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POSITIVE SOLUTIONS FOR A NONLINEAR PERIODIC BOUNDARY-VALUE PROBLEM WITH A PARAMETER

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Abstract. Using topological degree theory with a partially ordered structure of space, sufficient conditions for the existence and multiplicity of positive solutions for a second-order nonlinear periodic boundary-value problem are established. Inspired by ideas in Guo and Lakshmikantham [6], we study the dependence of positive periodic solutions as a parameter approaches infinity,

$$\lim_{\lambda\to+\infty}\|x_\lambda\| = +\infty, \quad \text{or} \quad \lim_{\lambda\to+\infty}\|x_\lambda\| = 0.$$

1. Introduction

In recent years, periodic boundary value problems have been studied extensively in the literature; see, for example, [1, 5, 9, 13, 15, 20] and references therein. Many techniques have been developed for studying the existence and multiplicity of periodic solutions (see [2, 3, 4, 7, 10, 16, 17, 18, 19]). In this article, we apply topological degree theory combined with partially ordered structure of a space to establish the existence and multiplicity of positive solutions to the periodic boundary-value problem

$$\begin{gathered} \lambda\mathbb{L}x = -g(t)f(t,x), \quad 0 \le t \le 2\pi, \\ x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \end{gathered} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\mathbb{L}x = x'' - \rho^2 x$, $\rho > 0$ is a constant. In addition, f and g satisfy

- (H1) $f \in C[0,+\infty) \times [0,+\infty), [0,+\infty))$;
- (H2) $g(t) \in L^p[0,2\pi]$ for some $1 \le p \le +\infty$ and there exists $m > 0$ such that $g(t) \ge m$ a.e. on $[0,2\pi]$.

For the case of $g(t) \in C[0,2\pi]$, not $g(t) \in L^p[0,2\pi]$, and $f(t,x)$ is replaced by $f(x)$, problem (1.1) reduces to the problem studied by Graef, Kong, and Wang in [5]. By using the fixed-point theorem of cone expansion and compression of norm type, the authors obtained some sufficient conditions for the existence, multiplicity, and nonexistence of positive solutions for problem (1.1).

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In the present article, some new criteria for the existence and multiplicity of positive solutions are established. In particular, we examine the dependence of positive solution $x_\lambda(t)$ on the parameter λ; i.e.,

$$\lim_{\lambda\to+\infty}\|x_\lambda\|=+\infty \quad \text{or} \quad \lim_{\lambda\to+\infty}\|x_\lambda\|=0.$$

We remark that our methods are entirely different from those used in [5, 7, 9, 10, 18, 20] and the results obtained in this paper generalize some of their results, to some degree. Moreover, some of our hypotheses on f involve

$$\limsup_{x\to0^+}\max_{t\in[0,2\pi]}\frac{f(t,x)}{x}, \quad \liminf_{x\to\infty}\min_{t\in[0,2\pi]}\frac{f(t,x)}{x}.$$

Our conditions strictly include the sublinear and superlinear cases.

The work is organized in the following fashion. In Section 2, we provide some necessary background. In particular, we shall introduce some lemmas and definitions associated with topological degree theory and partially ordered structure of space. The main results will be stated and proved in Section 3. The final section of the paper considers the dependence of positive solution $x_\lambda(t)$ on the parameter λ.

At the end of this section, it is worth to mention that some excellent results by Guo and Lakshmikantham, which can be found in [6].

Theorem 1.1. *Let E be a Banach space and let $K\subset E$ be a cone in E. Let operator $A:K\to K$ is completely continuous and $A\theta=\theta$, where θ is the zero element of E. Suppose that one of the two conditions (i)*

$$\lim_{x\in K,\,\|x\|\to0}\frac{\|Ax\|}{\|x\|}=0, \quad \lim_{x\in K,\,\|x\|\to+\infty}\frac{\|Ax\|}{\|x\|}=+\infty$$

and (ii)

$$\lim_{x\in K,\,\|x\|\to0}\frac{\|Ax\|}{\|x\|}=+\infty, \quad \lim_{x\in K,\,\|x\|\to+\infty}\frac{\|Ax\|}{\|x\|}=0$$

is satisfied. Then the following two conclusions hold.

(1) *Every $\mu>0$ is an eigenvalue of A, which corresponds to positive eigenvector; i.e., there exists $x_\mu>\theta$ such that $Ax_\mu=\mu x_\mu$;*
(2) *$\lim_{\mu\to+\infty}\|x_\mu\|=+\infty$ under condition (i), and $\lim_{\mu\to+\infty}\|x_\mu\|=0$ under condition (ii).*

From the proof of Theorem 1.1, it is not difficult to see that the conditions are different from those used in [6, 7, 11, 14, 21], which can be used to prove the dependence of positive solution $x_\mu(t)$ on the parameter μ.

2. Definitions and lemmas

In this section, we provide some background materials associated with topological degree theory and partially ordered structure of space. The following definitions can be found in the book by Guo and Lakshmikantham [6].

Definition 2.1. Let E be a real Banach space over $\mathbb{R}$. A nonempty closed set $P\subset E$ is said to be a cone provided that

(i) $au+bv\in P$ for all $u,v\in P$ and all $a\geq0,b\geq0$ and
(ii) $u,-u\in P$ implies $u=0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Lemma 2.2. *Let K be a closed convex set in a Banach space X and let D be a bounded open set such that $D_k := D \cap K \neq \emptyset$. Let $T : \bar{D}_k \to K$ be a compact map. Suppose that $x \neq T(x)$ for all $x \in \partial D_k$.*

- (P1) *(Solution property) If $i_k(T, D_k) \neq 0$, then T has a fixed point in D_k.*
- (P2) *(Normality) If $u \in D_k$, then $i_k(\hat{u}, D_k) = 1$, where $\hat{u}(x) = u$ for $x \in \bar{D}_k$.*
- (P3) *(Additivity) If V^1, V^2 are disjoint relatively open subsets of D_k such that $x \neq T(x)$ for $x \in \bar{D}_k \setminus (V^1 \cup V^2)$, then*
$$i_k(T, D_k) = i_k(T, V^1) + i_k(T, V^2).$$
- (P4) *(Homotopy invariance) Let $h : [0, 1] \times \bar{D}_k \to K$ be compact such that $x \neq h(t, x)$ for $x \in \partial D_k$ and $t \in [0, 1]$.*

Then $i_k(h(0, \dots), D_k) = i_k(h(1, \dots), D_k)$.

From these properties, one can have the following consequence.

Lemma 2.3 ([12])**.** *Let K be a cone in a real Banach space X. Let D be an open bounded subset of X with $D_k = D \cap K \neq \emptyset$ and $\bar{D}_k \neq K$. Assume that $A : \bar{D}_k \to K$ is completely continuous such that $x \neq Ax$ for $x \in \partial D_k$. Then the following results hold:*

1. *If $\|Ax\| \leq \|x\|$, $x \in \partial D_k$, then $i_k(A, D_k) = 1$.*
2. *If there exists $e \in K \backslash \{0\}$ such that $x \neq Ax + \lambda e$ for all $x \in \partial D_k$ and all $\lambda > 0$, then $i_k(A, D_k) = 0$.*
3. *Let U be open in K such that $\bar{U} \subset D_k$. If $i_k(A, D_k) = 1$ and $i_k(A, U_k) = 0$, then A has a fixed point in $D_k \backslash \bar{U}_k$. The same result holds if $i_k(A, D_k) = 0$ and $i_k(A, U_k) = 1$.*

Remark 2.4. In Lemma 2.2, using (2) gives better results than use of the common assumption $\|Tx\| \geq \|x\|$ for $x \in \partial D_k$.

Lemma 2.5 ([6])**.** *Let K be a cone in a real Banach space E. Assume Ω_1, Ω_2 are bounded open sets in E with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$. If*
$$A : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$$
is completely continuous such that either

- (i) *$\|Ax\| \leq \|x\|$ for all $x \in K \cap \partial\Omega_1$ and $\|Ax\| \geq \|x\|$ for all $x \in K \cap \partial\Omega_2$, or*
- (ii) *$\|Ax\| \geq \|x\|$ for all $x \in K \cap \partial\Omega_1$ and $\|Ax\| \leq \|x\|$ for all $x \in K \cap \partial\Omega_2$,*

then A has at least one fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

To obtain some of the norm inequalities in our main results we employ Hölder's inequality.

Lemma 2.6. *Let $f \in L^p[a, b]$ with $p > 1$, $g \in L^q[a, b]$ with $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1[a, b]$ and*
$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$
Let $f \in L^1[a, b]$, $g \in L^\infty[a, b]$. Then $fg \in L^1[a, b]$ and
$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty.$$

3. Main results

Let X be the space $C[0,2\pi]$ endowed with the norm $\|x\| = \max_{0\le t\le 2\pi} |x(t)|$. By a solution of problem (1.1), we mean a function $x \in C[0,2\pi] \cap C^2(0,2\pi)$ which satisfies (1.1).

To establish the existence of multiple positive solutions in $C[0,2\pi] \cap C^2(0,2\pi)$ of problem (1.1), we construct a cone K in X by

$$K = \Big\{x \in X : x(t) \ge 0 \text{ on } [0,2\pi] \text{ and } \min_{0\le t\le 2\pi} x(t) \ge \sigma\|x\|\Big\},$$

where

$$\sigma = \frac{2e^{\pi\rho}}{1+e^{2\pi\rho}}. \tag{3.1}$$

Let the map $T_\lambda : K \to X$ be defined by

$$(T_\lambda x)(t) = \lambda^{-1}\int_0^{2\pi} G(t,s)g(s)f(x(s))ds, \tag{3.2}$$

here

$$G(t,s) = \begin{cases} \frac{e^{\rho(t-s)}+e^{\rho(2\pi-t+s)}}{2\rho(e^{2\rho\pi}-1)}, & 0 \le s \le t \le 1,\\ \frac{e^{\rho(s-t)}+e^{\rho(2\pi-s+t)}}{2\rho(e^{2\rho\pi}-1)}, & 0 \le t \le s \le 1. \end{cases} \tag{3.3}$$

It follows that

$$\frac{e^{\rho\pi}}{2\rho(e^{2\rho\pi}-1)} = \hat{G}(\pi) \le G(t,s) \le \hat{G}(0) = \frac{1+e^{\rho 2\pi}}{2\rho(e^{2\rho\pi}-1)}, \quad t,s \in [0,2\pi] \tag{3.4}$$

where

$$\hat{G}(x) = \frac{e^{\rho x}+e^{\rho(2\pi-x)}}{2\rho(e^{2\rho\pi}-1)}, \quad x \in [0,2\pi].$$

Further, by (3.3) and (3.4), we have

$$\sigma G(s,s) \le G(t,s) \le G(s,s), \quad t \in [0,2\pi], \tag{3.5}$$

where

$$G(s,s) = \frac{1+e^{\rho 2\pi}}{2\rho(e^{2\rho\pi}-1)}.$$

Noticing $\rho > 0$, then it is easy to see from (3.4) and (3.5) that there exists $\tau > 0$ such that

$$G(t,s) \ge \tau, \quad \forall t,s \in [0,2\pi]. \tag{3.6}$$

Lemma 3.1 ([5]). *Assume that* (H1), (H2) *hold. Then $x \in K$ is a positive fixed point of T_λ if and only if x is a positive solution of problem* (1.1).

We define

$$\Omega_r = \{x \in K : \min_{t\in[0,2\pi]} x(t) < \sigma r\} = \{x \in X : \sigma\|x\| \le \min_{t\in[0,2\pi]} x(t) < \sigma r\}.$$

This allows f to satisfy weaker conditions than previously where the index was shown to be zero on the sets $K_r = \{x \in K : \|x\| < r\}$.

The following results are similar to [12, Lemma 2.5].

Lemma 3.2. *Ω_r has the following properties:*

(a) *Ω_r is open relative to K;*
(b) *$K_{\sigma r} \subset \Omega_r \subset K_r$;*
(c) *$x \in \partial\Omega_r$ if and only if $\min_{t\in[0,2\pi]} x(t) = \sigma r$;*
(d) *if $x \in \partial\Omega_r$, then $\sigma r \le x(t) \le r$ for $t \in [0,2\pi]$.*

Now for convenience we introduce the following notation. Let

$$f_{\sigma r}^{r}=\min\Big\{\min_{t\in[0,2\pi]}\frac{f(t,x)}{r}:x\in[\sigma r,r]\Big\},\quad f_0^r=\max\{\max_{t\in[0,2\pi]}\frac{f(t,x)}{r}:x\in[0,r]\},$$

$$f^{\delta}=\lim_{x\to\delta}\sup\max_{t\in[0,2\pi]}\frac{f(t,x)}{x},\quad f_{\delta}=\lim_{x\to\delta}\inf\min_{t\in[0,1]}\frac{f(t,x)}{x},\quad(\delta:=\infty,\ \text{or }0),$$

$$l=\min\{\big[\lambda^{-1}\|G\|_q\|g\|_p\big]^{-1},\big[\lambda^{-1}\|G\|_1\|g\|_\infty\big]^{-1},\big[\lambda^{-1}\hat{G}(0)\|g\|_1\big]^{-1}\},$$

$$L=\big[2m\lambda^{-1}\tau\pi\big]^{-1}\sigma.$$

We now give our results on the existence of multiple positive solutions of problem (1.1). We consider the following three cases for $g\in L^p[0,1]: p>1,\quad p=1$, and $p=\infty$. Case $p>1$ is treated in the following theorem.

Theorem 3.3. *Suppose that* (H1), (H2) *and one of the following two conditions hold:*

(H3) *There exist $\xi_1,\xi_2,\xi_3\in(0,\infty)$, with $\xi_1<\sigma\xi_2$ and $\xi_2<\xi_3$ such that*

$$f_0^{\xi_1}<l,\quad f_{\sigma\xi_2}^{\xi_2}>L,\quad f_0^{\xi_3}<l.$$

(H4) *There exist $\xi_1,\xi_2,\xi_3\in(0,\infty)$, with $\xi_1<\xi_2<\xi_3$ such that*

$$f_{\sigma\xi_1}^{\xi_1}>L,\quad f_0^{\xi_2}<l,\quad f_{\sigma\xi_3}^{\xi_3}>L.$$

Then, for all $\lambda>0$, problem (1.1) *has at least two positive solutions x_1,x_2 with $x_1\in\Omega_{\xi_2}\backslash\bar{K}_{\xi_1}$, $x_2\in K_{\xi_3}\backslash\bar{\Omega}_{\xi_2}$.*

Proof. We only consider the condition (H_3). If (H4) holds, then the proof is similar to that of the case when (H3) holds. Let T_λ be cone preserving, completely continuous operator that was defined by (3.2).

First, we show that $i_k(T_\lambda,K_{\xi_1})=1$. In fact, by (3.2) and $f_0^{\xi_1}<l$, we have for $x\in\partial K_{\xi_1}$,

$$\begin{aligned}(T_\lambda x)(t)&=\lambda^{-1}\int_0^{2\pi}G(t,s)g(s)f(s,x(s))ds\\&<l\xi_1\lambda^{-1}\int_0^{2\pi}G(t,s)g(s)ds\\&\le l\xi_1\lambda^{-1}\int_0^{2\pi}G(s,s)g(s)ds\\&\le l\xi_1\lambda^{-1}\|G\|_q\|g\|_p\le\xi_1;\end{aligned}\tag{3.7}$$

i.e., $\|T_\lambda x\|<\|x\|$ for $x\in\partial K_{\xi_1}$. By (1) of Lemma 2.2, we obtain that $i_k(T_\lambda,K_{\xi_1})=1$.

Secondly, we show that $i_k(T_\lambda,\Omega_{\xi_2})=0$. Let $e(t)\equiv1$ for $t\in[0,2\pi]$. Then $e\in\partial K_1$. We claim that

$$x\neq T_\lambda x+\zeta e,\quad\text{for }x\in\partial\Omega_{\xi_2}\text{ and }\zeta>0.\tag{3.8}$$

In fact, if not, there exist $x_0\in\partial\Omega_{\xi_2}$ and $\zeta_0>0$ such that $x_0=T_\lambda x_0+\zeta_0e$. Then, by (3.2) (3.6), (d) of Lemma 3.2 and $f_{\sigma\xi_2}^{\xi_2}>L$, for $t\in[0,2\pi]$, we have

$$\begin{aligned}x_0(t)&=(T_\lambda x_0)(t)+\zeta_0e\\&=\lambda^{-1}\int_0^{2\pi}G(t,s)g(s)f(s,x_0(s))ds+\zeta_0\end{aligned}$$

$$\begin{aligned}
&> \lambda^{-1} L\xi_2 \int_0^{2\pi} G(t,s)g(s)ds + \zeta_0 \\
&\geq 2\pi\lambda^{-1}\tau m L\xi_2 + \zeta_0 \\
&= \sigma\xi_2 + \zeta_0,
\end{aligned}$$

which implies that $\min_{t\in[0,2\pi]} x_0(t) > \sigma\xi_2 + \zeta_0 > \sigma\xi_2$. Since $\min_{t\in[0,2\pi]} x_0(t) = \sigma\xi_2$, by (c) of Lemma 3.2, this is a contradiction. Hence, by (2) of Lemma 2.2, it follows that $i_k(T_\lambda, \Omega_{\xi_2}) = 0$.

Finally, similar to the proof of $i_k(T_\lambda, K_{\xi_1}) = 1$, we can prove that $i_k(T_\lambda, K_{\xi_3}) = 1$. Since $\xi_1 < \sigma\xi_2$, we have $\bar{K}_{\xi_1} \subset K_{\sigma\xi_2} \subset \Omega_{\xi_2}$. Therefore, (3) of Lemma 2.2 implies that problem (1.1) has at least two positive solutions x_1, x_2 with $x_1 \in \Omega_{\xi_2}\backslash\bar{K}_{\xi_1}$, $x_2 \in K_{\xi_3} \setminus \bar{\Omega}_{\xi_2}$. $\square$

Remark 3.4. From the proof of Theorem 3.1, we can obtain that (1.1) has a third non-negative solution x_3 with $x_3 \in K_{\xi_1}$.

The proofs of the remaining results in this section are similar to the proof of Theorem 3.1. We will present only their sketches. The following result deals with the case $p = \infty$.

Corollary 3.5. *Suppose that* (H1)–(H3) *hold, or* (H1), (H2), (H4) *hold. Then, for all* $\lambda > 0$, *problem* (1.1) *has at least two positive solutions* x_1, x_2 *with* $x_1 \in \Omega_{\xi_2}\backslash\bar{K}_{\xi_1}$, $x_2 \in K_{\xi_3}\backslash\bar{\Omega}_{\xi_2}$.

Proof. Let $\|G\|_1\|g\|_\infty$ replace $\|G\|_p\|g\|_q$ and repeat the argument above. $\square$

Now we consider the case of $p = 1$.

Corollary 3.6. Suppose that (H1)–(H3) hold, or (H1), (H2), (H4) hold. Then, for all $\lambda > 0$, problem (1.1) has at least two positive solutions x_1, x_2 with $x_1 \in \Omega_{\xi_2}\backslash\bar{K}_{\xi_1}$, $x_2 \in K_{\xi_3}\backslash\bar{\Omega}_{\xi_2}$.

Proof. For $x \in \partial K_{\xi_1}$, from (3.2) and (3.4) it follows that

$$\begin{aligned}
(T_\lambda x)(t) &= \lambda^{-1}\int_0^{2\pi} G(t,s)g(s)f(x(s))ds \\
&< \lambda^{-1} l\xi_1 \int_0^{2\pi} G(t,s)g(s)ds \\
&\leq \lambda^{-1} l\xi_1 \int_0^{2\pi} G(s,s)g(s)ds \\
&\leq \lambda^{-1} l\xi_1 \hat{G}(0)\|g\|_1 \leq \xi_1.
\end{aligned}$$

Consequently, for $x \in \partial K_{\xi_1}$, we have $\|T_\lambda x\| < \|x\|$. By (1) of Lemma 2.2, this implies that $i(T_\lambda, K_{\xi_1}) = 1$.

Similarly, if $x \in \partial K_{\xi_3}$ we can obtain $i(T_\lambda, K_{\xi_3}) = 1$. And it also follows from (3.8) that $i_k(T_\lambda, \Omega_{\xi_2}) = 0$. This completes the proof. $\square$

As a special case of Theorem 3.1, we obtain the following result.

Corollary 3.7. *Assume* (H1), (H2) *and that there exist* ξ', $\xi \in (0,\infty)$ *with* $\xi' < \sigma\xi$ *such that one of the following two conditions hold:*

(H5) $f_0^{\xi'} < l$, $f_{\sigma\xi}^{\xi} > L$, $0 \leq f^\infty < l$.

(H6) $f_{\sigma\xi'}^{\xi'} > L$, $f_0^{\xi} < l$, $L < f_\infty \le \infty$.

Then, for all $\lambda > 0$, problem (1.1) has at least two positive solutions in K.

Proof. We show that (H5) implies (H3). Let $\alpha \in (f^\infty, l)$. Then there exists $r > \alpha$ such that $f(x) \le \alpha x$ for $x \in [r, \infty)$ since $0 \le f^\infty < l$. Let

$$\beta^* = \max\{f(x) : 0 \le x \le r\}, \quad \xi_3 > \max\{\frac{\beta^*}{l-\alpha}, \xi\}.$$

Then we have $f(x) \le \alpha x + \beta^* \le \alpha\xi_3 + \beta^* < l\xi_3$ for all $x \in [0, \xi_3]$. This implies that $f_0^{\xi_3} \le l$. Similarly (H6) implies (H4), and the Corollary is proved. $\square$

By an argument similar to that of Theorem 3.1 we obtain the following results.

Theorem 3.8. *Suppose* (H1), (H2) *and one of the following two conditions hold:*

(H7) *There exist $\xi_1, \xi_2 \in (0, \infty)$ with $\xi_1 < \xi_2$ such that $f_0^{\xi_1} \le l$ and $f_{\sigma\xi_2}^{\xi_2} \ge L$.*

(H8) *There exist $\xi_1, \xi_2 \in (0, \infty)$ with $\xi_1 < \xi_2$ such that $f_{\sigma\xi_1}^{\xi_1} \ge l$ and $f_0^{\xi_2} \le L$.*

Then, for all $\lambda > 0$, problem (1.1) has at least one positive solution in K.

As a special case of the above theorem, we obtain the following result.

Corollary 3.9. *Suppose* (H1), (H2) *and one of the following conditions hold:*

(H9) $0 \le f^0 < l$ *and* $L < f_\infty \le \infty$.

(H10) $0 \le f^\infty < l$ *and* $L < f_0 \le \infty$.

Then, for all $\lambda > 0$, problem (1.1) has at least one positive solution in K.

Theorem 3.1 can be generalized to obtain many solutions.

Theorem 3.10. *Suppose that* (H1), (H2) *hold. Then the following assertions hold.*

(1) *If there exists $\{\xi_i\}_{i=1}^{2m_0} \subset (0, \infty)$ with $\xi_1 < \sigma\xi_2 < \xi_2 < \xi_3 < \sigma\xi_4 < \cdots < \sigma\xi_{2m_0}$ such that*

$$f_0^{\xi_{2m-1}} < l, \quad f_{\sigma\xi_{2m}}^{\xi_{2m}} > L.$$

Then, for all $\lambda > 0$, problem (1.1) has at least $2m_0$ solutions in K.

(2) *If there exists $\{\xi_i\}_{i=1}^{2m_0} \subset (0, \infty)$ with $\xi_1 < \xi_2$ and $\xi_2 < \sigma\xi_3 < \xi_3 < \xi_4 < \sigma\xi_5 < \cdots < \sigma\xi_{2m_0+2}$ such that*

$$f_{\sigma\xi_{2m-1}}^{\xi_{2m-1}} > L, \quad f_0^{\xi_{2m}} < l.$$

Then, for all $\lambda > 0$, problem (1.1) has at least $2m_0 - 1$ solutions in K.

It is easy to see that our conditions include the sublinear and superlinear cases, so the results of this paper generalize and improve those in [5] to some degree.

4. Dependence of positive solution on the parameter

In this section, we consider the dependence of the positive solution $x_\lambda(t)$ on the parameter λ. In the following theorems we only consider the case of $p = 1$.

Theorem 4.1. *Assume that* (H1), (H2) *hold. Then the following two conditions hold.*

(H11) *If $f^0 = 0$ and $f_\infty = \infty$, then for every $\lambda > 0$ problem (1.1) has a positive solution $x_\lambda(t)$ satisfying $\lim_{\lambda\to\infty} \|x_\lambda\| = \infty$;*

(H12) *If $f_0 = \infty$ and $f^\infty = 0$, then for every $\lambda > 0$ problem (1.1) has a positive solution $x_\lambda(t)$ satisfying $\lim_{\lambda\to\infty} \|x_\lambda\| = 0$.*

Proof. We need to prove this theorem only under condition (H11) since the proof is similar when (H12) holds. Considering $f^0 = 0$, there exists $r_1 > 0$ such that

$$f(t,x) \le \varepsilon_1 x, \quad \forall t \in [0, 2\pi],\ 0 \le x \le r_1,$$

where $\varepsilon_1 > 0$ and satisfies $2\pi\lambda^{-1}\varepsilon_1 \hat{G}(0)\|g\|_1 \le 1$. Thus, for $x \in K \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned}
(T_\lambda x)(t) &= \lambda^{-1}\int_0^{2\pi} G(t,s)g(s)f(s,x(s))ds \\
&\le \lambda^{-1}\varepsilon_1\|x\|\int_0^{2\pi} G(t,s)g(s)ds \\
&\le 2\pi\lambda^{-1}\varepsilon_1\|x\|\hat{G}(0)\|g\|_1 \le \|x\|,
\end{aligned}$$

and therefore,

$$\|T_\lambda x\| \le \|x\|, \quad \forall t \in [0,2\pi],\ x \in K \cap \partial\Omega_{r_1}. \tag{4.1}$$

Next, turning to $f_\infty = \infty$, there exists $\tilde{r}$ satisfying $0 < r_1 < \tilde{r}$ such that

$$f(t,x) \ge \varepsilon_2 x, \quad \forall t \in [0,2\pi],\ x \ge \tilde{r},$$

where $\varepsilon_2 > 0$ and satisfies $2\pi\lambda^{-1}\varepsilon_2 m\sigma\tau \ge 1$.

Let $r_2 = \tilde{r}/\sigma$. Then, for $x \in K \cap \partial\Omega_{r_2}$, we have $x(t) \ge \sigma\|x\| = \sigma\tilde{r}/\sigma = \tilde{r}$, $t \in [0,2\pi]$. So, for $x \in K \cap \partial\Omega_{r_2}$, it follows from (3.7) that

$$\begin{aligned}
(T_\lambda x)(t) &= \lambda^{-1}\int_0^{2\pi} G(t,s)g(s)f(s,x(s))ds \\
&\ge \lambda^{-1}\varepsilon_2 m\sigma\|x\|\int_0^{2\pi} G(t,s)ds \\
&\ge 2\pi\lambda^{-1}\varepsilon_2 m\sigma\tau\|x\| \ge \|x\|,
\end{aligned}$$

and hence,

$$\|T_\lambda x\| \ge \|x\|, \quad \forall t \in [0,2\pi],\ x \in K \cap \partial\Omega_{r_2}. \tag{4.2}$$

Applying (i) of Lemma 2.3 to (4.1) and (4.2) yields that the operator T_λ has a fixed point $x_\lambda \in K \cap (\bar{\Omega}_{r_2}\backslash\Omega_{r_1})$. Thus it follows that for every $\lambda > 0$ problem (p) has a positive solution $x_\lambda(t)$.

It remains to prove $\|x_\lambda\| = +\infty$ as $\lambda \to +\infty$. In fact, if not, there exist a number $m > 0$ and a sequence $\lambda_n \to +\infty$ such that

$$\|x_{\lambda_n}\| \le m \quad (n = 1,2,3,\dots).$$

Furthermore, the sequence x_{λ_n} contains a subsequence that converges to a number $\eta (0 \le \eta \le m)$. For simplicity, suppose that $\{\|x_{\lambda_n}\|\}$ itself converges to η.

If $\eta > 0$, then $\|x_{\lambda_n}\| > \eta/2$ for sufficiently large n $(n > \mathbb{N})$, and therefore

$$\begin{aligned}
\lambda_n &= \frac{\|\int_0^{2\pi} G(t,s)g(s)f(s,x_{\lambda_n}(s))ds\|}{\|x_{\lambda_n}\|} \\
&\le \frac{\hat{G}(0)\int_0^{2\pi} g(s)f(s,x_{\lambda_n}(s))ds}{\|x_{\lambda_n}\|} \\
&\le \frac{\hat{G}(0)\mathbb{M}\|g\|_1}{\|x_{\lambda_n}\|} \\
&\le \frac{2\hat{G}(0)\mathbb{M}\|g\|_1}{\eta} \quad (n > \mathbb{N}),
\end{aligned}$$

where, $\mathbb{M} = \max_{t\in[0,2\pi,\, \|x\|\le m} f(t,x)$, which contradicts $\lambda_n \to +\infty$.

If $\eta = 0$, then $\|x_{\lambda_n}\| \to 0$ for sufficiently large n ($n > \mathbb{N}$), and therefore it follows from (H11) that for any $\varepsilon > 0$ there exists $r_3 > 0$ such that

$$f(t, x_{\lambda_n}) \le \varepsilon x_{\lambda_n}, \quad \forall t \in [0, 2\pi],\ 0 \le x_{\lambda_n} \le r_3,$$

and hence we obtain

$$\begin{aligned}
\lambda_n &= \frac{\| \int_0^{2\pi} G(t,s)g(s)f(s, x_{\lambda_n}(s))ds\|}{\|x_{\lambda_n}\|} \\
&\le \frac{\hat{G}(0)\int_0^{2\pi} g(s)f(s, x_{\lambda_n}(s))ds}{\|x_{\lambda_n}\|} \\
&\le \frac{\hat{G}(0)\varepsilon\|x_{\lambda_n}\|\int_0^{2\pi} g(s))ds}{\|x_{\lambda_n}\|} \\
&\le \frac{\hat{G}(0)\varepsilon\|x_{\lambda_n}\|\|g\|_1}{\|x_{\lambda_n}\|} \\
&= \hat{G}(0)\varepsilon\|g\|_1.
\end{aligned}$$

Since ε is arbitrary, we have $\lambda_n \to 0$ ($n \to +\infty$) in contradiction with $\lambda_n \to +\infty$. Therefore, $\|x_\lambda\| \to +\infty$ as $\lambda \to +\infty$ and our proof is complete. □

From the proof of Theorem 4.1, it is not difficult to see that the conditions are different from those used in [6, Theorem 2.3.7], which implies that the results of this paper are new and they improve [6, Theorem 2.3.7], to some degree.

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