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POSITIVE SOLUTIONS FOR BOUNDARY-VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS ON INFINITE INTERVALS

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Abstract. In this article, we consider the existence of positive solutions for a class of boundary value problems with integral boundary conditions on infinite intervals

$$(\varphi_p(x'(t)))' + \phi(t)f(t, x(t), x'(t)) = 0, \quad 0 < t < +\infty,$$
$$x(0) = \int_0^{+\infty} g(s)x'(s)ds, \quad \lim_{t\to+\infty} x'(t) = 0,$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. By applying the Avery-Peterson fixed point theorem in a cone, we obtain the existence of three positive solutions to the above boundary value problem and give an example at last.

1. Introduction

Boundary value problems (BVPs) on infinite intervals often appear in applied mathematics and physics and so on. The existence and multiplicity of positive solutions for such problems have become an important area of investigation in recent years. There are many papers concerning the existence of solutions on the half-line for the boundary value problems, see [1, 2, 4, 5, 8, 9, 11, 12, 15, 16, 17] and the references therein.

At the same time, we notice that a class of BVPs with integral boundary conditions appeared in heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics. Such problems include two-, three-, multi-point and nonlocal BVPs as special cases and attract more attention see [7, 10, 14] and the references therein. For more information about the general theory of integral equations and their relation with BVPs, we refer to the book of Corduneanu [6] and Agarwal and O'Regan [2].

Recently, Lian et al [12], studied the existence of positive solutions for the boundary-value problem with a p-Laplacian operator

$$(\varphi_p(x'(t)))' + \phi(t)f(t, x(t), x'(t)) = 0, \quad 0 < t < +\infty,$$
$$\alpha x(0) - \beta x'(0) = 0, x'(\infty) = 0.$$

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Guo and Yu [8] establish the existence of three positive solutions for m-point BVPs on infinite intervals

$$
\begin{gathered}
(\varphi_p(x'(t)))' + \phi(t) f(t, x(t), x'(t)) = 0, \quad 0 < t < +\infty, \\
x(0) = \sum_{i=1}^{m-2} \alpha_i x'(\eta_i), \quad \lim_{t\to+\infty} x'(t) = 0,
\end{gathered}
$$

using the Avery-Peterson fixed point theorem in a cone.

Due to the fact that an infinite interval is noncompact, the discussion about BVPs on the half-line is more complicated, in particular, for the BVPs with integral boundary conditions on infinite intervals, few works were done.

Motivated by the results [8, 12], we will study the following BVPs with integral conditions:

$$
\begin{gathered}
(\varphi_p(x'(t)))' + \phi(t) f(t, x(t), x'(t)) = 0, \quad 0 < t < +\infty, \\
x(0) = \int_0^{+\infty} g(s) x'(s) ds, \quad \lim_{t\to+\infty} x'(t) = 0,
\end{gathered} \tag{1.1}
$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, $f(t, u, v) : \mathbb{R}_+^3 \to \mathbb{R}_+$ is a continuous function, $\mathbb{R}_+ = [0, +\infty)$, $g \in L^1[0, +\infty)$ is nonnegative.

In this article, we use the following conditions:

(H1) $\phi \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\phi \not\equiv 0$ on any interval of the form $(t_0, +\infty)$ and

$$
\int_0^{+\infty} \phi(s) ds < +\infty, \quad \int_0^{+\infty} \varphi^{-1}\Big(\int_\tau^{+\infty} \phi(s) ds\Big) d\tau < +\infty;
$$

(H2) $f(t, (1+t)u, v) \in C(\mathbb{R}_+^3, \mathbb{R}_+)$, $f(t, 0, 0) \neq 0$ on any subinterval of $(0, +\infty)$ and when u, v are bounded $f(t, (1+t)u, v)$ is bounded on $[0, +\infty)$.

2. Preliminary results

In this section, we present some definitions, theorems and lemmas, which will be needed in the proof of the main results. We first give the Avery-Peterson fixed point theorem.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

(i) $x \in P$ and $\lambda \geq 0$ imply $\lambda x \in P$;
(ii) $x \in P$ and $-x \in P$ imply $x = 0$.

Definition 2.2. Given a cone P in a real Banach space E. A continuous map ψ is called a concave (resp. convex) functional on P if for all $x, y \in P$ and $0 \leq \lambda \leq 1$, it holds $\psi(\lambda x + (1-\lambda)y) \geq \lambda\psi(x) + (1-\lambda)\psi(y)$, (resp. $\psi(\lambda x + (1-\lambda)y) \leq \lambda\psi(x) + (1-\lambda)\psi(y)$).

Let $\alpha, \gamma, \theta, \psi$ be nonnegative continuous maps on P with α concave, γ, θ convex. Then for positive numbers a, b, c, d, we define the following subsets of P:

$$
\begin{gathered}
P(\gamma^d) = \{x \in P : \gamma(x) < d\}; \\
P(\alpha_b, \gamma^d) = \{x \in \overline{P(\gamma^d)} : b \leq \alpha(x)\}; \\
P(\alpha_b, \theta^c, \gamma^d) = \{x \in \overline{P(\gamma^d)} : b \leq \alpha(x), \theta(x) \leq c\}; \\
R(\psi_a, \gamma^d) = \{x \in \overline{P(\gamma^d)} : a \leq \psi(x)\}.
\end{gathered}
$$

It is obvious that $P(\gamma^d)$, $P(\alpha_b, \gamma^d)$, $P(\alpha_b, \theta^c, \gamma^d)$ are convex and $R(\psi_a, \gamma^d)$ is closed.

Theorem 2.3 ([3]). *Let P be a cone of a real Banach space E. Let γ, θ be nonnegative convex functional on P satisfying*

$$\psi(\lambda x) \le \lambda\psi(x), \quad \forall 0 \le \lambda \le 1,$$
$$\alpha(x) \le \psi(x), \quad \|x\| \le M\gamma(x) \quad \forall x \in \overline{P(\gamma^d)}$$

with M, d some positive numbers. Suppose that $T : \overline{P(\gamma^d)} \to \overline{P(\gamma^d)}$ is completely continuous and there exist positive numbers a, b, c with $a < b$ such that

(1) $\{x \in P(\alpha_b, \theta^c, \gamma^d) : \alpha(x) > b\} \ne \emptyset$ *and* $\alpha(Tx) > b$ *for* $x \in P(\alpha_b, \theta^c, \gamma^d)$*;*
(2) $\alpha(Tx) > b$ *for* $x \in P(\alpha_b, \gamma^d)$ *with* $\theta(Tx) > c$*;*
(3) $0 \notin R(\psi_a, \gamma^d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\psi_a, \gamma^d)$ *with* $\psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma^d)}$ such that $\gamma(x_i) \le d$, $i = 1, 2, 3$; $\psi(x_1) < a$; $\psi(x_2) > a$ with $\alpha(x_2) < b$; $\alpha(x_3) > b$.

Consider the space

$$X = \big\{x \in C^1[0, +\infty), \sup_{0 \le t < +\infty} \frac{|x(t)|}{1+t} < +\infty, \lim_{t \to +\infty} x'(t) = 0\big\} \tag{2.1}$$

with the norm $\|x\| = \max\{\|x\|_1, \|x'\|_\infty\}$, where $\|x\|_1 = \sup_{0 \le t < +\infty} |x(t)|/(1+t)$, $\|x'\|_\infty = \sup_{0 \le t < +\infty} |x'(t)|$. By using the standard arguments, we can obtain that $(X, \|\cdot\|)$ is a Banach space. Define the $P \subset X$ by

$$\begin{aligned} P = \Big\{&x \in X : x(t) \ge 0, t \in [0, +\infty), \\ &x(0) = \int_0^{+\infty} g(s)x'(s)ds, x \text{ is concave on } [0, +\infty)\Big\}. \end{aligned} \tag{2.2}$$

Remark 2.4. If x satisfies (1.1), then $(\varphi_p(x'(t)))' = -\phi(t)f(t, x(t), x'(t)) \le 0$ on $[0, +\infty)$, which implies that x is concave on $[0, +\infty)$. Moreover, if $\lim_{t \to +\infty} x'(t) = 0$, then $x'(t) \ge 0, t \in [0, +\infty)$ and so x is monotone increasing on $[0, +\infty)$.

Let $k > 1$ be a constant. For $x \in P$, define the nonnegative continuous functionals:

$$\begin{gathered} \alpha(x) = \frac{k}{k+1} \min_{\frac{1}{k} \le t < +\infty} x(t), \quad \gamma(x) = \sup_{0 \le t < +\infty} x'(t), \\ \psi(x) = \theta(x) = \sup_{0 \le t < +\infty} \frac{x(t)}{1+t}, \quad A = \int_0^{+\infty} g(s)ds, \\ C = \varphi_p^{-1}\Big(\int_0^{+\infty} \phi(s)ds\Big), \quad C_1(t) = \int_0^t \varphi_p^{-1}\Big(\int_\tau^{+\infty} \phi(s)ds\Big)d\tau. \end{gathered} \tag{2.3}$$

Since the Arzela-Ascoli theorem does not apply in the space X, we need a modified compactness criterion to prove T is compact. In the following, we present an explicit one. For more general cases, we refer the readers to [2, 17] and the reference therein.

Definition 2.5. For $l > 0$, let $V = \{x \in X : \|x\| < l\}$, and

$$V_1 := \{\frac{x(t)}{1+t}, x \in V\} \cup \{x'(t), x \in V\}$$

which is called equiconvergent at infinity if for all $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that for all $x \in V_1$,

$$|\frac{x(t_1)}{1+t_1} - \frac{x(t_2)}{1+t_2}| < \varepsilon, \quad |x'(t_1) - x'(t_2)| < \varepsilon, \quad \forall t_1, t_2 \geq T.$$

Lemma 2.6 ([13]). *If $\{\frac{x(t)}{1+t}, x \in V\}$ and $\{x'(t), x \in V\}$ are both equicontinuous on any compact interval of $[0, +\infty)$ and equiconvergent at infinity . Then V is relatively compact on X.*

Lemma 2.7. *Let $g \in L^1[0, +\infty)$ and g is nonnegative, if $v(t)$ is nonnegative and continuous on $[0, +\infty)$ and $\lim_{t\to+\infty} v(t)$ exists. Then there exists at least one η, $0 \leq \eta \leq +\infty$ such that*

$$\int_0^{+\infty} g(s)v(s)ds = v(\eta)\int_0^{+\infty} g(s)ds. \tag{2.4}$$

Proof. It is obvious that the function $v(t)$ exists and has maxima and minima which are nonnegative and noted by M^*, m^* on $[0, +\infty)$, then for all $t \in [0, +\infty)$, we have $m^* \leq v(t) \leq M^*$, so

$$m^* \int_0^{+\infty} g(s)ds \leq \int_0^{+\infty} g(s)v(s)ds \leq M^* \int_0^{+\infty} g(s)ds.$$

If $\int_0^{+\infty} g(s)ds = 0$, the result is clear; If $\int_0^{+\infty} g(s)ds > 0$, there is

$$m^* \leq \frac{\int_0^{+\infty} g(s)v(s)ds}{\int_0^{+\infty} g(s)ds} \leq M^*.$$

Therefore, there exists at least one η, $0 \leq \eta \leq +\infty$ such that

$$\int_0^{+\infty} g(s)v(s)ds = v(\eta)\int_0^{+\infty} g(s)ds. \tag{2.5}$$

□

Lemma 2.8. *Let $y \in C[\mathbb{R}_+, \mathbb{R}_+]$, and $\int_0^{+\infty} y(t)dt < \infty$, then the boundary-value problem*

$$\begin{gathered} (\varphi_p(x'(t)))' + y(t) = 0, \quad 0 < t < +\infty, \\ x(0) = \int_0^{+\infty} g(s)x'(s)ds, \quad \lim_{t\to+\infty} x'(t) = 0, \end{gathered} \tag{2.6}$$

has a unique solution

$$x(t) = \int_0^{+\infty} g(s)\varphi_p^{-1}\Big(\int_s^{+\infty} y(\tau)d\tau\Big)ds + \int_0^t \varphi_p^{-1}\Big(\int_s^{+\infty} y(\tau)d\tau\Big)ds.$$

Define the operator $T : P \to C^1[0, +\infty)$ by

$$\begin{aligned} (Tx)(t) = &\int_0^{+\infty} g(s)\varphi_p^{-1}\Big(\int_s^{+\infty} \phi(\tau)f(\tau, x(\tau), x'(\tau))d\tau\Big)ds \\ &+ \int_0^t \varphi_p^{-1}\Big(\int_s^{+\infty} \phi(\tau)f(\tau, x(\tau), x'(\tau))d\tau\Big)ds, \quad t \in [0, +\infty). \end{aligned} \tag{2.7}$$

Lemma 2.9. *For $x \in P$, $\|x\|_1 \leq M\|x'\|_\infty$, where $M = \max\big\{\int_0^{+\infty} g(s)ds, 1\big\}$.*

Proof. Since $x \in P$, by Lemma 2.7,

$$\begin{aligned}\frac{x(t)}{1+t} &= \frac{1}{1+t}(\int_0^t x'(s)ds + \int_0^{+\infty} g(s)x'(s)ds) \\ &\le \frac{t+\int_0^{+\infty} g(s)ds}{1+t}\|x'\|_\infty \le M\|x'\|_\infty.\end{aligned}$$

The result follows. □

Lemma 2.10 ([8])**.** *For $x \in P$, $\alpha(x) \ge \frac{1}{k+1}\theta(x)$.*

Lemma 2.11. *Let* (H1)–(H2) *hold. Then $T : P \to P$ is completely continuous.*

Proof. It is easy to see that $T : P \to P$ is well defined. Now we prove that T is continuous and compact respectively. Let $x_n \to x$ as $n \to +\infty$ in P, then there exists r_0 such that $\sup_{n\in N\setminus\{0\}} \|x_n\| < r_0$. Set $B_{r_0} = \sup\{f(t,(1+t)u,v),(t,u,v) \in [0,+\infty) \times [0,r_0]^2\}$. Then we have

$$\int_0^{+\infty} \phi(s)|f(s,x_n,x_n') - f(s,x,x')|ds \le 2B_{r_0}\int_0^{+\infty} \phi(s)ds.$$

Therefore, by the Lebesgue dominated convergence theorem,

$$\begin{aligned}|\varphi_p((Tx_n)'(t)) - \varphi_p((Tx)'(t))| &= \Big|\int_t^{+\infty} \phi(s)(f(s,x_n,x_n^{'}) - f(s,x,x'))ds\Big| \\ &\le \int_0^{+\infty} \phi(s)|f(s,x_n,x_n^{'}) - f(s,x,x')|ds \\ &\to 0, \quad \text{as } n \to +\infty.\end{aligned}$$

Furthermore, $\|Tx_n - Tx\| \le M\|(Tx_n)' - (Tx)'\|_\infty \to 0$, as $n \to +\infty$. Hence, T is continuous.

T is compact provided that it maps bounded sets into relatively compact sets. Let Ω be any bounded subset of P. Then there exists $r > 0$ such that $\|x\| \le r$ for all $x \in \Omega$. Obviously,

$$\|(Tx)'\|_\infty = \varphi_p^{-1}(\int_0^{+\infty} \phi(s)f(s,x(s),x'(s))ds) \le C\varphi_p^{-1}(B_r)$$

for all $x \in \Omega$. Hence, $\|T\Omega\| \le MC\varphi_p^{-1}(B_r)$. So $T\Omega$ is bounded.

Moreover, for any $L \in (0,+\infty)$ and $t_1, t_2 \in [0,L]$,

$$\begin{aligned}&\Big|\frac{(Tx)(t_1)}{1+t_1} - \frac{(Tx)(t_2)}{1+t_2}\Big| \\ &\le \int_0^{+\infty} g(s)\varphi_p^{-1}\Big(\int_s^{+\infty} \phi(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\Big|\frac{1}{1+t_1} - \frac{1}{1+t_2}\Big| \\ &\quad + \int_0^{t_2} \varphi_p^{-1}\Big(\int_s^{+\infty} \phi(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\Big|\frac{1}{1+t_1} - \frac{1}{1+t_2}\Big| \\ &\quad + \frac{1}{1+t_1}\Big|\int_{t_1}^{t_2} \varphi_p^{-1}\Big(\int_s^{+\infty} \phi(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\Big| \\ &\le \varphi_p^{-1}(B_r)(AC + C_1(L))|t_1 - t_2| + |C_1(t_1) - C_1(t_2)|) \\ &\to 0, \quad \text{uniformly as } t_1 \to t_2,\end{aligned}$$

and

$$
\begin{aligned}
|\varphi_p((Tx)'(t_1)) - \varphi_p((Tx)'(t_2))| &= \Big|\int_{t_1}^{t_2} \phi(s)(f(s,x,x')ds\Big| \\
&\le B_r\Big|\int_{t_1}^{t_2} \phi(s)ds\Big| \to 0, \quad \text{uniformly as } t_1 \to t_2,
\end{aligned}
$$

for all $x \in \Omega$. So $T\Omega$ is equicontinuous on any compact interval of $[0,+\infty)$.

Finally, for any $x \in \Omega$,

$$
\begin{aligned}
\lim_{t\to+\infty} \Big|\frac{(Tx)(t)}{1+t}\Big| &= \lim_{t\to+\infty} \frac{1}{1+t}\int_0^t \varphi_p^{-1}\Big(\int_s^{+\infty} \phi(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds \\
&\le M\varphi_p^{-1}(B_r)\lim_{t\to+\infty}\varphi_p^{-1}\Big(\int_t^{+\infty}\phi(s)ds\Big) = 0, \\
\lim_{t\to+\infty} |(Tx)'(t)| &= \lim_{t\to+\infty}\varphi_p^{-1}(\int_t^{+\infty}\phi(s)f(s,x(s),x'(s))ds) \\
&\le \varphi_p^{-1}(B_r)\lim_{t\to+\infty}\varphi_p^{-1}\Big(\int_t^{+\infty}\phi(s)ds\Big) = 0.
\end{aligned}
$$

So $T\Omega$ is equiconvergent at infinity. By using Lemma 2.6, we obtain that $T\Omega$ is relatively compact, that is, T is a compact operator. Hence, $T : P \to P$ is completely continuous. The proof is complete. $\square$

3. Main results

For the main result of this article we sue the hypothesis

(H3) $f(t,(1+t)u,v) \le \varphi_p(d/C)$, for $(t,u,v) \in [0,+\infty)\times[0,Md]\times[0,d]$;

(H4) $f(t,(1+t)u,v) > \varphi_p(b/N)$, for $(t,u,v) \in [\frac{1}{k},k]\times[\frac{b}{k},\frac{(k+1)^2b}{km}]\times[0,d]$;

(H5) $f(t,(1+t)u,v) < \varphi_p(a/MC)$, for $(t,u,v) \in [0,+\infty)\times[0,a]\times[0,d]$; where

$$
m = \min\{A,1\}, \quad N = \frac{1}{(k+1)^2}\int_{\frac{1}{k}}^{k}(g(s)+1)\varphi_p^{-1}\Big(\int_s^k \phi(\tau)d\tau\Big)ds.
$$

Theorem 3.1. *Let $A > 0$. Suppose* (H1)–(H5) *hold. Suppose further that there exist numbers a,b,d such that $0 < ka < b \le Mmkd/(k+1)^2$. Then* (1.1) *has at least three positive solutions x_1,x_2,x_3 such that*

$$
\begin{gathered}
\sup_{0\le t<+\infty} x_i'(t) \le d, \quad i = 1,2,3; \\
\sup_{0\le t<+\infty}\frac{x_1(t)}{1+t} < a, \quad a < \sup_{0\le t<+\infty}\frac{x_2(t)}{1+t} < \frac{(k+1)^2b}{km}, \quad \min_{\frac{1}{k}<t<k} x_2(t) < \frac{(k+1)}{k}b; \\
\sup_{0\le t<+\infty}\frac{x_3(t)}{1+t} < Md, \quad \min_{\frac{1}{k}\le t<k} x_3(t) > \frac{(k+1)}{k}b.
\end{gathered} \tag{3.1}
$$

Proof. Let $X,P,\alpha,\gamma,\theta,\psi$ and T be defined as (2.1)-(2.3) and (2.7) respectively. It is easy to prove that the fixed points of T coincide with the solution of BVP (1.1). So it is enough to show that T has three positive fixed points.

In fact, for any $x \in \overline{P(\gamma^d)}$, $\sup_{0\le t<+\infty} x'(t) \le d$ and so $\sup_{0\le t<+\infty}\frac{x(t)}{1+t} \le Md$. Condition (H3) implies that $f(t,x(t),x'(t)) \le \varphi_p(d/C)$ for all $t \in [0,+\infty)$.

Therefore,

$$\begin{aligned}\gamma(Tx) &= \sup_{0\le t<+\infty}(Tx)'(t) = (Tx)'(0)\\ &= \varphi_p^{-1}\Big(\int_0^{+\infty}\phi(s)f(s,x(s),x'(s))ds\Big)\\ &\le \frac{d}{C}\varphi_p^{-1}\Big(\int_0^{+\infty}\phi(s)ds\Big) = d.\end{aligned}$$

Hence $T:\overline{P(\gamma^d)}\to\overline{P(\gamma^d)}$ is completely continuous.

Obviously, $\alpha,\gamma,\theta,\psi$ satisfy the assumptions in Theorem 2.3. Next we show that conditions (1)-(3) in Theorem 2.3 hold.

Firstly, choose the function $x(t) = (1-\frac{1}{k+1}e^{-\frac{k}{A}t})\frac{(k+1)^2}{k}b, 0\le t<+\infty$. It can be checked that $x\in P(\alpha_b,\theta^c,\gamma^d)$ with $\alpha(x)>b$, where $c=\frac{(k+1)^2}{km}b$. So $\{x\in P(\alpha_b,\theta^c,\gamma^d)|\alpha(x)>b\}\neq\emptyset$. For any $x\in P(\alpha_b,\theta^c,\gamma^d)$, we obtain

$$\frac{b}{k}\le\frac{1}{1+k}\min_{\frac1k\le t\le k}x(t)\le\frac{x(t)}{1+k}\le\frac{x(t)}{1+t}\le\frac{(k+1)^2}{km}b,\quad t\in\Big[\frac1k,k\Big],$$

and $0\le x'(t)\le d, t\in[0,+\infty)$. In view of assumption (H4) together with Lemma 2.10, we obtain

$$\begin{aligned}\alpha(Tx) &\ge \frac{1}{k+1}\theta(Tx) = \frac{1}{k+1}\sup_{0\le t<+\infty}\frac{(Tx)(t)}{1+t}\\ &= \frac{1}{(k+1)}\sup_{0\le t<+\infty}\frac{1}{1+t}\Big[\int_0^{+\infty}g(s)\varphi_p^{-1}\Big(\int_s^{+\infty}\phi(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\ &\quad+\int_0^{t}\varphi_p^{-1}\Big(\int_s^{+\infty}\phi(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\Big]\\ &\ge \frac{1}{(k+1)^2}\Big[\int_{\frac1k}^{k}g(s)\varphi_p^{-1}\Big(\int_s^{k}\phi(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\\ &\quad+\int_{\frac1k}^{k}\varphi_p^{-1}\Big(\int_s^{k}\phi(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds\Big]\\ &> \frac{b}{N}\frac{1}{(k+1)^2}\int_{\frac1k}^{k}(g(s)+1)\varphi_p^{-1}\Big(\int_s^{k}\phi(\tau)d\tau\Big)ds = b.\end{aligned}$$

Hence, $\alpha(Tx)>b$ for $x\in P(\alpha_b,\theta^c,\gamma^d)$.

Next we will verify that the condition (2) of Theorem 2.3 is satisfied. Let $x\in P(\alpha_b,\gamma^d)$ with $\theta(Tx)>c$, it follows from Lemma 2.10 that

$$\alpha(Tx)\ge\frac{1}{k+1}\theta(Tx)>\frac{1}{k+1}c=\frac{1}{k+1}\frac{(k+1)^2}{km}b=\frac{(k+1)}{km}b>b,$$

Thus $\alpha(Tx)>b$ for all $x\in P(\alpha_b,\gamma^d)$ with $\theta(Tx)>c$.

Finally, we show that condition (3) of theorem 2.3 is satisfied. It is clear that $0\in R(\psi_a,\gamma^d)$. Suppose that $x\in R(\psi_a,\gamma^d)$ with $\psi(x)=a$, then by condition (H5) and Lemma 2.9, we obtain

$$\begin{aligned}\psi(Tx) &\le M\gamma(Tx) = M(Tx)'(0)\\ &= M\varphi_p^{-1}(\int_0^{+\infty}\phi(s)f(s,x(s),x'(s))ds)\end{aligned}$$

$$\leq M\cdot\frac{a}{MC}\varphi_p^{-1}(\int_0^{+\infty}\phi(s)ds=a.$$

Therefore, T has at least three fixed points $x_1,x_2,x_3\in\overline{P(\gamma^d)}$ such that

$$\psi(x_1)<a,\quad \psi(x_2)>a\quad\text{with } \alpha(x_2)<b,\ \alpha(x_3)>b.$$

In addition, condition (H2) guarantees that those fixed points are positive. So (1.1) has at least three positive solutions x_1,x_2,x_3 satisfying (3.1) and the proof is complete. □

4. Example

Consider the boundary-value problem with integral boundary value conditions

$$\begin{gathered}(|x'|x')'+e^{-t}f(t,x(t),x'(t))=0,\\ x(0)=\int_0^{+\infty}e^{-2s}x'(s)ds,\quad \lim_{t\to+\infty}x'(t)=0,\end{gathered}\tag{4.1}$$

where

$$f(t,u,v)=\begin{cases}\frac{|\sin t|}{100}+10^4\big(\frac{u}{1+t}\big)^{10}+\frac{1}{100}\big(\frac{v}{200}\big), & u\leq 1,\\ \frac{|\sin t|}{100}+10^4\big(\frac{1}{1+t}\big)^{10}+\frac{1}{100}\big(\frac{v}{200}\big), & u\geq 1.\end{cases}$$

Set $\phi(t)=e^{-t}$ and it is easy to verify that (H1) and (H2) hold. Choose $k=4$, $a=\frac14$, $b=2$, $d=200$. Then by simple calculations, we can obtain $M=1$, $m=\frac12$,

$$C=1, N=\frac{1}{25}\int_{\frac14}^4(e^{-2s}+1)\sqrt{e^{-s}-e^{-4}}ds\geq\frac{1}{25}\int_{\frac14}^4\sqrt{e^{-s}-e^{-4}}ds>\frac{1}{48}.$$

So the nonlinear term f satisfies

(1) $f(t,(1+t)u,v)\leq 0.01+10^4+0.01<4\times10^4=\varphi_3(d/C)$, for $(t,u,v)\in[0,+\infty)\times[0,200]^2$;
(2) $f(t,(1+t)u,v)\geq 10^4>96^2=\varphi_3(b/N)$, for $(t,u,v)\in[\frac14,4]\times[\frac12,25]\times[0,200]$;
(3) $f(t,(1+t)u,v)\leq 0.01+10^4\times\big(\frac14\big)^{10}+0.01<\frac{1}{16}=\varphi_3(a/MC)$, for $(t,u,v)\in[0,+\infty)\times[0,\frac14]\times[0,200]$.

Therefore, the conditions in Theorem 3.1 are all satisfied. So (4.1) has at least three positive solutions x_1,x_2,x_3 such that

$$\begin{gathered}\sup_{0\leq t<+\infty}x_i'(t)\leq 200,\quad i=1,2,3;\\ \sup_{0\leq t<+\infty}\frac{x_1(t)}{1+t}\leq\frac14,\quad \frac12<\sup_{0\leq t<+\infty}\frac{x_2(t)}{1+t}<25,\quad \min_{\frac1k\leq t\leq k}x_2(t)\leq\frac52;\\ \sup_{0\leq t<+\infty}\frac{x_3(t)}{1+t}\leq 200\quad \min_{\frac1k\leq t\leq k}x_3(t)>\frac52.\end{gathered}$$

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