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EXISTENCE OF MILD SOLUTIONS FOR A NEUTRAL FRACTIONAL EQUATION WITH FRACTIONAL NONLOCAL CONDITIONS

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Abstract. The existence of mild solutions in an appropriate space is established for a second-order abstract problem of neutral type with derivatives of non-integer order in the nonlinearity as well as in the initial conditions. We introduce some new spaces taking into account the minimum requirements of regularity.

1. Introduction

In this article we study the neutral second-order abstract differential problem

$$\begin{gathered}\frac{d}{dt}[u'(t)+g(t,u(t))]=Au(t)+f(t,u(t),D^{\alpha}u(t)),\quad t\in I=[0,T]\\ u(0)=u^0+p(u,D^{\beta}u(t)),\\ u'(0)=u^1+q(u,D^{\gamma}u(t))\end{gathered}\tag{1.1}$$

with $0\leq\alpha,\beta,\gamma\leq 1$. Here the prime denotes time differentiation and D^{κ}, $\kappa=\alpha,\beta,\gamma$ denotes fractional time differentiation (in the sense of Riemann-Liouville). The operator A is the infinitesimal generator of a strongly continuous cosine family $C(t), t\geq 0$ of bounded linear operators in the Banach space X and f,g are nonlinear functions from $\mathbb{R}^+\times X\times X$ to X and $\mathbb{R}^+\times X$ to X, respectively, u^0 and u^1 are given initial data in X. The functions $p:[C(I;X)]^2\to X$, $q:[C(I;X)]^2\to X$ are given continuous functions.

This problem has been studied in case α,β,γ are 0 or 1 (see [1, 2, 3, 4, 10, 11, 22]). Well-posedness has been established using different fixed point theorems and the theory of strongly continuous cosine families in Banach spaces. We refer the reader to [6, 20, 21] for a good account on the theory of cosine families. A similar problem to this one with Caputo derivative has been studied by the present author in [19]. Here the situation with the Riemann-Liouville fractional derivative is completely different. The singularity at zero inherent to the Riemann-Liouville derivative brings new challenges and difficulties. Moreover the underlying spaces are different. Indeed, Caputo derivative needs more regularity as it uses the first

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derivative of the function in question in its definition whereas the Riemann-Liouville does not require as much smoothness.

Problems with fractional derivatives are very convenient to model hereditary phenomena in many fields of sciences and engineering [5, 7, 8, 9, 12, 14, 15, 16, 17]. They can be used, for instance, as damping to reduce the effect of vibrations in mechanical structures or to reduce noise in signals.

Here we consider the neutral case ($g \not\equiv 0$) and prove existence and uniqueness of mild solutions under different conditions on the different data. Even in the case $g \equiv 0$ (and $p = q = 0$) these conditions are different from the ones assumed in [18]. In particular, this work may be viewed as an extension of the works in [10, 11] to the fractional order case and of (18) to the neutral case.

The next section of this paper contains some notation and preliminary results needed in our proofs. Section 3 treats the existence of a mild solution in an appropriate "fractional" space.

2. Preliminaries

In this section we present some notation, assumptions and results needed in our proofs later.

Definition 2.1. The integral

$$(I_{a+}^{\kappa} h)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{h(t)dt}{(x-t)^{1-\kappa}}, \quad x > a$$

is called the Riemann-Liouville fractional integral of h of order $\kappa > 0$ when the right side exists.

Here Γ is the usual Gamma function

$$\Gamma(z) := \int_0^\infty e^{-s} s^{z-1} ds, \quad z > 0.$$

Definition 2.2. The (left hand) Riemann-Liouville fractional derivative of h of order $\kappa > 0$ is defined by

$$(D_a^{\kappa} h)(x) = \frac{1}{\Gamma(n-\kappa)} (\frac{d}{dx})^n \int_a^x \frac{h(t)dt}{(x-t)^{\kappa-n+1}}, \quad x > a, \ n = [\kappa] + 1$$

whenever the right side is pointwise defined. In particular

$$(D_a^{\kappa} h)(x) = \frac{1}{\Gamma(1-\kappa)} \frac{d}{dx} \int_a^x \frac{h(t)dt}{(x-t)^{\kappa}}, \quad x > a, \ 0 < \kappa < 1$$

and

$$(D_a^{\kappa} h)(x) = \frac{1}{\Gamma(2-\kappa)} (\frac{d}{dx})^2 \int_a^x \frac{h(t)dt}{(x-t)^{\kappa-1}}, \quad x > a, \ 1 < \kappa < 2.$$

Lemma 2.3. *Let $0 < \alpha$, $\beta < 0$ and $\varphi \in L^1(a,b)$ be such that*

$$I^{n-\alpha}\varphi \in AC^n([a,b]) := \big\{\phi : [a,b] \to \mathbb{R} \text{ and } (D^{n-1}\phi)(x) \in AC[a,b]\big\}.$$

Then

$$I_{a+}^{\alpha} I_{a+}^{\beta} \varphi = I_{a+}^{\alpha+\beta} \varphi - \sum_{k=0}^{n-1} \frac{\varphi_{n+\beta}^{(n-k-1)}(a)}{\Gamma(\alpha-k)} (x-a)^{\alpha-k-1}$$

where $\varphi_{n+\beta}(x) = I_{a+}^{n+\beta}\varphi(x)$ and $n = [-\beta] + 1$.

If $0 < \kappa < 1$, $n = 1$ and $\gamma \le \kappa$ then

$$I_{a+}^{\kappa-\gamma} D_{a+}^{\kappa}\varphi = D_{a+}^{\gamma}\varphi - \frac{\varphi_{1-\kappa}(a)}{\Gamma(\kappa-\gamma)}(x-a)^{\kappa-\gamma-1}.$$

See [12, 13, 14, 16, 17] for more on fractional derivatives and fractional integrals. We will also need the following lemmas. The first one can be found in [17].

Lemma 2.4. *If $h(x) \in AC^n[a,b]$, $\alpha > 0$ and $n = [\alpha]+1$, then*

$$\begin{aligned}(D_a^{\alpha} h)(x) &= \sum_{k=0}^{n-1} \frac{h^{(k)}(a)}{\Gamma(1+k-\alpha)}(x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{h^{(n)}(t)dt}{(x-t)^{\alpha-n+1}}\\ &=: \sum_{k=0}^{n-1} \frac{h^{(k)}(a)}{\Gamma(1+k-\alpha)}(x-a)^{k-\alpha} + ({}^C D_a^{\alpha} h)(x), \quad x > a.\end{aligned}$$

The expression $({}^C D_a^{\alpha} h)(x)$ is known as the fractional derivative of h of order α in the sense of Caputo.

We will assume the following condition.

(H1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators in the Banach space X.

The associated sine family $S(t)$, $t \in \mathbb{R}$ is defined by

$$S(t)x := \int_0^t C(s)x\,ds, \quad t \in \mathbb{R},\ x \in X.$$

It is known (see [20, 21, 22]) that there exist constants $M \ge 1$ and $\omega \ge 0$ such that

$$|C(t)| \le Me^{\omega|t|},\ t \in \mathbb{R} \quad \text{and} \quad |S(t) - S(t_0)| \le M\Big|\int_{t_0}^t e^{\omega|s|}ds\Big|,\ t, t_0 \in \mathbb{R}.$$

For simplicity we will write $|C(t)| \le \tilde{M}$ and $|S(t)| \le \tilde{N}$ on $I = [0,T]$ (of course $\tilde{M} \ge 1$ and $\tilde{N} \ge 1$ depend on T).

Let us define

$$E := \{x \in X : C(t)x \text{ is once continuously differentiable on } \mathbb{R}\}.$$

Lemma 2.5 ([20, 21, 22])**.** *Assume* (H1) *is satisfied. Then*

(i) *$S(t)X \subset E$, $t \in \mathbb{R}$,*
(ii) *$S(t)E \subset D(A)$, $t \in \mathbb{R}$,*
(iii) *$\frac{d}{dt}C(t)x = AS(t)x$, $x \in E$, $t \in \mathbb{R}$,*
(iv) *$\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax$, $x \in D(A)$, $t \in \mathbb{R}$.*

Lemma 2.6 ([20, 21, 22])**.** *Suppose that* (H1) *holds, $v : \mathbb{R} \to X$ is a continuously differentiable function, and $q(t) = \int_0^t S(t-s)v(s)\,ds$. Then, $q(t) \in D(A)$, $q'(t) = \int_0^t C(t-s)v(s)\,ds$ and $q''(t) = \int_0^t C(t-s)v'(s)\,ds + C(t)v(0) = Aq(t) + v(t)$.*

Now we make clear what we mean by a mild solution of (1.1).

Definition 2.7. A continuous function u, such that $D^{\eta}u$ ($\eta = \max\{\alpha, \beta, \gamma\}$) exists and is continuous on I, satisfying the integro-differential equation

$$\begin{aligned}u(t) &= C(t)[u^0 + p(u, D^{\beta}u(t))]\\ &\quad + S(t)[u^1 + q(u, D^{\gamma}u(t)) - g(0, u^0 + p(u, D^{\beta}u(t)))]\end{aligned}$$

$$- \int_0^t C(t-s)g(s,u(s))ds + \int_0^t S(t-s)f(s,u(s),D^\alpha u(s))\,ds, \quad t \in I$$

is called a mild solution of problem (1.1).

3. Existence of mild solutions

In this section we prove the existence and uniqueness of a mild solution in the space

$$C_\eta^{RL}([0,T]) := \{v \in C([0,T]) : D^\eta v \in C([0,T])\} \tag{3.1}$$

equipped with the norm $\|v\|_\eta := \|v\|_C + \|D^\eta v\|_C$ where $\|.\|_C$ is the sup norm in $C([0,T])$ and $\eta = \max\{\alpha,\beta,\gamma\}$. For the initial data we define

$$E_\eta := \{x \in X : D^\eta C(t)x \text{ is continuous on } \mathbb{R}^+\}. \tag{3.2}$$

Lemma 3.1. *If $R(t)$ is a linear operator such that $I^{1-\nu}R(t)x \in C^1([0,T])$, $T>0$, then, for $0<\nu<1$, we have*

$$D^\nu \int_0^t R(t-s)x\,ds = \int_0^t D^\nu R(t-s)x\,ds + \lim_{t\to 0^+} I^{1-\nu}R(t)x, \quad x \in X,\ t \in [0,T].$$

Proof. By Definition 2.2 and Fubini's theorem we have

$$\begin{aligned}
&D^\nu \int_0^t R(t-s)x\,ds \\
&= \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t \frac{d\tau}{(t-\tau)^\nu}\int_0^\tau R(\tau-s)x\,ds \\
&= \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t ds\int_s^t \frac{R(\tau-s)x}{(t-\tau)^\nu}d\tau \\
&= \frac{1}{\Gamma(1-\nu)}\int_0^t ds\frac{\partial}{\partial t}\int_s^t \frac{R(\tau-s)x}{(t-\tau)^\nu}d\tau + \frac{1}{\Gamma(1-\nu)}\lim_{s\to t^-}\int_s^t \frac{R(\tau-s)x}{(t-\tau)^\nu}d\tau.
\end{aligned}$$

These steps are justified by the assumption $I^{1-\nu}R(t)x \in C^1([0,T])$. Moreover, a change of variable $\sigma = \tau - s$ leads to

$$\begin{aligned}
&D^\nu \int_0^t R(t-s)x\,ds \\
&= \frac{1}{\Gamma(1-\nu)}\int_0^t ds\frac{\partial}{\partial t}\int_0^{t-s} \frac{R(\sigma)x}{(t-s-\sigma)^\nu}d\sigma + \frac{1}{\Gamma(1-\nu)}\lim_{t\to 0^+}\int_0^t \frac{R(\sigma)x}{(t-\sigma)^\nu}d\sigma.
\end{aligned}$$

This is exactly the formula stated in the lemma. $\square$

Lemma 3.2. *If g is a continuous function such that $I^{1-\nu}g(t) \in C^1([0,T])$, $T>0$ and $R(t)$ is continuous, then, for $0<\nu<1$, we have*

$$D^\nu \int_0^t R(t-s)g(s)\,ds = \int_0^t R(t-s)D^\nu g(s)\,ds, \quad t \in [0,T].$$

Proof. By Definition 2.2, we have

$$\begin{aligned}
D^\nu \int_0^t R(t-s)g(s)\,ds &= \frac{d}{dt}I^{1-\nu}\int_0^t R(t-s)g(s)\,ds \\
&= \frac{d}{dt}I^{1-\nu}\int_0^t R(s)g(t-s)\,ds
\end{aligned}$$

$$= \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t \frac{d\tau}{(t-\tau)^\nu} \int_0^\tau R(s) g(\tau - s)\, ds.$$

Then, Fubini's theorem and the continuity of g allow us to write

$$\begin{aligned}
D^\nu \int_0^t R(t-s) g(s)\, ds &= \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t R(s) \int_s^t \frac{g(\tau - s)}{(t-\tau)^\nu} d\tau ds \\
&= \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t R(s) \int_0^{t-s} \frac{g(\sigma)}{(t-s-\sigma)^\nu} d\sigma ds \\
&= \int_0^t R(s) \frac{d}{dt} I^{1-\nu} g(t-s)\, ds \\
&= \int_0^t R(s) D^\nu g(t-s)\, ds
\end{aligned}$$

which is the desired relation. Note that we have used the continuity of g to deduce that the value (or the limit) of the inner integral in the second line of the relation is zero at $s = t$. □

Corollary 3.3. *For the sine family $S(t)$ associated with the cosine family $C(t)$, $x \in X$, $t \in [0, T]$ and $0 < \nu < 1$, we have*

$$D^\nu \int_0^t S(t-s) x\, ds = \int_0^t D^\nu S(t-s) x\, ds = \int_0^t I^{1-\nu} C(t-s) x\, ds.$$

Proof. First, from Lemma 2.4 as $S(t)x$ is absolutely continuous on $[0, T]$, we have

$$\begin{aligned}
\frac{d}{dt} I^{1-\nu} S(t)x = D^\nu S(t)x &= \frac{1}{\Gamma(1-\nu)} \Big[\frac{S(0)x}{t^\nu} + \int_0^t (t-s)^{-\nu} \frac{dS(s)}{ds} x\, ds \Big] \\
&= \frac{1}{\Gamma(1-\nu)} \int_0^t (t-s)^{-\nu} C(s) x\, ds \\
&= I^{1-\nu} C(t)x.
\end{aligned}$$

Now from the continuity of $C(t)$ it is clear that $I^{1-\nu}C(t)x$ is continuous on $[0, T]$ and therefore $I^{1-\nu}S(t)x \in C^1([0, T])$. We can therefore apply Lemma 2.6 to obtain

$$D^\nu \int_0^t S(t-s) x\, ds = \int_0^t D^\nu S(t-s) x\, ds + \lim_{t \to 0^+} I^{1-\nu} S(t)x, \quad x \in X,\ t \in [0, T].$$

Next, we claim that $\lim_{t \to 0^+} I^{1-\nu} S(t)x = 0$. This follows easily from

$$|I^{1-\nu} S(t)x| \le \frac{1}{\Gamma(1-\nu)} \int_0^t (t-s)^{-\nu} |S(s)x| ds \le \frac{t^{1-\nu}}{\Gamma(2-\nu)} \sup_{0 \le t \le T} |S(t)x|.$$

□

On the functins f, g, p and q we assume the following conditions.

(H2) (i) $f(t, ., .) : X \times X \to X$ is continuous for a.e. $t \in I$.

(ii) For every $(x, y) \in X \times X$, the function $f(., x, y) : I \to X$ is strongly measurable.

(iii) There exists a nonnegative continuous function $K_f(t)$ and a continuous nondecreasing positive function Ω_f such that

$$\|f(t, x, y)\| \le K_f(t) \Omega_f(\|x\| + \|y\|)$$

for $(t, x, y) \in I \times X \times X$.

(iv) For each $r > 0$, the set $f(I \times B_r(0, X^2))$ is relatively compact in X.

(H3 (i) $g \in C_\eta^{RL}([0,T])$.

(ii) There exist a nonnegative continuous function $K_g(t)$ and a continuous nondecreasing positive function Ω_g such that

$$\|g(t,x)\|_\eta \le K_g(t)\Omega_g(\|x\|)$$

for $(t,x) \in I \times X$.

(iii) The family of functions $\{t \to g(t,u)\, u \in B_r(0, C(I;X))\}$ is equicontinuous on I.

(iv) For each $r > 0$, the set $g(I \times B_r(0,X))$ is relatively compact in X.

(H4) $u^0 + p : [C(I;X)]^2 \to E_\eta$ (takes its values in E_η, see (3.2)) and $q : [C(I;X)]^2 \to X$ are completely continuous.

The positive constants N_p, N_q, $\tilde{N}_p$ and N_g will denote bounds for $\|u^0 + p(u,v)\|$, $\|q(u,v)\|$, $\|D^\eta C(t)[u^0 + p(u, I^{1-\beta}v(t))]\|$ and the term $\|g(0, u^0 + p(u, I^{\eta-\beta}v(t)))\|$, respectively. Note that $\|D^\eta C(t)[u^0 + p(u, I^{1-\beta}v(t))]\|$ is finite by the assumption in the next theorem. By $B_r(x, X)$ we will denote the closed ball in X centered at x and of radius r. Let also

$$C_1 := \tilde{M}N_p + \Big(\tilde{N} + \frac{\tilde{M}T^{1-\eta}}{\Gamma(2-\eta)}\Big)N_q + \Big(\tilde{N} + \frac{\tilde{M}T^{1-\eta}}{\Gamma(2-\eta)}\Big)(\|u^1\| + N_g) + \tilde{N}_p \tag{3.3}$$

$$C_3 := \max\Big\{1, \frac{T^{\eta-\alpha}}{\Gamma(\eta-\alpha+1)}\Big\}, \quad C_2 := C_1C_3 \tag{3.4}$$

$$H(t) := 2\tilde{M}K_g(t) + \tilde{N}K_f(t) + \tilde{M}(I^{1-\eta}K_f)(t). \tag{3.5}$$

We are now ready to state and prove our main result.

Theorem 3.4. *Assume that* (H1)–(H4) *hold. If* $0 < \eta < 1$ *and*

$$\int_0^t H(s)\,ds < \frac{1}{C_3}\int_{C_2}^\infty \frac{ds}{\Omega_f(s) + \Omega_g(s)}$$

then problem (1.1) *admits a mild solution* $u \in C_\eta^{RL}([0,T])$.

Proof. Note that by our assumptions and for $u \in C_\eta^{RL}([0,T])$ (see (3.1)), the maps

$$\begin{aligned} \Phi(u,v)(t) := {}& C(t)[u^0 + p(u, I^{\eta-\beta}v(t))] \\ & + S(t)[u^1 + q(u, I^{\eta-\gamma}v(t)) - g(0, u^0 + p(u, I^{\eta-\beta}v(t)))] \\ & - \int_0^t C(t-s)g(s,u(s))ds + \int_0^t S(t-s)f(s,u(s), I^{\eta-\alpha}v(s))\,ds, \end{aligned} \tag{3.6}$$

for $t \in I$, and

$$\begin{aligned} \Psi(u,v)(t) := {}& D^\eta C(t)[u^0 + p(u, I^{\eta-\beta}v(t))] \\ & + D^\eta S(t)[u^1 + q(u, I^{\eta-\gamma}v(t)) - g(0, u^0 + p(u, I^{\eta-\beta}v(t)))] \\ & - \int_0^t C(t-s)D^\eta g(s,u(s))\,ds \\ & + \int_0^t I^{1-\eta}C(t-s)f(s,u(s), I^{\eta-\alpha}v(s))\,ds, \quad t \in I \end{aligned} \tag{3.7}$$

are well-defined, and map $[C([0,T])]^2$ into $C([0,T])$. We obtained these mappings from the definition of a mild solution taking into account Lemma 3.2 (note that

$I^{1-\eta}g \in C^1([0,T])$ by our assumption (H3)(i)) and Corollary 3.3. In addition to that we passed from $D^\kappa u$, $\kappa = \alpha, \beta, \gamma$ to $D^\eta u =: v$ through the formula

$$D^\kappa_{a+}u(t) = I^{\eta-\kappa}_{a+}D^\eta_{a+}u(t)(t) + \frac{u_{1-\eta}(0)}{\Gamma(\eta-\kappa)}t^{\eta-\kappa-1}$$

and noticing that $u_{1-\eta}(0) = 0$.

We would like to apply the Leray-Schauder Alternative (which states that either the set of solutions (below) is unbounded or we have a fixed point in D (containing zero) a convex subset of X provided that the mappings Φ and Ψ are completely continuous). To this end we first prove that the set of solutions (u_λ, v_λ) of

$$(u_\lambda, v_\lambda) = \lambda(\Phi(u_\lambda, v_\lambda), \Psi(u_\lambda, v_\lambda)),\ 0 < \lambda < 1 \tag{3.8}$$

is bounded. Then, we prove that this map is completely continuous. Therefore there remains the alternative which is the existence of a fixed point. We observe first from (3.6) that

$$\begin{aligned} \|u_\lambda(t)\| &\le \tilde{M}N_p + \tilde{N}(\|u^1\| + N_q + N_g) + \tilde{M}\int_0^t K_g(s)\Omega_g(\|u_\lambda(s)\|)\,ds \\ &\quad + \tilde{N}\int_0^t K_f(s)\Omega_f(\|u_\lambda(s)\| + \frac{s^{\eta-\alpha}}{\Gamma(\eta-\alpha+1)}\sup_{0\le\tau\le s}\|v_\lambda(\tau)\|)\,ds \end{aligned}$$

and from (3.7),

$$\begin{aligned} &\|v_\lambda(t)\| \\ &\le \tilde{N}_p + \frac{\tilde{M}t^{1-\eta}}{\Gamma(2-\eta)}[\|u^1\| + N_q + N_g] + \tilde{M}\int_0^t K_g(s)\Omega_g(\|u_\lambda(s)\|)\,ds \\ &\quad + \tilde{M}\int_0^t (I^{1-\eta}K_f)(s)\Omega_f\Big(\sup_{0\le\tau\le s}\|u_\lambda(\tau)\| + \frac{s^{\eta-\alpha}}{\Gamma(\eta-\alpha+1)}\sup_{0\le\tau\le s}\|v_\lambda(\tau)\|\Big)\,ds \end{aligned}$$

where $\tilde{N}_p$ and N_g are bounds for the expressions $D^\eta C(t)[u^0 + p(u, I^{\eta-\beta}v(t))]$ and $g(0, u^0 + p(u, I^{\eta-\beta}v(t)))$, respectively. Clearly

$$\begin{aligned} &\|u_\lambda(t)\| + \|v_\lambda(t)\| \\ &\le C_1 + 2\tilde{M}\int_0^t K_g(s)\Omega_g(\|u_\lambda(s)\|)\,ds + \int_0^t (\tilde{N}K_f(s) + \tilde{M}(I^{1-\eta}K_f)(s)) \\ &\quad \times \Omega_f\Big(\sup_{0\le\tau\le s}\|u_\lambda(\tau)\| + \frac{s^{\eta-\alpha}}{\Gamma(\eta-\alpha+1)}\sup_{0\le\tau\le s}\|v_\lambda(\tau)\|\Big)\,ds \end{aligned} \tag{3.9}$$

where C_1 is given in (3.3). If we put

$$\Theta_\lambda(t) = \max\{1, T^{\eta-\alpha}/\Gamma(\eta-\alpha+1)\}\sup_{0\le\tau\le t}(\|u_\lambda(\tau)\| + \|v_\lambda(\tau)\|),$$

then (3.9) yields

$$\begin{aligned}
\Theta_\lambda(t) &\le \max\{1, T^{\eta-\alpha}/\Gamma(\eta-\alpha+1)\}\Big\{C_1 + 2\tilde{M}\int_0^t K_g(s)\Omega_g(\Theta_\lambda(s))\,ds \\
&\quad + \int_0^t \Big(\tilde{N}K_f(s) + \tilde{M}(I^{1-\eta}K_f)(s)\Big)\Omega_f(\Theta_\lambda(s))\,ds\Big\} \\
&\le \max\big\{1, \frac{T^{\eta-\alpha}}{\Gamma(\eta-\alpha+1)}\big\}\Big\{C_1 + \int_0^t H(s)[\Omega_g(\Theta_\lambda(s)) + \Omega_f(\Theta_\lambda(s))]ds\Big\} \\
&\le C_2 + C_3\int_0^t H(s)[\Omega_g(\Theta_\lambda(s)) + \Omega_f(\Theta_\lambda(s))]ds
\end{aligned} \tag{3.10}$$

where $H(s)$, C_2 and C_3 are as in the paragraph preceding the statement of the theorem (see (3.4) and (3.5)). Let us denote by $\varphi_\lambda(t)$ the right hand side of (3.10). Then $\varphi_\lambda(0) = C_2$, $\Theta_\lambda(t) \le \varphi_\lambda(t)$ and

$$\varphi'_\lambda(t) \le C_3 H(t)[\Omega_g(\varphi_\lambda(t)) + \Omega_f(\varphi_\lambda(t))], \quad t \in I.$$

We infer that

$$\int_{C_2}^{\varphi_\lambda(t)} \frac{ds}{\Omega_f(s) + \Omega_g(s)} \le C_3\int_0^t H(s)\,ds. \tag{3.11}$$

This relation, together with our hypotheses, shows that $\Theta_\lambda(t)$ and thereafter the set of solutions of (3.8) is bounded in $[C(I;X)]^2$.

It remains to show that the maps Φ and Ψ are completely continuous. From our hypotheses it is immediate that

$$\begin{aligned}
\Phi_1(u,v)(t) &:= C(t)[u^0 + p(u, I^{\eta-\beta}v(t))] \\
&\quad + S(t)[u^1 + q(u, I^{\eta-\gamma}v(t)) - g(0, u^0 + p(u, I^{\eta-\beta}v(t)))]
\end{aligned}$$

is completely continuous. To apply Ascoli-Arzela Theorem we need to check that

$$(\Phi - \Phi_1)(B_r^2) := \{(\Phi - \Phi_1)(u,v) : (u,v) \in B_r^2\}$$

is equicontinuous on I. Let us observe that

$$\begin{aligned}
&\|(\Phi - \Phi_1)(u,v)(t+h) - (\Phi - \Phi_1)(u,v)(t)\| \\
&\le \int_0^t \|(C(t+h-s) - C(t-s))g(s,u(s))\|ds \\
&\quad + \int_t^{t+h} \|C(t+h-s)g(s,u(s))\|ds \\
&\quad + \int_0^t \|(S(t+h-s) - S(t-s))f(s,u(s),I^{\eta-\alpha}v(s))\|ds \\
&\quad + \int_t^{t+h} \|S(t+h-s)f(s,u(s),I^{\eta-\alpha}v(s))\|ds.
\end{aligned}$$

By (H1) and (H3), for $t \in I$ and $\varepsilon > 0$ given, there exists $\delta > 0$ such that

$$\|(C(s+h) - C(s))g(t-s, u(t-s))\| < \varepsilon$$

for $s \in [0,t]$ and $u \in B_r$, when $|h| < \delta$. This together with (H2), (H3) and the fact that $S(t)$ is Lipschitzian imply that

$$\|(\Phi - \Phi_1)(u,v)(t+h) - (\Phi - \Phi_1)(u,v)(t)\|$$

$$
\begin{aligned}
&\le \varepsilon t + \tilde{M}\Omega_g(r)\int_t^{t+h} K_g(s)\,ds \\
&\quad + N_1 h\Omega_f\Big(r + \frac{T^{\eta-\alpha}r}{\Gamma(\eta-\alpha+1)}\Big)\int_0^t K_f(s)\,ds \\
&\quad + \tilde{N}\Omega_f\Big(r + \frac{T^{\eta-\alpha}r}{\Gamma(\eta-\alpha+1)}\Big)\int_t^{t+h} K_f(s)\,ds
\end{aligned}
$$

for some positive constant N_1. The equicontinuity is therefore established.

On the other hand, for $t \in I$, as $(s,\xi) \to C(t-s)\xi$ is continuous from $[0,t] \times \overline{g(I \times X)}$ to X and $[0,t] \times \overline{g(I \times X)}$ is relatively compact in X. The set

$$
\big\{\Phi_2 u(t) := \int_0^t C(t-s)g(s,u(s))\,ds,\ u \in B_r(0,X)\big\}
$$

is relatively compact in X as well. We infer that Φ_2 is completely continuous. As for $\Phi_3 := \Phi - \Phi_1 + \Phi_2$ we decompose it as follows

$$
\begin{aligned}
\Phi_3(u,v)(t) &= \sum_{i=1}^{k-1}\int_{s_i}^{s_{i+1}} (S(s)-S(s_i))f(t-s,u(t-s),I^{\eta-\alpha}v(t-s))\,ds \\
&\quad + \sum_{i=1}^{k-1}\int_{s_i}^{s_{i+1}} S(s_i)f(t-s,u(t-s),I^{\eta-\alpha}v(t-s))\,ds
\end{aligned}
$$

and select the partition $\{s_i\}_{i=1}^k$ of $[0,t]$ in such a manner that, for a given $\varepsilon > 0$

$$
\|(S(s)-S(s'))f(t-s,u(t-s),I^{\eta-\alpha}v(t-s))\| < \varepsilon,
$$

for $(u,v) \in B_r^2(0,X)$, when $s,s' \in [s_i,s_{i+1}]$ for some $i = 1,\dots,k-1$. This is possible in as much as

$$
\{f(t-s,u(t-s),I^{\eta-\alpha}v(t-s)),\ s \in [0,t],\ (u,v) \in B_r^2(0,X)\}
$$

is bounded (by (H2) (iii)) and the operator S is uniformly Lipschitz on I. This leads to

$$
\Phi_3(u,v)(t) \in \varepsilon B_T(0,X) + \sum_{i=1}^{k-1}(s_{i+1}-s_i)\overline{\mathrm{co}(U(t,s_i,r))}
$$

where

$$
\begin{aligned}
&U(t,s_i,r) \\
&:= \big\{S(s_i)f(t-s,u(t-s),I^{\eta-\alpha}v(t-s)),\ s \in [0,t],\ (u,v) \in B_r^2(0,X)\big\}
\end{aligned}
$$

and $\mathrm{co}\,U(t,s_i,r))$ designates its convex hull. Therefore $\Phi_3(B_r^2)(t)$ is relatively compact in X. By Ascoli-Arzela Theorem, $\Phi_3(B_r^2)$ is relatively compact in $C(I;X)$ and consequently Φ_3 is completely continuous. Similarly we may prove that Ψ is completely continuous.

We conclude that (Φ,Ψ) admits a fixed point in $[C([0,T])]^2$. $\square$

Remark 3.5. In the same way we may treat the more general case

$$
\begin{gathered}
\frac{d}{dt}[u'(t)+g(t,u(t))] = Au(t) + f(t,u(t),D^{\alpha_1}u(t),\dots,D^{\alpha_n}u(t)), \\
u(0) = u^0 + p(u,D^{\beta_1}u(t),\dots,D^{\beta_m}u(t)), \\
u'(0) = u^1 + q(u,D^{\gamma_1}u(t),\dots,D^{\gamma_r}u(t))
\end{gathered}
$$

where $0 \le \alpha_i, \beta_j, \gamma_k \le 1$, $i = 1, \dots, n$, $j = 1, \dots, m$, $k = 1, \dots, r$.

Example. Consider the problem

$$\begin{gathered}
\frac{\partial}{\partial t}[u_t(t,x) + G(t,x,u(t,x))] \\
= u_{xx}(t,x) + F(t,x,u(t,x), D^\alpha u(t,x)), \quad t \in I = [0,T],\ x \in [a,b] \\
u(t,a) = u(t,b) = 0, \quad t \in I \\
u(0,x) = u^0(x) + \int_0^T P(u(s), D^\beta u(s))(x)\,ds, \quad x \in [a,b] \\
u'(0) = u^1(x) + \int_0^T Q(u(s), D^\gamma u(s))(x)\,ds, \quad x \in [a,b]
\end{gathered} \tag{3.12}$$

in the space $X = L^2([0,\pi])$. This problem can be reformulated in the abstract setting (1.1). To this end we define the operator $Ay = y''$ with domain

$$D(A) := \{y \in H^2([0,\pi]) : y(0) = y(\pi) = 0\}.$$

The operator A has a discrete spectrum with $-n^2$, $n = 1, 2, \dots.$ as eigenvalues and $z_n(s) = \sqrt{2/\pi}\sin(ns)$, $n = 1, 2, \dots$, as their corresponding normalized eigenvectors. So we may write

$$Ay = -\sum_{n=1}^{\infty} n^2 (y, z_n) z_n, \quad y \in D(A).$$

Since $-A$ is positive and self-adjoint in $L^2([0,\pi])$, the operator A is the infinitesimal generator of of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$ which has the form

$$C(t)y = \sum_{n=1}^{\infty} \cos(nt)(y, z_n) z_n, \quad y \in X.$$

The associated sine family is

$$C(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}(y, z_n) z_n, \quad y \in X.$$

One can also consider more general non-local conditions by allowing the Lebesgue measure ds to be of the form $d\mu(s)$ and $d\eta(s)$ for non-decreasing functions μ and η (or even more general: μ and η of bounded variation); that is,

$$\begin{gathered}
u(0,x) = u^0(x) + \int_0^T P(u(s), D^\beta u(s))(x) d\mu(s), \\
u_t(0,x) = u^1(x) + \int_0^T Q(u(s), D^\gamma u(s))(x) d\eta(s).
\end{gathered}$$

These (continuous) non-local conditions cover, of course, the discrete cases

$$\begin{gathered}
u(0,x) = u^0(x) + \sum_{i=1}^{n} \alpha_i u(t_i, x) + \sum_{i=1}^{m} \beta_i D^\beta u(t_i, x), \\
u_t(0,x) = u^1(x) + \sum_{i=1}^{r} \gamma_i u(t_i, x) + \sum_{i=1}^{k} \lambda_i D^\gamma u(t_i, x)
\end{gathered}$$

which have been extensively studied by several authors in the integer order case.

For $u, v \in C([0,T];X)$ and $x \in [a,b]$, defining the operators

$$\begin{gathered}
p(u,v)(x) := \int_0^T P(u(s),v(s))(x)\,ds, \\
q(u,v)(x) := \int_0^T Q(u(s),v(s))(x)\,ds, \\
g(t,u)(x) := G(t,x,u(t,x)), \\
f(t,u,v)(x) := F(t,x,u(t,x),v(t,x)),
\end{gathered}$$

allows us to write (3.12) abstractly as

$$\begin{gathered}
\frac{d}{dt}[u'(t) + g(t,u(t),u'(t))] = Au(t) + f(t,u(t),D^{\alpha}u(t)), \\
u(0) = u^0 + p(u, D^{\beta}u(t)), \\
u'(0) = u^1 + q(u, D^{\gamma}u(t)).
\end{gathered}$$

Under appropriate conditions on F, G, P and Q which make (H2)–(H4) hold for the corresponding functions f, g, p and q, Theorem 3.4 ensures the existence of a mild solution to problem (3.12).

Some special cases of this problem may be found in models of some phenomena with hereditary properties (see [5, 7, 8, 9, 15]).

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