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DIRICHLET-TO-NEUMANN OPERATOR ON THE PERTURBED UNIT DISK

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Abstract. This article concerns the Laplacian on a perturbed unit disk $\Omega_\epsilon = \{z = r\exp(i\theta) : r < 1 + \epsilon f(\theta)\}$, with dynamical boundary condition whose solution can be represented by a Dirichlet-to-Neumann semigroup. By neglecting the terms of order ϵ^2, we obtain a simple expression which allows us to use Chernoff's theorem for its approximation. As a motivation for this research, we present an example which shows the feasibility of applying Chernoff's Theorem.

1. Introduction

In many inverse problems the reconstruction of the Dirichlet-to-Neumann operator has paramount importance. For example, in [1, 2, 10] a nondestructive testing technique is used to determine the distribution of electrical conductivity within some inaccessible region from electrostatic measurements at its boundary. It has been able to detect the presence and the approximate location of some object of unknown shape buried in a medium of constant background conductivity. In the present work we elaborate the same shape of the buried object, that is a star-shaped perturbation of a circle (see Figure 1). On this domain our problem is generated by an elliptic equation with a time dependent boundary condition as follows.

Let $\Omega_\epsilon = \{z = r\exp(i\theta) : r < 1 + \epsilon f(\theta)\}$ be the perturbed unit disk. In the cylinder $\Gamma_\epsilon = (0,\infty)\times\Omega_\epsilon$ with boundary $\partial\Gamma_\epsilon = (0,\infty)\times\partial\Omega_\epsilon$, we consider the linear elliptic differential equation with dynamic boundary conditions

$$\begin{gathered} \Delta u = 0, \quad \text{in } \Gamma_\epsilon, \\ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Gamma_\epsilon, \\ u(0) = g, \quad \text{on } \partial\Omega_\epsilon, \end{gathered} \tag{1.1}$$

We will see in the next section that by making the change of variables to polar coordinates and neglecting the terms of order ϵ^2, the system (1.1) can be transformed

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into the system

$$
\begin{gathered}
\Delta_{\gamma_\epsilon} v = 0, \quad \text{in } \Gamma, \\
\frac{\partial v}{\partial t} + \frac{\partial v}{\partial \nu_{\gamma_\epsilon}} = 0, \quad \text{on } \partial\Gamma, \\
v(0) = h, \quad \text{on } \partial D,
\end{gathered} \tag{1.2}
$$

where D is the unit disk, Γ the cylinder $(0,\infty)\times D$ with boundary $\partial\Gamma = (0,\infty)\times\partial D$,

$$
\Delta_{\gamma_\epsilon} = \operatorname{div}(\gamma_\epsilon \nabla), \quad \gamma_\epsilon = \begin{bmatrix} 1 & -\epsilon f' \\ -\epsilon f' & 1 \end{bmatrix}
$$

and $h(\rho,\theta) = g(\rho(1+\epsilon f(\theta)),\theta)$. The Dirichlet-to-Neumann operator $\Lambda_{\gamma_\epsilon}$ for this new system is defined by

$$
[\Lambda_{\gamma_\epsilon} h](\omega) = -\frac{\partial v}{\partial \nu_{\gamma_\epsilon}} = -\omega \cdot \gamma_\epsilon \nabla v\big|_{\partial D}, \tag{1.3}
$$

where ω is the outer normal vector at $\omega \in \partial D$, which coincides with ω.

It is well known that the operator $\Lambda_{\gamma_\epsilon}$ generates a (C_0) semigroup in $L^2(\partial\Omega_\epsilon)$ (see [9]) or in $C(\partial\Omega_\epsilon)$ (see [7]). In a simple case, where $\gamma_\epsilon = I$ and Ω_ϵ is the unit ball B in $\mathbb{R}^n$ this semigroup has a trivial explicit representation given by Lax [8], which is

$$
e^{-t\Lambda_I} h(\omega) = w(e^{-t}\omega), \quad \text{for any } \omega \in \partial D.
$$

where w is the harmonic lifting of h; i.e., the solution of

$$
\begin{gathered}
\Delta_I w = 0, \quad \text{in } D, \\
w\big|_{\partial D} = h, \quad \text{on } \partial D,
\end{gathered} \tag{1.4}
$$

In [6], it is shown that this representation cannot be generalized if Ω_ϵ has other geometry than a ball in $\mathbb{R}^n$. This consideration motivates the authors of [5] to use a new technique for having an approximation of the solution via Chernoff's Theorem. In fact, Chernoff [3] proves the following result.

Theorem 1.1. *If X is a Banach space and $\{V(t)\}_{t\geq 0}$ is a family of contractions on X with $V(0) = I$. Suppose that the derivative $V'(0)f$ exists for all f in a set $\mathcal{D}$ and the closure Λ of $V'(0)\mid_{\mathcal{D}}$ generates a C_0-semigroup $S(t)$ of contractions. Then, for each $f \in X$,*

$$
\lim_{n\to\infty} \|V(\frac{t}{n})^n f - S(t)f\| = 0, \tag{1.5}
$$

uniformly for t in compact subsets of $\mathbb{R}_+$.

In the next section we define all the ingredients of the above Theorem in the framework of our problem and we prove how we can approximate the solution of the system (1.2) by mean of this Theorem. Section 3 is devoted to give a bench mark example for justifying the feasibility of our method. Finally we generalize in section 4 this problem to 3D-consideration and take Ω_ϵ as a perturbed unit sphere. Because of the extreme complexity of the system we restrict ourself to the discrete values of ϕ, $\{\phi_j\ j = 1,\dots,N\}$, which achieves to N times 2D system, each of which can be computed as above.

2. Modified system

By making the change of variables

$$\rho = \frac{r}{1+\epsilon f(\theta)}$$

and θ remains invariant, the Laplacian in the new coordinates will be (see [4]),

$$\begin{aligned}\Delta_{\rho,\theta} &= \Big(\frac{1}{(1+\epsilon f)^2} + \frac{\epsilon^2 (f')^2}{(1+\epsilon f)^4}\Big)\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\Big(\frac{1}{(1+\epsilon f)^2} + \frac{2\epsilon^2 (f')^2}{(1+\epsilon f)^4} - \frac{\epsilon f''}{(1+\epsilon f)^3}\Big)\frac{\partial}{\partial \rho} \\ &\quad - \frac{1}{\rho}\Big(\frac{2\epsilon f'}{(1+\epsilon f)^3}\Big)\frac{\partial^2}{\partial\rho\partial\theta} + \frac{1}{\rho^2}\frac{1}{(1+\epsilon f)^2}\frac{\partial^2}{\partial\theta^2}.\end{aligned} \tag{2.1}$$

By factoring the term $\frac{1}{(1+\epsilon f)^2}$ and neglecting the terms of order ϵ^2, we obtain

$$\Delta^{(\epsilon)}_{\rho,\theta} = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}(1-\epsilon f'')\frac{\partial}{\partial\rho} - \frac{1}{\rho}(2\epsilon f')\frac{\partial^2}{\partial\rho\partial\theta} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}. \tag{2.2}$$

Here we have replaced $\frac{1}{1+\epsilon f}$ by $1-\epsilon f+\dots$. Now let

$$\gamma_\epsilon = \begin{bmatrix} 1 & -\epsilon f' \\ -\epsilon f' & 1 \end{bmatrix}$$

and $\Delta_{\gamma_\epsilon} = div(\gamma_\epsilon \nabla)$. Since in polar coordinates,

$$\nabla = {}^t\big(\frac{\partial}{\partial\rho}, \frac{1}{\rho}\frac{\partial}{\partial\theta}\big), \quad \mathrm{div}(F_1,F_2) = \frac{1}{\rho}\Big(\frac{\partial}{\partial\rho}\rho F_1 + \frac{\partial}{\partial\theta}F_2\Big),$$

we obtain $\Delta^{(\epsilon)}_{\rho,\theta} = \Delta_{\gamma_\epsilon}$. Consequently the Dirichlet-to-Neumann operator will be

$$[\Lambda_{\gamma_\epsilon} h](1,0) = -(1,0)\cdot \begin{bmatrix} 1 & -\epsilon f' \\ -\epsilon f' & 1 \end{bmatrix}\begin{bmatrix} v_\rho \\ \frac{v_\theta}{\rho}\end{bmatrix}_{(\rho,\theta)=(1,0)}. \tag{2.3}$$

Remark 2.1. Since γ_ϵ is not continuous at the origin, even for $f \in C^1([0,2\pi])$, in the sequel we replace Ω_ϵ by $\Omega_\epsilon \setminus \{0\}$, because for any periodic function $f \in C^\infty([0,2\pi])$, we have $\gamma_\epsilon \in C^\infty(\Omega\setminus\{0\})$.

Now, let us define $\partial X := C(\partial D)$ and the operator $V_\epsilon(t)$ by

$$V_\epsilon(t)h(\omega) := v(e^{-t\gamma_\epsilon}\omega)\Big|_{\partial D}, \quad \text{for any } \ h\in\partial X, \tag{2.4}$$

where v is the solution of (1.3).

Theorem 2.2. *The family of the operators $\{V_\epsilon(t)\}_{t\geq 0}$ defined above satisfies the assumptions of the Chernoff's Theorem 1.1.*

Proof. For ϵ small enough, the eigenvalues of γ_ϵ, $\lambda = 1\pm\epsilon f'(\theta)$ are positive implies that for any $\omega\in\partial D$, we have $e^{-t\gamma_\epsilon}\omega \in D$. Hence the fact that $V_\epsilon(t)$ is a contraction follows from the maximum principle. It is obvious that $V_\epsilon(0)=I$ on the Banach space ∂X. Since

$$D(\Lambda_{\gamma_\epsilon}) = \{h\in\partial X : \Lambda_{\gamma_\epsilon} h \in \partial X\},$$

for any $h\in D(\Lambda_{\gamma_\epsilon})$, $\frac{d}{dt}V_\epsilon(t)h\Big|_{t=0} = -\omega\cdot\gamma_\epsilon\nabla v(\omega) = \Lambda_{\gamma_\epsilon}h$. This shows that we can apply Theorem 1.1 for the family of operators $\{V_\epsilon(t)\}_{t\geq 0}$. $\square$

3. An example

Let us denote $\Delta_0 u = \operatorname{div}(\nabla u)$ and $\Delta_1 u = \operatorname{div}(f' \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \nabla u)$ in such a way that $\Delta_{\gamma_\epsilon} = \Delta_0 - \epsilon\Delta_1$. Now let us define u_0 and u_1 such that by neglecting the terms of order ϵ^2, $v = u_0 + \epsilon u_1$ satisfies the system (1.2). In fact by taking u_0 the solution of the Dirichlet problem

$$\begin{aligned} \Delta_0 u_0 &= 0, \quad \text{in } D, \\ u_0\big|_{\partial D} &= g_0, \quad \text{on } \partial D, \end{aligned} \tag{3.1}$$

and u_1 the solution of the Poisson problem

$$\begin{aligned} \Delta_0 u_1 &= \Delta_1 u_0, \quad \text{in } D, \\ u_1\big|_{\partial D} &= g_1, \quad \text{on } \partial D. \end{aligned} \tag{3.2}$$

Since

$$\begin{aligned} \Delta_{\gamma_\epsilon}(u_0 + \epsilon u_1) &= (\Delta_0 - \epsilon\Delta_1)(u_0 + \epsilon u_1) \\ &= \Delta_0 u_0 + \epsilon(\Delta_0 u_1 - \Delta_1 u_0) - \epsilon^2 \Delta_1 u_1 \end{aligned}$$

by taking $h = g_0 + \epsilon g_1$, $w_\epsilon = u_0 + \epsilon u_1$ approximates the solution of (1.2). Now by choosing $g_0(\theta) = \cos(2\theta)$, the solution of (1.3) will be $u_0(\rho, \theta) = \rho^2 \cos(2\theta)$, hence

$$\begin{aligned} \Phi(\rho, \theta) = \Delta_1 u_0 &= \operatorname{div}(f' \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \nabla u_0) \\ &= (-8f'(\theta)\sin(2\theta) + 2f''(\theta)\cos(2\theta)) \end{aligned}$$

Taking $f(\theta) = \cos(4\theta)$ (see Figure 1), one gets $\Phi(\rho, \theta) = -32\cos(6\theta)$. Putting $g_1(\theta) = \cos(6\theta)$ in (2.3) its solution $u_1(\rho, \theta) = \rho^2 \cos(6\theta)$.

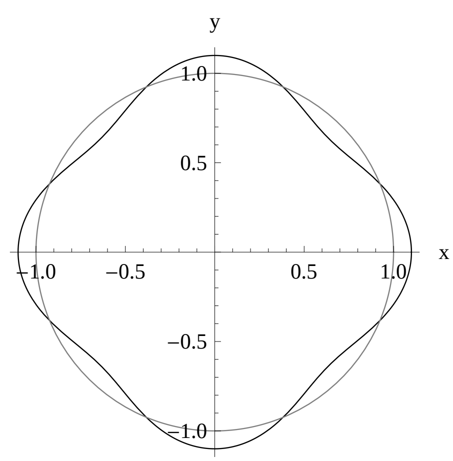

Figure 1. Ω_ϵ for $f(\theta) = \cos(4\theta)$ and $\epsilon = 0.1$

Hence for $f(\theta) = \cos(4\theta)$ and $h(\theta) = \cos(2\theta) + \epsilon\cos(6\theta)$, the function

$$w_\epsilon(\rho, \theta) = \rho^2 [\cos(2\theta) + \epsilon\cos(6\theta)] \tag{3.3}$$

approximates the solution of (1.2).

Since for $f(\theta) = \cos(4\theta)$,
$$e^{t\gamma_\epsilon} = e^t \begin{bmatrix} \cosh\psi(t) & \sinh\psi(t)) \\ \sinh\psi(t) & \cosh\psi(t) \end{bmatrix},$$
where $\psi(t) = 4\epsilon t\sin(4\theta)$. In this example the approximating family given by (2.4) can be constructed as
$$[V_\epsilon(t)h](1,\theta) = w_\epsilon\Big(e^{-t} \begin{bmatrix} \cosh\psi(t) & -\sinh\psi(t) \\ -\sinh\psi(t) & \cosh\psi(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\Big)$$
and by replacing in it in (3.3), we obtain
$$\begin{aligned} &[V_\epsilon(t)h](1,\theta) \\ &= w_\epsilon\left(e^{-t}(\cosh\psi(t)), e^{-t}(-\sinh\psi(t))\right) \\ &= \left(e^{-t}(\cosh\psi(t))\right)^2 \Big(\cos\Big[2e^{-t}(-\sinh\psi(t))\Big] + \epsilon\cos\Big[6e^{-t}(-\sinh\psi(t))\Big]\Big). \end{aligned} \tag{3.4}$$
Consequently,
$$\frac{d}{dt}[V_\epsilon(0)h](1,\theta) = -2(1+\epsilon)$$
which is exactly the directional derivative obtained by the following standard formula
$$[\Lambda_{\gamma_\epsilon}h](1,0) = -(1,0)\cdot \begin{bmatrix} 1 & 4\epsilon\sin 4\theta \\ 4\epsilon\sin 4\theta & 1 \end{bmatrix} \begin{bmatrix} 2\rho(\cos 2\theta + \epsilon\cos 6\theta) \\ \rho(-2\sin 2\theta - 6\epsilon\sin 6\theta) \end{bmatrix}_{(\rho,\theta)=(1,0)}.$$

This proves that $V_\epsilon(t)$ given by (3.4), can be used in Chernoff's Theorem in order to compute an approximating expression for the system (1.2).

4. Generalization to the perturbed unit sphere

Let
$$\Omega_\epsilon := \{(r,\theta,\phi) \in [0, 1+\epsilon f(\theta,\phi)] \times [0,2\pi) \times [-\pi/2, \pi/2)\}$$
be a perturbed unit sphere. Then the spherical coordinate the system (1.1) can be written as
$$\begin{gathered} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) = 0, \quad \text{in } \Gamma_\epsilon, \\ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} = 0, \quad \text{on } \partial\Gamma_\epsilon, \\ u(0) = g, \quad \text{on } \partial\Omega_\epsilon, \end{gathered} \tag{4.1}$$
where $\Gamma_\epsilon = [0,\infty) \times \Omega_\epsilon$. As in Section 2, we change variables:
$$\rho = \frac{r}{1+\epsilon f(\theta,\phi)}$$
where θ and ϕ remain invariant. Then the Laplacian in the new coordinates will be $\Delta_{\rho,\theta,\phi}$, which has a rather complicated expression, but by neglecting the terms of order ϵ^2, we transform $\Delta_{\rho,\theta,\phi}$ to $\Delta^{(\epsilon)}_{\rho,\theta,\phi}$. Our goal at this stage is to solve the problem,
$$\begin{gathered} \Delta^{(\epsilon)}_{\rho,\theta,\phi} v = 0, \quad \text{in } \Gamma, \\ \frac{\partial v}{\partial t} + \frac{\partial v}{\partial \rho} = 0, \quad \text{on } \partial\Gamma, \\ v(0) = g, \quad \text{on } \partial\Omega, \end{gathered} \tag{4.2}$$

where $\Gamma = [0, \infty) \times \Omega$, with Ω being the unit ball in $\mathbb{R}^3$. This system is still quite complicate to be solved; so by taking N discrete values of ϕ, $\{\phi_j := jh : j = 1, \dots, N\}$ with $h = \pi/N$, replacing $\frac{\partial v}{\partial \phi}$ by $(v(\rho, \theta, \phi_{j+1}) - v(\rho, \theta, \phi_j))/h$ and replacing $\frac{\partial^2 v}{\partial \phi^2}$ by $(v(\rho, \theta, \phi_{j+1}) - 2v(\rho, \theta, \phi_j) + v(\rho, \theta, \phi_{j-1}))/h^2$ we retrieve a system of N equations of (ρ, θ) parameters, each of which has an expression as we have already discussed in the preceding sections.

We postpone the numerical resolution of this system for a future article, with numerical illustrations.

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