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MULTIPLE POSITIVE SOLUTIONS FOR A THIRD-ORDER THREE-POINT BVP WITH SIGN-CHANGING GREEN'S FUNCTION

JIAN-PING SUN, JUAN ZHAO

Abstract. This article concerns the third-order three-point boundary-value problem

$$u'''(t) = f(t, u(t)), \quad t \in [0, 1],$$
$$u'(0) = u(1) = u''(\eta) = 0.$$

Although the corresponding Green's function is sign-changing, we still obtain the existence of at least $2m-1$ positive solutions for arbitrary positive integer m under suitable conditions on f.

1. Introduction

Third-order differential equations arise from a variety of areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [5].

Recently, the existence of single or multiple positive solutions to some third-order three-point boundary-value problems (BVPs for short) has received much attention from many authors. For example, in 1998, by using the Leggett-Williams fixed point theorem, Anderson [2] proved the existence of at least three positive solutions to the problem

$$-x'''(t) + f(x(t)) = 0, \quad t \in [0, 1],$$
$$x(0) = x'(t_2) = x''(1) = 0,$$

where $t_2 \in [\frac{1}{2}, 1)$. In 2003, Anderson [1] obtained some existence results of positive solutions for the problem

$$x'''(t) = f(t, x(t)), \quad t_1 \le t \le t_3,$$
$$x(t_1) = x'(t_2) = 0, \quad \gamma x(t_3) + \delta x''(t_3) = 0.$$

The main tools used were the Guo-Krasnosel'skii and Leggett-Williams fixed point theorems. In 2005, Sun [13] studied the existence of single and multiple positive

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solutions for the singular BVP

$$u'''(t) - \lambda a(t)F(t, u(t)) = 0, \quad t \in (0,1),$$
$$u(0) = u'(\eta) = u''(1) = 0,$$

where $\eta \in [\frac{1}{2}, 1)$, λ was a positive parameter and $a(t)$ was a nonnegative continuous function defined on $(0,1)$. His main tool was the Guo-Krasnosel'skii fixed point theorem. In 2008, by using the Guo-Krasnosel'skii fixed point theorem, Guo, Sun and Zhao [6] obtained the existence of at least one positive solution for the problem

$$u'''(t) + h(t)f(u(t)) = 0, \quad t \in (0,1),$$
$$u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta),$$

where $0 < \eta < 1$ and $1 < \alpha < 1/\eta$. For more results concerning the existence of positive solutions to third-order three-point BVPs, one can refer to [3, 4, 9, 10, 12, 14].

It is necessary to point out that all the above-mentioned works are achieved when the corresponding Green's functions are positive, which is a very important condition. A natural question is that whether we can obtain the existence of positive solutions to some third-order three-point BVPs when the corresponding Green's functions are sign-changing. It is worth mentioning that Palamides and Smyrlis [8] discussed the existence of at least one positive solution to the singular third-order three-point BVP with an indefinitely signed Green's function

$$u'''(t) = a(t)f(t, u(t)), \quad t \in (0,1),$$
$$u(0) = u(1) = u''(\eta) = 0, \quad \eta \in (\frac{17}{24}, 1).$$

Their technique was a combination of the Guo-Krasnosel'skii fixed point theorem and properties of the corresponding vector field. The following equality

$$\max_{t\in[0,1]} \int_0^1 G(t,s)a(s)f(s,u(s))ds = \int_0^1 \max_{t\in[0,1]} G(t,s)a(s)f(s,u(s))ds \tag{1.1}$$

played an important role in the process of their proof. Unfortunately, the equality (1.1) is not right. For a counterexample, one can refer to our paper [11].

Motivated greatly by the above-mentioned works, in this paper we study the following third-order three-point BVP

$$\begin{gathered} u'''(t) = f(t, u(t)), \quad t \in [0,1], \\ u'(0) = u(1) = u''(\eta) = 0, \end{gathered} \tag{1.2}$$

where $f \in C([0,1] \times [0, +\infty),\ [0, +\infty))$ and $\eta \in (\frac{1}{2}, 1)$. Although the corresponding Green's function is sign-changing, we still obtain the existence of at least $2m - 1$ positive solutions for arbitrary positive integer m under suitable conditions on f.

In the remainder of this section, we state some fundamental concepts and the Leggett-Williams fixed point theorem [7].

Let E be a real Banach space with cone P. A map $\sigma : P \to (-\infty, +\infty)$ is said to be a concave functional if

$$\sigma(tx + (1-t)y) \ge t\sigma(x) + (1-t)\sigma(y)$$

for all $x, y \in P$ and $t \in [0,1]$. Let a and b be two numbers with $0 < a < b$ and σ be a nonnegative continuous concave functional on P. We define the following convex sets

$$P_a = \{x \in P : \|x\| < a\},$$
$$P(\sigma, a, b) = \{x \in P : a \le \sigma(x),\ \|x\| \le b\}.$$

Theorem 1.1 (Leggett-Williams fixed point theorem)**.** *Let $A : \overline{P_c} \to \overline{P_c}$ be completely continuous and σ be a nonnegative continuous concave functional on P such that $\sigma(x) \le \|x\|$ for all $x \in \overline{P_c}$. Suppose that there exist $0 < d < a < b \le c$ such that*

(i) *$\{x \in P(\sigma, a, b) : \sigma(x) > a\} \neq \emptyset$ and $\sigma(Ax) > a$ for $x \in P(\sigma, a, b)$;*
(ii) *$\|Ax\| < d$ for $\|x\| \le d$;*
(iii) *$\sigma(Ax) > a$ for $x \in P(\sigma, a, c)$ with $\|Ax\| > b$.*

Then A has at least three fixed points x_1, x_2, x_3 in $\overline{P_c}$ satisfying

$$\|x_1\| < d,\ a < \sigma(x_2),\ \|x_3\| > d,\ \sigma(x_3) < a.$$

2. Preliminaries

In this article, we assume that Banach space $E = C[0,1]$ is equipped with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$.

For any $y \in E$, we consider the BVP

$$\begin{gathered} u'''(t) = y(t), \quad t \in [0,1], \\ u'(0) = u(1) = u''(\eta) = 0. \end{gathered} \tag{2.1}$$

After a simple computation, we obtain the following expression of Green's function $G(t,s)$ of the BVP (2.1): for $s \ge \eta$,

$$G(t,s) = \begin{cases} -\frac{(1-s)^2}{2}, & 0 \le t \le s \le 1, \\ \frac{t^2-2st+2s-1}{2}, & 0 \le s \le t \le 1 \end{cases}$$

and for $s < \eta$,

$$G(t,s) = \begin{cases} \frac{-t^2-s^2+2s}{2}, & 0 \le t \le s \le 1, \\ -st+s, & 0 \le s \le t \le 1. \end{cases}$$

Obviously, $G(t,s) \ge 0$ for $0 \le s < \eta$, and $G(t,s) \le 0$ for $\eta \le s \le 1$. Moreover, for $s \ge \eta$,

$$\max\{G(t,s) : t \in [0,1]\} = G(1,s) = 0$$

and for $s < \eta$,

$$\max\{G(t,s) : t \in [0,1]\} = G(0,s) = -\frac{s^2}{2} + s.$$

To obtain the existence of positive solutions for (1.2), we need to construct a suitable cone in E. Let u be a solution of (1.2). Then it is easy to verify that $u(t) \ge 0$ for $t \in [0,1]$ provided that $u'(1) \le 0$. In fact, since f is nonnegative, we know that $u'''(t) \ge 0$ for $t \in [0,1]$, which together with $u''(\eta) = 0$ implies that

$$u''(t) \le 0 \text{ for } t \in [0,\eta] \quad \text{and} \quad u''(t) \ge 0 \text{ for } t \in [\eta,1]. \tag{2.2}$$

In view of (2.2) and $u'(0) = 0$, we have

$$u'(t) \le 0 \text{ for } t \in [0,\eta] \quad \text{and} \quad u'(t) \le u'(1) \text{ for } t \in [\eta,1]. \tag{2.3}$$

If $u'(1) \le 0$, then it follows from (2.3) that $u'(t) \le 0$ for $t \in [0,1]$, which together with $u(1) = 0$ implies that $u(t) \ge 0$ for $t \in [0,1]$. Therefore, we define a cone in E as follows:

$$\hat{P} = \{y \in E : y(t) \text{ is nonnegative and decreasing on } [0,1]\}.$$

Lemma 2.1 ([11])**.** *Let $y \in \hat{P}$ and $u(t) = \int_0^1 G(t,s)y(s)ds$, $t \in [0,1]$. Then $u \in \hat{P}$ and u is the unique solution of* (2.1)*. Moreover, u satisfies*

$$\min_{t\in[1-\theta,\theta]} u(t) \ge \theta^* \|u\|,$$

where $\theta \in (\frac{1}{2}, \eta)$ and $\theta^* = (\eta - \theta)/\eta$.

3. Main results

In the remainder of this paper, we assume that $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and satisfies the following two conditions:

- (D1) For each $x \in [0,+\infty)$, the mapping $t \mapsto f(t,x)$ is decreasing;
- (D2) For each $t \in [0,1]$, the mapping $x \mapsto f(t,x)$ is increasing.

Let

$$P = \{u \in \hat{P} : \min_{t\in[1-\theta,\theta]} u(t) \ge \theta^* \|u\|\}.$$

Then it is easy to check that P is a cone in E. Now, we define an operator A on P by

$$(Au)(t) = \int_0^1 G(t,s)f(s,u(s))ds, \quad t \in [0,1].$$

Obviously, if u is a fixed point of A in P, then u is a nonnegative solution of (1.2). For convenience, we denote

$$H_1 = \int_0^\eta \big(-\frac{s^2}{2} + s\big)ds, \quad H_2 = \min_{t\in[1-\theta,\theta]} \int_{1-\theta}^\theta G(t,s)ds.$$

Theorem 3.1. *Assume that there exist numbers d, a and c with $0 < d < a < \frac{a}{\theta^*} \le c$ such that*

$$f(t,u) < \frac{d}{H_1}, \quad t \in [0,\eta],\ u \in [0,d], \tag{3.1}$$

$$f(t,u) > \frac{a}{H_2}, \quad t \in [1-\theta,\theta],\ u \in [a, \frac{a}{\theta^*}], \tag{3.2}$$

$$f(t,u) < \frac{c}{H_1}, \quad t \in [0,\eta],\ u \in [0,c]. \tag{3.3}$$

Then (1.2) *has at least three positive solutions u, v and w satisfying*

$$\|u\| < d, \quad a < \min_{t\in[1-\theta,\theta]} v(t), \quad d < \|w\|, \quad \min_{t\in[1-\theta,\theta]} w(t) < a.$$

Proof. For $u \in P$, we define

$$\sigma(u) = \min_{t\in[1-\theta,\theta]} u(t).$$

It is easy to check that σ is a nonnegative continuous concave functional on P with $\sigma(u) \le \|u\|$ for $u \in P$ and that $A : P \to P$ is completely continuous.

We first assert that if there exists a positive number r such that $f(t,u) < \frac{r}{H_1}$ for $t \in [0,\eta]$ and $u \in [0,r]$, then $A : \overline{P_r} \to P_r$. Indeed, if $u \in \overline{P_r}$, then

$$\begin{aligned}
\|Au\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)f(s,u(s))ds \\
&\le \int_0^1 \max_{t\in[0,1]} G(t,s)f(s,u(s))ds \\
&= \int_0^\eta \max_{t\in[0,1]} G(t,s)f(s,u(s))ds + \int_\eta^1 \max_{t\in[0,1]} G(t,s)f(s,u(s))ds \\
&= \int_0^\eta (-\frac{s^2}{2} + s)f(s,u(s))ds \\
&< \frac{r}{H_1} \int_0^\eta (-\frac{s^2}{2} + s)ds = r;
\end{aligned}$$

that is, $Au \in P_r$.

Hence, we have shown that if (3.1) and (3.3) hold, then A maps $\overline{P_d}$ into P_d and $\overline{P_c}$ into $_c$.

Next, we assert that $\{u \in P(\sigma, a, \frac{a}{\theta^*}) : \sigma(u) > a\} \neq \emptyset$ and $\sigma(Au) > a$ for all $u \in P(\sigma, a, \frac{a}{\theta^*})$. In fact, the constant function $\frac{a+\frac{a}{\theta^*}}{2}$ belongs to $\{u \in P(\sigma, a, \frac{a}{\theta^*}) : \sigma(u) > a\}$.

On the one hand, for $u \in P(\sigma, a, \frac{a}{\theta^*})$, we have

$$a \le \sigma(u) = \min_{t\in[1-\theta,\theta]} u(t) \le u(t) \le \|u\| \le \frac{a}{\theta^*} \tag{3.4}$$

for all $t \in [1-\theta, \theta]$.

Also, for any $u \in P$ and $t \in [1-\theta, \theta]$, we have

$$\begin{aligned}
&\int_0^{1-\theta} G(t,s)f(s,u(s))ds + \int_\theta^\eta G(t,s)f(s,u(s))ds + \int_\eta^1 G(t,s)f(s,u(s))ds \\
&\ge \int_0^{1-\theta} (1-t)sf(s,u(s))ds - \int_\eta^1 \frac{(1-s)^2}{2} f(s,u(s))ds \\
&\ge f(\eta,u(\eta))[\int_0^{1-\theta} (1-t)sds - \int_\eta^1 \frac{(1-s)^2}{2} ds] \\
&\ge f(\eta,u(\eta))[\int_0^{1-\theta} (1-t)sds - \int_\theta^1 \frac{(1-s)^2}{2} ds] \\
&= f(\eta,u(\eta))[\frac{(1-t)(1-\theta)^2}{2} - \frac{(1-\theta)^3}{6}] \\
&\ge f(\eta,u(\eta))[\frac{(1-\theta)(1-\theta)^2}{2} - \frac{(1-\theta)^3}{6}] \\
&= f(\eta,u(\eta))\frac{(1-\theta)^3}{3} \ge 0,
\end{aligned}$$

which together with (3.2) and (3.4) implies

$$\sigma(Au) = \min_{t\in[1-\theta,\theta]} \int_0^1 G(t,s)f(s,u(s))ds$$

$$\begin{aligned}&\geq \min_{t\in[1-\theta,\theta]}\int_{1-\theta}^{\theta}G(t,s)f(s,u(s))ds\\&> \frac{a}{H_2}\min_{t\in[1-\theta,\theta]}\int_{1-\theta}^{\theta}G(t,s)ds=a\end{aligned}$$

for $u\in P(\sigma,a,\frac{a}{\theta^*})$.

Finally, we verify that if $u\in P(\sigma,a,c)$ and $\|Au\|>a/\theta^*$, then $\sigma(Au)>a$. To see this, we suppose that $u\in P(\sigma,a,c)$ and $\|Au\|>a/\theta^*$. Then it follows from $Au\in P$ that

$$\sigma(Au)=\min_{t\in[1-\theta,\theta]}(Au)(t)\geq\theta^*\|Au\|>a.$$

To sum up, all the hypotheses of the Leggett-Williams fixed point theorem are satisfied. Therefore, A has at least three fixed points; that is, (1.2) has at least three positive solutions u,v and w satisfying

$$\|u\|<d,\quad a<\min_{t\in[1-\theta,\theta]}v(t),\quad d<\|w\|,\quad \min_{t\in[1-\theta,\theta]}w(t)<a.$$

□

Theorem 3.2. *Let m be an arbitrary positive integer. Assume that there exist numbers d_i $(1\leq i\leq m)$ and a_j $(1\leq j\leq m-1)$ with $0<d_1<a_1<\frac{a_1}{\theta^*}<d_2<a_2<\frac{a_2}{\theta^*}<\cdots<d_{m-1}<a_{m-1}<\frac{a_{m-1}}{\theta^*}<d_m$ such that*

$$f(t,u)<\frac{d_i}{H_1},\quad t\in[0,\eta],\ u\in[0,d_i],\ 1\leq i\leq m, \tag{3.5}$$

$$f(t,u)>\frac{a_j}{H_2},\quad t\in[1-\theta,\theta],\ u\in[a_j,\frac{a_j}{\theta^*}],\ 1\leq j\leq m-1. \tag{3.6}$$

Then (1.2) has at least $2m-1$ positive solutions in $\overline{P_{d_m}}$.

Proof. We use induction on m. First, for $m=1$, we know from (3.5) that $A:\overline{P_{d_1}}\to P_{d_1}$. Then it follows from Schauder fixed point theorem that (1.2) has at least one positive solution in $\overline{P_{d_1}}$.

Next, we assume that this conclusion holds for $m=k$. To show that this conclusion also holds for $m=k+1$, we suppose that there exist numbers d_i $(1\leq i\leq k+1)$ and a_j $(1\leq j\leq k)$ with $0<d_1<a_1<\frac{a_1}{\theta^*}<d_2<a_2<\frac{a_2}{\theta^*}<\cdots<d_k<a_k<\frac{a_k}{\theta^*}<d_{k+1}$ such that

$$f(t,u)<\frac{d_i}{H_1},\quad t\in[0,\eta],\ u\in[0,d_i],\ 1\leq i\leq k+1, \tag{3.7}$$

$$f(t,u)>\frac{a_j}{H_2},\quad t\in[1-\theta,\theta],\ u\in[a_j,\frac{a_j}{\theta^*}],\ 1\leq j\leq k. \tag{3.8}$$

By assumption, (1.2) has at least $2k-1$ positive solutions u_i $(i=1,2,\dots,2k-1)$ in $\overline{P_{d_k}}$. At the same time, it follows from Theorem 3.1, (3.7) and (3.8) that (1.2) has at least three positive solutions u,v and w in $\overline{P_{d_{k+1}}}$ such that

$$\|u\|<d_k,\quad a_k<\min_{t\in[1-\theta,\theta]}v(t),\quad d_k<\|w\|,\quad \min_{t\in[1-\theta,\theta]}w(t)<a_k.$$

Obviously, v and w are different from u_i $(i=1,2,\dots,2k-1)$. Therefore, (1.2) has at least $2k+1$ positive solutions in $\overline{P_{d_{k+1}}}$, which shows that this conclusion also holds for $m=k+1$. □

Example 3.3. We consider the BVP

$$\begin{gathered} u'''(t) = f(t, u(t)), \quad t \in [0,1], \\ u'(0) = u(1) = u''(\frac{2}{3}) = 0, \end{gathered} \tag{3.9}$$

where

$$f(t,u) = \begin{cases} (1-t)(u+1)^2, & (t,u) \in [0,1] \times [0,1], \\ (1-t)[122(u-1)+4], & (t,u) \in [0,1] \times [1,2], \\ 14(1-t)(u+1)^2, & (t,u) \in [0,1] \times [2,20], \\ 6174(1-t), & (t,u) \in [0,1] \times [20,+\infty). \end{cases}$$

Let $\theta = 3/5$. Then $\theta^* = 1/10$. A simple calculation shows that $H_1 = 14/81$ and $H_2 = 1/25$. If we choose $d = 1$, $a = 2$, $c = 1068$, then all the conditions of Theorem 3.1 are satisfied. Therefore, it follows from Theorem 3.1 that (3.9) has at least three positive solutions.

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Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, China

E-mail address, Jian-Ping Sun: `jpsun@lut.cn`

E-mail address, Juan Zhao: `z_1111z@163.com`