Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 147, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

UNIQUENESS AND ASYMPTOTIC BEHAVIOR OF BOUNDARY BLOW-UP SOLUTIONS TO SEMILINEAR ELLIPTIC PROBLEMS WITH NON-STANDARD GROWTH

SHUIBO HUANG, WAN-TONG LI, QIAOYU TIAN

Abstract. In this article, we analyze uniqueness and asymptotic behavior of boundary blow-up non-negative solutions to the semilinear elliptic equation

$$\begin{gathered}\Delta u = b(x)f(u), \quad x \in \Omega,\\ u(x) = \infty, \quad x \in \partial\Omega,\end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $b(x)$ is a non-negative function on Ω and f is non-negative on $[0, \infty)$ satisfying some structural conditions. The main novelty of this paper is that uniqueness is established only by imposing a control on their growth on the weights $b(x)$ near $\partial\Omega$ and the nonlinear term f at infinite, rather than requiring them to have a precise asymptotic behavior. Our proof is based on the method of sub and super-solutions and the Safonov iteration technique.

1. Introduction and statement of main results

This article is concerned with uniqueness and asymptotic behavior of boundary blow-up solutions to the semilinear elliptic equation

$$\begin{gathered}\Delta u = b(x)f(u), \quad u \ge 0, \quad x \in \Omega,\\ u(x) = \infty, \quad x \in \partial\Omega,\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \ge 3)$ is a bounded smooth domain. The boundary condition is to be understood as $\lim_{\delta(x)\to 0} u(x) = \infty$ for $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. By a solution to (1.1) we mean a function $u \in C^1_{\rm loc}(\Omega)$, which satisfies $\Delta u = b(x)f(u)$ in the weak sense and $\lim_{\delta(x)\to 0} u(x) = \infty$, such solutions are often referred to as large solutions, boundary blow-up solutions or explosive solutions.

We now explain our assumptions on the weight function $b(x)$. Let $\mathcal{K}_{[C_\ell, C^\ell]}$ denote the set of all positive, non-decreasing functions $k \in L^1(0,\vartheta) \cap C^1(0,\vartheta)$ which satisfy

$$\lim_{t\to 0+} \frac{K(t)}{k(t)} = 0, \quad \liminf_{t\to 0+} \frac{d}{dt}\Big(\frac{K(t)}{k(t)}\Big) = C_\ell, \quad \limsup_{t\to 0+} \frac{d}{dt}\Big(\frac{K(t)}{k(t)}\Big) = C^\ell,$$

2000 *Mathematics Subject Classification.* 35J65, 35J60, 74G30, 35B40.
Key words and phrases. Boundary blow-up solutions; uniqueness; asymptotic behavior.

Submitted July 20, 2012. Published August 21, 2012.
Supported by grants 11031003 from the NSF of China, and lzujbky-2011-k27 FRFCU.

where $K(t) = \int_0^t k(s)ds$. When $C_\ell = C^\ell = \ell$, denote $\mathcal{K}_\ell = \mathcal{K}_{[C_\ell, C^\ell]}$, for further details on $\mathcal{K}_\ell$, we refer to [38, 43, 12, 5, 26].

The basic structural assumptions of weight function $b(x)$ are the following:

(B1) $b \in C^\alpha(\Omega)$ with $\alpha \in (0,1)$, is non-negative on Ω.

(B2) There exist $k \in \mathcal{K}_{[C_\ell, C^\ell]}$ with $0 \le C_\ell \le C^\ell \le 1$ and positive constants $0 < C_k \le C^k$ such that

$$\liminf_{\delta(x)\to 0} \frac{b(x)}{k^2(\delta)} = C_k, \quad \limsup_{\delta(x)\to 0} \frac{b(x)}{k^2(\delta)} = C^k. \tag{1.2}$$

We assume the nonlinear term f satisfies:

(F1) $f(t) \ge 0$, $f(t) > 0$ for large $t > 0$, $f(0) = 0$, $f(t)$ is locally Lipschitz continuous on $[0,\infty)$ and differentiable for large t.

(F2) $\int_t^\infty \frac{ds}{f(s)} < \infty$ for large $t > 0$.

(F3) There exist positive constants Λ_1, Λ_2 with $\max\{1, \Lambda_1\} \le \Lambda_2 \le \Lambda_1 + 1$ such that

$$\liminf_{t\to\infty} f'(t) \int_t^\infty \frac{ds}{f(s)} = \Lambda_1, \quad \limsup_{t\to\infty} f'(t) \int_t^\infty \frac{ds}{f(s)} = \Lambda_2. \tag{1.3}$$

Note that when $\Lambda_1 = \Lambda_2 \ge 1$, by (1.3), we have

$$\lim_{t\to\infty} f'(t) \int_t^\infty \frac{ds}{f(s)} = \Lambda_1, \tag{1.4}$$

which already appeared in [21, 50, 48, 49, 3, 22, 51] in order to describe the variation of f at infinity. It is worth mentioning that, f is rapidly varying at infinity (see Definition 2.3 below) if $\Lambda_1 = \Lambda_2 = 1$. However, f is normalized regularly varying at infinity (see Definition 2.2 below) with index $\Lambda_1/(\Lambda_1 - 1)$ if $\Lambda_1 = \Lambda_2 > 1$.

Let us mention that condition similar to (1.4) have been used to describe the variation of $b(x)$ at zero. More precisely, set

$$B(t) = \int_t^\infty \frac{1}{A(s)} ds, \quad A(t) = \Big(\int_0^t b^{\frac{1}{p+1}} \Big)^{\frac{p+1}{p-1}},$$

then

$$\lim_{t\to 0} \Big(A'(t) \int_t^\infty \frac{1}{A(s)} ds \Big) = \lim_{t\to 0} \frac{B(t)B''(t)}{B^2(t)},$$

which appears in [38, 27, 2].

Singular boundary value problem (1.1) arises naturally from a number of different areas and has a long history. Indeed, elliptic boundary blow-up problems arise in completely different fields as Riemannian geometry [28, 29], population dynamics[36, 18], stochastic control problem with state constraints[32, 33] and fluid dynamics[15].

There is a great amount of research devoted to study boundary blow-up problems related with (1.1). Generally speaking, the existence problem is relatively well understood but the uniqueness problem is only partially understood. Furthermore, besides their own intrinsic interest, the uniqueness results provide us with the dynamics of the positive solutions in a large number of sublinear and superlinear indefinite parabolic problems in the absence of steady-state solutions, when the dynamics is governed by the metasolutions of the model, see [36, 37, 39] and the references therein.

When $f(u) = u^p, p > 1$ and weight function $b(x)$ was permitted to vanish on $\partial\Omega$, with a precise rate of the form

$$\lim_{\delta \to 0} \frac{b(x)}{\delta^\alpha(x)} = \beta,$$

for some positive constants α, β, García-Melián et al [23], Du and Huang [18] obtained the uniqueness of boundary blow-up solutions to (1.1). Further improvements of the results of [23, 18] can be found in [9, 8].

Recently, by making use of an iteration technique due to Safonov, the uniqueness also be established in [16, 4] provided $f(u) = u^p, p > 1$ and $b(x)$ satisfies

$$C_1\delta^\alpha(x) \le b(x) \le C_2\delta^\alpha(x), x \in \Omega_\eta, \tag{1.5}$$

where $\eta > 0$, $0 < C_1 \le C_2$, $\alpha > 0$ are constants and $\Omega_\eta = \{x \in \Omega, 0 < \delta(x) < \eta\}$. For more general nonlinear term f, García-Melián proved the uniqueness of (1.1) with $b(x) \in C(\overline{\Omega})$ satisfies (1.5), f satisfies (1.4)[21], or f satisfies $\lim_{u\to\infty} f(u)/u^p = 1$, $p > 1$ and $f(u)/u$ is increasing for $u > 0$ [20].

In a different direction, by using Karamata's theory for regularly varying functions, Cîrstea and Du [7] showed that uniqueness of boundary blow-up solution to (1.1) with $f \in RV_{\rho+1}, \rho > 0$ still holds provided (1.5) was relaxed to

$$C_1k^2(\delta(x)) \le b(x) \le C_2k^2(\delta(x)), x \in \Omega_\eta, \tag{1.6}$$

where C_1, C_2, α are positive constants and $k \in \mathcal{K}_\ell$. Zhang and Mi [51] also shown the uniqueness of (1.1) with $b(x)$ satisfies (1.6) and f satisfies (1.4).

Our main objective of this paper is to establish uniqueness and boundary behavior of boundary blow-up solutions to (1.1). A point worth emphasizing is that one could not expect that the solutions are well-behaved near $\partial\Omega$ if the weight function and nonlinear terms are not. Therefore, we can only obtain a control on boundary blow-up solutions's growth near $\partial\Omega$ under the assumptions (1.2) and (1.3).

It is worth pointing out that uniqueness of boundary blow-up solutions to elliptic problems (1.1) has been obtained frequently in the literature by means of boundary estimates (with the exception of [14, 38]). More precisely, proving uniqueness is reduced to showing that every boundary blow-up solution has the same explosion rate at the boundary, which can be obtained if $b(x)$ has a prescribed behavior near $\partial\Omega$ and f has a prescribed behavior near infinity. Consequently, the quotient of any two solutions tends to one as $\delta(x)$ tends to zero. The uniqueness is the direct result of an additional monotonicity condition, like

$$\frac{f(t)}{t} \text{ is increasing for } t > 0. \tag{1.7}$$

Note that, under the assumption of (1.2) and (1.3), we only can obtain a control on boundary blow-up solutions's growth near boundary, instead of a definite behavior of them near boundary, we will overcome the difficulty by Safonov iterative technique. Furthermore, we only have (see Remark 1.3 below),

$$\frac{f(t)}{t^p} \text{ is increasing for } t \ge t_0,\ 1 \le q < \Lambda, \text{ where } \Lambda = \begin{cases} \frac{\Lambda_1}{\Lambda_2 - 1}, & \Lambda_2 > 1, \\ \infty, & \Lambda_2 = 1\,, \end{cases} \tag{1.8}$$

instead of (1.7) holds. For related but different uniqueness results, see [41, 42, 13, 25, 19, 17, 14] and the references therein.

We begin by stating our result on boundary behavior and uniqueness of boundary blow-up solutions to (1.1) when there is no competition between nonlinear term f and weight function b.

Theorem 1.1. *Suppose that* (F1)–(F3), (B1), (B2) *are satisfied. Then* (1.1) *has unique positive solution* $u(x)$ *satisfying,*

$$\liminf_{\delta\to 0}\frac{u(x)}{\phi(\xi^+K^2(\delta))}\geq 1,\quad \limsup_{\delta\to 0}\frac{u(x)}{\phi(\xi^-K^2(\delta))}\leq 1, \tag{1.9}$$

if $(\Lambda_1-1)+C_\ell>0$, *where*

$$\int_{\phi(t)}^{\infty}\frac{ds}{f(s)}=t, \tag{1.10}$$

and

$$\xi^+=\frac{C^k}{4(\Lambda_1-1)+2C_\ell},\quad \xi^-=\frac{C_k}{4(\Lambda_2-1)+2C^\ell}.$$

Remark 1.2. According to Proposition 2.6 below, ϕ is the solution of the one-dimensional problem

$$\begin{gathered}\phi'(t)=-f(\phi(t)),\quad t\in(0,\infty),\\ \phi(0)=\infty,\end{gathered} \tag{1.11}$$

where f satisfies (F1)–(F3). It is interesting to note that (1.11) is independent of the weight function $b(x)$, and is not the one-dimensional version of (1.1).

Remark 1.3. Using Proposition 2.5 below, we have

$$\begin{aligned}\Lambda_1-1&=\liminf_{t\to\infty}\Big(f'(t)\int_t^\infty\frac{ds}{f(s)}-1\Big)\leq\liminf_{t\to\infty}\frac{f(t)}{t}\int_t^\infty\frac{ds}{f(s)}\\ &\leq\limsup_{t\to\infty}\frac{f(t)}{t}\int_t^\infty\frac{ds}{f(s)}\leq\limsup_{t\to\infty}\Big(f'(t)\int_t^\infty\frac{ds}{f(s)}-1\Big)=\Lambda_2-1.\end{aligned} \tag{1.12}$$

Then, by (1.3) we find that for large t,

$$\Big(f'(t)-p\frac{f(t)}{t}\Big)\int_t^\infty\frac{ds}{f(s)}\geq\Lambda_1-p(\Lambda_2-1), \tag{1.13}$$

while,

$$\Big(\frac{f(t)}{t^p}\Big)'=\frac{1}{t^p}\Big(f'(t)-p\frac{f(t)}{t}\Big).$$

This fact, combineed with (1.13), shows that $f(t)/t^p$ is increasing for $t\geq t_0$ if $1<p<\Lambda$, where Λ appears in (1.8).

Remark 1.4. By Remark 1.3, we easily get that $f(t)$ satisfies the following Keller-Osserman condition

$$\int_t^\infty\frac{ds}{\sqrt{2F(s)}}<\infty,\quad F(t)=\int_0^t f(s)ds.$$

Then, by Theorem 1.1 in [9], we know that (1.1) has at least one boundary blow-up solution. Other related results on the existence of the minimal solution to (1.1), see [31, 10, 6, 47, 2, 9, 42] and the references therein.

Remark 1.5. In particular, according to Proposition 2.5 below, we know that $\phi \in NRVZ_{1-\Lambda_1}$ if $\Lambda_1 = \Lambda_2$. Then

$$\begin{aligned}\Big(\frac{C^k}{4(\Lambda_1 - 1) + 2C_\ell}\Big)^{1-\Lambda_1} &\leq \liminf_{\delta\to 0} \frac{u(x)}{\phi(K^2(\delta))} \leq \limsup_{\delta\to 0} \frac{u(x)}{\phi(K^2(\delta))} \\ &\leq \Big(\frac{C_k}{4(\Lambda_2 - 1) + 2C^\ell}\Big)^{1-\Lambda_1},\end{aligned}$$

provided $\Lambda_1 = \Lambda_2 > 1$, and

$$\lim_{\delta\to 0} \frac{u(x)}{\phi(K^2(\delta))} = 1, \tag{1.14}$$

provided $\Lambda_1 = \Lambda_2 = 1$, $0 < C_\ell \leq C^\ell$. This fact shows that boundary blow-up solution to (1.1) has a exact boundary behavior whereas the weight function not if f is rapidly varying at infinity ($\Lambda_1 = \Lambda_2 = 1$), which differs from the case that f is regularly varying at infinity ($\Lambda_1 = \Lambda_2 > 1$).

Remark 1.6. If $f = u^p, p > 1$, it is easy to find that

$$\Lambda_1 = \Lambda_2 = \frac{p}{p-1},\ \phi(t) = \Big(\frac{1}{(p-1)t}\Big)^{1/(p-1)}.$$

Then, (1.9) implies that, for small $\delta > 0$,

$$u(x) \geq \Big(\frac{C^k(p-1)}{4 + 2(p-1)C_\ell}\Big)^{-1/(p-1)} \Big(\frac{1}{(p-1)K^2(\delta)}\Big)^{1/(p-1)},$$

and

$$u(x) \leq \Big(\frac{C_k(p-1)}{4 + 2(p-1)C^\ell}\Big)^{-1/(p-1)} \Big(\frac{1}{(p-1)K^2(\delta)}\Big)^{1/(p-1)}.$$

Remark 1.7. Let $f = e^u$, it follows that $\Lambda_1 = \Lambda_2 = 1$, $\phi(t) = -\log t$. Therefore,

$$\lim_{\delta\to 0} \frac{u(x)}{\log K(\delta)} = -2,$$

provided $0 < C_\ell \leq C^\ell$. Note that $f = e^u$ does not satisfy $f(0) = 0$, but this is no importance for the results.

The next objective is to consider the case that $\Lambda_2 = 1$.

Theorem 1.8. *Suppose that* (F1)–(F3) *hold with* $\Lambda_2 = 1$, b *satisfies* (B1) *and*

$$\liminf_{\delta(x)\to 0} \frac{b(x)}{k^2(\delta)\big(\frac{K(\delta)}{k(\delta)}\big)'} = \mathscr{C}_k, \quad \limsup_{\delta(x)\to 0} \frac{b(x)}{k^2(\delta)\big(\frac{K(\delta)}{k(\delta)}\big)'} = \mathscr{C}^k, \tag{1.15}$$

where $k \in \mathcal{K}_0$ *with* $\Big(\frac{K(\delta)}{k(\delta)}\Big)'' \geq 0$. *Furthermore,* K *satisfies*

$$\lim_{t\to 0} \frac{K(t)}{k(t)\big(\frac{K(t)}{k(t)}\big)'} = 0, \tag{1.16}$$

and

$$\liminf_{t\to 0} \Big(\frac{1 - f'(\phi(K^2(t))) \int_{\phi(K^2(t))}^\infty \frac{ds}{f(s)}}{\big(\frac{K(t)}{k(t)}\big)'}\Big) = \mathscr{C}_\ell > \frac{1}{2}, \tag{1.17}$$

$$\limsup_{t\to 0} \Big(\frac{1 - f'(\phi(K^2(t))) \int_{\phi(K^2(t))}^\infty \frac{ds}{f(s)}}{\big(\frac{K(t)}{k(t)}\big)'}\Big) = \mathscr{C}^\ell. \tag{1.18}$$

Then (1.1) *has a unique boundary blow-up solution* $u(x)$ *satisfying* (1.14).

Remark 1.9. As we already said, by a standard argument, the uniqueness of boundary blow-up solution to (1.1) will be a direct consequence of boundary estimate provided $\Lambda_2 = 1$, since any boundary blow-up solution has the same boundary behavior near the boundary. Hence we focus on boundary behavior of boundary blow-up solution to (1.1) when $\Lambda_2 = 1$.

Remark 1.10. Thanks to Theorem 1.8, we find that when f is rapidly varying at infinity, which grows faster than any power functions, then the vanishing rate of weight function b at boundary $\partial\Omega$ enters into competition with the growth of f at infinity. This phenomena was firstly studied by Cîrstea in [5], where b satisfies (1.6) with $k \in \mathcal{K}_0$, instead of (1.15).

Remark 1.11. The transformation $u = \phi(v)$ changes (1.1) into

$$\begin{gathered} -\Delta v + \Pi(v)\frac{|\nabla v|^2}{v} = b(x), \quad x \in \Omega, \\ v(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.19}$$

where

$$\Pi(t) = -\frac{t\phi''(t)}{\phi'(t)}.$$

Obviously, for small t,

$$\Pi(t) = tf'(\phi(t)) = f'(\phi(t))\int_{\phi(t)}^{\infty}\frac{ds}{f(s)},$$

which, together with (1.3), implies $\liminf_{t\to 0}\Pi(t) = \Lambda_1$, $\limsup_{t\to 0}\Pi(t) = \Lambda_2$.

Here, we will prove Theorem 1.1 and 1.8 directly, unlike earlier works [34, 35, 21], considering boundary value problem (1.19) satisfied by $v = \psi(u)$, where ψ defined by

$$\psi(t) = \int_t^{\infty}\frac{ds}{f(s)}.$$

The distribution of this paper is as follows. In Section 2, we collect some preliminary results. Theorem 1.1 will be proved in Section 3. Section 4 is devoted to prove Theorem 1.8. Some illustrative examples are analyzed in Section 5.

2. Preliminaries

We start by recalling some definitions and qualities about regular variation theory. For detailed accounts of the theory of regular variation, its extensions and many of its applications, we refer to [1, 45, 44, 24, 46, 40].

2.1. **Regular variation theory.**

Definition 2.1. A positive measurable function f defined on $[D,\infty)$ for some $D > 0$, is called regularly varying (at infinity) with index $\rho \in \mathbb{R}$ (written $f \in RV_\rho$) if for all $\xi > 0$

$$\lim_{u\to\infty}\frac{f(\xi u)}{f(u)} = \xi^{\rho}.$$

When the index of regular variation ρ is zero, we say that the function is slowly varying.

Definition 2.2. A function $f(u)$ defined for $u > B$ is called a normalized regularly varying function of index q (in short $f \in NRV_\rho$) if it is C^1 and satisfies

$$\lim_{u\to\infty} \frac{uf'(u)}{f(u)} = \rho.$$

Note that $f \in NRV_{\rho+1}$ if and only if f is C^1 and $f' \in RV_\rho$.

The notion of regular variation can be extended to any real number. We say that $f(u)$ is regularly varying (respectively, normalized regularly varying) at the origin from the right with index $\rho \in \mathbb{R}$, denoted by $f \in RVZ_\rho$ (respectively, $f \in NRVZ_\rho$), if $f(1/u) \in RV_{-\rho}$ (respectively, $f(1/u) \in NRV_{-\rho}$).

Definition 2.3. A positive measurable function f defined on (A, ∞) for some $A > 0$ is called rapidly varying at infinity if for each $p > 1$,

$$\lim_{u\to\infty} \frac{f(u)}{u^p} = \infty.$$

For the sake of convenience, we introduce several classes of functions.

Let $RV_{[\rho_1,\rho_2]}$ denote the set of all positive measurable function f defined on $[D, \infty)$ for some $D > 0$, satisfying

$$\liminf_{u\to\infty} \frac{f(\xi u)}{f(u)} \ge \xi^{\rho_1}, \quad \limsup_{u\to\infty} \frac{f(\xi u)}{f(u)} \le \xi^{\rho_2}, \quad \xi > 0.$$

In particular, when $\rho_1 = \rho_2$, f is called regularly varying at infinity with index ρ_1. One can show that all regularly varying functions belong to this class. This is also true for all positive measurable functions which are on (A, ∞) bounded away from both 0 and ∞.

It is sometimes necessary to transfer attention from infinity to the origin. More precisely, let $RVZ_{[\rho_1,\rho_2]}$ denote the set of all positive measurable function f defined on $[D, \infty)$ for some $D > 0$, satisfy

$$\liminf_{u\to 0} \frac{f(\xi u)}{f(u)} \ge \xi^{\rho_1}, \quad \limsup_{u\to 0} \frac{f(\xi u)}{f(u)} \le \xi^{\rho_2}, \quad \xi > 0.$$

Let $NRV_{[\rho_1,\rho_2]}$ denote the set of all C^1 functions satisfying

$$\liminf_{u\to\infty} \frac{uf'(u)}{f(u)} \ge \rho_1 \quad \limsup_{u\to\infty} \frac{uf'(u)}{f(u)} \le \rho_2\,, \quad \rho \in \mathbb{R}.$$

Clearly, when $\rho_1 = \rho_2$, f is called normalized regularly varying at infinity with index ρ_1 and $NRV_\rho \subset NRV_{[\rho_1,\rho_2]}$ for any $\rho \in [\rho_1, \rho_2]$.

Similarly, $NRVZ_{[\rho_1,\rho_2]}$ denotes the set of all C^1 functions satisfying

$$\liminf_{u\to 0} \frac{uf'(u)}{f(u)} \ge \rho_1\,, \quad \limsup_{u\to 0} \frac{uf'(u)}{f(u)} \le \rho_2\,, \quad \rho \in \mathbb{R}.$$

2.2. **Comparison principle.** The following comparison principle will play an important role in the proof of our main theorem.

Proposition 2.4. *Let f be continuous on $(0, \infty)$ such that $f(u)/u$ is increasing for $u > 0$, and $b(x) \in C(\Omega)$ be a non-negative function. Assume that $u_1, u_2 \in C^2(\Omega)$ are positive functions such that*

$$\Delta u_1 - b(x)f(u_1) \le 0 \le \Delta u_2 - b(x)f(u_2), \quad x \in \Omega,$$
$$\limsup_{\delta(x)\to 0}(u_2 - u_1)(x) \le 0.$$

Then we have $u_1 \geq u_2$ in Ω.

The proof of the above proposition can be found in [9, 11], see also [35] for a version corresponding to the p-Laplacian case.

2.3. **General l'Hôpital rule.** For the sake of computation, we mention here the general l'Hôpital rule which appears in [34] and is used throughout the paper.

Proposition 2.5. *Suppose $f(x)$ and $g(x)$ are differentiable functions defined on (α, β) for with $g'(x) \neq 0$ for all $x \in (\alpha, \beta)$, where $-\infty \leq \alpha < \beta \leq \infty$. If $\liminf_{t\to\beta} \frac{f'(t)}{g'(t)}$ and $\limsup_{t\to\beta} \frac{f'(t)}{g'(t)}$ exist and $\lim_{t\to\beta} g(t) = \infty$. Then*

$$\liminf_{t\to\beta} \frac{f'(t)}{g'(t)} \leq \liminf_{t\to\beta} \frac{f(t)}{g(t)} \leq \limsup_{t\to\beta} \frac{f(t)}{g(t)} \leq \limsup_{t\to\beta} \frac{f'(t)}{g'(t)}.$$

The proof of this results follows by slight modification of the usual proof of l'Hôpital rule, hence we omit it.

2.4. **Properties of f and ϕ.** In this subsection we quote some results about f and ϕ which are used in subsequent sections.

Proposition 2.6. *Suppose that f satisfies (F1)–(F3). Then*

(i) $\frac{\Lambda_1 - 1}{\Lambda_2} = \liminf_{t\to\infty} \frac{f(t)}{tf'(t)} \leq \limsup_{t\to\infty} \frac{f(t)}{tf'(t)} \leq \frac{\Lambda_2 - 1}{\Lambda_1}$.

(ii) *f is rapidly varying at infinity if $\Lambda_2 = 1$.*

(iii) *ϕ is well defined on $(0, \infty)$, $\phi(t) > 0$, $t > 0$, $\phi(0) = \infty$, $\phi(\infty) = 0$, $\phi'(t) = -f(\phi(t))$, $\phi''(t) = f(\phi(t))f'(\phi(t))$.*

(iv) $-\phi' \in NRVZ_{[-\Lambda_2, -\Lambda_1]}$, $\phi \in NRVZ_{[1-\Lambda_2, 1-\Lambda_1]}$.

(v) *$t^p\phi(t)$ is increasing for $t \geq t_0$ if $p > \Lambda_2 - 1$, is decreasing for $t \geq t_0$ if $p < \Lambda_1 - 1$.*

Proof. (i). Direct computations show that

$$\begin{aligned}
\frac{\Lambda_1 - 1}{\Lambda_2} &\leq \frac{1}{\Lambda_2} \liminf_{t\to\infty} \frac{f(t)\int_t^\infty \frac{d\tau}{f(\tau)}}{t} \leq \liminf_{t\to\infty} \frac{f(t)\int_t^\infty \frac{d\tau}{f(\tau)}}{tf'(t)\int_t^\infty \frac{d\tau}{f(\tau)}} \\
&= \liminf_{t\to\infty} \frac{f(t)}{tf'(t)} \leq \limsup_{t\to\infty} \frac{f(t)\int_t^\infty \frac{d\tau}{f(\tau)}}{tf'(t)\int_t^\infty \frac{d\tau}{f(\tau)}} \\
&\leq \frac{1}{\Lambda_1} \limsup_{t\to\infty} \frac{f(t)\int_t^\infty \frac{d\tau}{f(\tau)}}{t} \leq \frac{\Lambda_2 - 1}{\Lambda_1}.
\end{aligned}$$

(ii). By (i), we find that there exists a positive constant t_0, such that for all $t > t_0$,

$$\frac{f'(t)}{f(t)} > (q+1)t^{-1}.$$

Integrating the above inequality from t_0 to t, we have

$$\int_{t_0}^t \frac{f'(s)ds}{f(s)} = \ln f(t) - \ln f(t_0) > (q+1)(\ln t - \ln t_0).$$

That is,

$$\frac{f(t)}{t^p} > \frac{f(t_0)t}{t_0^{p+1}},$$

which implies that f is rapidly varying at infinity.

(iii). For the proof of this results, see [3, 50, 21].

(iv). By (iii), it can be easily seen that, for small $t > 0$,

$$-\frac{t\phi''(t)}{\phi'(t)} = f'(\phi(t)) \int_{\phi(t)}^{\infty} \frac{ds}{f(s)},$$

which implies that

$$-\liminf_{t\to 0} \frac{t\phi''(t)}{\phi'(t)} = \Lambda_1, \ -\limsup_{t\to 0} \frac{t\phi''(t)}{\phi'(t)} = \Lambda_2.$$

That is, $-\phi'(t) \in RV_{[-\Lambda_2,-\Lambda_1]}$. Consequently, $f(\phi(t)) \in RV_{[-\Lambda_2,-\Lambda_1]}$ and

$$\begin{aligned}
1-\Lambda_1 &= \liminf_{t\to 0} \frac{t\phi''(t)+\phi'(t)}{\phi'(t)} \le \liminf_{t\to 0} \frac{t\phi'(t)}{\phi(t)} \\
&\le \limsup_{t\to 0} \frac{t\phi'(t)}{\phi(t)} \le \limsup_{t\to 0} \frac{t\phi''(t)+\phi'(t)}{\phi'(t)} = 1-\Lambda_2.
\end{aligned}$$

(v) A simple calculation yields

$$(t^p\phi(t))' = t^{p-1}\phi(t)\left(\frac{t\phi'(t)}{\phi(t)} + p\right).$$

This fact, together with (iv), shows that $(t^p\phi(t))' > 0$ if $p > \Lambda_2 - 1$ and $(t^p\phi(t))' < 0$ if $p < \Lambda_1 - 1$. □

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. As remarked in the introduction, the main point is that the behavior of the solutions can be characterized in terms of a one-dimensional first-order equation. For clarity, we divide the lengthy proof into two steps.

3.1. **Asymptotic Behavior.**

Proof. We now diminish $\eta > 0$ to ensure that, for all $\varepsilon \in (0, C_k/2)$, $\delta \in (0,\eta)$ and $\beta \in (0,\delta)$,

(i) $k(x)$ is non-increasing on $(0, 2\eta)$.
(ii) $(C_k - \varepsilon)k^2(\delta(x) - \beta) \le b(x) < (C^k + \varepsilon)k^2(\delta + \beta)$ in the set $\Omega_{2\eta} = \{x \in \Omega, 0 < \delta(x) < 2\eta\}$.
(iii) $\|\nabla\delta(x)\| = 1$ for every $x \in \Omega_{2\eta}$.
(iv) $\delta(x)$ is C^2-function in the set $\Omega_{2\eta}$.

For $\beta \in (0,\delta)$, define

$$u_\beta^{\pm}(x) = \phi(\xi_\varepsilon^{\pm} K^2(\delta(x) \pm \beta)), \quad x \in \Omega_\beta^{\pm},$$

where ϕ is given by (1.10), $\Omega_\beta^- = \Omega_{2\eta}\backslash\bar{\Omega}_\beta$, $\Omega_\beta^+ = \Omega_{2\eta-\beta}$ and

$$\xi_\varepsilon^+ = \frac{C^k + 2\varepsilon}{4(\Lambda_1 - 1) + 2C_\ell}, \quad \xi_\varepsilon^- = \frac{C_k - 2\varepsilon}{4(\Lambda_2 - 1) + 2C^\ell}.$$

Then

$$\begin{aligned}
&\Delta u_\beta^+ - b(x)f(u_\beta^+) \\
&\ge 4(\xi_\varepsilon^+)^2\phi''(\xi_\varepsilon^+K^2(\delta+\beta))K^2(\delta+\beta)k^2(\delta+\beta) + 2\xi_\varepsilon^+\phi'(\xi_\varepsilon^+K^2(\delta+\beta))k^2(\delta+\beta) \\
&\quad + 2\xi_\varepsilon^+\phi'(\xi_\varepsilon^+K^2(\delta+\beta))K(\delta+\beta)k'(\delta+\beta)
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\xi_\varepsilon^+ \phi'(\xi_\varepsilon K^2(\delta+\beta))K(\delta+\beta)k(\delta+\beta)\Delta\delta - (C^k+\varepsilon)k^2(\delta+\beta)f(u_\beta^+) \\
&= k^2(\delta+\beta)f(u_\beta^+)[A_1^+(\delta+\beta) + A_2^+(\delta+\beta) + A_3^+(\delta+\beta) + A_4^+(\delta+\beta)\Delta\delta \\
&\quad - (C^k+\varepsilon)],
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta u_\beta^- - b(x)f(u_\beta^-) \\
&\le 4(\xi_\varepsilon^-)^2\phi''(\xi_\varepsilon^- K^2(\delta-\beta))K^2(\delta-\beta)k^2(\delta-\beta) + 2\xi_\varepsilon^-\phi'(\xi_\varepsilon K^2(\delta-\beta))k^2(\delta-\beta) \\
&\quad + 2\xi_\varepsilon^-\phi'(\xi_\varepsilon K^2(\delta-\beta))K(\delta-\beta)k'(\delta-\beta) \\
&\quad + 2\xi_\varepsilon^-\phi'(\xi_\varepsilon K^2(\delta-\beta))K(\delta-\beta)k(\delta-\beta)\Delta\delta - (C_k-\varepsilon)k^2(\delta-\beta)f(u_\beta^-) \\
&= k^2(\delta-\beta)f(u_\beta^-)[A_1^-(\delta-\beta) + A_2^-(\delta-\beta) + A_3^-(\delta-\beta) + A_4^-(\delta-\beta)\Delta\delta \\
&\quad - (C_k-\varepsilon)],
\end{aligned}
$$

where

$$
\begin{gathered}
A_1^\pm(t) = 4(\xi_\varepsilon^\pm)^2 \frac{\phi''(\xi_\varepsilon^\pm K^2(t))K^2(t)}{f(\phi(\xi_\varepsilon^\pm K^2(t))}, \quad A_2^\pm(t) = 2\xi_\varepsilon^\pm \frac{\phi'(\xi_\varepsilon^\pm K^2(t))}{f(\phi(\xi_\varepsilon^\pm K^2(t))}, \\
A_3^\pm(t) = 2\xi_\varepsilon^\pm \frac{\phi'(\xi_\varepsilon^\pm K^2(t))K(t)k'(t)}{k^2(t)f(\phi(\xi_\varepsilon^\pm K^2(t))}, \quad A_4^\pm(t) = 2\xi_\varepsilon \frac{\phi'(\xi_\varepsilon^\pm K^2(t))K(t)}{k(t)f(\phi(\xi_\varepsilon^\pm K^2(t))}.
\end{gathered}
$$

By Proposition 2.5, we obtain

$$
\begin{gathered}
\liminf_{t\to 0} A_1^\pm(t) = 4\xi_\varepsilon^\pm \Lambda_1, \quad \limsup_{t\to 0} A_1^\pm(t) = 4\xi_\varepsilon^\pm \Lambda_2, \\
\lim_{t\to 0} A_2^\pm(t) = -2\xi_\varepsilon^\pm, \quad \lim_{t\to 0} A_4^\pm(t) = 0, \\
\begin{aligned}
\liminf_{t\to 0} A_3^\pm(t) &= -2\xi_\varepsilon^\pm \liminf_{t\to 0} \frac{K(t)k'(t)}{k^2(t)} = 2\xi_\varepsilon^\pm \liminf_{t\to 0}\Big(\Big(\frac{K(t)}{k(t)}\Big)' - 1\Big) \\
&= 2\xi_\varepsilon^\pm (C_\ell - 1), \\
\limsup_{t\to 0} A_3^\pm(t) &= -2\xi_\varepsilon^\pm \limsup_{t\to 0} \frac{K(t)k'(t)}{k^2(t)} = 2\xi_\varepsilon^\pm \limsup_{t\to 0}\Big(\Big(\frac{K(t)}{k(t)}\Big)' - 1\Big) \\
&= 2\xi_\varepsilon^\pm (C^\ell - 1).
\end{aligned}
\end{gathered}
$$

The above computation leads to

$$
\begin{gathered}
\liminf_{\delta+\beta\to 0}[A_1^+(\delta+\beta) + A_2^+(\delta+\beta) + A_3^+(\delta+\beta) + A_4^+(\delta+\beta)\Delta\delta - (C^k+\varepsilon)] = \varepsilon, \\
\limsup_{\delta-\beta\to 0}[A_1^-(\delta-\beta) + A_2^-(\delta-\beta) + A_3^-(\delta-\beta) + A_4^-(\delta-\beta)\Delta\delta - (C_k-\varepsilon)] = -\varepsilon.
\end{gathered}
$$

Thus diminish η if necessary such that

$$
\begin{gathered}
\Delta u_\beta^+ - b(x)f(u_\beta^+) > 0, \ x \in \Omega_\beta^+, \\
\Delta u_\beta^- - b(x)f(u_\beta^-) < 0, \ x \in \Omega_\beta^-.
\end{gathered}
$$

It is obvious that

$$
\begin{gathered}
u_\beta^+(x) \le N(\eta) + u(x), x \in \{x \in \Omega : \delta(x) = 2\eta - \beta\}, \\
\lim_{\delta\to 0}[u_\beta^+(x) - N(\eta) - u(x)] = -\infty,
\end{gathered}
$$

where $N(\eta) = \phi(\xi_\varepsilon K^2(\eta))$, u is a positive solution to (1.1), which implies that

$$
u_\beta^+(x) \le N(\eta) + u(x), \quad x \in \partial\Omega_\beta^+.
$$

Clearly,

$$\Delta(u_\beta^+(x) - N(\eta)) = \Delta u_\beta^+(x) \ge b(x)f(u_\beta^+) \ge b(x)f(u_\beta^+(x) - N(\eta)).$$

This fact, combined with Proposition 2.4, shows that

$$u_\beta^+(x) \le N(\eta) + u(x), \quad x \in \Omega_\beta^+. \tag{3.1}$$

On the other hand,

$$\begin{gathered} u(x) \le M(2\eta) + u_\beta^-, \quad x \in \{x \in \Omega : \delta(x) = 2\eta\}, \\ \lim_{\delta \to \beta} [M(2\eta) + u_\beta^- - u(x)] = \infty, \end{gathered}$$

where $M(2\eta) = \max_{\delta(x) \ge 2\eta} u(x)$. That is $u(x) \le M(2\eta) + u_\beta^-, x \in \partial\Omega_\beta^-$, which combined with Proposition 2.4 and

$$\Delta(M(2\eta) + u_\beta^-) = \Delta u_\beta^- \le b(x)f(u_\beta^-) \le b(x)f(M(2\eta) + u_\beta^-),$$

shows that

$$u(x) \le M(2\eta) + u_\beta^-, \quad x \in \Omega_\beta^-. \tag{3.2}$$

Using (3.1) and (3.2), we infer that

$$u_\beta^+(x) - N(\eta) \le u(x) \le M(2\eta) + u_\beta^-, \quad x \in \Omega_\beta^- \cap \Omega_\beta^+,$$

where $\Omega_\beta^- \cap \Omega_\beta^+ = \{x \in \Omega, \beta < \delta(x) < 2\eta - \beta\}$. This yields that for any $x \in \Omega_\beta^- \cap \Omega_\beta^+$,

$$\begin{gathered} \frac{u(x)}{\phi(\xi_\varepsilon^- K^2(\delta(x) - \beta))} - \frac{M(2\eta)}{\phi(\xi_\varepsilon^- K^2(\delta(x) - \beta))} \le 1, \\ \frac{u(x)}{\phi(\xi_\varepsilon^+ K^2(\delta(x) + \beta))} + \frac{N(\eta)}{\phi(\xi_\varepsilon^+ K^2(\delta(x) + \beta))} \ge 1. \end{gathered}$$

Letting $\beta \to 0$, we arrive at

$$\frac{u(x)}{\phi(\xi_\varepsilon^- K^2(\delta))} - \frac{M(2\eta)}{\phi(\xi_\varepsilon^- K^2(\delta))} \le 1, \tag{3.3}$$

$$\frac{u(x)}{\phi(\xi_\varepsilon^+ K^2(\delta))} + \frac{N(\eta)}{\phi(\xi_\varepsilon^+ K^2(\delta))} \ge 1. \tag{3.4}$$

In view of Proposition 2.6 and boundedness of $N(\eta)$, $M(2\eta)$, letting $\delta \to 0$ and $\varepsilon \to 0$ in (3.3) and (3.4), we derive that (1.9) holds. □

3.2. **Uniqueness.** The aim of the present section is proving that any two positive solutions $u_1(x)$, $u_2(x)$ to (1.1) satisfy $u_1(x)/u_2(x) \to 1$ as $\delta(x) \to 0$, which together with (1.7), leads to uniqueness. The proof is a refinement of the iterative technique attributed to Safonov, which has been used in [21, 22, 30, 51].

Proof. Step 1. We first remark that, thanks to Proposition 2.5, given any two positive strong solutions to (1.1), it follows that the quotient of any two solutions is bounded and bounded away from zero.

Step 2. Let u_1, u_2 be arbitrary positive solutions to (1.1). To prove the uniqueness it suffices to show that

$$\lim_{\delta \to 0} \frac{u_1(x)}{u_2(x)} = 1.$$

Firstly, we show that

$$\lambda = \limsup_{\delta \to 0} \frac{u_1(x)}{u_2(x)} \le 1.$$

The argument proceeds by contradiction; that is $\lambda > 1$. Given a small $\varepsilon > 0$ such that $\varepsilon \in (0, \max\{\lambda - 1, C_k\})$; thus, there exist $\delta_\epsilon > 0$ and x_0 such that

(i)
$$\frac{u_1(x)}{u_2(x)} < \lambda + \varepsilon, \quad x \in \Omega_{\delta_\epsilon}, \tag{3.5}$$

(ii) $b(x) \geq (C_k - \varepsilon)k^2(\delta)$.
(iii) $\frac{u_1(x_0)}{u_2(x_0)} > \lambda - \varepsilon$, $x_0 \in \Omega_{2\delta_\epsilon/3}$.
(iv) $(\lambda - \varepsilon)u_2(x) > t_0$, where t_0 is such that $f(t)/t^p$ is increasing for $t \geq t_0$ and some $p \in (1, \frac{\Lambda_1}{\Lambda_2 - 1})$.
(v) $u(x) \geq \phi(\xi^+ K^2(\delta))$, $x \in \Omega_{\delta_\epsilon}$.
(vi) $f(\phi(K^2(\delta)))K^2(\delta) \leq (\Lambda_2 - 1)\phi(K^2(\delta))$, $x \in \Omega_{\delta_\epsilon}$.

Define
$$\Omega_0 = \{x \in \Omega : u_1(x) > (\lambda - \varepsilon)u_2(x)\} \cap B_\rho(x_0),$$
where $B_\rho(x_0) = \{x \in \Omega : |x - x_0| < \rho\}$ and $\rho = \delta(x_0)/2$. In the set Ω_0, we find
$$\begin{aligned}
\Delta(u_1 - (\lambda - \varepsilon)u_2) &= b(x)[f(u_1) - (\lambda - \varepsilon)f(u_2)] \\
&\geq b(x)[f((\lambda - \varepsilon)u_2) - (\lambda - \varepsilon)f(u_2)] \\
&\geq b(x)[(\lambda - \varepsilon)^p - (\lambda - \varepsilon)]f(u_2) \\
&\geq (C_k - \varepsilon)[(\lambda - \varepsilon)^p - (\lambda - \varepsilon)]k^2(\delta)f(u_2) \\
&\geq (C_k - \varepsilon)[(\lambda - \varepsilon)^p - (\lambda - \varepsilon)]k^2(\delta(x))f\left(\phi(\xi^+ K^2(\delta))\right) \\
&\geq (C_k - \varepsilon)[(\lambda - \varepsilon)^p - (\lambda - \varepsilon)]k^2(\rho/2)f\left(\xi^+ \phi(K^2(3\rho/2))\right) \\
&\geq C(\lambda - \varepsilon)K^2(\rho)f(\phi(K^2(C\rho))),
\end{aligned} \tag{3.6}$$
where C is a positive constant which can be taken independently of ε varying from line to line.

Define $\vartheta(x) = (\rho^2 - |x - x_0|^2)/2N$. Obviously, $\vartheta(x)$ satisfies
$$-\Delta\vartheta(x) = 1, x \in B_\rho(x_0), \ \vartheta(x) = 0, x \in \partial B_\rho(x_0),$$
which together with (3.6), we arrive at
$$\Delta(u_1 - (\lambda - \varepsilon)u_2 + M_1\vartheta) \geq 0, x \in \Omega_0,$$
where $M_1 = C(\lambda - \varepsilon)K^2(\rho)f(\phi(K^2(C\rho)))$. Then, according to maximum principle, we find that there exists $x_1 \in \partial\Omega_0$ such that
$$u_1(x_0) - (\lambda - \varepsilon)u_2(x_0) + M_1\vartheta(x_0) \leq u_1(x_1) - (\lambda - \varepsilon)u_2(x_1) + M_1\vartheta(x_1). \tag{3.7}$$
If $x_1 \in B_\rho(x_0)$, then $u_1(x_1) = (\lambda - \varepsilon)u_2(x_1)$, taking into account (3.7), we infer that $\vartheta(x_0) < \vartheta(x_1)$, which is impossible. Thus $x_1 \in \partial B_\rho(x_0)$, namely, $\vartheta(x_1) = 0$, this fact, combined with (3.7), implies
$$M_1\rho^2/2N = M_1\vartheta(x_0) \leq u_1(x_1) - (\lambda - \varepsilon)u_2(x_1). \tag{3.8}$$
On the other hand, by $\delta(x_1) < 3\delta(x_0)/2 \leq \delta_\varepsilon$,
$$M_1\rho^2/2N > C(\lambda - \varepsilon)K^2(\rho)f(\phi(K^2(C\rho))) \geq C(\lambda - \varepsilon)u_2(x_1), \tag{3.9}$$
which, combined with (3.8) and (3.9), shows that
$$u_1(x_1) \geq (1 + C)(\lambda - \varepsilon)u_2(x_1). \tag{3.10}$$

Thus, taking into account (3.5), we obtain $\lambda+\varepsilon \geq (1+C)(\lambda-\varepsilon)$, letting $\varepsilon \to 0$, we arrive at $1 \geq (1+C)$, which is impossible. This contradiction leads to $\lambda \leq 1$. A symmetric argument proves that $\lambda \geq 1$.

Step 3. The uniqueness follows from a standard argument. For completeness we include the short proof. Let $u_{\min}(x)$, $u_{\max}(x)$ are minimal and maximal solutions to (1.1), separately, in the sense that any other solutions $u(x)$ to (1.1) must satisfy $u_{\min}(x) \leq u(x) \leq u_{\max}(x)$. Subsequently, we will show that $u_{\min}(x) = u_{\max}(x)$.

Then, taking into account step 2, we have

$$\lim_{\delta(x)\to 0} \frac{u_{\min}(x)}{u_{\max}(x)} = 1.$$

Thus given $\varepsilon > 0$, there is $\eta_0 > 0$ such that

$$(1-\varepsilon)u_{\max}(x) \leq u_{\min}(x), \quad x \in \Omega_{\eta_0}.$$

By (1.7), we have

$$\begin{aligned}\Delta((1-\varepsilon)u_{\max}(x)) &= (1-\varepsilon)\Delta u_{\max}(x)\\ &= (1-\varepsilon)b(x)f(u_{\max}) \geq b(x)f((1-\varepsilon)u_{\max}(x)).\end{aligned}$$

Let ω be the unique solution of

$$\begin{gathered}\Delta\omega = b(x)f(\omega), \quad x \in \mathcal{O},\\ \omega = u_{\min}(x), \quad x \in \partial\mathcal{O},\end{gathered}$$

where $\mathcal{O} = \{x \in \Omega : \delta(x,\partial\Omega) \geq \eta_0\}$. By the comparison principle, it follows that

$$(1-\varepsilon)u_{\max}(x) \leq u_{\min}(x), \quad x \in \mathcal{O},$$

On the other hand, in view of the uniqueness of ω, we derive that $\omega(x) = u_{\min}(x)$, $x \in \mathcal{O}$. Consequently,

$$(1-\varepsilon)u_{\max}(x) \leq \omega(x), \quad x \in \Omega,$$

which implies that $u_{\max}(x) \leq u_{\min}(x)$, $x \in \Omega$. By the definition of $u_{\max}(x)$ and $u_{\min}(x)$, we have $u_{\max}(x) = u_{\min}(x)$. □

4. Proof of Theorem 1.8

Proof. Fix $\varepsilon \in (0, \max\{1-2\mathscr{C}^\ell, \mathscr{C}_k\})$ and choose $\varsigma > 0$ such that

(i) $\delta(x)$ is a C^2 function in the set Ω_ς.
(ii) $k(x)$ is non-decreasing in $(0,\delta)$.
(iii) $(\mathscr{C}_k - \varepsilon)k^2(\delta-\beta)\left(\frac{K(\delta-\beta)}{k(\delta-\beta)}\right)' < b(x) < (\mathscr{C}^k+\varepsilon)k^2(\delta+\beta)\left(\frac{K(\delta+\beta)}{k(\delta+\beta)}\right)'$ in the set Ω_ς.

Define $u_\beta^\pm(x) = \phi(\xi_\varepsilon^\pm \kappa(\delta(x)\pm\beta)), x \in \Omega_\beta^\pm$ for any $\beta \in (0,\varsigma)$, where $\kappa(t) = K^2(t)$,

$$\xi_\varepsilon^+ = \frac{\mathscr{C}^k+\varepsilon}{1-2\mathscr{C}_\ell-\varepsilon}, \quad \xi_\varepsilon^- = \frac{\mathscr{C}_k-\varepsilon}{1-2\mathscr{C}^\ell+\varepsilon}.$$

Then

$$\begin{aligned}&\Delta u_\beta^+ - b(x)f(u_\beta^+)\\ &\geq (\xi_\varepsilon^+)^2\phi''(\xi_\varepsilon^+\kappa(\delta(x)+\beta))(\kappa'(\delta(x)+\beta))^2 + \xi_\varepsilon^+\phi'(\xi_\varepsilon^+\kappa(\delta+\beta))\kappa''(\delta(x)+\beta)\\ &\quad + \xi_\varepsilon^+\phi'(\xi_\varepsilon^+\kappa(\delta(x)+\beta))\kappa'(\delta(x)+\beta)\Delta\delta - (\mathscr{C}^k+\varepsilon)k^2(\delta+\beta)\left(\frac{K(\delta+\beta)}{k(\delta+\beta)}\right)' f(u_\beta^+)\end{aligned}$$

$$
\begin{aligned}
&= \xi_\varepsilon^+ \phi'(\xi_\varepsilon^+ \kappa(\delta(x)+\beta))\kappa(\delta(x)+\beta)\left(\frac{\kappa'(\delta(x)+\beta)}{\kappa(\delta(x)+\beta)}\right)' \\
&\quad \times \left[B_1^+(\delta+\beta) - B_2^+(\delta+\beta) + B_3^+(\delta+\beta)(\mathscr{C}^k+\varepsilon)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta u_\beta^- - b(x) f(u_\beta^-) \\
&\le (\xi_\varepsilon^-)^2 \phi''(\xi_\varepsilon^- \kappa(\delta(x)-\beta))(\kappa'(\delta(x)-\beta))^2 + \xi_\varepsilon^- \phi'(\xi_\varepsilon^- \kappa(\delta-\beta))\kappa''(\delta(x)-\beta) \\
&\quad + \xi_\varepsilon^- \phi'(\xi_\varepsilon^- \kappa(\delta(x)-\beta))\kappa'(\delta(x)-\beta)\Delta\delta - (\mathscr{C}^k+\varepsilon)k^2(\delta-\beta)\left(\frac{K(\delta-\beta)}{k(\delta-\beta)}\right)' f(u_\beta^-) \\
&= \xi_\varepsilon^- \phi'(\xi_\varepsilon^- \kappa(\delta(x)-\beta))\kappa(\delta(x)-\beta)\left(\frac{\kappa'(\delta(x)-\beta)}{\kappa(\delta(x)-\beta)}\right)' \\
&\quad \times \left[B_1^-(\delta-\beta) - B_2^-(\delta-\beta) + B_3^-(\delta-\beta)(\mathscr{C}_k-\varepsilon)\right],
\end{aligned}
$$

where

$$
B_1^\pm(t) = 1 - \frac{\frac{\kappa(t)}{\kappa'(t)}}{\left(\frac{\kappa(t)}{\kappa'(t)}\right)'}\Delta\delta, \quad B_2^\pm(t) = \frac{1 + \frac{\xi_\varepsilon^\pm \kappa(t)\phi''(\xi_\varepsilon^\pm \kappa(t))}{\phi'(\xi_\varepsilon^\pm \kappa(t))}}{\left(\frac{\kappa(t)}{\kappa'(t)}\right)'},
$$

$$
B_3^\pm(t) = -\frac{k^2(t)\left(\frac{K(t)}{k(t)}\right)'}{\xi_\varepsilon^\pm \kappa(t)\left(\frac{\kappa'(t)}{\kappa(t)}\right)'}.
$$

By (1.16), we have $\lim_{t\to 0} B_1^\pm(t) = 1$, using (1.17) and (1.18), we find

$$
\liminf_{t\to 0} B_2^\pm(t) = 2\mathscr{C}_\ell, \quad \limsup_{t\to 0} B_2^\pm(t) = 2\mathscr{C}^\ell, \quad \lim_{t\to 0} B_3^\pm(t) = -1/2\xi_\varepsilon^\pm.
$$

We can use the same line of arguments as in the proof of Theorem 1.1 to obtain this results, here we omit the details of the proof. □

5. Examples

We now give some examples of nonlinearity f which satisfy the assumptions of the main theorem in this paper.

Example 5.1. Let $f(t) = t^\rho + \sin t^\rho + 2$, $\rho > 0$, thus

$$
\liminf_{t\to\infty} \frac{f(\xi t)}{f(t)} = \limsup_{t\to\infty} \frac{f(\xi t)}{f(t)} = \xi^\rho, \quad \xi > 0.
$$

Namely, $f(t) \in RV_\rho$. However,

$$
\liminf_{t\to\infty} \frac{tf'(t)}{f(t)} = 0, \quad \limsup_{t\to\infty} \frac{tf'(t)}{f(t)} = 2\rho,
$$

which implies that $f \in NRV_{[0,2\rho]}$.

Example 5.2. Let $f(t)$ is a positive, differentiable function satisfying

$$
C_1 t^{\rho_1} \le f'(t) \le C_2 t^{\rho_2}, \quad f(0) = 0, \text{ for large } t > 0, \tag{5.1}
$$

where $C_1 \le C_1$, $0 < \rho_1 \le \rho_2$ are positive constant. Then, by (5.1), we find

$$
\frac{C_1}{1+\rho_1} t^{1+\rho_1} \le f(t) \le \frac{C_2}{1+\rho_2} t^{1+\rho_2}. \tag{5.2}
$$

Taking into account (5.1) and (5.2), we obtain

$$\frac{C_1(1+\rho_2)}{C_2\rho_2}t^{\rho_1-\rho_2} \le f'(t)\int_t^\infty \frac{ds}{f(s)} \le \frac{C_2(1+\rho_1)}{C_1\rho_1}t^{\rho_2-\rho_1},$$

which implies that $0 \le \Lambda_1 \le \Lambda_2$, however, we do not obtain a finite upper bound for Λ_2. In particular, if $\rho_1 = \rho_2$,

$$\frac{C_1(1+\rho_1)}{C_2\rho_1} \le \Lambda_1 \le \Lambda_2 \le \frac{C_2(1+\rho_1)}{C_1\rho_1}.$$

Example 5.3. Let $f \in NRV_{[1+\theta_1,1+\theta_2]}$ satisfies $f(0) = 0$, where $0 < \theta_1 \le \theta_2$. It follows that

$$\lim_{t\to\infty} \frac{t}{f(t)} = 0,$$

which together with Proposition 2.5, shows that

$$\begin{aligned}
\frac{1}{\theta_2} &\le \liminf_{t\to\infty} \frac{1}{\frac{tf'(t)}{f(t)} - 1} \le \liminf_{t\to\infty} \frac{f(t)\int_t^\infty \frac{ds}{f(s)}}{t} \\
&\le \limsup_{t\to\infty} \frac{f(t)\int_t^\infty \frac{ds}{f(s)}}{t} \le \limsup_{t\to\infty} \frac{1}{\frac{tf'(t)}{f(t)} - 1} \le \frac{1}{\theta_1}.
\end{aligned}$$

This inequality, combined with

$$f'(t)\int_t^\infty \frac{ds}{f(s)} = \frac{tf'(t)}{f(t)} \frac{f(t)\int_t^\infty \frac{ds}{f(s)}}{t},$$

implies that

$$\frac{1+\theta_1}{\theta_2} \le \Lambda_1 \le \Lambda_2 \le \frac{1+\theta_2}{\theta_1}.$$

Example 5.4. Let $f = e^{g(t)}$, where $g(t) \in NRV_{[\theta_1,\theta_2]}$ with $0 < \theta_1 \le \theta_2$. Obviously,

$$\lim_{t\to\infty} \frac{t}{g(t)e^{g(t)}} = 0.$$

Hence, in view of Proposition 2.5, we have

$$\begin{aligned}
\frac{1}{\theta_2} &\le \liminf_{t\to\infty} \frac{1}{\frac{tg'(t)-g(t)}{g^2(t)} + \frac{tg'(t)}{g(t)}} \le \liminf_{t\to\infty} \frac{g(t)e^{g(t)}}{t}\int_t^\infty \frac{ds}{e^{g(s)}} \\
&\le \limsup_{t\to\infty} \frac{g(t)e^{g(t)}}{t}\int_t^\infty \frac{ds}{e^{g(s)}} \le \limsup_{t\to\infty} \frac{1}{\frac{tg'(t)-g(t)}{g^2(t)} + \frac{tg'(t)}{g(t)}} \le \frac{1}{\theta_1}.
\end{aligned}$$

Consequently, by

$$f'(t)\int_t^\infty \frac{ds}{f(s)} = \frac{tg'(t)}{g(t)} \frac{g(t)e^{g(t)}}{t}\int_t^\infty \frac{ds}{e^{g(s)}},$$

we derive that

$$\frac{\theta_1}{\theta_2} \le \Lambda_1 \le \Lambda_2 \le \frac{\theta_2}{\theta_1}.$$

Acknowledgement. We would like to express our deep thanks to Professor Zhijun Zhang at Yantai University for sending us his papers and his valuable suggestions.

References

[1] N. H. Bingham, C. M. Goldie, J. L. Teugels; *Regular Variation*, Cambridge University Press, Cambridge, 1987.

[2] S. Cano-Casanova, J. López-Gómez; *Existence, uniqueness and blow-up rate of large solutions for a canonical class of one-dimensional problems on the half-line*, J. Differential Equations 244 (2008) 3180-3203.

[3] Y. Chen, M. Wang; *Uniqueness results and asymptotic behavior of solutions with boundary blow-up for logistic-type porous media equations*, Z. Angew. Math. Phys. 61 (2010) 277-292.

[4] M. Chuaqui, C. Cortázar, M. Elgueta, J. García-Melián; *Uniqueness and boundary behaviour of large solutions to elliptic problems with singular weights*, Comm. Pure Appl. Anal. 3 (2004) 653-662.

[5] F. Cîrstea; *Elliptic equations with competing rapidly varying nonlinearities and boundary blow-up*, Adv. Differential Equations 12 (2007) 995-1030.

[6] F. Cîrstea, M. Ghergu, V. Rădulescu; *Combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane-Emden-Fowler type*, J. Math. Pures Appl. 84 (2005) 493-508.

[7] F. Cîrstea, Y. Du; *General uniqueness results and variation speed for blow-up solutions of elliptic equations*. Proc. London Math. Soc. 91 (2005) 459-482.

[8] F. Cîrstea, V. Rădulescu; *Uniqueness of the blow-up boundary solution of logistic equations with absorbtion*, C. R. Acad. Sci. Paris, Ser. I. 335 (2002) 447-452.

[9] F. Cîrstea, V. Rădulescu; *Existence and uniqueness of blow-up solutions for a class of logistic equations*, Commun. Contemp. Math. 4 (2002) 559-586.

[10] F. Cîrstea, V. Rădulescu; *Blow-up solutions for semilinear elliptic problems*, Nonlinear Anal. 48 (2002) 541-554.

[11] F. Cîrstea, V. Rădulescu; *Extremal singular solutions for degenerate logistic-type equations in anisotropic media*, C. R. Acad. Sci. Paris, Ser. I 339 (2004) 119-124.

[12] F. Cîrstea, V. Rădulescu; *Nonlinear problems with boundary blow-up: a Karamata regular variation theory approach*, Asymptotic Anal. 46 (2006) 275-298.

[13] O. Costin, L. Dupaigne; *Boundary blow-up solutions in the unit ball: asymptotics, uniqueness and symmetry*, J. Differential Equations 249 (2010) 931-964.

[14] O. Costin, L. Dupaigne, O. Goubet; *Uniqueness of large solutions*, J. Math. Anal. Appl. 395 (2012) 806-812.

[15] J. I. Díaz, M. Lazzo, P. G. Schmidt; *Large solutions for a system of elliptic equations arising from fluid dynamics*, SIAM J. Math. Anal. 37 (2005) 490-513.

[16] Y. Du; *Asymptotic behavior and uniqueness results for boundary blow-up solutions*, Differential Integral Equations 17 (2004) 819-834.

[17] Y. Du, Z. Guo; *Uniqueness and layer analysis for boundary blow-up solutions*, J. Math. Pures Appl. 83 (2004) 739-763.

[18] Y. Du, Q. Huang; *Blow-up solutions for a class of semilinear elliptic and parabolic equations*, SIAM J Math. Anal. 31 (1999) 1-18.

[19] H. Dong, S. Kim, M. Safonov; *On uniqueness of boundary blow-up solutions of a class of nonlinear elliptic equations*. Comm. Partial Differential Equations 33 (2008) 177-188.

[20] J. García-Melián; *Nondegeneracy and uniqueness for boundary blow-up elliptic problems*, J. Differential Equations 223 (2006) 208-227.

[21] J. García-Melián; *Boundary behavior of large solutions to elliptic equations with singular weights*, Nonlinear Anal. 67 (2007) 818-826.

[22] J. García-Melián; *Uniqueness of positive solutions for a boundary blow-up problem*, J. Math. Anal. Appl. 360 (2009) 530-536.

[23] J. García-Melián, R. L. Albornoz, J. S. Lis; *Uniqueness and asymptotic behavior for solutions of semilinear problems with boundary blow-up*, Proc. Amer. Math. Soc. 129 (2001) 3593-3602.

[24] J. L. Geluk, L. de Haan; *Regular Variation, Extensions and Tauberian Theorems*, CWI Tract, Centrum Wisk. Inform. Amsterdam, 1987.

[25] Z. Guo, J. Shang; *Remarks on uniqueness of boundary blow-up solutions*, Nonlinear Anal. 66 (2007) 484-497.

[26] S. Huang, Q. Tian; *Second order estimates for large solutions of elliptic equations*, Nonlinear Anal. 74 (2011) 2031-2044.

[27] S. Huang, Q. Tian, S. Zhang, J. Xi, Z. Fan; *The exact blow-up rates of large solutions for semilinear elliptic equations*. Nonlinear Anal. 73 (2010) 3489-3501.
[28] S. Kichenassamy; *Boundary blow-up and degenerate equations*, J. Funct. Anal. 215 (2004) 271-289.
[29] S. Kichenassamy; *Boundary behavior in the Loewner-Nirenberg problem*, J. Funct. Anal. 222 (2005) 98-113.
[30] S. Kim; *A note on boundary blow-up problem of* $\Delta u = u^p$, IMA preprint No. 1872, 2002.
[31] A. V. Lair; *A necessary and sufficient condition for existence of large solutions to semilinear elliptic equations*, J. Math. Anal. Appl. 240 (1999) 205-218.
[32] J. M. Lasry, P. L. Lions; *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem*, Math. Ann. 283 (1989) 583-630.
[33] T. Leonori, A. Porretta; *The boundary behavior of blow-up solutions related to a stochastic control problem with state constraint*, SIAM J. Math. Anal. 39 (2007/08) 1295-1327.
[34] G. M. Lieberman; *Asymptotic behavior and uniqueness of blow-up solutions of elliptic equations*, Methods Appl. Anal. 15 (2008) 243-262.
[35] G. M. Lieberman; *Asymptotic behavior and uniqueness of blow-up solutions of quasilinear elliptic equations*. J. Anal. Math. 115 (2011) 213-249.
[36] J. López-Gómez; *Large solutions, metasolutions, and asymptotic behaviour of the regular positive solutions of sublinear parabolic problems*. Electron. J. Differ. Equ. Conf., 05 (2000) 135-171.
[37] J. López-Gómez; *Dynamics of parabolic equations: from classical solutions to metasolutions*, Differential Integral Equations 16 (2003) 813-828.
[38] J. López-Gómez; *Optimal uniqueness theorems and exact blow-up rates of large solutions*, J. Differential Equations 224 (2006) 385-439.
[39] J. López-Gómez, M. Molina-Meyerb; *Superlinear indefinite systems: Beyond Lotka-Volterra models*, J. Differential Equations 221 (2006) 343-411.
[40] V. Maric; *Regular Variation and Differential Equations*, Lecture Notes in Math., vol. 1726, Springer-Verlag, Berlin, 2000.
[41] M. Marcus, L. Véron; *Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations*. Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (1997) 237-274.
[42] M. Marcus, L. Véron; *Existence and uniqueness results for large solutions of general nonlinear elliptic equations*. J. Evol. Equ. 3 (2003) 637-652.
[43] A. Mohammed; *Boundary asymptotic and uniqueness of solutions to the p-Laplacian with infinite boundary values*, J. Math. Anal. Appl. 325 (2007) 480-489.
[44] V. Rădulescu; *Singular phenomena in nonlinear elliptic problems. From blow-up boundary solutions to equations with singular nonlinearities*, M. Chipot (Ed.), Handbook of Differential Equations: Stationary Partial Differential Equations, vol. 4 (2007) 483-591.
[45] S. I. Resnick; *Extreme Values, Regular Variation, and Point Processes*, Springer Verlag, Berlin/New York, 1987.
[46] E. Seneta; *Regularly Varying Functions*, Lecture Notes in Math. Vol. 508, Springer Verlag, Berlin, New York, 1976.
[47] S. Tao, Z. Zhang; *On the existence of explosive solutions for semilinear elliptic problems*, Nonlinear Anal. 48 (2002) 1043-1050.
[48] Z. Zhang; *Boundary behavior of solutions to some singular elliptic boundary value problems*, Nonlinear Anal. 67 (2007) 818-826.
[49] Z. Zhang, B. Li; *The boundary behavior of the unique solution to a singular Dirichlet problem*, J. Math. Anal. Appl. 391 (2012) 278-290.
[50] Z. Zhang, Y. Ma, L. Mi, X. Li; *Blow-up rates of large solutions for elliptic equations*, J. Differential Equations 249 (2010) 180-199.
[51] Z. Zhang, L. Mi; *Blow-up rates of large solutions for semilinear elliptic equations*. Commun. Pure Appl. Anal. 10 (2011) 1733-1745.

Shuibo Huang
School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, China
E-mail address: huangshuibo2008@163.com

Wan-Tong Li
School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, China
E-mail address: wtli@lzu.edu.cn

Qiaoyu Tian
Department of Mathematics, Gansu Normal University for Nationalities, Hezuo, Gansu 747000, China
E-mail address: tianqiaoyu2004@163.com