.3) implies

$$\|u^+\|_{L^2} \le C_j\|v^+ + v^-\|_{L_t^{p_t+1}} = C_j\|v\|_{L_t^{p_t+1}}.$$

Hence,

$$I(\mathbf{u}) \le \frac{1}{2}\sigma_j^{\mu+\nu-2}(\|\mathbf{u}^+\|_E^2 - \|\mathbf{u}^-\|_E^2) - C_j\sigma_j^{(p_s+1)(\mu-1)}\|u^+\|_{L^2}^{p_s+1}$$

or

$$I(\mathbf{u}) \le \frac{1}{2}\sigma_j^{\mu+\nu-2}(\|\mathbf{u}^+\|_E^2 - \|\mathbf{u}^-\|_E^2) - C_j\sigma_j^{(p_t+1)(\nu-1)}\|u^+\|_{L^2}^{p_t+1}.$$

Thus we may choose $\|\mathbf{u}^+\|_E = \sigma_j$ large enough in order to obtain $\sigma_k > \rho_k$ and it is possible to choose $\mu, \nu > 1$ such that $(p_t+1)(\mu-1) > \mu+\nu-2$ and $(p_s+1)(\nu-1) > \mu + \nu - 2$ if

$$\frac{1}{p_t + 1} + \frac{1}{p_s + 1} < 1,$$

$p_t, p_s > 1$ makes sure that the estimate above holds. Then, $I(\mathbf{u}) \le 0$.

Taking $\|\mathbf{u}^+\|_E \le \sigma_j$ and $\|\mathbf{u}^-\|_E = M_j$, we obtain

$$I(z) \le \sigma_j^{\mu+\nu-2}(\sigma_j^2 - M_j^2) \le 0$$

if $M_j \ge \sigma_j$.

Then we choose β_j large so that the second inequality holds. □

Proof of existence of infinitely many solutions in Theorem 1.1. Combining Lemma 3.2, Lemma 3.3 with Lemma 3.4, I satisfies the conditions (L1)–(L3). By Theorem 3.1, I has a sequence of critical values in $[\alpha_j, \beta_j]$ and $\alpha_j \to +\infty$ as $j \to +\infty$, then there exist a sequence critical points of I, which are infinite many solutions of (1.1). We finish the proof.

4. Nonexistence result

In this section, we show the nonexistence of solution in Theorem 1.1. To obtain this nonexistence result, we introduce some lemmas.

Assume that the Euler-Lagrange equations are

$$div(\frac{\partial L}{\partial p_i^k}) - \frac{\partial L}{\partial u_k} = 0, \quad k = 1, \dots, s. \quad (4.1)$$

$i = 1, \dots, N$; where $\mathbf{u} = (u_k)$, $\mathbf{p} = (p_i^k)$, $p_i^k = \frac{\partial u_k}{\partial x^i}$, and Ω is a bounded and smooth domain in $\mathbb{R}^N$. We have the following result.

Lemma 4.1 ([18])**.** *Let $L \in C^1(\Omega \times \mathbb{R}^s \times \mathbb{R}^{N\times s})$ and $\mathbf{u} = (u_1, \dots, u_s) : \Omega \to \mathbb{R}^s$ be a solution of* (4.1) *with $u_k \in C^2(\Omega)$. Let $a_{kl}, h^i \in C^1(\Omega)$. Then*

$$\begin{aligned}&\operatorname{div}(h^i L - h^j \frac{\partial u_k}{\partial x^j}\frac{\partial L}{\partial p_i^k} - a_{kl}u_l \frac{\partial L}{\partial p_i^k})\\ &= \frac{\partial h^i}{\partial x^i}L + h^i\frac{\partial L}{\partial x^i} - (\frac{\partial u_k}{\partial x^j}\frac{\partial h^j}{\partial x^i} + u_l\frac{\partial a_{kl}}{\partial x^i})\frac{\partial L}{\partial p_i^k} - a_{kl}(\frac{\partial u_l}{\partial x^i}\frac{\partial L}{\partial p_i^k} + u_l\frac{\partial L}{\partial u_k}), \quad \text{in } \Omega.\end{aligned} \tag{4.2}$$

Furthermore,

$$\begin{aligned}&\oint_{\partial\Omega}((h^i L - h^j \frac{\partial u_k}{\partial x^j}\frac{\partial L}{\partial p_i^k} - a_{kl}u_l\frac{\partial L}{\partial p_i^k}), n)ds\\ &= \int_\Omega(\frac{\partial h^i}{\partial x^i}L + h^i\frac{\partial L}{\partial x^i} - (\frac{\partial u_k}{\partial x^j}\frac{\partial h^j}{\partial x^i} + u_l\frac{\partial a_{kl}}{\partial x^i})\frac{\partial L}{\partial p_i^k} - a_{kl}(\frac{\partial u_l}{\partial x^i}\frac{\partial L}{\partial p_i^k} + u_l\frac{\partial L}{\partial u_k}))dx,\end{aligned} \tag{4.3}$$

where n is the outward normal on $\partial\Omega$.

By choosing suitable functions in the above lemma, we obtain the following result.

Lemma 4.2. *Let $(u, v) \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2$ be a solution of problem* (1.1)*. Then u and v satisfy the identity*

$$\begin{aligned}\oint_{\partial\Omega}(\nabla u\cdot\nabla v)(x,n)ds = \int_\Omega\{&(2 + a_{11} + a_{22} - N)(\nabla u \cdot \nabla v - \frac{\lambda uv}{|y|^2})\\ &+ (\frac{N-t}{p_t+1} - a_{22})\frac{|v|^{p_t+1}}{|y|^t} + (\frac{N-s}{p_s+1} - a_{11})\frac{|u|^{p_s+1}}{|y|^s}\}dx,\end{aligned} \tag{4.4}$$

where a_{11} and a_{22} are constants to be chosen latter.

Proof. For our system (1.1), we define

$$L = \Sigma_{i=1}^N p_i^1 p_i^2 - \lambda\frac{uv}{|y|^2} - \frac{1}{p_t+1}\frac{|v|^{p_t+1}}{|y|^t} - \frac{1}{p_s+1}\frac{|u|^{p_s+1}}{|y|^s}, \tag{4.5}$$

where $p_i^1 = \frac{\partial u}{\partial x^i}$, $p_i^2 = \frac{\partial v}{\partial x^i}$, $x^i = y^i$ if $i \le k$ and (u, v) is a classical solution of system (1.1).

We give explicitly the values of parameters for using Lemma 4.1: $k = 1, 2$, $a_{11}(x) = a_{11}, a_{22}(x) = a_{22}$, $a_{12}(x) = a_{21}(x) = 0$ and $h^i(x) = x^i$ where $i = 1, \dots, N$. For the purpose of deleting the singularity of L at the domain $U = \{x = (y, z) \in \Omega : y = 0\}$, assume that $N_\delta(U) = \{x \in \Omega : dist(x, U) \le \delta\}$ and $\Omega_\delta = \Omega \setminus N_\delta(U)$, where $\delta > 0$. And we have $\partial\Omega_\delta = (\partial\Omega \setminus \partial N_\delta(U)) \cup (\partial N_\delta(U) \setminus \partial\Omega)$.

Since $u(x) = v(x) = 0$, $x \in \partial\Omega$, we have

$$x^j\frac{\partial u}{\partial x^j}\frac{\partial v}{\partial x^i}n_i = \frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^i}x^j n_j,$$

which follows from $\frac{\partial u}{\partial x^i} = \frac{\partial u}{\partial n}n_i$, $\frac{\partial v}{\partial x^i} = \frac{\partial v}{\partial n}n_i$. Then the left-hand side of (4.3) is

$$\begin{aligned}&\oint_{\partial N_\delta(U)\setminus\partial\Omega}(\nabla u\cdot\nabla v - \lambda\frac{uv}{|y|^2} - \frac{1}{p_t+1}\frac{|v|^{p_t+1}}{|y|^t} - \frac{1}{p_s+1}\frac{|u|^{p_s+1}}{|y|^s})(x,n)ds\\ &- \oint_{\partial N_\delta(U)\setminus\partial\Omega}((\Sigma_{j=1}^N x^j\frac{\partial u}{\partial x^j}\frac{\partial v}{\partial x^i} + \Sigma_{j=1}^N x^j\frac{\partial v}{\partial x^j}\frac{\partial u}{\partial x^i} + a_{11}u\frac{\partial v}{\partial x^i} + a_{22}v\frac{\partial u}{\partial x^i}), n)ds\end{aligned}$$

$$- \oint_{\partial\Omega\setminus\partial N_\delta(U)} (\nabla u \cdot \nabla v)(x,n)ds.$$

We claim that the first two terms in the quantity above go to zero as $\delta \to 0$. In fact, $|\nabla u|$, $|\nabla v|$, u and v are bounded and $\lim_{\delta\to 0} |\partial N_\delta(U) \setminus \partial\Omega| = 0$, we have the first two terms go to zero. For the third term, since $\lim_{\delta\to 0} \partial\Omega \setminus \partial N_\delta(U) = \partial\Omega \setminus \{(0,z) \in \partial\Omega\}$ and $|\partial\Omega \setminus \{(0,z) \in \partial\Omega\}| = |\partial\Omega|$, we obtain that this term tends to

$$- \oint_{\partial\Omega} (\nabla u \cdot \nabla v)(x,n)ds.$$

Hence, the left hand of (4.3) tends to

$$- \oint_{\partial\Omega} (\nabla u \cdot \nabla v)(x,n)ds.$$

In the following, we do estimate the right hand side of (4.3). After calculating, the right hand side of (4.3) with integrate domain being Ω_δ,

$$\begin{aligned}&\int_{\Omega_\delta} \big\{(N-2-a_{11}-a_{22})(\nabla u \cdot \nabla v - \frac{\lambda uv}{|y|^2}) - (\frac{N-t}{p_t+1} - a_{22})\frac{|v|^{p_t+1}}{|y|^t}\\ &- (\frac{N-s}{p_s+1} - a_{11})\frac{|u|^{p_s+1}}{|y|^s}\big\}dx.\end{aligned}$$

Since $\lim_{\delta\to 0^+} \Omega_\delta = \Omega \setminus U$ and $|\Omega \setminus U| = |\Omega|$, the right-hand side of (4.3) tends to

$$\begin{aligned}&\int_{\Omega} \{(N-2-a_{11}-a_{22})(\nabla u \cdot \nabla v - \frac{\lambda uv}{|y|^2}) - (\frac{N-t}{p_t+1} - a_{22})\frac{|v|^{p_t+1}}{|y|^t}\\ &- (\frac{N-s}{p_s+1} - a_{11})\frac{|u|^{p_s+1}}{|y|^s}\}dx.\end{aligned}$$

Thus, using the Lemma 4.1, this yields (4.4). □

Now we use Lemma 4.2 to obtain the following nonexistence result.

Theorem 4.3. *Suppose that Ω is star-sharped with respect to the origin. Let $0 \le \lambda < \frac{(k-2)^2}{4}$ if $k > 2$, $\lambda = 0$ if $k = 2$, $0 \le t, s < 2$ and $p_t, p_s > 1$ satisfying*

$$\frac{1}{p_t+1}(1-\frac{t}{N}) + \frac{1}{p_s+1}(1-\frac{s}{N}) \le \frac{N-2}{N}. \tag{4.6}$$

Then system (1.1) *does not have any classical positive solution.*

Proof. Suppose (u,v) is classical positive solution of system (1.1), then $u(x) = v(x) = 0$, $x \in \partial\Omega$. Since Ω is star-shaped with respect to the origin, then $(x^0, n) \ge 0$ for all $x^0 \in \partial\Omega$ and $(x^0,n) > 0$ on some subset of $\partial\Omega$ of positive measure (see [16]). And applying Hopf's Lemma (see [7]), we have

$$\frac{\partial u}{\partial n}(x^0) = (\nabla u, n)|_{x=x^0} < 0, \quad \frac{\partial v}{\partial n}(x^0) = (\nabla v, n)|_{x=x^0} < 0,$$

and $\frac{\partial u}{\partial x^i} = \frac{\partial u}{\partial n} n_i$, $\frac{\partial v}{\partial x^i} = \frac{\partial v}{\partial n} n_i$ when $x \in \partial\Omega$, which implies

$$(\nabla u \cdot \nabla v)|_{x=x^0} = \frac{\partial u}{\partial n}\frac{\partial v}{\partial n}|_{x=x^0} > 0.$$

Then the left-hand side of (4.4) in Lemma 4.2 is

$$\oint_{\partial\Omega} (\nabla u \cdot \nabla v)(x,n)ds > 0,$$

and now by choosing $a_{11} = N - 2 - \frac{N-t}{p_t+1}$, $a_{22} = \frac{N-t}{p_t+1}$ in (4.4), it yields

$$\int_\Omega [\frac{N-s}{p_s+1} - (N-2-\frac{N-t}{p_t+1})] \frac{|u|^{p_s+1}}{|y|^s} dx > 0.$$

Therefore, we obtain p_s, p_t satisfy the formulation

$$\frac{N-s}{p_s+1} - (N-2-\frac{N-t}{p_t+1}) > 0,$$

which contradicts (4.6). This complete the proof of Theorem 4.3. $\square$

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