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EXISTENCE OF LATTICE SOLUTIONS TO SEMILINEAR ELLIPTIC SYSTEMS WITH PERIODIC POTENTIAL

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Abstract. Under the assumption that the potential W is invariant under a general discrete reflection group $G' = TG$ acting on $\mathbb{R}^n$, we establish existence of G'-equivariant solutions to $\Delta u - W_u(u) = 0$, and find an estimate. By taking the size of the cell of the lattice in space domain to infinity, we obtain that these solutions converge to G-equivariant solutions connecting the minima of the potential W along certain directions at infinity. When particularized to the nonlinear harmonic oscillator $u'' + \alpha \sin u = 0$, $\alpha > 0$, the solutions correspond to those in the phase plane above and below the heteroclinic connections, while the G-equivariant solutions captured in the limit correspond to the heteroclinic connections themselves. Our main tool is the G'-positivity of the parabolic semigroup associated with the elliptic system which requires only the hypothesis of symmetry for W. The constructed solutions are positive in the sense that as maps from $\mathbb{R}^n$ into itself leave the closure of the fundamental alcove (region) invariant.

1. Introduction

The study of the system

$$\Delta u - W_u(u) = 0, \quad \text{for } u : \mathbb{R}^n \to \mathbb{R}^n, \tag{1.1}$$

where $W : \mathbb{R}^n \to \mathbb{R}$ and $W_u := (\partial W/\partial u_1, \dots, \partial W/\partial u_n)^T$ under symmetry hypotheses on the potential W, was initiated in Bronsard, Gui, and Schatzman [4], where existence for the case $n = 2$ with the symmetries of the equilateral triangle was settled, and followed later by Gui and Schatzman [13], where the case $n = 3$ for the symmetry group of the tetrahedron was established. The corresponding solutions are known as the *triple junction* and the *quadruple junction* respectively. This class of solutions is characterized by the fact that they *connect* the global minima of the potential W in certain directions as $|x| \to +\infty$. In [1], Alikakos and Fusco established an abstract theorem for the existence of such solutions, together with an estimate, for general dimension n and *any reflection point group* G acting on $\mathbb{R}^n$. Finally in [2] one of us gave a simpler proof of the result in [1].

Reflections are special linear isometries, and as such leave the Laplacian invariant. The groups of symmetries of the equilateral triangle and the tetrahedron are

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reflection groups. This motivated the study in [1] for general reflection groups. By including also translations, which obviously leave the Laplacian invariant, we are led naturally to *discrete reflection groups* G' acting on $\mathbb{R}^n$. We note that $G' = TG$ where T is the translation group of G' and G the point group fixing the origin O. T generates a *lattice* in $\mathbb{R}^n$ via the orbit $\{\tau O : \tau \in T\}$. There are two natural problems that one can identify in this context:

(I) The periodic potential problem;
(II) The periodic solution problem.

In Problem (I), we require that $W(gu) = W(u)$, for every $u \in \mathbb{R}^n$ and every $g \in G'$. Both domain and target have G' acting on them and the solutions are G'-equivariant. Actually it is more appropriate to scale G' in domain and target differently by introducing a parameter $R > 0$:

$$G'_R := \{g_R : g_R(x) := Rg(x/R), g \in G'\} \tag{1.2}$$

and then seek (G'_R, G') *equivariant* solutions:

$$u(g_R x) = gu(x) \quad \text{for every } x \in \mathbb{R}^n \text{ and every } g \in G'. \tag{1.3}$$

In Problem (II), G'_R is acting in the domain and G in the target. We require that $W(gu) = W(u)$, for every $u \in \mathbb{R}^n$ and $g \in G$, and we seek a solution in the class of G-equivariant maps modulo the translations of the discrete group G'_R:

$$u(\tau_R g x) = gu(x) \quad \text{for every } x \in \mathbb{R}^n, \text{ every } \tau \in T, \text{ and every } g \in G. \tag{1.4}$$

These two classes of solutions are best visualized in terms of an elementary example, the harmonic oscillator $u'' + \alpha \sin u = 0$, $\alpha > 0$, that we explain in detail below. Type (I) correspond to the solutions above and below the separatrices in the phase plane, while type (II) correspond to the periodic solutions inside a single cell. The heteroclinic solutions (separatrices) correspond to the solutions constructed in [4, 13, 1].

In this paper we focus on problem (I). We construct an equivariant solution to (1.1) which maps each cell in the domain lattice to the corresponding cell in the target lattice. Moreover, the map u_R in the limit $R \to \infty$, converges to another solution to (1.1) equivariant for the point group G. In terms of the harmonic oscillator above, the parameter R relates to the "time" x that it takes for the solution u_R to transverse a cell, and thus as $R \to \infty$ the solution converges to the heteroclinic connection[1].

One of the main tools in [1, 2], is the positivity invariance of the gradient flow $u_t = \Delta u - W_u(u)$, under appropriate boundary conditions. A map u is *positive* if $u(\overline{F}) \subset \overline{F}$, where F is the fundamental region of the group. Positivity is built in the minimization process (that is, in constructing u_R) as a constraint. This constraint is then removed via the gradient flow which is shown to preserve it. The solutions we construct are global minimizers in the class of positive maps. It is not known if the property of positivity is automatic for global minimizers.

In the point reflection group case treated in [1] one can reduce the problem of positivity to a half-space determined by a reflection plane (instead of F), intersected by a ball of radius R, and homogeneous Neuman conditions on the circumference. In the discrete reflection group case treated in the present paper, the fundamental alcove is a bounded simplex and one has to deal with the whole object all at once.

[1] N. D. A. is indebted to Peter Bates for pointing this out to him, as well as the different scalings.

If S_i denotes the ith face of the fundamental alcove and Π_i its supporting plane then the boundary conditions take the form

$$
\begin{gathered}
x \in S_i \Rightarrow u(x) \in \Pi_i \\
x \in S_i \Rightarrow \frac{\partial u}{\partial r_i}(x) \perp \Pi_i, \text{ where } r_i \perp \Pi_i\,.
\end{gathered}
\tag{1.5}
$$

The main contribution in the present paper is the positivity result under (1.5) (Theorem 5.1). Otherwise the paper proceeds as in [2]. We mention in passing that the boundary conditions in (1.5) are conformal and for smooth domains have been studied by Hamilton [14].

2. Notation

As it was mentioned in the Introduction, G' denotes a discrete reflection group acting on $\mathbb{R}^n$, G its associated point group, and G'_R the image of G' by the homothetic transformation $x \to Rx$, where $R > 0$. G'_R is the scaled discrete group of the blown-up lattice; its elements are defined in (1.2).

A discrete reflection group G' is generated by a finite set of reflections. The hyperplanes $\{x \in \mathbb{R}^n : gx = x\}$ associated to these reflections $g \in G'$, bound a region called *fundamental alcove* (see [17]) of G' and denoted by F', with the following properties:

(i) F' is an open polyhedron with acute angles [2],
(ii) $F' \cap gF' = \emptyset$, for $I \neq g \in G'$, where I is the identity,
(iii) $\mathbb{R}^n = \cup\{g\overline{F'} : g \in G'\}$.

Similarly the associated point group G has a *fundamental region*, that is, a subset $F \subset \mathbb{R}^n$ with the following properties:

(i) F is open and convex,
(ii) $F \cap gF = \emptyset$, for $I \neq g \in G$, where I is the identity,
(iii) $\mathbb{R}^n = \cup\{g\overline{F} : g \in G\}$.

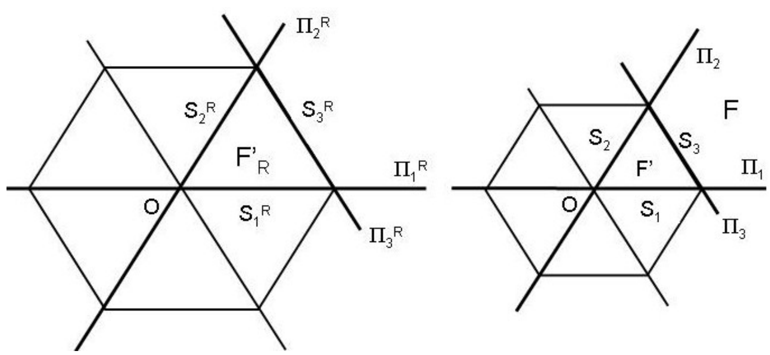

Figure 1. Fundamental alcoves for the discrete groups G'_R and G' of the hexagonal lattice

F' and F can be chosen such that the origin O is a vertex of F' and $F' \subset F$. As regards the scaled group G'_R, it is clear that its fundamental alcove denoted by F'_R

[2]Thanks to Christos Athanasiadis of the University of Athens who confirmed this fact to us.

is the set $\{Rx : \ x \in F'\}$. We shall denote the closed faces of F' (respectively F'_R) by S_i, $i = 1, \cdots, N$ (respectively S_i^R), and by Π_i (respectively Π_i^R) their supporting planes.

In the same way we defined in (1.3) (G'_R, G') equivariant solutions, a map u is called G equivariant when:

$$u(gx) = gu(x) \quad \text{for every } x \in \mathbb{R}^n \text{ and every } g \in G. \tag{2.1}$$

Given $a \in \mathbb{R}^n$, the *stabilizer* of a, denoted by G'_a, is the subgroup of G' that fixes a. Finally, we shall denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product, by $|\cdot|$ the Euclidean norm, by $d(\cdot, \cdot)$ the Euclidean distance, and by $B(x, \epsilon)$ the open ball of center x and radius ϵ with respect to this distance.

3. Main Theorems

We begin by stating the hypotheses.

(H1) (Symmetry) The potential W, of class C^3, is invariant under a discrete reflection group G' acting on $\mathbb{R}^n$; that is,

$$W(gu) = W(u) \quad \text{for all } g \in G' \text{ and } u \in \mathbb{R}^n. \tag{3.1}$$

(H2) (nondegenerate global minimum) Let $F' \subset \mathbb{R}^n$ be a fundamental alcove of G'. We assume that W is non-negative and has a unique zero a in $\overline{F'}$. Furthermore, there holds $v^\top \partial^2 W(u) v \geq 2c^2 |v|^2$, for $v \in \mathbb{R}^n$ and $|u - a| \leq \bar{q}$, for some c, $\bar{q} > 0$.

(H3) (Q-monotonicity) We restrict ourselves to potentials W for which there is a continuous function $Q : \mathbb{R}^n \to \mathbb{R}$ that satisfies

$$Q(u + a) = |u| + H(u), \tag{3.2}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function such that $H(0) = 0$ and $H_u(0) = 0$, and

$$Q \text{ is convex}, \tag{3.3a}$$

$$Q(u) > 0, \quad \text{on } \mathbb{R}^n \setminus \{a\}, \tag{3.3b}$$

and, moreover,

$$\langle Q_u(u), W_u(u) \rangle \geq 0, \quad \text{in } D' \setminus \{a\}, \tag{3.4}$$

where we have set

$$D' := \operatorname{Int}\left(\cup_{g \in G'_a} g\overline{F'}\right). \tag{3.5}$$

Theorem 3.1. *Under Hypothesis* (H1)*, for every $R > 0$ there exists a (G'_R, G') equivariant classical solution u_R to system* (1.1) *such that $u_R(F'_R) \subset F'$. Furthermore, if* (H2)–(H3) *also hold, then there exist positive constants R_0, k, K, such that for $R > R_0$ and $x \in D'_R := \{Rx : x \in D'\}$*

$$|u_R(x) - a| \leq K e^{-kd(x, \partial D'_R)}. \tag{3.6}$$

Theorem 3.2. *Under Hypotheses* (H1)–(H3)*, there exists a G equivariant classical solution u to system* (1.1) *such that*

(i) *$u(\overline{F}) \subset \overline{F'}$ and $u(D) \subset D'$, where we have set $D := \cup_{R>0}\{Rx : x \in D'\}$.*
(ii) *$|u(x) - a| \leq K e^{-kd(x, \partial D)}$, for $x \in D$ and for positive constants k, K.*

It would be interesting to relate Theorem 3.2 to the main result in [1]. Theorem 3.1 was conjectured by G. Fusco in a personal communication.

The harmonic oscillator is the elementary example which best illustrates the above theorems. For the ODE,

$$u'' + \alpha \sin u = 0, \quad \text{with } \alpha > 0, \tag{3.7}$$

the corresponding potential is $W(u) = \alpha(1 + \cos u)$ which is invariant under the discrete group G' acting on $\mathbb{R}$, generated by the reflections s_0 and s_π with respect to the points 0 and π. The associated point group of G' is $G = \{I, s_0\}$, where I denotes the identity. The fundamental alcove of G' is $F' = (0, \pi)$ while the fundamental region of G is $F = (0, \infty)$. Clearly, W has in $\overline{F'}$ a unique minimum attained at $a = \pi$, and since $a \in \partial F'$, we have $D' = (0, 2\pi)$ and $D = (0, \infty)$. Finally, Q-monotonicity is also verified by taking $Q(u + \pi) = |u|$.

Now we shall prove that solutions u to (3.7), above the separatrices in the plane phase and satisfying the initial condition $u(0) = 0$ are (G'_R, G') equivariant for some $R > 0$ depending on $u'(0)$. For such solutions, $\lambda := u'(0) > 2\sqrt{\alpha}$ holds, and by integrating $u'u'' = -\alpha(sinu)u'$, we obtain

$$u' = \sqrt{2W(u) + \lambda^2 - 4\alpha}\,. \tag{3.8}$$

As a consequence, u is strictly increasing with $\lim_{x\to+\infty} u(x) = +\infty$. This ensures the existence of a minimal $T > 0$, depending on λ, such that $u(T) = 2\pi$. Then, utilizing the periodicity of W, it is easy to see that u is (G'_R, G') equivariant, for $R := T/2\pi$. Actually, since T is strictly decreasing from $+\infty$ to 0, when $\lambda \in (2\sqrt{\alpha}, +\infty)$, there exists for every $R > 0$ a unique (G'_R, G') equivariant solution to (3.7), called u_R.

According to Theorem 3.1, u_R satisfies $u_R((0, T/2)) \subset (0, \pi)$, which is obvious, and $|u_R(x) - \pi| \le K\mathrm{e}^{-kx}$, for $x \in F'_R = (0, T/2)$ and R big enough. Considering then the sequence $u_R : (-T/2, T/2) \to \mathbb{R}$, and passing to the limit when $R \to \infty$, we capture the heteroclinic solution u of Theorem 3.2, which is G equivariant (i.e. odd) and satisfies:

(i) $u([0, \infty)) \subset [0, \pi]$;
(ii) $|u(x) - \pi| \le K\mathrm{e}^{-kx}$, for $x \in (0, +\infty)$.

In $\mathbb{R}^2$, we give other examples of potentials W satisfying our hypotheses for the discrete group G' generated by the reflections with respect to the lines $u_2 = 0$, $u_1 = \pi$ and $u_1 = u_2$. The fundamental alcove bounded by these three lines is: $F' = \{(u_1, u_2) \in (0, \pi) \times (0, \pi) : u_1 > u_2\}$. Considering the auxilliary function $\phi(u) = \cos^2 u + \cos u + 1/4$, we construct the potentials:

$$W_1(u_1, u_2) = \phi(u_1) + \phi(u_2), \quad W_2(u_1, u_2) = \phi(u_1) + \phi(u_2) + \phi(u_1)\phi(u_2)$$

which have a unique minimum, nondegenerate, in $\overline{F'}$ at $a = (2\pi/3, 2\pi/3)$. It can then be verified that W_1 and W_2 satisfy Q-monotonicity in $D' = (0, \pi) \times (0, \pi)$ with $Q(u) = |u - a|$, where $u = (u_1, u_2)$.

The Hypotheses (H1)–(H3) are exact analogs of the hypotheses introduced in [1] for the point group G. Actually our hypothesis on Q here is less restrictive since no symmetry assumptions are imposed (cf. [2]). Q-monotonicity is a restrictive hypothesis on the potential. We note that for $n = 1$ and even symmetry, for a double-well potential W, and $D = F = \{u > 0\}$, Q-monotonicity implies that for $u > 0$, $W_u(u)(u - a_1) \ge 0$ holds. Nevertheless Q-monotonicity allows for a large class of nontrivial potentials (cf. [1]). Very recently, in ongoing work, Fusco (cf. [9])

has been able to remove the Q-monotonicity from the hypotheses in establishing the main result in [1]. Since the main contribution of the present work is the positivity result, which does not require the Q-monotonicity hypothesis, we decided not to take into account these new developments.

4. Minimization

Thanks to the proposition below, we can identify the class of (G'_R, G') equivariant Sobolev maps $W^{1,2}_{\mathrm{E\,loc}}(R)(\mathbb{R}^n;\mathbb{R}^n)$, with the class

$$K^R := \{u \in W^{1,2}(F'_R;\mathbb{R}^n) : \mathrm{Tr}\, u(S_i^R) \subset \Pi_i \ ,\forall i \ 1 \le i \le N\}. \tag{4.1}$$

of $W^{1,2}(F'_R;\mathbb{R}^n)$ maps such that the restriction of the trace (denoted by Tr) to each of the faces S_i^R of F'_R, takes values almost everywhere in the corresponding hyperplane Π_i.

Proposition 4.1. *The map which associates to each $u \in W^{1,2}_{\mathrm{E\,loc}}(R)(\mathbb{R}^n;\mathbb{R}^n)$ its restriction to F'_R is one-to-one and onto K^R.*

Proof. We denote by $(x_1,\dots,x_n)$ the coordinates of $x \in \mathbb{R}^n$, and by $(u^1,\dots,u^n)$ the components of a map u with values in $\mathbb{R}^n$. Without loss of generality we assume that $R = 1$ and that $\Pi = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n = 0\}$ is the supporting plane of the face S of F'. Furthermore, we denote by $g(x_1,\dots,x_n) = (x_1,\dots,-x_n)$, the reflection with respect to Π. To a map $\phi : F' \to \mathbb{R}^n$, we associate its extension by reflection to gF', denoted by $\overline{\phi}$. Clearly, we have for every $x \in gF'$:

$$\overline{\phi}(x) := g\phi(gx) = (\phi^1(x_1,\dots,-x_n),\dots,\phi^{n-1}(x_1,\dots,-x_n),-\phi^n(x_1,\dots,-x_n)). \tag{4.2}$$

It is easy to check that $u \in W^{1,2}(F';\mathbb{R}^n) \Rightarrow \overline{u} \in W^{1,2}(gF';\mathbb{R}^n)$. In addition, if the sequence $\phi_m \in C^\infty(\overline{F'};\mathbb{R}^n)$ converges to u in $W^{1,2}(F';\mathbb{R}^n)$, then the sequence $\overline{\phi_m} \in C^\infty(\overline{gF'};\mathbb{R}^n)$ also converges to $\overline{u}$ in $W^{1,2}(gF';\mathbb{R}^n)$.

Now let us consider a map $u \in W^{1,2}_{\mathrm{E\,loc}}(\mathbb{R}^n;\mathbb{R}^n)$ and let us prove that $\mathrm{Tr}\, u(S) \subset \Pi$. Obviously $u = \overline{u}$ and $(\mathrm{Tr}\, u)\big|_S = (\mathrm{Tr}\, \overline{u})\big|_S$, where $\big|_S$ denotes the restriction of a map to S. Then writing that $\lim_{m\to\infty} \phi_m\big|_S = \lim_{m\to\infty} \overline{\phi_m}\big|_S$ in $L^2(S;\mathbb{R}^n)$ and utilizing (4.2), we find that $\lim_{m\to\infty}(\phi^n)_m\big|_S = 0$ in $L^2(S,\mathbb{R})$ which means that $\mathrm{Tr}\, u(S) \subset \Pi$. Therefore, the map of Proposition 4.1. takes its values in K^R and clearly by equivariance it is one-to-one. To complete the proof, it remains to show that it is onto.

Indeed, if $u \in K^R$, we can extend it by reflection to a map called U, defined in $\Omega := \mathrm{Int}\left(\overline{F'} \cup g\overline{F'}\right)$, setting:

$$U(x) = \begin{cases} u(x), & \text{for } u \in F', \\ \overline{u}(x), & \text{for } u \in gF'. \end{cases}$$

Since $\mathrm{Tr}\, u(S) \subset \Pi$, it can be shown as before that $(\mathrm{Tr}\, \overline{u}^{\,n})\big|_S = 0$ a.e.. Next, noting that by definition of $\overline{u}$, $(\mathrm{Tr}\, u^i)\big|_S = (\mathrm{Tr}\, \overline{u}^{\,i})\big|_S$ for $1 \le i < n$, we conclude that U defines a $W^{1,2}(\Omega;\mathbb{R}^n)$ map symmetric with respect to Π. Repeating this process, U can be extended to an equivariant map in the whole space. $\square$

The Proposition above allows us to define for (G'_R, G') equivariant Sobolev maps the functional associated to (1.1) by

$$J_{F'_R}(u) = \int_{F'_R} \big\{\frac{1}{2}|\nabla u|^2 + W(u)\big\}\mathrm{d}x, \tag{4.3}$$

and to consider the minimization problem

$$\min_{A^R} J_{F'_R}, \quad \text{where } A^R := \big\{u \in K^R : u(\overline{F'_R}) \subset \overline{F'}\big\}. \tag{4.4}$$

In the class A^R, we have imposed the *positivity* constraint:

$$u(\overline{F'_R}) \subset \overline{F'}. \tag{4.5}$$

Note that the convexity of F' implies that A^R is convex and closed in $W^{1,2}(F'_R;\mathbb{R}^n)$, and so a minimizer $u_R \in A^R$ exists.

5. The gradient flow and positivity

To show that the positivity constraint built in A^R does not affect the Euler-Lagrange equation we will utilize the gradient flow

$$\begin{gathered} \frac{\partial u}{\partial t} = \Delta u - W_u(u) \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^n, \end{gathered} \tag{5.1}$$

with initial condition $u_0 \in W^{1,2}_{\mathrm{E\,loc}}(R)(\mathbb{R}^n;\mathbb{R}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$. Since W is C^3 (cf. (H1)), the results in [15, Ch. 3, §3.3, §3.5] apply and provide a unique solution to (5.1) in $C([0,\infty); W^{1,2}_{\mathrm{E\,loc}}(R)(\mathbb{R}^n;\mathbb{R}^n))$. For every bounded domain $\Omega \subset \mathbb{R}^n$, this solution is for $t > 0$, as a function of x, in $C^{2+\alpha}(\overline{\Omega};\mathbb{R}^n)$, for some $0 < \alpha < 1$. Moreover, if the initial condition u_0 is also assumed globally Lipschitz in $\mathbb{R}^n$, the flow $u \in C([0,\infty); C(\overline{\Omega};\mathbb{R}^n)) \cap C(0,\infty; C^2(\overline{\Omega};\mathbb{R}^n))$ for every bounded domain Ω.

Theorem 5.1. *Let W be a potential satisfying* (H1). *If the initial condition $u_0 \in W^{1,2}_{\mathrm{E\,loc}}(R)(\mathbb{R}^n;\mathbb{R}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ is assumed to be positive (cf.* (4.5)*), then*

$$u(\cdot,t;u_0)(\overline{F'_R}) \subset \overline{F'}, \ \textit{for } t \geq 0 \quad \textit{(positivity)},$$

and, moreover,

$$u(\cdot,t;u_0)(F'_R) \subset F', \quad \textit{for } t > 0 \quad \textit{(strictly positivity)}$$

Proof. Without loss of generality we assume $R = 1$. We first prove the Theorem when $u_0 \in W^{1,2}_{\mathrm{E\,loc}}(R)(\mathbb{R}^n;\mathbb{R}^n)$ is globally Lipschitz, in which case u is smooth and satisfies for $t > 0$ the Boundary Conditions (1.5). We shall give a detailed proof for $n = 2$ and an outline in higher dimensions $n \geq 3$, just pointing out the new elements. Next, we shall consider the general case of an initial condition $u_0 \in W^{1,2}_{\mathrm{E\,loc}}(R)(\mathbb{R}^n;\mathbb{R}^n) \cap L^\infty_{loc}(\mathbb{R}^n;\mathbb{R}^n)$.

Proof for $n = 2$, and $u_0 \in W^{1,2}_{\mathrm{E\,loc}}(R)(\mathbb{R}^2;\mathbb{R}^2)$ and globally Lipschitz. In $\mathbb{R}^2$, F' is either a triangle with acute angles or a rectangle. We shall consider the case of a triangle, the proof being similar for a rectangle. Let us denote by $P_0 = O$, P_1 and P_2 the vertices of F' and by $S_1 = P_0P_1$, $S_2 = P_0P_2$, $S_3 = P_1P_2$, its sides supposed to be closed. Π_1, Π_2 and Π_3 are the corresponding supporting lines of S_1, S_2, S_3, while for $1 \leq i \leq 3$, r_i is the outer unit normal vector to the side S_i. Finally, we also need to define the closed segment $P_{ij}(t)P_{ji}(t) := u(P_iP_j, t)$ which is the image of the side P_iP_j by $u(.,t)$.

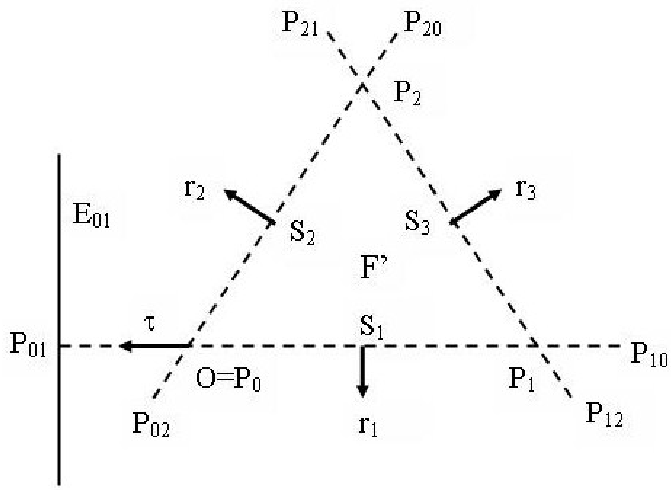

FIGURE 2. The fundamental alcove F' and the extensions $P_{ij}P_{ji}$.

We shall suppose that there exists a $t_0 > 0$ for which $u(\overline{F'}, t_0) \not\subseteq \overline{F'}$, and seek a contradiction in the two following cases:
Case 1: $u(\partial F', t) \subset \overline{F'}$ for every $t \geq 0$. Without loss of generality we can suppose that there exists $x_0 \in F'$ such that for instance:

$$\langle u(x_0, t_0) - P_0 \ , r_1 \rangle > 0 \,. \tag{5.2}$$

Following [1], we set $h_1(x,t) = \langle u(x,t) - P_0 \ , r_1 \rangle$ which is solution of

$$\frac{\partial h_1}{\partial t} = \Delta h_1 - \langle W_u(u) \ , r_1 \rangle, \tag{5.3}$$

and noting that

$$\langle W_u(u), r_1 \rangle = \langle \int_0^1 W_{uu}\big(u + (s-1)h_1 r_1\big) r_1 \mathrm{d}s, \, r_1 \rangle h_1, \tag{5.4}$$

due to the periodicity of W, (5.3) can be written as

$$\frac{\partial h_1}{\partial t} = \Delta h_1 + c_1 h_1, \tag{5.5}$$

with c_1 continuous and bounded on $\mathbb{R}^2 \times [0, \infty)$. To have an equation with a non-positive first order coefficient, we apply a well-known trick, considering instead of h_1 the function $\overline{h_1}(x,t) = e^{-\lambda t} h_1(x,t)$ which is solution to $\frac{\partial \overline{h_1}}{\partial t} = \Delta \overline{h_1} + (c_1 - \lambda)\overline{h_1}$. Choosing $\lambda \geq c_1$, the maximum principle applies to $\overline{h_1}$ and gives

$$0 < \max \big\{ \overline{h_1}(x,t) : x \in \overline{F'}, \, t \in [0, t_0] \big\} = \overline{h_1}(x_1, t_1) \tag{5.6}$$

for some $x_1 \in F'$ and some $t_1 \in [0, t_0]$, since we supposed that $h_1(x_0, t_0) > 0$, $h_1(\partial F', t) \subset (-\infty, 0]$ for all $t \geq 0$ (hypothesis in case 1), and $h_1(\overline{F'}, 0) \subset (-\infty, 0]$ (positivity of u_0). As a consequence $\overline{h_1} \equiv \overline{h_1}(x_1, t_1) > 0$ which contradicts the fact that $h_1(P_0, t) = 0$, for every $t \geq 0$.
Case 2: $u(\partial F', t_2) \not\subseteq \overline{F'}$, for some $t_2 > 0$. This case is more difficult since we have to deal with the extensions $P_{ij}(t)P_{ji}(t)$. As before, we consider for $i = 1, 2, 3$, the

projection h_i of u with respect to the vector r_i which is solution to $\frac{\partial h_i}{\partial t} = \Delta h_i + c_i h_i$, with c_i bounded. We will also need the functions $\overline{h_i}(x,t) = e^{-\lambda t} h_i(x,t)$ which are solutions to $\frac{\partial \overline{h_i}}{\partial t} = \Delta \overline{h_i} + (c_i - \lambda)\overline{h_i}$, and choose $\lambda \geq c_i$, for $i = 1, 2, 3$.

Without loss of generality we can suppose that for instance

$$\max \big\{ e^{-\lambda t} d(P_i, P_{ij}(t)) : t \in [0, t_2], 0 \leq i, j \leq 2, i \neq j \big\} = e^{-\lambda t_3} d(P_0, P_{01}(t_3)), \tag{5.7}$$

for some $t_3 \in (0, t_2]$. By projecting u onto the direction $\tau := -\frac{1}{|\overrightarrow{P_0P_1}|}\overrightarrow{P_0P_1}$ we define as before the functions $h(x,t) = \langle u(x,t) - P_0\ , \tau \rangle$ and $\overline{h}(x,t) = e^{-\lambda t} h(x,t)$. Since the angles of F' are acute, τ can be written $\tau = \alpha r_1 + \beta r_2$ with α, $\beta \geq 0$, and therefore $h = \alpha h_1 + \beta h_2$ and $\overline{h} = \alpha \overline{h_1} + \beta \overline{h_2}$. Clearly, $\frac{\partial \overline{h}}{\partial t} = \Delta \overline{h} + \overline{c}\, \overline{h}$ holds with $\overline{c} \leq 0$. In order to successfully apply the maximum principle to $\overline{h}$, we note that by (5.7) and since F' has acute angles, there exists a closed half-plane E_{01} with the following properties:

(i) $e^{-\lambda t_3} P_{01}(t_3) \in \partial E_{01}$,
(ii) $\overrightarrow{P_0P_1} \perp \partial E_{01}$,
(iii) $e^{-\lambda t} P_{ij}(t) \in E_{01}$, for all $t \in [0, t_2]$ and all (i,j) such that $0 \leq i \neq j \leq 2$.

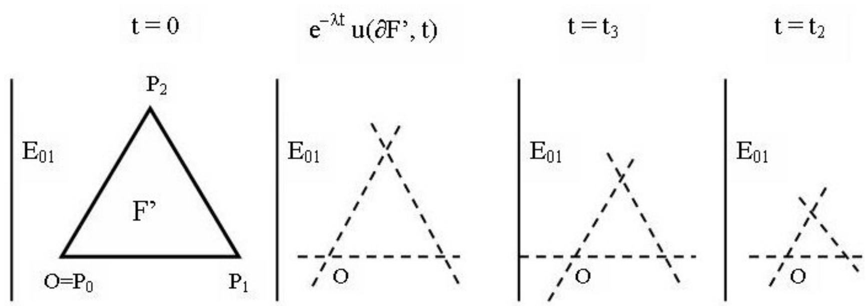

FIGURE 3. The separating half-plane E_{01} and the extensions $P_{ij}P_{ji}$ according to the time

Now, let $\mu := \max\big\{\overline{h}(x,t) : x \in \overline{F'},\ t \in [0, t_2]\big\} \geq e^{-\lambda t_3} d(P_0, P_{01}(t_3)) > 0$. If $\mu > e^{-\lambda t_3} d(P_0, P_{01}(t_3))$, then necessarily μ is attained at an interior point $x_2 \in F'$, for some $t_4 > 0$. Thus for $x \in \overline{F'}$ and $t \in [0, t_4]$, we must have $\overline{h}(x,t) = \mu$ which contradicts the fact that $h(P_0, t) = 0$, for every $t \geq 0$. If $\mu = e^{-\lambda t_3} d(P_0, P_{01}(t_3))$, then there necessarily exists an interior point x_3 of the segment P_0P_1 such that $\overline{h}(x_3, t_3) = \mu$. In this case, we consider the extension of u, h and $\overline{h}$ to the union of two fundamental alcoves $\Omega := \operatorname{Int}\big(\overline{F'} \cup g\overline{F'}\big)$ where g denotes the reflection with respect to the line P_0P_1. By equivariance of u, $\mu = \max\big\{\overline{h}(x,t) : x \in \overline{\Omega},\ t \in [0, t_2]\big\}$ also holds and since x_3 is an interior point of Ω, we reach the same contradiction as previously.

So far we have proved the first statement of the Theorem (positivity) for smooth initial condition in $\mathbb{R}^2$. To complete the proof in this case it remains to show that $u(\cdot, t; u_0)(F') \subset F'$, for $t > 0$ (strong positivity). To see this, suppose by contradiction that there exist $x_4 \in F'$ and $t_5 > 0$ such that $u(x_4, t_5) \in \Pi_1$ for instance; i.e., $h_1(x_4, t_5) = 0$. Then, $max\big\{h_1(x,t) : x \in \overline{F'},\ t \in [0, t_5]\big\} = h_1(x_4, t_5)$

due to the positivity of u. According to the maximum principle, this implies that for $x \in \overline{F'}$ and $t \in [0, t_5]$, we have $h_1(x,t) = 0$ which contradicts the fact that $u(P_2,t) = P_2$, for every $t \geq 0$.

Proof for $n \geq 3$, and $u_0 \in W^{1,2}_{\mathrm{E\,loc}}(R)(\mathbb{R}^n;\mathbb{R}^n)$ and globally Lipschitz. To prove the Theorem in higher dimension we proceed as in $\mathbb{R}^2$. Essential is the fact that the angles between the faces of F' are acute (see [17]). This implies in particular that any lower dimensional face has also acute angles. As before, we shall suppose that there exists a $t_0 > 0$ for which $u(\overline{F'}, t_0) \not\subseteq \overline{F'}$, and seek a contradiction in the two following cases:

Case 1: $u(\partial F', t) \subset \overline{F'}$, for every $t \geq 0$. This case presents no difficulty. As previously, to reach a contradiction, we consider the projection:

$$h_1(x,t) = \langle u(x,t) - P_0 \ , r_1 \rangle,$$

where r_1 is the outer unit normal vector to the face S_1 and P_0 a vertex of S_1.

Case 2: $u(\partial F', t_0') \not\subseteq \overline{F'}$, for some $t_0' > 0$. Let S_1, S_2,..., S_N be the closed faces of F'. In the same way we considered in $\mathbb{R}^2$ the extensions $P_{ij}(t)P_{ji}(t)$ of the sides of F', in higher dimension we have to deal with the extensions of its faces. Without loss of generality and omitting the $e^{-\lambda t}$ term for reason of simplicity, we can assume that for instance:

$$\max\big\{d(u(x,t),S_i) : x \in S_i \ , t \in [0,t_0'], 1 \leq i \leq N\big\} = d(u(x_1,t_1),S_1) = \epsilon > 0\,, \tag{5.8}$$

for some $x_1 \in S_1$ and $t_1 \in (0, t_0']$. Setting $u_1 := u(x_1,t_1)$, there exists since S_1 is compact and convex, a unique point $v_1 \in \partial S_1$ such that $d(u_1,v_1) = \epsilon$. We shall consider the projection h of u with respect to the vector $\rho := \frac{1}{|u_1 - v_1|}(u_1 - v_1)$:

$$h(x,t) = \langle u(x,t) - v_1, \rho\rangle. \tag{5.9}$$

Since the angles of F' are acute, the half-space $E_0 := \big\{x \in \mathbb{R}^n : \langle x - v_1, \rho\rangle \leq 0\big\}$ contains $\overline{F'}$ and setting $E_1 := \big\{x \in \mathbb{R}^n : \langle x - u_1, \rho\rangle \leq 0\big\}$ we have $d(\overline{F'}, \partial E_1) \geq \epsilon$. In particular, by (5.8), $u(S_i,t) \subset E_1$, for every $i = 1, \dots, N$ and every $t \in [0, t_0']$; that is, the half-space E_1 contains all the extensions of the faces of F'.

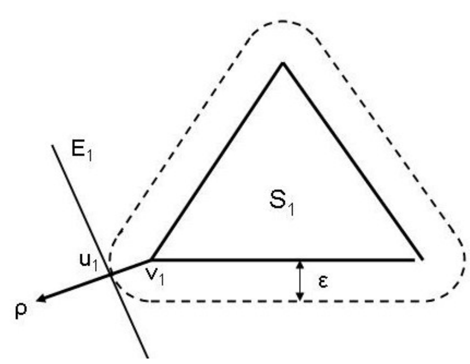

FIGURE 4. The separating half-space E_1

Now, let $\mu := \max\big\{h(x,t) : x \in \overline{F'},\ t \in [0,t_0']\big\} \geq h(x_1,t_1) = \epsilon$ and let us apply the maximum principle to h. If $\mu > \epsilon$, then necessarily μ is attained at an interior point of F', and it is easy in this case to reach a contradiction. Otherwise $\mu = \epsilon$ and the maximum of h is attained at (x_1, t_1). We have the following hierarchy:

(i) if ρ is not parallel to an $n-2$ dimensional face of S_1, then $x_1 \in Int(S_1)$;
(ii) if ρ is parallel to an $n-2$ dimensional face of S_1 but not parallel to a lower dimensional face, then either $x_1 \in Int(S_1)$ or x_1 belongs to the interior of this $n-2$ dimensional face ...
(iii) if ρ is parallel to an edge of S_1, then x_1 belongs to the interior of this edge or to the interior of any higher dimensional face of S_1 containing this edge.

In all cases, if we consider the extension of u and h to $\Omega := \operatorname{Int}\big(\cup_{g\in\Gamma} g\overline{F'}\big)$ where $\Gamma := G'_{x_1}$ is the stabilizer of x_1, we will have by equivariance

$$\mu = \max\big\{h(x,t) : x \in \overline{\Omega},\, t \in [0, t'_0]\big\} = h(x_1, t_1),$$

and since x_1 is an interior point of Ω it is easy to reach a contradiction. This proves the positivity of the gradient flow for dimensions $n \geq 3$. The proof of strong positivity is straightforward by applying the same arguments as on the plane.

Proof for a general $u_0 \in W^{1,2}_{\mathrm{E\,loc}}(R)(\mathbb{R}^n;\mathbb{R}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$. The idea is to approximate u_0 by smooth positive equivariant maps an then utilize the continuous dependence for the flow of initial condition. We need the following result[3].

Proposition 5.2. *If $u \in W^{1,2}_{\mathrm{E\,loc}}(R)(\mathbb{R}^n;\mathbb{R}^n)$ is such that $u(\overline{F'_R}) \subset \overline{F'}$ then there exists a sequence $(u_m) \subset C^\infty(\overline{F'_R};\mathbb{R}^n)$ with the following properties:*

(i) $(\operatorname{Tr} u_m)(S_i^R) \subset \Pi_i$, *for every face S_i^R of F'_R,*
(ii) $u_m(\overline{F'_R}) \subset \overline{F'}$,
(iii) u_m *converges to u in $W^{1,2}(F'_R;\mathbb{R}^n)$ as $m \to \infty$.*

Proof. Without loss of generality we can suppose that $R = 1$. In what follows we will utilize the standard mollifier $\eta_m(x) := m^n(\int \eta)^{-1}\eta(mx)$, $m \geq 1$, where

$$\eta(x) = \begin{cases} e^{(|x|^2-1)^{-1}} & \text{for } x \in \mathbb{R}^n,\ |x| < 1,\\ 0 & \text{for } x \in \mathbb{R}^n,\ |x| \geq 1. \end{cases}$$

Let us consider the restriction of u to a smooth domain Ω containing $\overline{F'}$ and let us extend it to a $W^{1,2}(\mathbb{R}^n;\mathbb{R}^n)$ map called $\overline{u}$. Setting $\overline{u_m} := \overline{u} * \eta_m$, it is clear that $\overline{u_m} \in C^\infty(\mathbb{R}^n;\mathbb{R}^n) \cap W^{1,2}(\mathbb{R}^n;\mathbb{R}^n)$ and that $\overline{u_m}$ converges to $\overline{u}$ in $W^{1,2}(\mathbb{R}^n;\mathbb{R}^n)$ as $m \to \infty$. Therefore, if we define (u_m) as the restrictions of $(\overline{u_m})$ to F', (iii) will be trivially satisfied. Before proving (i) and (ii), we need to introduce some notation.

Let us denote by S a face of F', by r the outer unit normal vector to S, by $\Pi = \{x \in \mathbb{R}^n : \langle x, r\rangle = k\}$ the supporting plane of S, and by g the reflection with respect to Π. Π defines the closed half-spaces E^- and $E^+ = \{x \in \mathbb{R}^n : \langle x, r\rangle \leq k\}$ which contains F'.

For $x \in S$, we compute

$$\begin{aligned} u_m(x) &= \int_{B(x,1/m)} \eta_m(x-y)u(y)\,\mathrm{d}y\\ &= \int_{B(x,1/m)\cap E^+} \eta_m(x-y)u(y)\,\mathrm{d}y + \int_{B(x,1/m)\cap E^-} \eta_m(x-z)u(z)\,\mathrm{d}z\\ &= \int_{B(x,1/m)\cap E^+} \eta_m(x-y)u(y)\,\mathrm{d}y + \int_{B(x,1/m)\cap E^+} \eta_m(gx-gy)u(gy)\,\mathrm{d}y \end{aligned}$$

[3] In a former version of [1] such an argument appeared.

$$= \int_{B(x,1/m)\cap E^+} \eta_m(x-y)(u(y)+gu(y))\,\mathrm{d}y$$

and from this expression we see that

$$\langle u_m(x), r\rangle = \int_{B(x,1/m)\cap E^+} \eta_m(x-y)2k\mathrm{d}y = k \quad \text{(by equivariance of u)}$$

which means that (u_m) satisfies (i). To prove (ii), take $x \in \overline{F'}$ and compute as before

$$\begin{aligned}
u_m(x) &= \int_{B(x,1/m)} \eta_m(x-y)u(y)\,\mathrm{d}y \\
&= \int_{B(x,1/m)\cap E^+} \eta_m(x-y)u(y)\,\mathrm{d}y + \int_{B(x,1/m)\cap E^-} \eta_m(x-z)u(z)\,\mathrm{d}z \\
&= \int_{B(x,1/m)\cap E^+} \eta_m(x-y)u(y)\,\mathrm{d}y + \int_{B':=g(B(x,1/m)\cap E^-)} \eta_m(x-gy)gu(y)\,\mathrm{d}y \\
&= \int_{(B(x,1/m)\cap E^+)\setminus B'} \eta_m(x-y)u(y)\,\mathrm{d}y + \int_{B'} (\eta_m(x-y)-\eta_m(x-gy))u(y)\,\mathrm{d}y \\
&\quad + \int_{B'} \eta_m(x-gy)(u(y)+gu(y))\,\mathrm{d}y\,.
\end{aligned}$$

From this decomposition, the equivariance of u, the fact that $u(E^+) \subset E^+$ and the properties of the mollifier, we deduce that

$$\begin{aligned}
\langle u_m(x), r\rangle &\le k \int_{(B(x,1/m)\cap E^+)\setminus B'} \eta_m(x-y)\,\mathrm{d}y + k\int_{B'} (\eta_m(x-y)-\eta_m(x-gy))\,\mathrm{d}y \\
&\quad + 2k\int_{B'} \eta_m(x-gy)\,\mathrm{d}y
\end{aligned}$$

and

$$\begin{aligned}
\langle u_m(x), r\rangle &\le k \int_{(B(x,1/m)\cap E^+)\setminus B'} \eta_m(x-y)\,\mathrm{d}y + k\int_{B'} \eta_m(x-y)\,\mathrm{d}y \\
&\quad - k\int_{B(x,1/m)\cap E^-} \eta_m(x-y)\,\mathrm{d}y + 2k\int_{B(x,1/m)\cap E^-} \eta_m(x-y)\,\mathrm{d}y
\end{aligned}$$

Thus $\langle u_m(x), r\rangle \le k$ which means that $u(x) \in E^+$ and completes the proof. □

Taking into account Propositions 4.1 and 5.2, it is possible to construct for every initial condition $u_0 \in W^{1,2}_{\mathrm{E\,loc}}(R)(\mathbb{R}^n;\mathbb{R}^n)$ a sequence of positive maps $(u_m) \subset W^{1,2}_{\mathrm{E\,loc}}(R)(\mathbb{R}^n;\mathbb{R}^n)$, globally Lipschitz in $\mathbb{R}^n$ (and actually C^∞ in the closure of every fundamental alcove) which converges to u_0 in $W^{1,2}_{loc}(\mathbb{R}^n;\mathbb{R}^n)$. Utilizing then the continuous dependence for the flow of the initial condition one can therefore prove the positivity results of Theorem 5.1. in the general case. □

6. Proofs of Theorem 3.1 and Theorem 3.2

In this section we follow [2]. By taking for initial condition in Theorem 5.1 the minimizer u_R constructed in Section 4, we have $u(\cdot,t;u_R) \in A^R$, for $t \ge 0$. Since

u_R is a global minimizer of $J_{F'_R}$ in A^R, and since $u(\cdot,t;u_R)\in C^1(0,\infty;C^{2+\alpha}(\overline{F'_R}))$, is a classical solution to (5.1) for $t>0$, we conclude from

$$\frac{\mathrm{d}}{\mathrm{d}t}J_{F'_R}(u(\cdot,t)) = -\int_{F'_R}|u_t|^2\,\mathrm{d}x \tag{6.1}$$

that $|u_t(x,t)|=0$, for all $x\in\mathbb{R}^n$ and $t>0$. Hence, for $t>0$, $u(\cdot,t)$ is satisfying

$$\Delta u(x,t) - W_u(u(x,t)) = 0. \tag{6.2}$$

By taking $t\to 0+$ and utilizing the continuity of the flow in $W^{1,2}(F'_R;\mathbb{R}^n)$ at $t=0$, $u(\cdot,\cdot;u_R)\in C([0,\infty);W^{1,2}(F'_R;\mathbb{R}^n))$, we obtain that u_R is a (G'_R,G') equivariant classical solution to system (1.1) satisfying also $u_R(F'_R)\subset F'$.

To prove the estimate in the second statement of Theorem 3.1 we need the following lemma.

Lemma 6.1. *Let u_R be as above, $H':=\operatorname{Int}\big(\cup_{g\in G}\overline{F'}\big)$ and $H'_R:=\operatorname{Int}\big(\cup_{g\in G}\overline{F'_R}\big)$. Then for $R>1$ the following hold:*

(i) $\|u_R\|_{L^\infty(H'_R;\mathbb{R}^n)}\le M$, *where* $M:=\max_{u\in\overline{H'}}|u|$,

(ii) $Q(u_R(x))\le\overline{Q}$, *for* $x\in H'_R$, *where* $\overline{Q}:=\max_{u\in\overline{H'}}Q(u)$,

(iii) $J_{F'_R}(u_R)\le CR^{n-1}$, *where C is a positive constant independent of R,*

(iv) $\Delta Q(u_R(x))\ge 0$, *in* $W^{1,2}_{\rm loc}(D'_R;\mathbb{R}^n)$, *where D'_R is as in Theorem 3.1.*

Proof. (i) and to (ii) are trivial. For (iii), define

$$b_R(x) := \begin{cases} ((x/R)-a)(1-d(x;\partial F'_R))+a, & \text{for } x\in\overline{F'_R} \text{ with } d(x;\partial F'_R)\le 1,\\ a, & \text{for } x\in F'_R \text{ with } d(x;\partial F'_R)\ge 1.\end{cases}$$

b_R is continuous and C^∞ piecewise in $\overline{F'_R}$ thus it defines a $W^{1,2}(F'_R;\mathbb{R}^n)$ map which also satisfies $b_R(\overline{F'_R})\subset\overline{F'}$, by convexity of F'. According to Proposition 4.1 it can be extended equivariantly on the whole space since for $x\in S_i^R$, we have $b_R(x)=x/R\in S_i$. An easy computation shows that $|\nabla b_R|$ and $W(b_R)$ are bounded in F'_R, independently of R, and clearly these quantities vanish when $d(x;\partial F'_R)\ge 1$. This implies that $J_{F'_R}(b_R)\le CR^{n-1}$ and since $J_{F'_R}(u_R)\le J_{F'_R}(b_R)$ by definition of u_R, we obtain the desired estimate.

For (iv), see [2, Lemma 4.1]. □

Estimate (iii) in Lemma 6.1 and the subharmonicity of $Q(u_{4R})$ in D'_{4R} allow us to obtain via an iterated application of the *De Giorgi oscillation Lemma* the following pointwise estimate, for $R\ge R_0$:

$$\sup_{B(x_R,R^*)} Q(u_{4R}(x))\le\bar{q}, \tag{6.3}$$

where $R^*=R/2^k$ (with k integer independent of R), $x_R:=4Rx_0$ (with $x_0\in D'$ fixed), and $\overline{q}$ as in Hypothesis 2. Utilizing then the comparison arguments in [2, Section 5], it is possible to show that the ball $B(x_R,R^*)$ in (6.3) can be replaced by a large set D'_{R^*} which includes all of D'_{4R} with the exception of a strip along the boundary $\partial D'_{4R}$ of width d_0 independent of R, for $R\ge R_0$; that is,

$$D'_{R^*}\supset\{x\in D'_{4R}: d(x,\partial D'_{4R})\ge d_0\}. \tag{6.4}$$

This result states that the minimizer $u_{4R}(x)$ on a set of large measure is close to a, the zero of W in D', for $R\to\infty$. From it, by applying, once again comparison arguments, follows the estimate of Theorem 3.1:

$$|u_R(x) - a| \leq K\mathrm{e}^{-kd(x,\partial D'_R)} \tag{6.5}$$

which holds for $R > R_0$ and $x \in D'_R$. Finally, the uniform bound (i) provided by Lemma 6.1 and elliptic regularity, via a diagonal argument, allow us to pass to the limit along a subsequence in R and capture a function

$$u(x) = \lim_{R_n \to \infty} u_{R_n}\big|_{H'_{R_n}}, \tag{6.6}$$

where $\big|_{H'_R}$ denotes the restriction of a map to $H'_R = \mathrm{Int}\big(\cup_{g\in G}\overline{F'_R}\big)$. Since the convergence in (6.6) is uniform up to the second derivatives on compact sets, one can then see that the limit u satisfies the exponential estimate of Theorem 3.2 and is also a G equivariant solution to

$$\Delta u - W_u(u) = 0 \quad \text{in } \mathbb{R}^n.$$

Clearly, positivity is verified (i.e. $u(\overline{F}) \subset \overline{F'}$) and with the help of the exponential estimate, strong positivity can be established in D', that is $u(D) \subset D'$.

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